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ONE-DIMENSIONAL LINEAR RECURSIONS WITH MARKOV-DEPENDENT COEFFICIENTS

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For a class of stationary Markov-dependent sequences $(A_n, B_n) \in \mathbb{R}^2$, we consider the random linear recursion $S_n = A_n + B_n S_{n-1}$, $n \in \mathbb{Z}$, and show that the distribution tail of its stationary solution has a power law decay.

1. Introduction and statement of results. Consider the stochastic difference equation

$$S_n = A_n + B_n S_{n-1}, \qquad n \in \mathbb{N}, S_n \in \mathbb{R}, \tag{1.1}$$

with real-valued random coefficients A_n and B_n.

If the sequence of random pairs $(A_n, B_n)_{n\in\mathbb{Z}}$ is stationary and ergodic, $E(\log|B_0|) < 0$, and $E(\log|A_0|^+) < \infty$, where $x^+ = \max(0, x)$, then for any initial random value S_0, the limit law of S_n is the same as that of the random variable $R = A_0 + \sum_{n=1}^{\infty} A_{-n} \prod_{i=0}^{n-1} B_{-i}$, and it is the unique initial distribution under which $(S_n)_{n\geq 0}$ is stationary (cf. [6]). Letting $\xi_n = A_{-n}$ and $\rho_n = B_{-n}$ for $n \in \mathbb{Z}$, we get

$$R = \xi_0 + \sum_{n=1}^{\infty} \xi_n \prod_{i=0}^{n-1} \rho_i. \tag{1.2}$$

The stochastic difference equation (1.1) has been studied by many authors and has a remarkable variety of applications (see, e.g., [8, 20, 23] for an extensive account). The distribution tail of the random variable R is the topic of, for example, [9–12, 14], all assuming that $(\rho_n, \xi_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence, and of [7], where it is assumed that $(\rho_n)_{n\in\mathbb{Z}}$ is a finite Markov chain.

We study here the asymptotic behavior of the distribution tail of R in the case that the sequence $(\zeta_n)_{n\in\mathbb{Z}} = (\xi_n, \rho_n)_{n\in\mathbb{Z}}$ is an "observable part" of a Markov-modulated process. By Markov-modulated process we mean the following:

DEFINITION 1.1. Let $(\mathcal{S}, \mathcal{T})$ be a measurable space and let $(x_n)_{n\in\mathbb{Z}}$ be a stationary Markov chain with transition kernel $H(x, \cdot)$ defined on it.

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A Markov-modulated process (MMP) associated with $(x_n)_{n\in\mathbb{Z}}$ is a stationary Markov chain $(x_n, \zeta_n)_{n\in\mathbb{Z}}$ defined on a product space $(\mathcal{S}\times\Upsilon, \mathcal{T}\otimes\Xi)$, whose transitions depend only on the position of (x_n). That is, for any $n\in\mathbb{Z}$, $A\in\mathcal{T}$, $B\in\Xi$,

$$P\big(x_n\in A, \zeta_n\in B|\sigma\big((x_i,\zeta_i): i<n\big)\big) = \int_A H(x,dy)\mathbb{G}(x,y,B)|_{x=x_{n-1}},$$

where $\mathbb{G}(x,y,\cdot) = P(\zeta_1\in\cdot|x_0=x, x_1=y)$ is a kernel on $(\mathcal{S}\times\mathcal{S}\times\Xi)$.

For MMP $(x_n,\zeta_n)_{n\in\mathbb{Z}}$, where $\zeta_n=(\xi_n,\rho_n)$, satisfying Assumption 1.2 below we show that for some $\kappa>0$, the limits $\lim_{t\to\infty} t^\kappa P(R>t)$ and $\lim_{t\to\infty} t^\kappa P(R<-t)$ exist and are not both zero. Under our assumption, the parameter κ is determined by

$$\Lambda(\kappa)=0, \qquad \text{where } \Lambda(\beta)=\lim_{n\to\infty}\frac{1}{n}\log E\left(\prod_{i=0}^{n-1}|\rho_i|^\beta\right). \tag{1.3}$$

This extends both the one-dimensional version of a result of Kesten which is valid for i.i.d. variables $(\xi_n,\rho_n)_{n\in\mathbb{Z}}$ (cf. [14], Theorem 5, see also an alternative approach developed by Goldie in [9]) as well as the recent result of de Saporta [7] where it is assumed that $(\rho_n)_{n\in\mathbb{Z}}$ is a finite irreducible and aperiodic Markov chain independent of the process $(\xi_n)_{n\in\mathbb{Z}}$ which is an i.i.d. sequence.

In the joint paper with Eddy Mayer–Wolf and Ofer Zeitouni [16], in the context of an application to random walks in random environments, we treated the particular case where $P(\xi_0=1,\rho_0>0)=1$ and $(\rho_n)_{n\in\mathbb{Z}}$ is a point-wise transformation of a stationary Markov chain $(x_n)_{n\in\mathbb{Z}}$ which is either finite-state and irreducible (possibly periodic) or such that some power of its transition kernel is dominated from above and below by a probability measure (and thus is aperiodic).

The general case is more involved and requires additional arguments to deal with it. We consider here the following Markov-modulated model where the coefficients $(\xi_n,\rho_n)_{n\in\mathbb{Z}}$ of the linear recursion (1.1) are not necessarily positive, the underlying Markov chain $(x_n)_{n\in\mathbb{Z}}$ is defined in a general state space and may be periodic, and the sequences $(\xi_n)_{n\in\mathbb{Z}}$ and $(\rho_n)_{n\in\mathbb{Z}}$ are not assumed to be independent.

Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}$.

ASSUMPTION 1.2. *There is a stationary Markov chain $(x_n)_{n\in\mathbb{Z}}$ on a measurable space $(\mathcal{S},\mathcal{T})$ with transition kernel $H(x,\cdot)$ such that $(x_n,\zeta_n)_{n\in\mathbb{Z}}$, where*

$$\zeta_n=(\xi_n,\rho_n), \qquad n\in\mathbb{Z},$$

is a MMP on $(\mathcal{S}\times\mathbb{R}^2, \mathcal{T}\times\mathcal{B}^{\otimes 2})$ and:

(A1) *The σ-field $\mathcal{T}$ is countably generated.*

(A2) *The kernel $H(x,\cdot)$ is irreducible, that is, there exists a σ-finite measure φ on $(\mathcal{S},\mathcal{T})$ such that for all $x\in\mathcal{S}$, $\sum_{n=1}^{\infty}H^n(x,A)>0$ whenever $\varphi(A)>0$.*

(A3) *There exist a probability measure μ on $(\mathcal{S},\mathcal{T})$, a number $m_1\in\mathbb{N}$, and a measurable density kernel $h(x,y):\mathcal{S}^2\to[0,\infty)$ such that*

$$H^{m_1}(x,A)=\int_A h(x,y)\mu(dy),$$

and the family of functions $\{h(x,\cdot):\mathcal{S}\to[0,\infty)\}_{x\in\mathcal{S}}$ is uniformly integrable with respect to the measure μ.

(A4) $P(|\xi_0|<c_\xi)=1$ *for some* $c_\xi>0$.

(A5) $P(c_\rho^{-1}<|\rho_0|<c_\rho)=1$ *for some* $c_\rho>1$.

(A6) *Let* $\Lambda(\beta)=\limsup_{n\to\infty}\frac{1}{n}\log E(\prod_{i=0}^{n-1}|\rho_i|^{\beta_1})$. *Then there exist constants* $\beta_1>0$ *and* $\beta_2>0$ *such that* $\Lambda(\beta_1)\geq 0$ *and* $\Lambda(\beta_2)<0$.

(A7) *There do not exist a constant $\alpha>0$ and a measurable function $\beta:\mathcal{S}\times\{-1,1\}\to[0,\alpha)$ such that*

$$P\big(\log|\rho_1|\in\beta(x_0,\eta)-\beta\big(x_1,\eta\cdot\operatorname{sign}(\rho_1)\big)+\alpha\cdot\mathbb{Z}\big)=1,$$

for $\eta\in\{-1,1\}$.

REMARK 1.3. The assumption that the sequence $(x_n,\zeta_n)_{n\in\mathbb{Z}}$ is stationary is explicit in Definition 1.1 of Markov-modulated processes. It turns out (see Lemma 2.1 below) that under assumptions (A1)–(A3), the Markov chain $(x_n)_{n\in\mathbb{Z}}$ has a unique stationary distribution. This distribution induces a (unique) stationary probability measure for the sequence (Markov chain) $(x_n,\zeta_n)_{n\in\mathbb{Z}}$, which we denote by P. The expectation according to the stationary measure P is denoted by E.

Note that condition (A6) implies by Jensen's inequality that $E(\log|\rho_0|)<0$. Thus, by a theorem of Brandt [6], the series in (1.2) converges absolutely, P-a.s. It will be shown later (see Lemma 2.3 below) that the both lim sup's in (A6) is in fact a limit, and thus this condition guarantees, by convexity, the existence of a unique κ in (1.3).

Assumption (A7) ensures that $\log|\rho_n|$ is nonarithmetic (in the sense of the following definition) relative to both the underlying process $(x_n)_{n\in\mathbb{Z}}$ as well as to the auxiliary chain $(\hat{x}_n)_{n\in\mathbb{Z}}$ introduced in Section 4.

DEFINITION 1.4 ([2, 22]). Let $(x_n,q_n)_{n\in\mathbb{Z}}$ be a MMP. The process $(q_n)_{n\in\mathbb{Z}}$ is said to be nonarithmetic relative to the Markov chain $(x_n)_{n\in\mathbb{Z}}$ if there do not exist a constant $\alpha>0$ and a measurable function $\beta:\mathcal{S}\to[0,\alpha)$ such that

$$P\big(q_0\in\beta(x_{-1})-\beta(x_0)+\alpha\cdot\mathbb{Z}\big)=1.$$

We will next state our results for the coefficients $(\xi_n, \rho_n)_{n\in\mathbb{Z}}$ satisfying Assumption 1.2. We will denote

$$P_x^-(\cdot) = P(\cdot|x_{-1} = x) \quad \text{and} \quad E_x^-(\cdot) = E(\cdot|x_{-1} = x), \tag{1.4}$$

and keep the notation $P_x(\cdot)$ and $E_x(\cdot)$ for $P(\cdot|x_0 = x)$ and $E(\cdot|x_0 = x)$, respectively.

The case of positive coefficients $(\xi_n, \rho_n)_{n\in\mathbb{Z}}$ is qualitatively different from and technically simpler than the general one [e.g., it turns out that in this case $\lim_{t\to\infty} t^\kappa P(R > t)$ is *always* positive], and it will be convenient to treat it separately.

THEOREM 1.5. *Let Assumption 1.2 hold and denote by π the stationary distribution of the Markov chain $(x_n)_{n\in\mathbb{Z}}$.*

If $P(\xi_0 > 0, \rho_0 > 0) = 1$ then for π-almost every $x \in \mathcal{S}$, the following limit exists and is strictly positive:

$$K(x) = \lim_{t\to\infty} t^\kappa P_x^-(R > t),$$

where the parameter κ is given by (1.3) *and the random variable R is defined in* (1.2).

An application of Theorem 1.5 and estimates (1.7), (1.8) to random walks in random environments can be found in [16]. The main step of the proof follows Goldie's argument (cf. [9], Theorem 2.3) closely and relies on the application of a version (due to Alsmeyer, cf. [2]) of the Markov renewal theorem due to Kesten (cf. [15], see also [4, 22] and references to related articles in [15]).

For coefficients $(\xi_n, \rho_n)_{n\in\mathbb{Z}}$ with arbitrary signs we have:

THEOREM 1.6. *Let Assumption 1.2 hold and denote by π the stationary distribution of the Markov chain $(x_n)_{n\in\mathbb{Z}}$.*

Then, with κ given by (1.3) *and R defined in* (1.2),

(a) *For π-almost every $x \in \mathcal{S}$, the following limits exist*:

$$K_1(x) = \lim_{t\to\infty} t^\kappa P_x(R > t) \quad \text{and} \quad K_{-1}(x) = \lim_{t\to\infty} t^\kappa P_x(R < -t). \tag{1.5}$$

(b) $\pi(K_1(x) + K_{-1}(x) > 0) \in \{0, 1\}$.

(c) *If Condition* G (*see Definition 1.7 below*) *is satisfied then it holds that $\pi(K_1(x) = K_{-1}(x)) = 1$. Moreover, if Condition* G *is not satisfied then either $\pi(K_1(x) > 0$ and $K_{-1}(x) > 0) \in \{0, 1\}$ or there exists a (possibly trivial) partition of $\mathcal{S}$ into two disjoint measurable sets A and B such that π-a.s., $K_1(x) > 0$ and $K_{-1}(x) = 0$ for $x \in A$ whereas $K_1(x) = 0$ and $K_{-1}(x) > 0$ for $x \in B$.*

DEFINITION 1.7. We say that Condition G holds if there does not exist a (possibly trivial) partition of $\mathcal{S}$ into two disjoint measurable sets A_1 and A_{-1} such that for $i \in \{-1, 1\}$,

$$P(x_0 \in A_i, x_1 \in A_{-i}, \rho_1 > 0) = P(x_0 \in A_i, x_1 \in A_i, \rho_1 < 0) = 0.$$

Condition G is a generalization of the condition of l-irreducibility introduced in [7]. Note that this condition is not satisfied if $P(\rho_0 > 0)$ (take $A_1 = \mathcal{S}$ and $A_{-1} = \varnothing$). Proposition 4.1 shows that Condition G is equivalent to the assertion that the Markov chain $\hat{x}_n = (x_n, \gamma_n)$, where $\gamma_n = \operatorname{sign}(\rho_0 \cdots \rho_{n-1})$, is irreducible under Assumption 1.2.

The proof of Theorem 1.6 is basically by applying a Markovian adaptation of the implicit renewal theory of Goldie [9] (see Section 3) to the Markov chain $\hat{x}_n$ and the random walk $V_n = \sum_{i=0}^{n-1} \log|\rho_i|$. The Markov chain $\hat{x}_n$ carries the necessary information about the sign of the products of ρ_i and at the same time, as we shall see in Section 4, inherits all essential properties of the Markov chain x_n.

In order to show that $K_1(x) + K_{-1}(x) > 0$ in Theorem 1.6, we need an extra nondegeneracy assumption which guarantees that the random variable R is not a deterministic function of the initial state x_{-1}. Again following [9] and using the renewal theory developed in [2], we complement Theorem 1.6 by the following necessary and sufficient condition for R to be nondegenerate under P_x^- and for the limit to be positive. This condition is a natural generalization of the criterion that appears in the case where the random variables (ξ_n, ρ_n) are i.i.d. (cf. [14] and [9]). Note that the condition is trivially satisfied under the assumptions of [7] [because $(\xi_n)_{n\in\mathbb{Z}}$ is assumed to be independent of $(x_n)_{n\in\mathbb{Z}}$].

THEOREM 1.8. *Let Assumption 1.2 hold and denote by π the stationary distribution of the Markov chain $(x_n)_{n\in\mathbb{Z}}$. Then*:

(a) $\pi(K_1(x) + K_{-1}(x) > 0) = 0$ *if and only if there exists a measurable function $\Gamma : \mathcal{S} \to \mathbb{R}$ such that*

$$P\big(\xi_0 + \Gamma(x_0)\rho_0 = \Gamma(x_{-1})\big) = 1. \tag{1.6}$$

(b) *There exists a constant $C_1 > 0$ such that for π-almost every $x \in \mathcal{S}$,*

$$t^\kappa P_x^-(|R| > t) \le C_1 \qquad \forall t > 0. \tag{1.7}$$

In particular, $\lim_{t\to\infty} t^\kappa P(R > t) = E(K_1(x_{-1}))$, $\lim_{t\to\infty} t^\kappa P(R < -t) = E(K_{-1}(x_{-1}))$, *and the limits are finite.*

(c) *If (1.6) does not hold for any measurable function $\Gamma : \mathcal{S} \to \mathbb{R}$, then there exist positive constants C_2 and t_c such that for π-almost every $x \in \mathcal{S}$,*

$$t^\kappa P_x^-(|R| > t) \ge C_2 \qquad \forall t > t_c. \tag{1.8}$$

In particular, $\lim_{t\to\infty} t^\kappa P(R > t)$ *and* $\lim_{t\to\infty} t^\kappa P(R < -t)$ *are not both zero.*

REMARK 1.9. Throughout this paper we work with the probability measures $P_x^-(\cdot) = P(\cdot|x_{-1} = x)$ defined in (1.4) rather than with $P_x(\cdot) = P(\cdot|x_0 = x)$. Since

$$P_x(R > t) = E\left(P_x^-\left(R > \frac{t-a}{b}\right)\Big|\xi_0 = a, \rho_0 = b\right),$$

the bounded convergence theorem and part (b) of Theorem 1.6 show that all our results hold also for the usual conditional measure P_x.

However, treating the linear recursion (1.1) in the setup of Markov-modulated processes, it is not so natural to work with the conditional probabilities P_x. In order to elucidate this point, let us consider the following two examples:

(i) The random variable R is conditionally independent of the "past," that is, of $\sigma((\xi_n, \rho_n)_{n<0})$, given x_{-1} but not given x_0.

(ii) Let $\tau > 0$ be a finite random time such that x_τ is distributed according to a probability measure ψ, and define $P_\psi(\cdot) := \int_{\mathcal{S}} P_x(\cdot)\psi(dx)$ and $R_\tau = \xi_\tau + \sum_{n=\tau+1}^{\infty} \xi_n \prod_{i=\tau}^{n-1} \rho_i$. Then in general, since the distribution of $x_{\tau-1}$ and hence that of (ξ_τ, ρ_τ) are unknown,

$$P(R_\tau \in \cdot) \neq P_\psi(R \in \cdot).$$

On the other hand, $P(R_{\tau+1} \in \cdot) = P_\psi^-(R \in \cdot)$ for $P_\psi^-(\cdot) := \int_{\mathcal{S}} P_x^-(\cdot)\psi(dx)$.

The rest of the paper is organized as follows. Section 2, divided into three subsections, is mostly devoted to the properties of the Markov chain $(x_n)_{n\in\mathbb{Z}}$ and of the random walk $V_n = \sum_{i=0}^{n-1} \log|\rho_i|$. Section 2.1 is devoted to the basic properties of the underlying Markov chain $(x_n)_{n\in\mathbb{Z}}$. In Section 2.2 we state a Perron–Frobenius type theorem (Proposition 2.4) which plays an important role in the subsequent proofs and in particular implies the existence and uniqueness of κ in (1.3) (see Lemma 2.3). The proof of Proposition 2.4 is deferred to the Appendix. Section 2.3 is devoted to the Markov renewal theory which is then used in Section 5, where it is applied to the Markov chain $\hat{x}_n = (x_n, \gamma_n)$ and the random walk V_n. Section 3 contains a reduction of Theorems 1.5 and 1.6 to a renewal theorem which is an adaptation of a particular case of Goldie's implicit renewal theorem (cf. [9]). Section 4 is devoted to study of the auxiliary Markov chain $\hat{x}_n$. The main goal here is to show that the renewal theorem obtained in Section 3 can be applied to the couple $(\hat{x}_n, V_n)$. The proofs of the main results (Theorems 1.5, 1.6 and 1.8) are then completed in Section 5.

2. Background and preliminaries. Similarly to the i.i.d. case (cf. [9] and [14]), the asymptotic behavior of the tail of R under Assumption 1.2 is determined by the properties of $V_n = \sum_{i=0}^{n-1} \log|\rho_i|$ and in particular is closely related to the renewal theory for this random walk. This section is devoted to the properties of the Markov chain $(x_n)_{n\in\mathbb{Z}}$ and of the associated random walk with Markov-dependent increments. The aim here is to provide for future use some

technical tools, namely the regeneration times N_i defined in Section 2.1 by the Athreya–Ney–Nummelin procedure, a Perron–Frobenius theorem for positive kernels stated in Section 2.2, and the Markov renewal theory recalled in Section 2.3.

2.1. *Some properties of the underlying Markov chain* $(x_n)_{n\in\mathbb{Z}}$. First, let us note that assumption (A3) implies that the transition kernel H is *quasi-compact*. Recall that a transition probability kernel $H(x,\cdot)$ on a measurable space $(\mathcal{X},\mathcal{T})$ is called quasi-compact if there exist constants $\varepsilon\in(0,1)$, $\delta\in(0,1)$, $m_1\in\mathbb{N}$, and a probability measure μ such that $H^{m_1}(x,A)<\varepsilon$ whenever $\mu(A)<\delta$, or alternatively, $H^{m_1}(x,A)>1-\varepsilon$ whenever $\mu(A)>1-\delta$. If a quasi-compact kernel H is the transition kernel of a Markov chain $(x_n)_{n\in\mathbb{Z}}$, then the chain is also called quasi-compact. The condition on transition kernels used in this definition was introduced by Doeblin (see, e.g., [24] for a historical account).

In the following lemma we summarize some properties of quasi-compact chains which will be useful in the sequel (see Theorem 3.7 in [21], Chapter 6, Section 3 for the first three assertions, Proposition 5.4.6 and Theorem 16.0.2 in [17] for the fourth, and Propositions 3.5, 3.6 in [21], Chapter 3, Section 3 for the last one).

LEMMA 2.1. *Let $(x_n)_{n\in\mathbb{Z}}$ be an irreducible quasi-compact Markov chain defined on a measurable space $(\mathcal{X},\mathcal{T})$. Then, there exist a number $d\in\mathbb{N}$ [the period of $(x_n)_{n\in\mathbb{Z}}$,] a sequence of d disjoint measurable sets $(\mathcal{X}_1,\mathcal{X}_2,\dots,\mathcal{X}_d)$ (a d-cycle), and probability measures π and ψ on $(\mathcal{X},\mathcal{T})$ such that*:

(i) *The following holds for all $i=1,\dots,d$, and $x\in\mathcal{X}_i$: $H(x,\mathcal{X}_j^c)=0$ for $j=i+1 \pmod d$.*

(ii) *π is the unique stationary distribution of (x_n), $\pi(\mathcal{X}_i)>0$ for $i=1,\dots,d$, and $\pi(\mathcal{X}_0)=1$, where $\mathcal{X}_0=\bigcup_{i=1}^d\mathcal{X}_i$.*

(iii) *$(x_n)_{n\in\mathbb{Z}}$ is Harris recurrent chain when restricted to the states of the set $\mathcal{X}_0$. That is, $P(x_n\in A \text{ i.o. for } n\ge 0 | x_0=x)=1$, for all $x\in\mathcal{X}_0$ and measurable $A\subseteq\mathcal{X}_0$ with $\pi(A)>0$.*

(iv) *$\psi(\mathcal{X}_1)=1$, and there exist constants $r\in(0,1)$ and $m\in\mathbb{N}$ such that*

$$H^m(x,A)>r\psi(A)\qquad \forall x\in\mathcal{X}_1,\ A\in\mathcal{T}. \tag{2.1}$$

(v) *The process $(x_n)_{n\in\mathbb{Z}}$ is ergodic under its stationary distribution.*

The minorization condition (2.1) with *some* recurrent set $\mathcal{X}_1$ is equivalent to the Harris recurrence (see, e.g., [18]). The particular form of the set $\mathcal{X}_1$ in (iv) as cyclic element is particularly advantageous and is due to the Doeblin condition.

We will next define a sequence of regeneration times $\{N_i\}_{i\ge 0}$ for the Markov chain $(x_n)_{n\in\mathbb{Z}}$ restricted to $(\mathcal{X}_0,\mathcal{T}_0)$, where $\mathcal{T}_0=\{A\in\mathcal{T}: A\subseteq\mathcal{X}_0\}$. Let the set $\mathcal{X}_1$ and the number m be the same as in (2.1), and let N_0 be the first hitting time of the set $\mathcal{X}_1$:

$$N_0=\inf\{n\ge -1 : x_n\in\mathcal{X}_1\}. \tag{2.2}$$

Note that $N_0 \le d-1$ and N_0 is a deterministic function of x_{-1} on the set $\mathcal{S}_0$. The randomized stopping times N_i, $i \ge 1$, can be defined in an enlarged (if needed) probability space by the following procedure (see [3, 5, 18]). Given a state $x_{N_0} \in \mathcal{S}_1$, generate x_{N_0+m} as follows: with probability r distribute x_{N_0+m} over $\mathcal{S}_0$ according to ψ and with probability $1-r$ according to $1/(1-r)\cdot\Theta(x_0,\cdot)$, where the substochastic kernel $\Theta(x,\cdot)$ is defined by

$$H^m(x,A) = \Theta(x,A) + r\mathbf{1}_{\mathcal{S}_1}(x)\psi(A), \qquad x \in \mathcal{S}_0, A \in \mathcal{T}_0. \tag{2.3}$$

Then, (unless $m=1$) sample the segment $(x_{N_0+1}, x_{N_0+2}, \dots, x_{N_0+m-1})$ according to the $(x_n)_{n\in\mathbb{Z}}$ chain's conditional distribution, given x_{N_0} and x_{N_0+m}. Generate x_{N_0+2m} and $x_{N_0+m+1}, x_{N_0+m+2}, \dots, x_{N_0+2m-1}$ in a similar way, and so on. Let $\{n_j\}_{j\ge1}$ be the successful times when the move of the chain $(x_{N_0+mn})_{n\ge0}$ is according to ψ, and set $N_j = N_0 + mn_j$, $j \ge 1$. Note that N_j is not the jth visit to $\mathcal{S}_1$.

By construction, the blocks $(x_{N_i+1}, x_{N_i+2}, \dots, x_{N_{i+1}})$ are one-dependent and for $i \ge 1$ they are identically distributed (x_{N_i}, $i \ge 1$, are independent and distributed according to the measure ψ). It follows from the construction that the random times $N_{i+1} - N_i$ are i.i.d. for $i \ge 0$, and that there exist constants $\vartheta \in \mathbb{N}$, $\delta > 0$, such that

$$P_x^-(N_1 \le \vartheta) > \delta \qquad \forall x \in \mathcal{S}_0. \tag{2.4}$$

We summarize the properties of the random times N_i in the following lemma.

LEMMA 2.2. *Let $(x_n)_{n\in\mathbb{Z}}$ be an irreducible quasi-compact Markov chain with state space $\mathcal{S}$, and let the set $\mathcal{S}_0$ be as in Lemma 2.1.*

Then there exists a strictly increasing sequence $(N_i)_{i\ge0}$ of random times such that:

(i) *$(N_{i+1} - N_i)_{i\ge0}$ are i.i.d.*

(ii) *The blocks $(x^{(0)}_{N_i+1}, \dots, x^{(0)}_{N_{i+1}})$ are one-dependent for $i \ge 0$ and identically distributed for $i \ge 1$, where $(x^{(0)}_n)_{n\in\mathbb{Z}}$ is the Markov chain induced by $(x_n)_{n\in\mathbb{Z}}$ on $(\mathcal{S}_0, \mathcal{T}_0)$.*

(iii) *$N_0 \le d-1, \forall x \in \mathcal{S}_0$, where d is the period of $(x_n)_{n\in\mathbb{Z}}$.*

(iv) *There exist constants $\vartheta \in \mathbb{N}$ and $\delta > 0$ such that (2.4) is satisfied.*

Throughout the rest of the paper we shall be concerned with the measurable space $(\mathcal{S}_0, \mathcal{T}_0)$, where $\mathcal{T}_0 = \{A \in \mathcal{T} : A \subseteq \mathcal{S}_0\}$, rather than with $(\mathcal{S}, \mathcal{T})$. Without loss of generality we may and shall assume that

$$P_x^-\big(|\xi_0| < c_\xi \text{ and } |\rho_0| \in (c_\rho, ^{-1}, c_\rho)\big) = 1 \qquad \forall x \in \mathcal{S}_0. \tag{2.5}$$

Otherwise we can restrict our attention to the Markov chain induced by $(x_n)_{n\in\mathbb{Z}}$ on the set of full measure π where the equality in (2.5) does hold. Clearly, Assumption 1.2 and Lemma 2.1 remain true for this Markov chain.

2.2. *A Perron–Frobenius theorem for positive bounded kernels.* The aim of this subsection is to state a Perron–Frobenius theorem for positive kernels (Proposition 2.4 below). Proposition 2.4 is an essential part of the subsequent proofs where it is applied to kernels of the form $K(x, A) = E_x^-(\prod_{i=0}^n |\rho_i|^\beta; x_n \in A)$ and $\widehat{\Theta}(x, A) = E_x^-(\mathbf{1}_{\{n<N_1\}} \prod_{i=0}^n |\rho_i|^\beta; x_n \in A)$, where the random time N_1 is defined in Section 2.1. The proof of Proposition 2.4 is deferred to Appendix A.

One immediate consequence of this proposition is the following lemma which proves the existence and uniqueness of the parameter κ in (1.3).

LEMMA 2.3. *Let Assumption 1.2 hold and let the set $\mathcal{S}_0$ be as defined in Lemma 2.1. Then,*

(a) *For any $\beta > 0$ and every $x \in \mathcal{S}_0$, the following limit exists and does not depend on x:*

$$\Lambda(\beta) = \lim_{n\to\infty} \frac{1}{n} \log E_x^- \left(\prod_{i=0}^{n-1} |\rho_i|^\beta \right). \tag{2.6}$$

Moreover, for some constants $c_\beta \geq 1$ that depend on β only,

$$c_\beta^{-1} e^{n\Lambda(\beta)} \leq E_x^- \left(\prod_{i=0}^{n-1} |\rho_i|^\beta \right) \leq c_\beta e^{n\Lambda(\beta)} \qquad \forall x \in \mathcal{S}_0, n \in \mathbb{N}. \tag{2.7}$$

(b) *There exists a unique $\kappa > 0$ such that $\Lambda(\kappa) = 0$, $\Lambda(\beta)(\beta - \kappa) \geq 0$ for all $\beta > 0$.*

We next proceed with Proposition 2.4, from which the lemma is derived at the end of this subsection.

A function $K : \mathcal{S}_0 \times \mathcal{T}_0 \to (0, \infty)$ is a *positive bounded kernel*, or simply *kernel*, if the following three conditions hold: (i) $K(\cdot, A)$ is a measurable function on $\mathcal{S}_0$ for all $A \in \mathcal{T}_0$, (ii) $K(x, \cdot)$ is a finite positive measure on $\mathcal{T}_0$ for all $x \in \mathcal{S}_0$, (iii) $\sup_{x\in\mathcal{S}_0} K(x, \mathcal{S}_0) < \infty$. Let B_b be the Banach space of bounded measurable real-valued functions on the measurable space $(\mathcal{S}_0, \mathcal{T}_0)$ with the norm $\|f\| = \sup_{x\in\mathcal{S}_0} |f(x)|$. Any positive bounded kernel $K(x, A)$ defines a bounded linear operator on B_b by setting $Kf(x) = \int_{\mathcal{S}_0} K(x, dy) f(y)$. We denote by r_K the spectral radius of the operator corresponding to the kernel K, that is

$$r_K = \lim_{n\to\infty} \sqrt[n]{\|K^n \mathbf{1}\|} = \lim_{n\to\infty} \sqrt[n]{\|K^n\|},$$

where $\mathbf{1}(x) \equiv 1$.

The following proposition generalizes Lemma 2.6 in [16] allowing us to deal with a more general class of underlying Markov chains $(x_n)_{n\in\mathbb{Z}}$.

PROPOSITION 2.4. *Let $K(x, \cdot)$ be a positive bounded kernel on $(\mathcal{S}_0, \mathcal{T}_0)$ and $s(x, y) : \mathcal{S}_0^2 \to \mathbb{R}$ be a measurable function such that $s(x, y) \in (c_1^{-1}, c_1)$ for some $c_1 > 1$ and all $(x, y) \in \mathcal{S}_0^2$. Assume that there exists a set $\mathcal{S}_1 \in \mathcal{T}_0$ such that:*

(i) *For some constants* $d \in \mathbb{N}$, $p > 0$,

$$\sum_{i=1}^{d} K^i(x, \mathcal{S}_1) \geq p \qquad \forall x \in \mathcal{S}_0.$$

(ii) *For some constant* $m \in \mathbb{N}$ *and probability measure* ψ *concentrated on* $\mathcal{S}_1$,

$$K^m(x, \mathcal{S}_1^c) = 0 \qquad \forall x \in \mathcal{S}_1,$$

where $\mathcal{S}_1^c$ *is the complement set of* $\mathcal{S}_1$, *and*

$$K^m(x, A) \geq \int_A s(x, y)\psi(dy) \qquad \forall x \in \mathcal{S}_1, A \in \mathcal{T}_0. \tag{2.8}$$

Further, assume that:

(iii) *There are a probability measure* μ *on* $(\mathcal{S}_0, \mathcal{T}_0)$ *and a constant* $m_1 \in \mathbb{N}$ *such that for all* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\mu(A) < \delta \qquad \textit{implies} \ \sup_{x \in \mathcal{S}_0} K^{m_1}(x, A) < \varepsilon. \tag{2.9}$$

[*This condition entails* $K(x, \cdot) \ll \mu$ *for all* $x \in \mathcal{S}_0$.]

Let $\mathcal{T}_1 = \{A \in \mathcal{T}_0 : A \subseteq \mathcal{S}_1\}$ *and let a kernel* $\widehat{\Theta}(x, A)$ *on* $(\mathcal{S}_1, \mathcal{T}_1)$ *be such that*

$$K^m(x, A) = \widehat{\Theta}(x, A) + r \int_A s(x, y)\psi(dy) \qquad \forall x \in \mathcal{S}_1, A \in \mathcal{T}_1,$$

for some $r \in (0, 1)$.

Then:

(a) *There exists a function* $f \in B_b$ *such that* $\inf_x f(x) > 0$ *and* $Kf = r_K f$.
(b) *There exists a function* $g \in B_b$ *such that* $\inf_x g(x) > 0$ *and* $\widehat{\Theta} g = r_{\widehat{\Theta}} g$.
(c) $r_{\widehat{\Theta}} \in (0, r_K^m)$.

The proof of the proposition is included in Appendix A.

PROOF OF LEMMA 2.3. Let $Q(x, y, B) = P(\rho_{-n} \in B | x_{n-1} = x, x_n = y)$, and for any $\beta \geq 0$ define the kernel $H_\beta(x, \cdot)$ on $(\mathcal{S}_0, \mathcal{T}_0)$ by

$$H_\beta(x, dy) = H(x, dy) \int_{\mathbb{R}} Q(x, y, dz)|z|^\beta. \tag{2.10}$$

Then for any $\beta \geq 0$,

$$E_x^- \left(\prod_{i=0}^{n-1} |\rho_i|^\beta \right) = H_\beta^n \mathbf{1}(x) \qquad \forall x \in \mathcal{S}_0. \tag{2.11}$$

The kernels H_β, $\beta \geq 0$, satisfy the conditions of Proposition 2.4. It follows from (A.2) with $K = H_\beta$ that for some constant $c_\beta \geq 1$ which depends on β only,

$$c_\beta^{-1} r_\beta^n \leq E_x^- \left(\prod_{i=0}^{n-1} |\rho_i|^\beta \right) \leq c_\beta r_\beta^n \qquad \forall x \in \mathcal{S}_0, n \in \mathbb{N}, \tag{2.12}$$

where $r_\beta = r_{H_\beta}$. This yields assertion (a) of the lemma. The claim of its part (b) follows then from the convexity of the function $\Lambda(\beta)$ which takes by assumption (A6) both positive and negative values. $\square$

2.3. *Markov renewal theory.* The proofs of our results rely on the use of the following version of the Markov renewal theorem which is due to Alsmeyer [2]. Recall Definition 1.1 of Markov-modulated processes and Definition 1.4 of nonarithmetic processes. Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}$ and let $(\mathcal{S}_0, \mathcal{T}_0)$ be a measurable space such that $\mathcal{T}$ is countably generated.

THEOREM 2.5. ([2], Theorem 1) *Let $(x_n)_{n\in\mathbb{Z}}$ be a Harris recurrent Markov chain on $(\mathcal{S}_0, \mathcal{T}_0)$ with stationary distribution π and let $(x_n, q_n)_{n\in\mathbb{Z}}$ be an associated with it MMP on $(\mathcal{S}_0 \times \mathbb{R}, \mathcal{T}_0 \times \mathcal{B})$ such that $\mu_0 := E(q_n) > 0$ and the process q_n is nonarithmetic relative to $(x_n)_{n\in\mathbb{Z}}$. Further, let $V_n = \sum_{i=0}^{n-1} q_i$ and let $g : \mathcal{S}_0 \times \mathbb{R} \to \mathbb{R}$ be any measurable function satisfying*

$$\text{for } \pi\text{-a.e. } z \in \mathcal{S}_0,\ g(z,\cdot) \text{ is Lebesgue-a.e. continuous}, \tag{2.13}$$

and

$$\int_{\mathcal{S}_0} \sum_{n\in\mathbb{Z}} \sup_{n\delta \le t < (n+1)\delta} |g(z,t)|\pi(dz) < \infty \qquad \text{for some } \delta > 0. \tag{2.14}$$

Then,

$$\lim_{t\to\infty} E_z^- \left(\sum_{n=0}^{\infty} g(x_{n-1}, t - V_n) \right) = \frac{1}{\mu_0} \int_{\mathcal{S}_0} \int_{\mathbb{R}} g(u,v)\, dv\, \pi(du), \tag{2.15}$$

for π-almost every $z \in \mathcal{S}_0$.

Under the assumptions of Theorem 2.5, let $\sigma_{-1} = -1$, $V_{-1} = 0$, and for $n \ge 0$, let $\sigma_n = \inf\{i > \sigma_{n-1} : V_i > V_{\sigma_{n-1}}\}$ be the ladder indexes of the random walk V_n. Set $\widetilde{V}_n = V_{\sigma_n}$. Further, for $n \ge 0$ let $\tilde{x}_n = x_{\sigma_n - 1}$ and $\tilde{q}_n = \widetilde{V}_n - \widetilde{V}_{n-1}$ ($\tilde{q}_0 = \sum_{i=0}^{\sigma_0 - 1} q_i$ and $\tilde{q}_n = \sum_{i=\sigma_{n-1}}^{\sigma_n - 1} q_n$ for $n \ge 1$). Denote by π_1 the unique stationary measure of the Markov chain $(\tilde{x}_n)_{n\ge 0}$ (existing by [2], Theorem 2) and by H_1 the transition kernel of $(\tilde{x}_n, \tilde{q}_n)_{n\ge 0}$.

For $t > 0$, set $\upsilon(t) = \inf\{n \ge 0 : V_n > t\}$, $Z(t) = x_{\upsilon(t)-1}$, and $W(t) = V_{\upsilon(t)} - t$. Note that $\upsilon(t)$ is a member of the sequence $(\sigma_n)_{n\ge 0}$.

COROLLARY 2.6 ([2], Corollary 2). *Let $(x_n, V_n)_{n\ge 0}$ be as in Theorem 2.5. Then, with $\mu_1 := \int_{\mathcal{S}_0} E_x^-(\tilde{q}_0)\pi_1(dx)$,*

$$\begin{aligned} &\lim_{t\to\infty} E_z^-(g(Z(t), W(t))) \\ &= \frac{1}{\mu_1} \int_{\mathcal{S}_0} \int_{\mathcal{S}_0 \times (0,\infty)} \int_{[0,s)} g(v,w)\, dw\, H_1(u, dv \times ds)\pi_1(du), \end{aligned}$$

holds for π_1-a.e. $z \in \mathcal{S}_0$ and for every measurable function $g : \mathcal{S}_0 \times [0, \infty) \to \mathbb{R}$ such that the function $b(z, y) := E_z^-(g(\tilde{x}_0, \tilde{q}_0 - y)\mathbf{1}_{\{\tilde{q}_0 > y\}})$ satisfies (2.13) and (2.14).

Theorem 2.5 will be applied in Section 5 to the underlying Markov chain $(x_n)_{n\in\mathbb{Z}}$ restricted to the space $(\mathcal{S}_0, \mathcal{T}_0)$ defined in Section 2.1 and to the random walk $V_n = \sum_{i=0}^{n-1} \log|\rho_i|$. In order to enable the application of the renewal theorem, we use a standard change of measure argument (involving a *similarity transform* of the transition kernel H) which defines a new stationary measure $\widetilde{P}$ for the MMP $(x_n, \zeta_n)_{n\in\mathbb{Z}}$ under which the Markov random walk $V_n = \sum_{i=0}^{n-1} \log|\rho_i|$ has positive drift, that is, the expectation $\widetilde{E}(\log|\rho_0|)$ with respect to $\widetilde{P}$ is strictly positive.

We next proceed with the construction of the measure $\widetilde{P}$. Observe that in virtue of Lemma 2.3, $r_\kappa = 1$, where r_κ is the spectral radius of the kernel H_κ on $(\mathcal{S}_0, \mathcal{T}_0)$ defined in (2.10). Therefore, by Proposition 2.4, there exists a positive measurable function $h(x) : \mathcal{S}_0 \to \mathbb{R}$ bounded away from zero and infinity such that

$$h(x) = \int_{\mathcal{S}_0} H_\kappa(x, dy)h(y). \tag{2.16}$$

Let $\zeta_n = (\xi_n, \rho_n)$, $n \in \mathbb{Z}$,

$$\widetilde{H}(x, dy) := \frac{1}{h(x)} H_\kappa(x, dy)h(y), \tag{2.17}$$

and let $\widetilde{P}$ be the stationary law of the Markov chain $(x_n, \zeta_n)_{n\in\mathbb{Z}}$ on $\mathcal{S} \times \mathbb{R}^2$ with transition kernel

$$\widetilde{P}\big(y_0 \in A \times B | \sigma(y_i : i < 0)\big) = \int_A \widetilde{H}(x, dz)\mathbb{G}(x, z, B)|_{x=x_{-1}},$$

where $A \in \mathcal{T}_0$, $B \in \mathcal{B}^{\otimes 2}$ and $\mathbb{G}(x, z, \cdot) = P(\zeta_n \in \cdot | x_{n-1} = x, x_n = z)$. That is, the law of $(\zeta_n)_{n\in\mathbb{Z}} = (\xi_n, \rho_n)_{n\in\mathbb{Z}}$ conditioned upon $(x_n)_{n\in\mathbb{Z}}$ is the same under P and $\widetilde{P}$, whereas the chain $(x_n)_{n\in\mathbb{Z}}$ has transition kernels H and $\widetilde{H}$, respectively. We will denote by $\widetilde{E}$ the expectation with respect to $\widetilde{P}$ and will use the notation

$$\widetilde{P}_x^-(\cdot) := \widetilde{P}(\cdot | x_{-1} = x) \quad \text{and} \quad \widetilde{P}_x(\cdot) := \widetilde{P}(\cdot | x_0 = x), \tag{2.18}$$

and, correspondingly, $\widetilde{E}_x^-(\cdot) := \widetilde{E}(\cdot | x_{-1} = x)$ and $\widetilde{E}_x(\cdot) := \widetilde{E}(\cdot | x_0 = x)$.

Let

$$c_h := \sup_{x,y\in\mathcal{S}_0} h(x)/h(y). \tag{2.19}$$

Since $c_h \in (0, \infty)$ and $c_h^{-1} H(x, A) \le \widetilde{H}(x, A) \le c_h H(x, A)$, we have:

- Conditions (A1)–(A3) of Assumption 1.2 hold for the kernel $\widetilde{H}$.
- The Markov chain $(x_n)_{n\in\mathbb{Z}}$ on $(\mathcal{S}_0, \mathcal{T}_0)$ with the kernel $\widetilde{H}$ is Harris recurrent and the minorization condition (2.1) holds in the following form:

$$\widetilde{H}^m(x, A) > r c_h^{-1} \psi(A) \qquad \forall x \in \mathcal{S}_1, A \in \mathcal{T}. \tag{2.20}$$

- The invariant measure π_h of the kernel $\widetilde{H}$ is equivalent to π (this follows, for example, from [18], Proposition 2.4).
- Assumptions (A7) and (2.5) hold for the sequence $(\xi_n, \rho_n)_{n\in\mathbb{Z}}$ under the measure $\widetilde{P}$.

LEMMA 2.7. *Let Assumption 1.2 hold. Then* $\widetilde{E}(\log|\rho_0|) > 0$.

PROOF. Let $V_0 = 0$ and

$$V_n = \sum_{i=0}^{n-1} \log|\rho_i|, \qquad n \in \mathbb{N}. \tag{2.21}$$

With c_h defined in (2.19) we obtain for any $x \in \mathcal{S}_0$ and $\gamma > 0$,

$$\begin{aligned} \widetilde{P}_x^-(e^{V_n} \le e^{-\gamma n^{1/4}}) &= \frac{1}{h(x)} E_x^-\big(e^{\kappa V_n} h(x_{n-1}); e^{V_n} \le e^{-\gamma n^{1/4}}\big) \\ &\le c_h E_x^-(e^{\kappa V_n}; e^{V_n} \le e^{-\gamma n^{1/4}}) \le c_h e^{-\kappa\gamma n^{1/4}}. \end{aligned} \tag{2.22}$$

Thus, $\lim_{n\to\infty} \widetilde{P}_x^-(V_n \le -\gamma n^{1/4}) = 0$, implying by the ergodic theorem that $\widetilde{E}(\log|\rho_0|) \ge 0$.

It remains to show that $\widetilde{E}(\log|\rho_0|) = 0$ is impossible. For any $x \in \mathcal{S}_0$, $\delta > 0$, and $\beta \in (0, \kappa)$ we get, using Chebyshev's inequality,

$$\begin{aligned} \widetilde{P}_x^-(|V_n| \le \delta n) &= \frac{1}{h(x)} E_x^-\big(e^{\kappa V_n} h(x_{n-1}); V_n \in [-\delta n, \delta n]\big) \\ &\le c_h e^{\kappa\delta n} P_x^-(V_n \ge -\delta n) \le c_h e^{(\kappa+\beta)\delta n} E_x^-\left(\prod_{i=0}^{n-1} |\rho_i|^\beta\right). \end{aligned}$$

It follows from Lemma 2.3 that for all $\delta > 0$ small enough and some suitable constants $A, b > 0$ that depend on δ,

$$\sup_{x\in\mathcal{S}_0} \widetilde{P}_x^-(|V_n| \le \delta n) \le Ae^{-bn}. \tag{2.23}$$

Therefore, the ergodic theorem implies that $\widetilde{E}(\log|\rho_0|) > 0$. □

3. Reduction to a renewal theorem. The main goal of this section is to prove the following Proposition 3.1 which reduces the limit problem for the tail of the random variable R to a renewal theorem [namely, to the checking that (3.3) below indeed holds a.s.]. Furthermore, some useful estimates are obtained here and collected in Lemma 3.2.

Let $\Pi_0 = 1$ and for $n \ge 1$,

$$\Pi_n = \prod_{i=0}^{n-1} \rho_i. \tag{3.1}$$

That is $\Pi_n = \gamma_{n-1} e^{V_n}$ where V_n is defined in (2.21) and

$$\gamma_n := \operatorname{sign}(\Pi_{n+1}), \qquad n \geq -1. \tag{3.2}$$

PROPOSITION 3.1. *Let Assumption 1.2 hold. Further, let the set $\mathcal{S}_0$ and the measure π be as in Lemma 2.1 and assume that for some $\eta \in \{-1, 1\}$ the following limit exists for π-almost every $z \in \mathcal{S}_0$:*

$$\widetilde{K}_\eta(z) := \lim_{t\to\infty} \widetilde{E}_z^- \left(\sum_{i=0}^{\infty} g_{\eta\gamma_{i-1}}(x_{i-1}, t - V_i) \right), \tag{3.3}$$

where the expectation is taken according to the measure $\widetilde{P}_z^-$ defined in (2.18) and the nonnegative functions $g_\gamma : \mathcal{S}_0 \times \mathbb{R} \to [0, \infty)$ are defined for $\gamma \in \{-1, 1\}$ by

$$g_\gamma(x, t) = \frac{e^{-t}}{h(x)} \int_0^{e^t} v^\kappa \left[P_x^-(\gamma R > v) - P_x^-\big(\gamma(R - \xi_0) > v\big) \right] dv. \tag{3.4}$$

Then, for π-almost every $z \in \mathcal{S}_0$, $\lim_{t\to\infty} t^\kappa P_z^-(\eta R > t) = h(z)\widetilde{K}_\eta(z)$.

We note that certain particular cases of this proposition are the basis for the proofs in [7] and in [16]. All these results are adaptations to various Markovian situations of a particular case of the "implicit renewal" theorem of Goldie (cf. Theorem 2.3 in [9]). For the sake of completeness, a proof of the proposition is provided at the end of this section.

We begin by proving the following technical lemma:

LEMMA 3.2. *Let Assumption 1.2 hold. Then the following assertions hold true*:

(a) *There exists constants $M_g > 0$ and $\varepsilon_g > 0$ such that for π-almost every $x \in \mathcal{S}_0$,*

$$|g_\eta(x, t)| \leq M_g e^{-\varepsilon_g |t|}, \tag{3.5}$$

for any $t \in \mathbb{R}$ and $\eta \in \{-1, 1\}$.

In particular, for any $\delta > 0$ there exists a constant $M(\delta) > 0$ such that

$$\sum_{n\in\mathbb{Z}} \sup_{n\delta \leq t < (n+1)\delta} \left\{ \max_{\eta\in\{-1,1\}} |g_\eta(x, t)| \right\} \leq M(\delta) \tag{3.6}$$

for π-almost every $x \in \mathcal{S}_0$.

(b) *For any $\delta > 0$ there exists a constant $M_u = M_u(\delta) > 0$ such that*

$$\sum_{i=0}^{\infty} \sup_{z\in\mathcal{S}_0} \widetilde{P}_z^-(V_i \in [-\delta, \delta]) \leq M_u.$$

(c) *There exists a constant $M_r > 0$ such that, for π-almost every $z \in \mathcal{S}_0$,*

$$\sum_{i=0}^{\infty} \widetilde{E}_z^- \Big(\max_{\eta \in \{-1,1\}} |g_\eta(x_{i-1}, t - V_i)| \Big) \le M_r \qquad \forall t \in \mathbb{R}. \tag{3.7}$$

PROOF. (a) First, assume that $t > 0$. Let

$$\tilde{c}_h = \max_{x \in S_0} 1/h(x). \tag{3.8}$$

For any $\varepsilon \in (0, 1)$, we get from (3.4):

$$\begin{aligned} |g_\eta(x,t)| &\le \tilde{c}_h e^{-t} \int_0^{e^t} v^\kappa \big| P_x^-(\eta R > v) - P_x^-\big(\eta(R - \xi_0) > v\big) \big| \, dv \\ &\le \tilde{c}_h e^{-\varepsilon t} \int_0^{e^t} v^{\kappa+\varepsilon-1} \big| P_x^-(\eta R > v) - P_x^-\big(\eta(R - \xi_0) > v\big) \big| \, dv \\ &\le \tilde{c}_h \kappa^{-1} e^{-\varepsilon t} E_x^- \big(|[(\eta R)^+]^{\kappa+\varepsilon} - [(\eta R - \eta\xi_0)^+]^{\kappa+\varepsilon}| \big), \end{aligned} \tag{3.9}$$

where the last inequality follows from [9], Lemma 9.4.

To bound the right-hand side in (3.9) we will exploit an argument similar to the proof of [9], Theorem 4.1. We have,

$$|g_\eta(x,t)| \le \tilde{c}_h \kappa^{-1} e^{-\varepsilon t} [I_1(x) + I_2(x) + I_3(x) + I_4(x)],$$

where

$$\begin{aligned} I_1(x) &:= E_x^- \big(\mathbf{1}_{\eta\xi_0 < \eta R \le 0} (\eta R - \eta\xi_0)^{\kappa+\varepsilon} \big), \\ I_2(x) &:= E_x^- \big(\mathbf{1}_{0 < \eta R \le \eta\xi_0} (\eta R)^{\kappa+\varepsilon} \big), \\ I_3(x) &:= E_x^- \big(\mathbf{1}_{\eta R > 0, \eta\xi_0 < 0} [(\eta R - \eta\xi_0)^{\kappa+\varepsilon} - (\eta R)^{\kappa+\varepsilon}] \big), \\ I_4(x) &:= E_x^- \big(\mathbf{1}_{0 \le \eta\xi_0 < \eta R} [(\eta R)^{\kappa+\varepsilon} - (\eta R - \eta\xi_0)^{\kappa+\varepsilon}] \big). \end{aligned}$$

It follows from (2.5) that the sum $I_1(x) + I_2(x)$ is bounded by $c_\xi^{\kappa+\varepsilon}$. It remains therefore to bound $I_3(x)$ and $I_4(x)$. For this purpose we will use the following inequalities valid for any $\gamma > 0$ and $A > 0$, $B > 0$ (this is exactly (9.26) and (9.27) in [9]):

$$(A + B)^\gamma \le 2^\gamma (A^\gamma + B^\gamma)$$

and

$$(A + B)^\gamma - A^\gamma \le \begin{cases} B^\gamma, & \text{if } 0 \le \gamma \le 1, \\ \gamma B (A + B)^{\gamma - 1}, & \text{if } \gamma > 1. \end{cases}$$

We obtain that $I_3(x) + I_4(x) \le a_\varepsilon$, where

$$a_\varepsilon := \begin{cases} c_\xi^{\kappa+\varepsilon}, & \text{if } \kappa + \varepsilon \le 1, \\ (\kappa + \varepsilon) c_\xi 2^{\kappa+\varepsilon-1} E_x(|R|^{\kappa+\varepsilon-1} + c_\xi^{\kappa+\varepsilon-1}), & \text{if } \kappa + \varepsilon > 1. \end{cases}$$

By Lemma 2.3 and the ellipticity condition (2.5), for any $\delta > 0$ small enough, there exists a constant $L_\delta > 0$ independent of x such that

$$E_x^-(|R|^{\kappa-\delta}) \le L_\delta \qquad \forall x \in \mathcal{S}_0. \tag{3.10}$$

This yields (3.5) for all $t > 0$ and appropriate constants $M_g, \varepsilon_g > 0$ that do not depend on t.

Further, (3.4) implies that $|g_\eta(x,0)| \le \tilde{c}_h$, where the constant $\tilde{c}_h$ is defined in (3.8), and that for $t < 0$ and any $\varepsilon \in (0,\kappa)$,

$$\begin{aligned}
|g_\eta(x,t)| &\le \tilde{c}_h e^{-t} \int_0^{e^t} v^\kappa \big| P_x^-(\eta R > v) - P_x^-\big(\eta(R-\xi_0) > v\big)\big|\, dv \\
&\le \tilde{c}_h e^{\varepsilon t} \int_0^{e^t} v^{\kappa-\varepsilon-1} \big| P_x^-(\eta R > v) - P_x^-\big(\eta(R-\xi_0) > v\big)\big|\, dv \\
&\le \tilde{c}_h \kappa^{-1} e^{\varepsilon t} E_x^-\big(|[(\eta R)^+]^{\kappa-\varepsilon} - [(\eta R - \eta\xi_0)^+]^{\kappa-\varepsilon}|\big),
\end{aligned}$$

where the last inequality follows, similarly to (3.9), from [9], Lemma 9.4. Thus, for $t < 0$,

$$|g_\eta(x,t)| \le \tilde{c}_h \kappa^{-1} e^{-\varepsilon|t|} E_x^-\big(|R|^{\kappa-\varepsilon} + (|R| + c_\xi)^{\kappa-\varepsilon}\big).$$

This completes the proof in view of (3.10).

(b) Follows from (2.23), since $\widetilde{P}_x^-(V_i \in [-\delta,\delta]) \le \widetilde{P}_x^-(V_i \in [-i\delta, i\delta])$ for any $x \in \mathcal{S}_0$ and $i \in \mathbb{N}$.

(c) Fix any $\delta > 0$ and denote for $t \in \mathbb{R}$ and $n \in \mathbb{Z}$, $I_{n,\delta}^t = [t + n\delta, t + (n+1)\delta)$. Then, it follows from the previous parts of the lemma that

$$\begin{aligned}
&\sum_{i=0}^\infty \widetilde{E}_z^-\big(|g_\eta(x_{i-1}, t - V_i)|\big) \\
&\quad = \sum_{i=0}^\infty \int_{\mathcal{S}_0} \int_{\mathbb{R}} |g_\eta(x, t-s)| \widetilde{P}_z^-(x_{i-1} \in dx, V_i \in ds) \\
&\quad \le \sum_{n\in\mathbb{Z}} \sup_{x\in\mathcal{S}_0, s\in I_{n,\delta}^t} |g_\eta(x,t-s)| \sum_{i=0}^\infty \sup_{z\in\mathcal{S}_0} \widetilde{P}_z^-(V_i \in t - I_{n,\delta}^s) \\
&\quad \le M(\delta) \sum_{i=0}^\infty \sup_{z\in\mathcal{S}_0} \widetilde{P}_z^-(V_i \in [-\delta,\delta]) \le M(\delta)\cdot M_u,
\end{aligned}$$

where the last but one inequality follows from (3.6) and from the fact that $\sup_{z\in\mathcal{S}_0} \sum_{i=0}^\infty \widetilde{P}_z^-(V_i \in t - I_{n,\delta}^s) \le \sup_{z\in\mathcal{S}_0} \sum_{i=0}^\infty \widetilde{P}_z^-(V_i \in [-\delta,\delta])$ (cf. [2], Lemma A.2). □

PROOF OF PROPOSITION 3.1. Let $U_0 = R$, and for $n \ge 1$,

$$R_n = \sum_{i=0}^{n-1} \xi_i \Pi_i, \qquad U_n = (R - R_n)/\Pi_n, \tag{3.11}$$

where Π_n are defined in (3.1). Recall that $\Pi_n = \gamma_{n-1} e^{V_n}$. Following Goldie [9], we write for any numbers $n \in \mathbb{N}$, $t \in \mathbb{R}$, $\eta \in \{-1, 1\}$ and any $z \in \mathcal{S}_0$,

$$P_z^-(\eta R > e^t) = \sum_{i=0}^{n-1} [P_z^-(\eta \gamma_{i-1} e^{V_i} U_i > e^t) - P_z^-(\eta \gamma_i e^{V_{i+1}} U_{i+1} > e^t)] + P_z^-(\eta \Pi_n U_n > e^t),$$

where the random variable U_i is defined in (3.11).

For $n \geq -1$ let $\hat{x}_n = (x_n, \gamma_n)$ and $\Omega = \mathcal{S} \times \{-1, 1\} \times \mathbb{R}$. To shorten the notation, we denote $\sum_{\gamma \in \{-1,1\}} \int_{\mathcal{S}} \int_{\mathbb{R}} F(\gamma, x, u) \mu(\gamma_{i-1} = \gamma, x_i \in dx, V_i \in du)$ by $\int_\Omega F(\gamma, x, u) \mu(\hat{x}_i \in (dx, \gamma), V_i \in du)$ for a measurable function F and a probability measure μ on Ω. We have, using the identity $U_i = \xi_i + \rho_i U_{i+1}$,

$$\begin{aligned}
&P_z^-(\eta \gamma_{i-1} e^{V_i} U_i > e^t) - P_z(\eta \gamma_i e^{V_{i+1}} U_{i+1} > e^t) \\
&\quad = \int_\Omega P\big(\eta \gamma U_i > e^{t-u} | \hat{x}_{i-1} = (x, \gamma), V_i = u\big) P_z^-\big(\hat{x}_{i-1} \in (dx, \gamma), V_i \in du\big) \\
&\qquad - \int_\Omega P\big(\eta \gamma \rho_i U_{i+1} > e^{t-u} | \hat{x}_{i-1} = (x, \gamma), V_i = u\big) \\
&\qquad\qquad \times P_z^-\big(\hat{x}_{i-1} \in (dx, \gamma), V_i \in du\big) \\
&\quad = \int_\Omega e^{-\kappa(t-u)} f_{\eta\gamma}(x, t-u) P_z^-\big(\hat{x}_{i-1} \in (dx, \gamma), V_i \in du\big),
\end{aligned}$$

where we denote

$$f_\gamma(x, t) = e^{\kappa t}\big[P_x^-(\gamma R > e^t) - P_x^-\big(\gamma(R - \xi_0) > e^t\big)\big] \qquad \text{for } \gamma \in \{-1, 1\}.$$

Thus, letting $\delta_n(z, \eta, t) = e^{\kappa t} P_z^-(\eta \gamma_{n-1} e^{V_n} U_n > e^t)$ we obtain

$$\begin{aligned}
\check{r}_z(\eta, t) &:= e^{\kappa t} P_z^-(\eta R > e^t) \\
&= \sum_{i=0}^{\infty} \int_\Omega f_{\eta\gamma}(x, t-u) e^{\kappa u} P_z^-\big(\hat{x}_{i-1} \in (dx, \gamma), V_i \in du\big) + \delta_n(z, \eta, t) \\
&= \sum_{i=0}^{n-1} \int_\Omega f_{\eta\gamma}(x, t-u) \frac{h(z)}{h(x)} \widetilde{P}_z^-\big(x_{i-1} \in (dx, \gamma), V_i \in du\big) + \delta_n(z, \eta, t).
\end{aligned}$$

We have $P(\lim_{n\to\infty} \delta_n(z, \eta, t) = 0) = 1$ for any fixed $t > 0$, $\eta \in \{-1, 1\}$, and $z \in \mathcal{S}_0$, because P-a.s., $\Pi_n U_n \to 0$ as n goes to infinity. Therefore P-a.s.,

$$r_z(\eta, t) = \sum_{i=0}^{\infty} \int_\Omega f_{\eta\gamma}(x, t-u) \frac{h(z)}{h(x)} \widetilde{P}_z^-\big(\hat{x}_{i-1} \in (dx, \gamma), V_i \in du\big).$$

We will use the following Tauberian lemma:

LEMMA 3.3 ([9], Lemma 9.3). *Let R be a random variable such that for some constants $\kappa > 0$ and $K \geq 0$, $\lim_{t\to\infty} t^{-1} \int_0^t u^\kappa P(R > u)\,du = K$. Then $\lim_{t\to\infty} t^\kappa P(R > t) = K$.*

It follows from Lemma 3.3 that in order to prove that for some $\eta \in \{-1, 1\}$, the limit $\lim_{t\to\infty} t^\kappa P(\eta R > t)$ exists and is strictly positive, it suffices to show that for π-a.s. every $z \in \mathcal{S}_0$, there exists

$$\lim_{t\to\infty} \check{r}_z(\eta, t) \in (0, \infty), \tag{3.12}$$

where the smoothing transform $\check{q}$ is defined for a measurable function $q : \mathbb{R} \to \mathbb{R}$ bounded on $(-\infty, t]$ for all t by $\check{q}(t) := \int_{-\infty}^t e^{-(t-u)} q(u)\,du$.

For $\gamma \in \{-1, 1\}$ let

$$\begin{aligned} g_\gamma(x, t) &:= \frac{1}{h(x)} \int_{-\infty}^t e^{-(t-u)} f_\gamma(x, u)\,du \\ &= \frac{1}{h(x)} \int_{-\infty}^t e^{-(t-u)} e^{\kappa u} \big[P_x^-(\gamma R > e^u) - P_x^-\big(\gamma(R - \xi_0) > e^u\big)\big]\,du \\ &= \frac{e^{-t}}{h(x)} \int_0^{e^t} v^\kappa \big[P_x^-(\gamma R > v) - P_x^-\big(\gamma(R - \xi_0) > v\big)\big]\,dv. \end{aligned}$$

Then, using (3.7) and the Fubini theorem, we obtain for any $z \in \mathcal{S}_0$,

$$\begin{aligned} \check{r}_z(\eta, t) &= \int_{-\infty}^t e^{-(t-w)} r_z(\eta, w)\,dw \\ &= \int_{-\infty}^t e^{-(t-w)} \sum_{i=0}^\infty \int_\Omega f_{\eta\gamma}(x, w-u) \frac{h(z)}{h(x)} \widetilde{P}_z^-\big(\hat{x}_{i-1} \in (dx, \gamma), V_i \in du\big)\,dw \\ &= \sum_{i=0}^\infty \int_\Omega g_{\eta\gamma}(x, t-u) h(z) \widetilde{P}_z^-\big(\hat{x}_{i-1} \in (dx, \gamma), V_i \in du\big) \\ &= h(z) \widetilde{E}_z^- \left(\sum_{i=0}^\infty g_{\eta\gamma_{i-1}}(x_{i-1}, t - V_i) \right). \end{aligned}$$

This completes the proof of Proposition 3.1. □

4. The auxiliary Markov chain $\hat{x}_n = (x_n, \gamma_n)$. To deal with the case where $P(\rho_0 < 0) > 0$ we introduce the Markov chain $\hat{x}_n = (x_n, \gamma_n)$, $n \geq -1$, where the random variables γ_n are defined in (3.2). It will turn out (cf. Proposition 4.1) that the space $\mathcal{S}_0 \times \{-1, 1\}$ can be partitioned into at most two measurable subsets such that the restriction of $\hat{x}_n$ to either one of them satisfies Assumption 1.2. Therefore, the Markov renewal theorem (Theorem 2.5) can be applied to the irreducible com-

ponents of the MMP $(\hat{x}_n, \log|\rho_n|)$. This fact is the key to the proof (given in the next section) that the limit in (3.3) exists π-a.s. and has the properties stated in Theorem 1.6.

Let $\widehat{H}$ be the transition kernel of $\hat{x}_n$ on the product space $\mathfrak{S} := \mathcal{S}_0 \times \{-1, 1\}$, and let $\hat{\pi}$ be the probability measure on $\mathfrak{S}$ defined by $\hat{\pi}(A \times \eta) = 1/2\pi(A)$ for any $\eta \in \{-1, 1\}$ and $A \in \mathcal{T}_0$. It is easy to see that $\hat{\pi}$ is a stationary distribution of the Markov chain $\hat{x}_n$.

PROPOSITION 4.1. *Let Assumption 1.2 hold and suppose in addition that* $P(\rho_0 < 0) > 0$. *Then, there exist two disjoint measurable subsets* $\mathfrak{S}_1$ *and* $\mathfrak{S}_{-1}$ *of* $\mathfrak{S}$ *such that*:

(i) *Either* $\hat{\pi}(\mathfrak{S}_1) = \hat{\pi}(\mathfrak{S}_{-1}) = 1/2$, *or* $\mathfrak{S}_1 = \varnothing$ *and* $\mathfrak{S}_{-1} = \mathfrak{S}$.
(ii) $\widehat{H}(\hat{x}, \mathfrak{S}_n) = 1$ *for every* $\hat{x} \in \mathfrak{S}_n$, $n = -1, 1$.
(iii) $\mathfrak{S}_1 = \varnothing$ *if and only if Condition* G *is satisfied*.
(iv) (A1)–(A3) *of Assumption 1.2 hold for the Markov chain* $(\hat{x}_n)_{n \geq -1}$ *restricted to either* $\mathfrak{S}_1$ (*provided that it is not the empty set*) *or* $\mathfrak{S}_{-1}$.

PROOF. (i)–(ii) Say that for $\hat{x} \in \mathfrak{S}$, $A \in \mathcal{T}_0$, $\gamma \in \{-1, 1\}$,

$$\hat{x} \not\succ A \times \{\gamma\} \qquad \text{if } \sum_{n=1}^{\infty} \widehat{H}^n(\hat{x}, A \times \{\gamma\}) = 0,$$

and $\hat{x} \succ A \times \{\gamma\}$ otherwise.

Since the Markov chain $(x_n)_{n \in \mathbb{Z}}$ is π-irreducible, for any $\hat{x} \in \mathfrak{S}$ and $A \in \mathcal{T}_0$ such that $\pi(A) > 0$ either $\hat{x} \succ A \times \{1\}$ or $\hat{x} \succ A \times \{-1\}$. For $\hat{x} \in \mathfrak{S}$ and $\eta \in \{-1, 1\}$ let:

$$\mathcal{F}_\eta(\hat{x}) = \big\{A \in \mathcal{T}_0 : \pi(A) > 0 \text{ and } \hat{x} \not\succ A \times \{\eta\}\big\},$$

and set $\mathcal{F}(\hat{x}) = \mathcal{F}_1(\hat{x}) \cup \mathcal{F}_{-1}(\hat{x})$. Note that $\mathcal{F}_1(\hat{x}) \cap \mathcal{F}_{-1}(\hat{x}) = \varnothing$.

Roughly speaking, the set $\mathfrak{S}_1$ is defined below as an element of $\mathcal{F}(x^*)$ of maximal $\hat{\pi}$-measure for some $x^* \in \mathfrak{S}$, and $\mathfrak{S}_{-1}$ as its complement in $\mathfrak{S}$.

To be precise, let

$$\varsigma_\eta(\hat{x}) = \sup\{\pi(A) : A \in \mathcal{F}_\eta(\hat{x})\}, \qquad \eta \in \{-1, 1\}, \hat{x} \in \mathfrak{S},$$

and $\varsigma(\hat{x}) = \varsigma_{-1}(\hat{x}) + \varsigma_1(\hat{x})$. If $\varsigma(\hat{x}) = 0$ for every $\hat{x} \in \mathfrak{S}$, set $\mathfrak{S}_1 = \varnothing$ and $\mathfrak{S}_{-1} = \mathfrak{S}$. Conclusions (i)–(ii) follow trivially in this case, in particular the chain (x_n, γ_n) is $\hat{\pi}$-irreducible.

Assume now that $\varsigma(x^*) > 0$ for some $x^* \in \mathfrak{S}$. We will next construct two sets A_η, $\eta \in \{-1, 1\}$, such that $A_\eta \in \mathcal{F}_\eta(x^*)$ and $\pi(A_\eta) = \varsigma_\eta(x^*)$. We will then show that $\varsigma(x^*) = \pi(A_1) + \pi(A_{-1}) = 1$ and will define (up to a $\hat{\pi}$-null set) $\mathfrak{S}_1 := (A_{-1} \times \{-1\}) \cup (A_1 \times \{1\})$.

For $\eta \in \{-1, 1\}$, let $A_{\eta,n} \in \mathcal{F}_\eta(x^*)$, $n \in \mathbb{N}$, be a sequence of [empty if $\varsigma_\eta(x^*) = 0$] sets in $\mathcal{F}_\eta(x^*)$ such that $\pi(A_{\eta,n}) > \varsigma_\eta(x^*) - 1/n$ for any $n \in \mathbb{N}$, and define $A_\eta = \bigcup_{n=1}^{\infty} A_{\eta,n}$. Since the collections of sets $\mathcal{F}_\eta(x^*)$ are closed with

respect to countable unions, $A_\eta \in F_\eta(x^*)$ and $\pi(A_1) + \pi(A_{-1}) = \varsigma(x^*)$. Put $A_0 = A_{-1} \cup A_1$, $B_0 = S_0 - A_0$, and set

$$\widetilde{\mathfrak{S}}_1 = (A_{-1} \times \{-1\}) \cup (A_1 \times \{1\}),$$
$$\widetilde{\mathfrak{S}}_{-1} = (A_{-1} \times \{1\}) \cup (A_1 \times \{-1\}) \cup (B_0 \times \{1\}) \cup (B_0 \times \{-1\}).$$

Thus, $\widetilde{\mathfrak{S}}_{-1}$ is the complement of $\widetilde{\mathfrak{S}}_1$ in the set $\mathfrak{S} = \mathcal{S}_0 \times \{-1, 1\}$. Since $A_\eta \in F_\eta(x^*)$ is the maximal set such that $x^* \not\succ A_\eta \times \{\eta\}$, it follows immediately that $x^* \not\succ A \times \{\eta\}$ and $x^* \succ A \times \{-\eta\}$ for any π-positive $A \subset A_\eta$.

We will now show, using the irreducibility of the Markov chain $(x_n)_{n\in\mathbb{Z}}$, that

$$\hat{\pi}(N_{-1} \cup N_1) = 0 \qquad \text{where } N_\eta := \{\hat{x} \in \widetilde{\mathfrak{S}}_{-1} : \hat{x} \succ A_\eta \times \{\eta\}\}. \tag{4.1}$$

Note that (4.1) yields $\pi(B_0) = 0$ because for all $x \in \mathcal{S}_0$ either $(x, 1) \in N_\eta$ or $(x, -1) \in N_\eta$, $\eta \in \{-1, 1\}$.

To see that (4.1) is true, observe that $N_\eta = \bigcup_{m\in\mathbb{N}}\{\hat{x} : \widehat{H}^m(\hat{x}, A_\eta \times \{\eta\}) > 0\}$ are measurable sets, and $\hat{\pi}(N_\eta) > 0$ implies that there exist $m \in \mathbb{N}$, $N_0 \in \mathcal{T}_0$, and $\gamma \in \{-1, 1\}$ such that

$$\widehat{H}^m(x^*, N_0 \times \{\gamma\}) > 0 \quad \text{and} \quad (x, \gamma) \succ A_\eta \times \{\eta\} \qquad \forall x \in N_0. \tag{4.2}$$

But (4.2) yields

$$\sum_{n=0}^{\infty} \widehat{H}^{m+n}(x^*, A_\eta \times \{\eta\})$$
$$\geq \sum_{n=0}^{\infty} \int_{N_0} \widehat{H}^m(x^*, dy \times \{\gamma\}) \widehat{H}^n\big((y, \gamma), A_\eta \times \{\eta\}\big) > 0,$$

which is impossible since $x^* \not\succ A_\eta \times \{\eta\}$ by our construction.

Finally, we observe that (4.1) implies that

$$\hat{\pi}(\overline{N}_{-1} \cup \overline{N}_1) = 0 \qquad \text{where } \overline{N}_\eta := \{\hat{x} \in \widetilde{\mathfrak{S}}_1 : \hat{x} \succ A_\eta \times \{-\eta\}\}. \tag{4.3}$$

Indeed, if $(x, \gamma) \in \overline{N}_\eta$ then $(x, -\gamma) \in N_\eta$ and hence $\hat{\pi}(\overline{N}_\eta) = \hat{\pi}(N_\eta) = 0$ for $\eta \in \{-1, 1\}$.

To complete the proof, we set

$$\mathfrak{S}_1 = (A_{-1} \times \{-1\}) \cup (A_1 \times \{1\}) - \overline{N}_{-1} \cup \overline{N}_1,$$

and

$$\mathfrak{S}_{-1} = (A_{-1} \times \{1\}) \cup (A_1 \times \{-1\}) - N_{-1} \cup N_1.$$

Since $\pi(B_0) = 0$, (4.1) and (4.3) imply that $\hat{\pi}(\mathfrak{S}_1) = \hat{\pi}(\mathfrak{S}_{-1}) = 1/2$ [recall that $\pi(A_1 \cap A_{-1}) = 0$] and that conclusion (ii) of the proposition holds as well.

(iii) The claim is immediate from the definition of the sets A_1 and A_{-1}.

(iv) Let $\hat{\mu}$ be the probability measure on $\mathfrak{S}$ defined by $\hat{\mu}(A \times \eta) = 1/2\mu(A)$, where $\mu(\cdot)$ is given by assumption (A3). Since $\widehat{H}^{m_1}((x, \gamma), A \times \{\eta\}) \leq$

$H^{m_1}(x, A)$, it follows from (A3) that there exists a measurable density kernel $\hat{h}(\hat{x}, \hat{y}) : \mathfrak{S}^2 \to [0, \infty)$ such that or any $\hat{x} \in \mathfrak{S}$, $\eta\{-1, 1\}$, $A \in \mathcal{T}_0$,

$$\widehat{H}^{m_1}(\hat{x}, A \times \{\eta\}) = \int_{A \times \{\eta\}} \hat{h}(\hat{x}, \hat{y}) \hat{\mu}(d\hat{y}), \tag{4.4}$$

and the family of functions $\{\hat{h}(\hat{x}, \cdot) : \mathfrak{S} \to [0, \infty)\}_{\hat{x} \in \mathfrak{S}}$ is uniformly integrable with respect to the measure $\hat{\mu}$. Thus assumptions (A1) and (A3) hold for the Markov chain $(x_n, \gamma_n)_{n \geq -1}$. Moreover, the Markov chain $(x_n, \gamma_n)_{n \geq -1}$, when restricted to either $\mathfrak{S}_1$ or $\mathfrak{S}_{-1}$, is clearly $\hat{\pi}$-irreducible which in combination with (4.4) shows (iv). □

5. Distribution tail of R. In this section we complete the proof of Theorems 1.5, 1.6 and 1.8.

5.1. *Proofs of Theorems 1.5 and 1.6 for $P(\rho_0 > 0) = 1$.* In view of Proposition 3.1, the following lemma completes the proof of Theorem 1.5 and of Theorem 1.6 in the case where $P(\rho_0 > 0) = 1$.

LEMMA 5.1. *Let Assumption 1.2 hold and suppose that $P(\rho_0 > 0) = 1$. Then the following assertions hold true for $\eta \in \{-1, 1\}$:*

(a) *The limit in (3.3) exists for π-a.e. $z \in \mathcal{S}_0$ and does not depend on z.*
(b) *If in addition $P(\xi_0 > 0) = 1$, then the limit is π-a.s. strictly positive.*
(c) $\pi(K_\eta(x) > 0) \in \{0, 1\}$.

PROOF. (a) In view of Lemma 2.7, estimate (3.6), and the properties of the measure $\widetilde{P}$ listed right before the statement of Lemma 2.7, we can apply Theorem 2.5 to the restriction of the underlying Markov chain $(x_n)_{n \in \mathbb{Z}}$ on $(\mathcal{S}_0, \mathcal{T}_0)$ with transition kernel $\widetilde{H}$, the associated with it random walk $V_n = \sum_{i=0}^{n-1} \log |\rho_i|$, and the functions g_η defined in (3.4). It follows from (2.15) that the limit in (3.3) is π_h-a.s. (and thus also π-a.s.) equal to

$$\widetilde{K}_\eta = \frac{1}{\tilde{a}} \int_{\mathcal{S}_0} \int_{\mathbb{R}} g_\eta(x, t) \pi_h(dx)\, dt, \tag{5.1}$$

where $\tilde{a} = \widetilde{E}(\log \rho_0)$.

(b) It follows from Proposition 3.1 and (5.1) that for π_h-almost every $z \in \mathcal{S}_0$ (compare with the formula (4.3) in [9]),

$$\begin{aligned} &\lim_{t \to \infty} t^\kappa P_z^-(R > t) \\ &\quad = h(z) \widetilde{K}_1(z) \\ &\quad = \frac{h(z)}{\tilde{a}} \int_{\mathcal{S}_0} \int_{\mathbb{R}} g_1(x, t) \pi_h(dx)\, dt \end{aligned} \tag{5.2}$$

$$= \frac{h(z)}{\tilde{a}} \int_{\mathcal{S}} \frac{1}{h(x)} \int_{\mathbb{R}} e^{-t} \int_0^{e^t} v^\kappa [P_x^-(R > v) - P_x^-(R - \xi_0 > v)]\, dv\, dt\, \pi_h(dx)$$

$$= \frac{h(z)}{\tilde{a}} \int_{\mathcal{S}} \frac{1}{h(x)} \int_0^\infty v^{\kappa-1} [P_x^-(R > v) - P_x^-(R - \xi_0 > v)]\, dv\, \pi_h(dx)$$

$$= \frac{h(z)}{\tilde{a}\kappa} \int_{\mathcal{S}} \frac{1}{h(x)} E_x^-[R^\kappa - (R - \xi_0)^\kappa] \pi_h(dx) > 0,$$

where the last but one equality is obtained by change of the order of the integration between dt and dv while the last one follows from [9], Lemma 9.4. Since π_h is equivalent to π and $P(R > R - \xi_0 > 0) = 1$, this completes the proof of the claim.

(c) The claim follows from Proposition 3.1 and the fact that the limit $\widetilde{K}_1$ in (3.3) does not depend on z. □

5.2. *Proof of Theorem 1.6 for* $P(\rho_0 < 0) > 0$. (a) Just as in the case $P(\rho_0 > 0)$, it follows from Theorem 2.5, applied separately to the irreducible components of the Markov chain $(\hat{x}_n)_{n \geq -1}$, the random walk V_n, and the function $g_{\eta\gamma_n}(x_{n-1}, t - V_n)$ defined in (3.4), that the limits in (3.3) and hence in (1.5) exist for π-almost every $x \in \mathcal{S}_0$.

(b)–(c) We shall continue to use the notation introduced in Section 4. Similarly to (2.10), define the kernel $\widehat{H}_\beta(x, \cdot)$ on $\mathfrak{S}$ by

$$\widehat{H}_\beta(\hat{x}, d\hat{y}) = \widehat{H}(\hat{x}, d\hat{y}) E(|\rho_0|^\beta | \hat{x}_{-1} = \hat{x}, \hat{x}_0 = \hat{y}),$$

and the function $\hat{h} : \mathfrak{S} \to (0, \infty)$ by the following rule:

$$\hat{h}(\hat{x}) = h(x) \qquad \text{for } x = (x, \gamma),$$

where $h : \mathcal{S}_0 \to \mathbb{R}$ is defined in (2.16).

For any $\hat{x} = (x, \gamma) \in \mathfrak{S}$,

$$\int_{\mathfrak{S}} \widehat{H}_\kappa(\hat{x}, d\hat{y}) \hat{h}(\hat{y}) = E_x^-(|\rho_0|^\kappa h(x_0)) = \int_{\mathcal{S}_0} H_\kappa(x, dy) h(y) = \hat{h}(\hat{x}).$$

Consequently, setting $\hat{\pi}_h(A \times \eta) = 1/2 \pi_h(A)$ for $A \in \mathcal{T}_0$ and $\eta \in \{-1, 1\}$, we have:

$$\int_{\mathfrak{S}} \Big(\int_{A \times \eta} \frac{1}{\hat{h}(\hat{x})} \widehat{H}_\kappa(\hat{x}, d\hat{y}) \hat{h}(\hat{y}) \Big) \hat{\pi}_h(d\hat{x})$$

$$= \int_{\mathcal{S}_0} \frac{1}{2h(x)} E_x^-\big(|\rho_0|^\kappa h(x_0); x_0 \in A\big) \pi_h(dx)$$

$$= \frac{1}{2} \int_{\mathcal{S}_0} H_\kappa(x, A) \pi_h(dx) = \frac{1}{2} \pi_h(A) = \hat{\pi}_h(A \times \eta).$$

We will use these facts to write down formulas similar to (5.2) for the limits $K_1(x)$ and $K_{-1}(x)$ in (1.5). Claims (b) and (c) of Theorem 1.6 are immediate consequences of these formulas.

First, assume that $\mathfrak{S}_1 = \varnothing$. That is, by part (iii) of Proposition 4.1, Condition G is satisfied. We get from Proposition 3.1 and (2.15) that for π-almost every $z \in \mathcal{S}_0$ and $\eta \in \{-1, 1\}$ (compare with (4.4) in [9]):

$$K_\eta(z) = \frac{1}{2\tilde{a}}\left[\int_{\mathcal{S}_0}\int_{\mathbb{R}} g_1(x,t)\pi_h(dx)\,dt + \int_{\mathcal{S}_0}\int_{\mathbb{R}} g_{-1}(x,t)\pi_h(dx)\,dt\right]$$
$$= \frac{1}{2\tilde{a}\kappa}\int_{\mathcal{S}_0}\frac{1}{h(x)}E_x^-(|R|^\kappa - |R-\xi_0|^\kappa)\pi_h(dx),$$

where $\tilde{a} = \widetilde{E}(\log|\rho_0|)$.

Assume now that $\mathfrak{S}_1 \neq \varnothing$, that is, Condition G is not satisfied. We get from Proposition 3.1 and (2.15) that π-a.s., if $(z, 1) \in \mathfrak{S}_\gamma$ (i.e. $z \in A_\gamma$), then

$$K_\eta(z) = \frac{1}{2\tilde{a}}\left[\int_{A_1}\int_{\mathbb{R}} g_{\eta\gamma}(x,t)\pi_h(dx)\,dt + \int_{A_{-1}}\int_{\mathbb{R}} g_{-\eta\gamma}(x,t)\pi_h(dx)\,dt\right].$$

This completes the proof of Theorem 1.6.

5.3. *Proof of part* (a) *of Theorem* 1.8. The "if" part of the claim is trivial. Indeed, if (1.6) holds for a measurable function $\Gamma : \mathcal{S}_0 \to \mathbb{R}$, then substituting $\xi_n = \Gamma(x_{n-1}) - \rho_n\Gamma(x_n)$ into the formula for R_n in (3.11) yields

$$R_n = \Gamma(x_{-1}) - \Gamma(x_{n-1})\prod_{i=0}^{n-1}\rho_i.$$

The Markov chain induced by $(x_n)_{n\in\mathbb{Z}}$ on $(\mathcal{S}_0, \mathcal{T}_0)$ is Harris recurrent by Lemma 2.3 and hence $P_x^-(|\Gamma(x_{n-1})| < M \text{ i.o.}) = 1$ for some $M > 0$. Since, P-a.s., R_n converges to R and $\prod_{i=0}^{n-1}\rho_i$ converges to zero, we obtain that with probability one $R = \Gamma(x_{-1})$. Hence for π-almost every $x \in \mathcal{S}$, $P_x^-(|R| > t) = 0$ for all t large enough.

Assume now that $\lim_{t\to\infty} t^\kappa P_x^-(|R| > t) = 0$ for π-almost every $x \in \mathcal{S}$. Our aim is to show that (1.6) holds for some measurable function $\Gamma : \mathcal{S} \to \mathbb{R}$. First, we will prove the following extension of Grincevičius' symmetrization inequality (cf. [13], see also [9], Proposition 4.2 and [7], Lemma 4). It will be shown in the sequel that if the right-hand side of (5.4) is a.s. zero, then (1.6) holds with the measurable function $\Gamma(x)$ defined in (5.3).

LEMMA 5.2. *Let* $y_n = (x_n, \xi_n, \rho_n)_{n\in\mathbb{Z}}$ *be a MMP associated with Markov chains* $(x_n)_{n\in\mathbb{Z}}$, $(\xi_n, \rho_n) \in \mathbb{R}^2$, *and let* R *be the random variable defined in* (1.2). *Further, for any* $x \in \mathcal{S}$, *let*

$$\Gamma(x) = \inf\{a \in \mathbb{R} : P_x^-(R \le a) > \tfrac{1}{2}\}. \tag{5.3}$$

Then, for any $t > 0$ and $z \in \mathcal{S}$,

$$P_z^-(|R| \geq t) \geq \tfrac{1}{2} P_z^-\big(|R_n + \Gamma(x_{n-1})\Pi_n| > t \text{ for some } n \geq 0\big), \tag{5.4}$$

where the random variables Π_n and R_n are defined in (3.1) *and* (3.11)*, respectively.*

PROOF. By its definition, $\Gamma(x)$ is a median of the random variable R under the measure P_x^-, that is $P_x^-(R \geq \Gamma(x)) \geq 1/2$ and $P_x^-(R \leq \Gamma(x)) \geq 1/2$. Moreover, $\Gamma(x)$ is a measurable function of x.

Fix now any $t > 0$ and let $\tau_1 = \inf\{n > 0 : R_n + \Gamma(x_{n-1})\Pi_n > t\}$. Since $\Gamma(x)$ is a median of the distribution $P_x^-(R \in \cdot)$, it follows from the definition (3.11) of the random variables R_n and the Markov property that

$$\begin{aligned} P_z^-(R \geq t) &\geq \sum_{n=0}^{\infty} \int_{\mathcal{S}} P_z^-(\tau_1 = n; x_{n-1} \in dx, \Pi_n > 0) P_x^-\big(R \geq \Gamma(x)\big) \\ &\quad + \sum_{n=0}^{\infty} \int_{\mathcal{S}} P_z^-(\tau_1 = n; x_{n-1} \in dx, \Pi_n < 0) P_x^-\big(R \leq \Gamma(x)\big) \\ &\geq \tfrac{1}{2} P_z^-(\tau_1 < \infty). \end{aligned}$$

Replacing the sequence ξ_n by the sequence $-\xi_n$ and consequently R by $-R$, we obtain [note that we can replace $\Gamma(x_n)$ by $-\Gamma(x_n)$ because the latter is a median of $-R$]:

$$P_z^-(-R \geq t) \geq \tfrac{1}{2} P_z^-(\tau_2 < \infty),$$

where $\tau_2 := \inf\{n > 0 : -R_n - \Gamma(x_{n-1})\Pi_n > t\}$. Combining together these two inequalities, we get (5.4). □

We will apply this lemma to the Markov chain $y_n^* = (x_n^*, Q_n^*, M_n^*)_{n\in\mathbb{Z}}$, defined below by a "geometric sampling," rather than to $y_n = (x_n, \xi_n, \rho_n)_{n\in\mathbb{Z}}$. The stationary sequence $(x_n^*)_{n\geq -1}$ [it is expanded then into the double-sided sequence $(x_n^*)_{n\in\mathbb{Z}}$] is a random subsequence of $(x_n)_{n\geq -1}$ that forms a Markov chain which inherits the properties of $(x_n)_{n\in\mathbb{Z}}$ and in addition is *strongly aperiodic*, that is, Lemma 2.1 holds for this chain with $d = m = 1$.

Let $(\eta_n)_{n\geq 0}$ be a sequence of i.i.d. variables independent of $(x_n, \xi_n, \rho_n)_{n\in\mathbb{Z}}$ (defined in a probability space enlarged if needed) such that $P(\eta_0 = 1) = 1/2$ and $P(\eta_0 = 0) = 1/2$, and define $\varrho_{-1} = -1$, $\varrho_n = \inf\{i > \varrho_{n-1} : \eta_i = 1\}$, $n \geq 0$. Further, for $n \geq -1$ let,

$$\begin{aligned} x_n^* &= x_{\varrho_n}, \\ Q_{n+1}^* &= \xi_{\varrho_n+1} + \xi_{\varrho_n+2}\rho_{\varrho_n+1} + \cdots + \xi_{\varrho_{n+1}}\rho_{\varrho_n+1}\rho_{\varrho_n+2}\cdots\varrho_{\varrho_{n+1}+1}, \\ M_{n+1}^* &= \rho_{\varrho_n+1}\rho_{\varrho_n+2}\cdots\rho_{\varrho_{n+1}}. \end{aligned} \tag{5.5}$$

The transition kernel of the Markov chain $(x_n^*)_{n\geq -1}$ is given by

$$H^*(x,\cdot)=\sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^n H^n(x,\cdot). \tag{5.6}$$

Hence, $(x_n^*)_{n\geq -1}$ is Harris recurrent on $\mathcal{S}_0$ and its stationary distribution is π. Moreover, the sequence $(y_n^*)_{n\geq 0}=(x_n^*, Q_n^*, M_n^*)_{n\geq 0}$ is a stationary Markov chain whose transitions depend only on the position of x_n^* and

$$R=Q_0^*+\sum_{n=1}^{\infty} Q_n^*\prod_{i=0}^{n-1} M_i^*. \tag{5.7}$$

Expand $(y_n^*)_{n\geq 0}$ into a double-sided stationary sequence $(y_n^*)_{n\in\mathbb{Z}}$.

The following corollary to Lemma 5.2 is immediate in view of (5.7).

COROLLARY 5.3. *Let Assumption 1.2 hold. Then, for any $t>0$ and $z\in\mathcal{S}$,*

$$P_z^-(|R|\geq t)\geq \tfrac{1}{2}P_z^-\big(|R_n^*+\Gamma(x_{n-1}^*)\Pi_n^*|>t \text{ for some } n\geq 0\big), \tag{5.8}$$

where $\Pi_n^*:=\prod_{i=0}^{n-1} M_i^*$ and $R_n^*:=\sum_{i=0}^{n-1} Q_i^*\Pi_i^*$.

Our aim now is to show that the right-hand side of (5.8) is bounded away from zero for π-almost every $z\in\mathcal{S}$. The main advantage of using the "geometrically sampled" MMP $(x_n^*, Q_n^*, M_n^*)_{n\in\mathbb{Z}}$ is that studying its one-step transitions one can obtain some information concerning all possible transitions of the original MMP $(x_n,\xi_n,\rho_n)_{n\in\mathbb{Z}}$. We will use this when passing from (5.14) to (5.15) below.

At some stage of the proof, we shall apply Corollary 2.6 to the Markov chain $(x_n^*)_{n\in\mathbb{Z}}$ and the random walk $V_n^*=\sum_{i=0}^{n-1}\log|M_i^*|$ considered under the measure $\widetilde{P}$ introduced in Section 2.3. Let $h_\beta:\mathcal{S}_0\to(0,\infty)$ be the eigenfunction of the operator H_β in the space B_b corresponding to the kernel defined in (2.10). This eigenfunction exists and is bounded away from zero by Proposition 2.4, and it corresponds to the eigenvalue r_β which coincides with the spectral radius r_{H_β} of the operator. Let

$$H_\beta^*(x,dy)=\sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^n H_\beta^n(x,\cdot). \tag{5.9}$$

Then, similarly to (2.11), $E_x^-\big(\prod_{i=1}^n |M_i^*|^\beta\big)=H_\beta^{*n}\mathbf{1}(x)$ for any $\beta\geq 0$ and $x\in\mathcal{S}_0$.

Transition kernel $\widetilde{H}^*$ of x_n^* under $\widetilde{P}$ is given by

$$\widetilde{H}^*(x,dy)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\widetilde{H}^n=\frac{1}{h(x)}H_\kappa^*(x,dy)h(y), \tag{5.10}$$

where as before $h(x)=h_\kappa(x)$. It follows from (5.9) that, as long as $r_\beta<2$,

$$H_\beta^* h_\beta(x)=\frac{r_\beta}{2-r_\beta}h_\beta(x),$$

and thus, as in Proposition 2.4, $r_\beta^* := \frac{r_\beta}{2-r_\beta}$ is the spectral radius of the operator H_β^* in B_b. In particular, $r_\kappa^* = 1$. Note also that the invariant distribution of $\widetilde{H}^*$ coincides with the invariant distribution π_h of $\widetilde{H}$.

To enable in the use of Corollary 2.6 we need the following two lemmas which ensure that its conditions are satisfied. First, the same proof as that of Lemma 2.7 yields:

LEMMA 5.4. *Let Assumption 1.2 hold. Then,* $\widetilde{E}(\log|M_1^*|) > 0$.

In addition, we have:

LEMMA 5.5. *Let Assumption 1.2 hold. Then, the process* $\log|M_n^*|$ *is nonarithmetic relative to the Markov chain* $(x_n^*)_{n\in\mathbb{Z}}$ *with transition kernel* $\widetilde{H}^*$ *defined in* (5.10) (*in the sense of Definition* 1.4).

PROOF. Since the process $\log|\rho_i|$ is nonarithmetic relative to the Markov chain $(x_n)_{n\in\mathbb{Z}}$ with kernel $\widetilde{H}$, the claim follows from Lemma A.6 in [2], which deals with the nonarithmetic condition relative to the "sampled" Markov chain $(x_n^*)_{n\in\mathbb{Z}}$.

□

We are now in position to complete the proof of part (a) of Theorem 1.8.

LEMMA 5.6. *Let Assumption 1.2 hold and suppose in addition that* $\lim_{t\to\infty} t^\kappa P_x^-(|R| > t) = 0$ *for* π*-almost every* $x \in \mathcal{S}$. *Then,* (1.6) *holds with the function* $\Gamma(x)$ *defined in* (5.3).

PROOF. For $n \in \mathbb{Z}$, let $\alpha_n = R_n^* + \Gamma(x_{n-1}^*)\Pi_n^*$ and write $\alpha_n = \alpha_{n-1} + \beta_n$, where

$$\begin{aligned}\beta_n &= Q_{n-1}^*\Pi_{n-1}^* + \Gamma(x_{n-1}^*)\Pi_n^* - \Gamma(x_{n-2}^*)\Pi_{n-1}^* \\ &= \Pi_{n-1}^*\big(Q_{n-1}^* + \Gamma(x_{n-1}^*)M_{n-1}^* - \Gamma(x_{n-2}^*)\big).\end{aligned}$$

Set

$$\delta_n := Q_n^* + \Gamma(x_n^*)M_n^* - \Gamma(x_{n-1}^*). \tag{5.11}$$

Thus, $\alpha_n = \alpha_{n-1} + \beta_n = \alpha_{n-1} + \Pi_{n-1}^*\delta_{n-1}$, and hence for any $\varepsilon > 0$ (cf. [9], page 157):

$$\begin{aligned}P_z^-(|\alpha_n| > t \text{ for some } n \geq 0) &\geq P_z^-(|\beta_n| > 2t \text{ for some } n \geq 0) \\ &\geq P_z^-(|\Pi_n^*| > 2t/\varepsilon \text{ and } |\delta_n| > \varepsilon \text{ for some } n \geq 1).\end{aligned}$$

Indeed, $|\beta_n| > 2t$ implies that either $|\alpha_{n-1}| > t$ or, if not, $|\alpha_n| \geq |\beta_n| - |\alpha_{n-1}| > 2t - t = t$.

Fix a number $\varepsilon > 0$ and let $\upsilon(t) = \inf\{n \geq 1 : |\Pi_n^*| > 2t/\varepsilon\}$. Then, setting

$$V_n^* := \log|\Pi_n^*| = \sum_{i=0}^{n-1} \log|M_i^*|, \tag{5.12}$$

we obtain from (5.8) and the Markov property that for any $z \in \mathcal{S}_0$,

$$\begin{aligned} &t^\kappa P_z^-(|R| \geq t) \\ &\quad \geq \frac{t^\kappa}{2} \int_{\mathcal{S}_0} P_z^-\big(x^*_{\upsilon(t)-1} \in dx, |\delta_{\upsilon(t)}| > \varepsilon, \upsilon(t) < \infty\big) \\ &\quad = \frac{t^\kappa}{2} E_z^-\big(P^-_{x^*_{\upsilon(t)-1}}(|\delta_0| > \varepsilon); \upsilon(t) < \infty\big) \\ &\quad = \frac{1}{2}\Big(\frac{\varepsilon}{2}\Big)^\kappa h(z) \widetilde{E}_z^-\big(e^{-\kappa(V^*_{\upsilon(t)} - \log(2t/\varepsilon))} P^-_{x^*_{\upsilon(t)-1}}(|\delta_0| > \varepsilon)/h\big(x^*_{\upsilon(t)-1}\big)\big), \end{aligned}$$

where the expectation $\widetilde{E}_z^-$ is according to the measure $\widetilde{P}_z^-$ defined in Section 2.3.

Thus, in virtue of part (b) of Theorem 1.6 it suffices to prove that under Assumption 1.2,

either

(i) for some $\varepsilon > 0$ and probability measure $\hat{\pi}$ absolutely continuous with respect to π, either the following limit exists and is strictly positive:

$$\lim_{t\to\infty} \widetilde{E}^-_{\hat{\pi}}\big(e^{-\kappa(V^*_{\upsilon(t)} - \log(2t/\varepsilon))} P^-_{x^*_{\upsilon(t)-1}}(|\delta_0| > \varepsilon)\big), \tag{5.13}$$

where $\widetilde{E}^-_{\hat{\pi}}(\cdot) := \int_{\mathcal{S}_0} \widetilde{E}_z^-(\cdot)\hat{\pi}(dz)$,

or, if not,

(ii) then, (1.6) holds with the function $\Gamma(x)$ defined in (5.3).

To bound the limit in (5.13) away from zero we will apply Corollary 2.6 to the Markov chain $(x_n^*)_{n\in\mathbb{Z}}$ on $(\mathcal{S}_0, \mathcal{T}_0)$ introduced in (5.5) and governed by the kernel $\widetilde{H}^*$ defined in (5.10), the random walk V_n^* defined in (5.12), and the function

$$g(x,t) = e^{-\kappa t} P_x^-(|\delta_0| > \varepsilon).$$

Let $\sigma_{-1} = -1$, $V_{-1}^* = 0$, and for $n \geq 0$, $\sigma_n = \inf\{i > \sigma_{n-1} : V_i^* > V^*_{\sigma_{n-1}}\}$. Further, let $\hat{\pi}$ be the stationary distribution of the Markov chain $\hat{x}_n := x^*_{\sigma_n}$ under $\widetilde{P}$ (which exists and is unique by [2], Theorem 4). The measure $\hat{\pi}$ is an irreducible measure of the Markov chain $(x_n^*)_{n\in\mathbb{Z}}$ with transition kernel $\widetilde{H}^*$ and hence is absolutely continuous with respect to its stationary distribution, which in turn is equivalent to the stationary distribution π of $(x_n^*)_{n\in\mathbb{Z}}$ with transition kernel H^*.

To apply Corollary 2.6 to the Markov chain $(x_n^*)_{n\in\mathbb{Z}}$ with kernel $\widetilde{H}^*$ and the random walk V_n^*, we need to check conditions (2.13) and (2.14) for the function

$$b(x,y) = \widetilde{E}_x^-\big(e^{-\kappa(\widehat{V}_0 - y)} \mathbf{1}_{\{\widehat{V}_0 > y\}} P^-_{\hat{x}_0}(|\delta_0| > \varepsilon)\big),$$

where $\widehat{V}_n := V^*_{\sigma_n}$.

Condition (2.13) follows from the following estimate valid for any $\delta > 0$:

$$|b(x, y+\delta) - b(x, y)| \le \widetilde{E}^-_x\big(|e^{\kappa\delta}\mathbf{1}_{\{\widehat{V}_0 > y+\delta\}} - \mathbf{1}_{\{\widehat{V}_0 > y\}}|\big)$$
$$= (e^{\kappa\delta} - 1)\widetilde{P}^-_x(\widehat{V}_0 > y+\delta) + \widetilde{P}^-_x(y \le \widehat{V}_0 < y+\delta).$$

As to condition (2.14), we have:

$$b(x, y) \le \begin{cases} e^{\kappa y}, & \text{if } y < 0, \\ \widetilde{E}^-_x\big(\mathbf{1}_{\{\widehat{V}_0 > y\}}\big) = \widetilde{P}^-_x(\widehat{V}_0 > y), & \text{if } y \ge 0. \end{cases}$$

Hence,

$$\int_{\mathcal{S}_0} \sum_{n\in\mathbb{Z}} \sup_{n\le y<n+1} |b(x, y)|\hat{\pi}(dy)$$
$$\le \sum_{n=0}^{\infty} e^{-\kappa n} + \int_{\mathcal{S}_0} \sum_{n=0}^{\infty} \widetilde{P}^-_x(\widehat{V}_0 > n)\hat{\pi}(dx) < \infty,$$

because by part (iv) of [2], Theorem 2,

$$\int_{\mathcal{S}_0} \sum_{n=0}^{\infty} \widetilde{P}^-_x(\widehat{V}_0 > n)\hat{\pi}(dx) \le \int_{\mathcal{S}_0} \widetilde{E}^-_x(\widehat{V}_0)\hat{\pi}(dx) < \infty.$$

(Part (iv) of [2], Theorem 2 implies that the constant μ_1 in the statement of Corollary 2.6 is finite. In our case, $\mu_1 = \int_{\mathcal{S}_0} \widetilde{E}^-_x(\widehat{V}_0)\hat{\pi}(dx)$.)

Let $\widehat{H}$ be the transition kernel of the Markov chain $(\hat{x}_n, \widehat{V}_n - \widehat{V}_{n-1})_{n\ge 0}$. It follows from Corollary 2.6 that for some $A \in (0, 1)$,

$$\lim_{t\to\infty} \widetilde{E}^-_{\hat{\pi}}\big(e^{-\kappa(V^*_{\upsilon(t)} - \log(2t/\varepsilon))} P^-_{x^*_{\upsilon(t)}-1}(|\delta_0| > \varepsilon)\big)$$
$$= \frac{1}{\mu_1}\int_{\mathcal{S}_0}\int_{\mathcal{S}_0\times(0,\infty)}\int_{[0,z)} e^{-\kappa w} P^-_y(\delta_0 > \varepsilon)\, dw\, \widehat{H}(x, dy\times dz)\hat{\pi}(dx)$$
$$= \frac{1}{\mu_1\kappa}\int_{\mathcal{S}_0}\int_{\mathcal{S}_0\times(0,\infty)} (1 - e^{-\kappa z}) P^-_y(|\delta_0| > \varepsilon)\widehat{H}(x, dy\times dz)\hat{\pi}(dx)$$
$$\ge A\int_{\mathcal{S}_0}\int_{\mathcal{S}_0} P^-_y(|\delta_0| > \varepsilon)\widehat{H}\big(x, dy\times(0,\infty)\big)\hat{\pi}(dx)$$
$$= A\int_{\mathcal{S}_0} P^-_y(|\delta_0| > \varepsilon)\hat{\pi}(dy) = AP^-_{\hat{\pi}}(|\delta_0| > \varepsilon).$$

It follows that if (5.13) is not true for any $\varepsilon > 0$ then

$$P^-_{\hat{\pi}}(\delta_0 = 0) = 1. \tag{5.14}$$

It remains to show that (5.14) implies that (1.6) holds for the function Γ defined in (5.3). By the definition of the kernel H^* in (5.6) and the quantity δ_n in (5.11), we get from (5.14) that

$$P^-_{\hat{\pi}}\big(R_n + \Gamma(x_n)\Pi_n - \Gamma(x_{-1}) = 0\big) = 1 \qquad \text{for all } n \in \mathbb{N}. \tag{5.15}$$

Taking respectively $n=0$ and $n=1$ in the last equality we obtain that $P^-_{\hat\pi}(\xi_0+\Gamma(x_0)\rho_0-\Gamma(x_{-1})=0)=P^-_{\hat\pi}(\xi_0+\xi_1\rho_0+\Gamma(x_1)\rho_0\rho_1-\Gamma(x_{-1})=0)=1$. It follows that

$$P^-_{\hat\pi}\big(\xi_1+\Gamma(x_1)\rho_1-\Gamma(x_0)=0\big)=1.$$

Similarly, by induction on n, one can show that

$$P^-_{\hat\pi}\big(\xi_n+\Gamma(x_n)\rho_n-\Gamma(x_{n-1})=0\big)=1 \qquad \text{for all } n\in\mathbb{N}.$$

Since the Markov chain (x_n) is π-recurrent and $\hat\pi$ is absolutely continuous with respect to π, we obtain (1.6). $\square$

5.4. *Proof of parts* (b) *and* (c) *of Theorem* 1.8. Let $\mathcal{S}_0$ be as defined in Lemma 2.1 and recall the regeneration times N_n defined in Section 2.1. Let $Q_0=\xi_0+\mathbf{1}_{\{N_1\geq 1\}}\sum_{i=0}^{N_1-1}\xi_{i+1}\prod_{j=0}^{i}\rho_j$ and $M_0=\prod_{i=0}^{N_1}\rho_i$, and for $n\geq 1$,

$$Q_n=\xi_{N_n+1}+\mathbf{1}_{\{N_{n+1}-N_n\geq 2\}}\sum_{i=N_n+1}^{N_{n+1}-1}\xi_{i+1}\prod_{j=N_n+1}^{i}\rho_j \quad\text{and}\quad M_n=\prod_{i=N_n+1}^{N_{n+1}}\rho_i.$$

The pairs (Q_n,M_n), $n\geq 0$, are one-dependent and for $n\geq 1$ they are identically distributed. Since the series in (1.2) converges absolutely, we obtain the representation

$$R=Q_0+M_0\big(Q_1+M_1\big(Q_2+M_2(Q_3+\cdots)\big)\big):=Q_0+M_0\widehat{R}. \tag{5.16}$$

Note that x_{N_1} is distributed according to the measure ψ introduced in Lemma 2.4 and hence $P(|\widehat{R}|>t)=P^-_\psi(|R|>t)$, where we denote as usual $P^-_\psi(\cdot):=\int_{\mathcal{S}}P^-_x(\cdot)\psi(dx)$. We have:

LEMMA 5.7. *The following limit exists and is strictly positive*:

$$\widehat{K}=\lim_{t\to\infty}t^\kappa P^-_\psi(|R|>t)=\lim_{t\to\infty}t^\kappa P(|\widehat{R}|>t). \tag{5.17}$$

PROOF. The measure ψ is an irreducible measure of the Markov chain $(x_n)_{n\in\mathbb{Z}}$ and hence it is absolutely continuous with respect to its stationary distribution π. Therefore, the claim follows by the bounded convergence theorem from part (a) of Theorem 1.6 and part (c) of Lemma 3.2. $\square$

We will show next that the contribution of Q_0 in R is negligible in the following precise sense [recall that ξ_n are assumed to be bounded by (2.5)]: for some $\beta>\kappa$,

$$\sup_{x\in\mathcal{S}_0}E^-_x\left(\left[\mathbf{1}_{\{N_1\geq 1\}}\sum_{i=0}^{N_1-1}\prod_{j=0}^{i}|\rho_j|\right]^\beta\right)<\infty. \tag{5.18}$$

Let $A(x) = E_x^-([\mathbf{1}_{\{N_1\geq 1\}}\sum_{i=0}^{N_1-1}\prod_{j=0}^{i}|\rho_j|]^\beta) < \infty$. Since for any positive numbers $\{a_i\}_{i=1}^n$ we have $(a_1+a_2+\cdots+a_n)^\beta \leq n^\beta(a_1^\beta+a_2^\beta+\cdots+a_n^\beta)$, we obtain for any $\beta>0$ and $x\in\mathcal{S}_0$:

$$\begin{aligned} A(x) &= E_x^-\left(\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\prod_{j=0}^{i}|\rho_j|\mathbf{1}_{\{N_1=n\}}\right)^\beta \\ &= \sum_{n=1}^{\infty}E_x^-\left(\sum_{i=0}^{n-1}\prod_{j=0}^{i}|\rho_j|\mathbf{1}_{\{N_1=n\}}\right)^\beta \qquad (5.19)\\ &\leq \sum_{n=1}^{\infty}n^\beta\sum_{i=0}^{n-1}E_x^-\left(\prod_{j=0}^{i}|\rho_j|^\beta\mathbf{1}_{\{N_1\geq n\}}\right). \end{aligned}$$

Let

$$\widetilde{\Theta}_\beta(x,dy) := \Theta(x,dy)E(|\rho_0\rho_1\rho_2\cdots\rho_{m-1}|^\beta|x_{-1}=x, x_{m-1}=y),$$

where the kernel $\Theta(x,dy)$ on $(\mathcal{S}_0,\mathcal{T}_0)$ is defined in (2.3), and let

$$\begin{aligned} K_\beta(x,dy) &:= H^m(x,dy)E(|\rho_0\rho_1\rho_2\cdots\rho_{m-1}|^\beta|x_{-1}=x, x_{m-1}=y)\\ &= H_\beta^m(x,dy), \end{aligned}$$

where the kernel H_β on $(\mathcal{S}_0,\mathcal{T}_0)$ is defined in (2.10).

Then for any $x\in\mathcal{S}_0$,

$$\widetilde{\Theta}_\beta\mathbf{1}(x) = E_x^-\left(\prod_{j=0}^{m-1}|\rho_j|^\beta\mathbf{1}_{\{N_1\geq m\}}\right) \quad\text{and}\quad K_\beta\mathbf{1}(x) = E_x^-\left(\prod_{j=0}^{m-1}|\rho_j|^\beta\right).$$

By Lemma 2.1 and (2.5), the kernels K_β and $\widetilde{\Theta}_\beta$ satisfy the conditions of Proposition 2.4 with $s(x,y) = E(|\rho_0\rho_1\rho_2\cdots\rho_{m-1}|^\beta|x_{-1}=x, x_{m-1}=y)$ and $c_1 = c_\rho^{-m}$. In virtue of Lemma 2.3, the spectral radius of H_κ and hence K_κ is equal to 1. Thus, by part (c) of Proposition 2.4, the spectral radius of $\widetilde{\Theta}_\kappa$ is strictly less than one. Since $r_{\widetilde{\Theta}_\beta}$ is a continuous function of β, we have for some $\beta>\kappa$:

$$r_{\widetilde{\Theta}_\beta} < 1. \qquad (5.20)$$

For $l\in\mathbb{N}$, denote $\hat{l} = m\cdot\max\{[l/m],1\}$, where m is as in (2.1). We obtain from (5.20) that for any $l\in\mathbb{N}$, $n>\max\{l,m\}$, $x\in\mathcal{S}_1$, and for suitable constants $A_\beta>0$, $\widetilde{\Lambda}_\beta<0$:

$$\begin{aligned} E_x^-\left(\prod_{j=0}^{l}|\rho_j|^\beta\mathbf{1}_{\{N_1\geq n\}}\right) &\leq c_\rho^m E_x^-\left(\prod_{j=0}^{\hat{l}-1}|\rho_j|^\beta\mathbf{1}_{\{N_1\geq\hat{n}\}}\right)\\ &\leq c_\rho^m\widetilde{\Theta}_\beta^{\hat{l}/m}\Theta^{(\hat{n}-\hat{l})/m}\mathbf{1}(x) \leq A_\beta e^{n\widetilde{\Lambda}_\beta}, \end{aligned}$$

where in the first inequality we use (2.5) and the fact that $\hat{n} \le n$ for $n > m$ (note also that $r_\theta < 1$ by Proposition 2.4 applied to the kernels H and Θ). This yields (5.18) in virtue of (5.19).

Fix some $\beta > \kappa$ which satisfies (5.18) and $\alpha \in (\frac{\kappa}{\beta}, 1)$. By (5.19) and the Chebyshev inequality, $\lim_{t\to\infty} t^\kappa P_x^-(|Q_0| \ge t^\alpha) = 0$ uniformly in x. Let

$$M_{0,1} = \mathbf{1}_{\{N_1-m=-1\}} + \mathbf{1}_{\{N_1-m\ge 0\}} \cdot \prod_{i=0}^{N_1-m} |\rho_i| \quad \text{and} \quad M_{0,2} = \prod_{i=N_1-m+1}^{N_1} |\rho_i|.$$

Then, $M_0 = M_{0,1} \cdot M_{0,2}$ and $c_\rho^{-m} M_0 \le M_{0,1} \le c_\rho^m M_0$, where c_ρ is introduced in assumption (A4).

Recall the random variable $\widehat{R}$ defined in (5.16) and note that $M_{0,1}$ and $\widehat{R}$ are independent under the measure P_x^- because only the $m-1$ last variables in the block $(x_0, x_1, \dots, x_{N_1-1}\}$ are dependent on x_{N_1}.

For any $\beta > \kappa$ such that (5.18) holds, we have

$$\begin{aligned} t^\kappa P_x^-(|R| > t) &\le t^\kappa P_x^-(|Q_0| + |M_0\widehat{R}| > t, |Q_0| < t^\alpha) + t^\kappa P_x^-(|Q_0| \ge t^\alpha) \\ &\le t^\kappa P_x^-(|M_0\widehat{R}| > t - t^\alpha) + \frac{t^\kappa}{t^{\alpha\beta}} E_x^-(|Q_0|^\beta) \\ &\le t^\kappa P_x^-(c_\rho^m |M_{0,1}\widehat{R}| > t - t^\alpha) + E_x^-(|Q_0|^\beta). \end{aligned}$$

The expectation $E_x^-(|Q_0|^\beta)$ is bounded on $\mathcal{S}_0$ by (5.18), while (5.17) and the fact that $\widehat{R}$ is independent of $M_{0,1}$ under P_x^- imply that for some $L > 0$,

$$t^\kappa P_x^-(c_\rho^m |M_{0,1}\widehat{R}| > t - t^\alpha) \le L\left(\frac{t}{t-t^\alpha}\right)^\kappa E_x^-(|M_{0,1}|^\kappa) \qquad \forall t > 1$$

yielding the upper bound in (1.7) since the expectation $E_x^-|(M_{0,1}|^\beta)$ is bounded on $\mathcal{S}_0$ in view of (5.18).

To get the lower bound in (1.8), write

$$\begin{aligned} t^\kappa P_x^-(|R| > t) &\ge t^\kappa P_x^-(|M_0\widehat{R}| - |Q_0| > t) \\ &\ge t^\kappa P_x^-(|M_0\widehat{R}| - |Q_0| > t, |Q_0| < t^\alpha) \\ &\ge t^\kappa P_x^-(|M_0\widehat{R}| > t + t^\alpha) - P_x^-(|Q_0| > t^\alpha) \\ &\ge t^\kappa P_x^-(|M_0\widehat{R}| > t + t^\alpha) - \frac{t^\kappa}{t^{\alpha\beta}} E_x^-(|Q_0|^\beta) \\ &\ge t^\kappa P_x^-(c_\rho^{-m} |M_{0,1}\widehat{R}| > t + t^\alpha) - \frac{t^\kappa}{t^{\alpha\beta}} E_x^-(|Q_0|^\beta), \end{aligned}$$

and note that $\frac{t^\kappa}{t^{\alpha\beta}} E_x^-(|Q_0|^\beta)$ converges to zero uniformly on x by (5.18) while by (5.17) we have for any $\lambda > 0$, some constant $J > 0$ that depends on λ, and all t

large enough:

$$t^{\kappa} P_x^-(c_\rho^{-m}|M_{0,1}\widehat{R}| > t) \geq t^{\kappa} P_x^-(\lambda \cdot c_\rho^{-m} \cdot |\widehat{R}| > t; |M_{0,1}| > \lambda)$$
$$\geq J P_x^-(|M_{0,1}| \geq \lambda).$$

To complete the proof it remains to show that for some $\lambda > 0$ there exists a number $\delta_1 > 0$ such that

$$P_x^-(|M_{0,1}| \geq \lambda) > \delta_1, \qquad \pi\text{-a.s.}$$

Toward this end observe that for every $x \in \mathcal{S}_0$, with $\vartheta \in \mathbb{N}$ defined in (2.4) and $c_\rho > 0$ defined in (A4) (we will assume, actually without loss of generality, that $\vartheta > m$),

$$P_x^-\big(|M_{0,1}| \geq c_\rho^{-(\vartheta - m)}\big) \geq P_x^-\left(|M_{0,1}| \geq \min_{m \leq i \leq \vartheta} c_\rho^{-(i-m)}; N_1 \leq \vartheta\right)$$
$$= P_x^-(N_1 \leq \vartheta) \geq \delta,$$

where $\delta > 0$ is defined in (2.4).

APPENDIX A: PROOF OF PROPOSITION 2.4

(a) First, we note that if a nonnegative eigenfunction $f \not\equiv 0$ of the operator $K : B_b \to B_b$ exists then necessarily $\inf_x f(x) > 0$. Indeed, assuming that $Kf = \lambda f$ for some $\lambda > 0$, we have for any $x \in \mathcal{S}_0$,

$$\sum_{i=1}^{d+m} \lambda^i f(x) = \sum_{i=1}^{d+m} K^i f(x) \geq \sum_{i=1}^{d} \int_{\mathcal{S}_1} \int_{\mathcal{S}_1} K^i(x, dz) K^m(z, dy) f(y)$$
$$\geq p \cdot c_1^{-1} \cdot \int_{\mathcal{S}_1} f(y) \psi(dy) > 0,$$

where the last inequality follows from the fact that $f(x) > 0$ for every $x \in \mathcal{S}_0$ (cf. [18], Proposition 5.1(ii)).

The proof of the existence of such $f \in B_b$ is an application of Nussbaum's extension of the Krein–Rutman theorem (cf. Theorem 2.2 in [19]). Theorem 2.2. In view of this theorem (this is explained in Appendix B) it is sufficient to show that there exists a double-indexed sequences of compact linear operators $Q_{n,i}$ on the space B_b such that

$$\limsup_{i \to \infty} \sqrt[i]{\|K^i - Q_{n,i}\|} \leq 1/n, \qquad n \in \mathbb{N}. \tag{A.1}$$

It even suffices to show that $\limsup_{i \to \infty} \sqrt[i]{\|K^{mi} - \widehat{Q}_{n,i}\|} \leq 1/n$ for some compact operators $\widehat{Q}_{n,i}$ on B_b, since we can then set $Q_{n,i} = K^{i - m j_i} \widehat{Q}_{n,j_i}$, where j_i is the integer part of i/m. For this purpose we shall adapt the Yosida–Kakutani's proof that Markov kernels satisfying Doeblin's condition are quasi-compact (cf. [24], Section 4.7).

(1) First, we observe that if $n(x, y)$ and $j(x, y)$ are jointly measurable bounded function, then the product of the two operators defined by the kernels $N(x, dy) = n(x, y)\mu(dy)$ and $J(x, dy) = j(x, y)\mu(dy)$ is compact in B_b. Indeed, we can approximate $n(x, y)$ in $L_1(\mathcal{S}_0 \times \mathcal{S}_0, \mathcal{T}_0 \times \mathcal{T}_0, \mu \times \mu)$ up to $1/i$ by a simple function $n_i(x, y)$ which is a finite linear combination of the indicator functions of "rectangle" sets $B_{i,k} \times C_{i,k}$, where $B_{i,k}, C_{i,k} \subset \mathcal{S}_0$. Then, the operators corresponding to the kernels $N_i(x, dy) = n_i(x, y)\mu(dy)$ are finite-dimensional and hence $JN = \lim_{i\to\infty} JN_i$, being the limit in operator norm of a sequence of compact operators, is compact.

(2) Fix $n \in \mathbb{N}$ and let $\delta = \delta(1/n)$ be defined as in condition (iii) of the proposition. Let $k(x, y)$ be a jointly measurable density of the kernel K^m with respect to μ (such a density exists since the σ-field $\mathcal{T}_0$ is assumed to be countably generated, see, e.g., [18], Lemma 2.5) and set

$$q_n(x, y) = \min\{k(x, y), \delta^{-1} \cdot \|K^m\|\}.$$

Let $D_x = \{y \in \mathcal{S}_0 : k(x, y) \neq q_n(x, y)\}$, thus $k(x, \cdot) \geq \delta^{-1}\|K^m\|$ on D_x. Since

$$\sup_x K^m(x, D_x) = \sup_x \int_{D_x} k(x, y)\mu(dy) \leq \|K^m\|,$$

then $\mu(D_x) \leq \delta$. Hence, letting $Q_n(x, dy) = q_n(x, dy)\mu(dy)$,

$$\|K^m - Q_n\| \leq \sup_x \int_{D_x} k(x, y)\mu(dy) = \sup_x K^m(x, D_x) \leq 1/n.$$

(3) Let $R_n = K^m - Q_n$. Then $K^{mi} = (Q_n + R_n)^i = \sum$ of 2^i terms each of them, except maybe those $i + 1$ where Q_n appear at most once, is compact by (1). But

$$\begin{aligned}\|R_n^i + Q_n R_n^{i-1} + R_n Q_n R_n^{i-2} + \cdots + R_n^{i-1} Q_n\| \\ \leq (1/n)^i + i \cdot \|Q_n\| \cdot (1/n)^{i-1} \leq (1/n)^i + i \cdot \|K^m\| \cdot (1/n)^{i-1},\end{aligned}$$

as required.

(b) The proof for the kernel $\widehat{\Theta}$ on $(\mathcal{S}_1, \mathcal{T}_1)$ is the same as for K, since the conditions of this proposition hold for $\widehat{\Theta}$ as well (with $d = m = 1$).

(c) Let $c_K > 1$ be a constant such that $f(x) \in (c_K^{-1}, c_K)$ for all $x \in \mathcal{S}_0$. Then, for any $x \in \mathcal{S}_0$, $c_K^{-1} f(x) \leq \mathbf{1}(x) \leq c_K f(x)$, and hence

$$c_K^{-2} r_K^n \leq K^n \mathbf{1}(x) \leq c_K^2 r_K^n \qquad \forall x \in \mathcal{S}_0. \tag{A.2}$$

Let $\widehat{K}(x, \cdot)$ be the restriction of the kernel K^m to the states of the set $\mathcal{S}_1$. It follows from (A.2) that the spectral radius of $\widehat{K}$ coincides with r_K^m.

By [18], Proposition 5.3 and [18], Theorem 5.2, the kernel $\widehat{\Theta}$ has an invariant measure $\pi_{\widehat{\Theta}}$. Since $r_K^m f \geq \widehat{\Theta} f$, the equality $r_{\widehat{\Theta}} = r_K^m$ would imply by [18], Proposition 5.3 and [18], Theorem 5.1 that $\pi_{\widehat{\Theta}}$-a.s., $\widehat{\Theta} f(x) = r_K^m f(x) = K^m f(x)$, which is impossible because $f(x) > 0$ and $K^m(x, dy) - \widehat{\Theta}(x, dy) \geq r c_1^{-1} \psi(dy)$ for any $x \in \mathcal{S}_1$. Hence $r_{\widehat{\Theta}} < r_K^m$.

APPENDIX B: THE NUSSBAUM FIXED POINT THEOREM

This appendix is devoted to the Nussbaum's extension of the Krein–Rutman fixed point theorem (cf. Theorem 2.2 in [19]) or, to be precise, to the version of this theorem which is actually used in (A.1).

Let X be a Banach space. For a bounded subset S of X, Kuratowski's measure of noncompactness $\alpha(S)$ is defined by

$$\alpha(S) = \inf\left\{ d > 0 : S = \bigcup_{i=1}^{n} S_i, n \in \mathbb{N}, \text{ and } D(S_i) \le d \text{ for } 1 \le i \le n \right\},$$

where $D(S) := \sup_{x,y \in S} \|x - y\|$ is the diameter of the set S.

A bounded linear operator K in X is called a *b-set-contraction* for a number $b \ge 0$ if $\alpha(K(S)) \le b\alpha(S)$ for every bounded subset S of X. A closed subset C of X is called a *cone* if the following holds: (i) if $x, y \in C$ and $\alpha, \beta \ge 0$ are nonnegative reals, then $\alpha x + \beta y \in C$. (ii) if $x \in C - \{0\}$, then $-x \notin C$.

THEOREM B.1 ([19], Theorem 2.2). *Let X be a Banach space, C be a cone in X, and K be a bounded linear operator in X such that $K(C) \subset C$. Let*

$$\|K\|_C := \sup\{\|Ku\| : u \in C, \|u\| \le 1\}$$

and $\alpha_C(K) := \inf\{b \ge 0 : K_C$ is ab-set-contraction$\}$, where $K_C : C \to C$ is the restriction of K to the cone C. Further, let

$$r_C(K) := \lim_{n\to\infty} \sqrt[n]{\|K^n\|_C} \quad \text{and} \quad \rho_C(K) := \lim_{n\to\infty} e\sqrt[n]{\alpha_C(K^n)}.$$

Assume that $\rho_C(K) < r_C(K)$. Then there exists an $x \in C - \{0\}$ such that $Kx = r_C(K)x$.

We want to apply this theorem in the situation of Proposition 2.4, namely to the Banach space B_b, the operator K defined by $Kf = \int_{\mathcal{S}_0} K(x, dy) f(y)$, and the cone C of nonnegative functions in B_b. Note that $r_C(K)$ coincides with the spectral radius r_K in this case. It follows from (2.8) and the assumption $s(x, y) \in (c_1^{-1}, c_1)$ that $r_K > c_1^{-1/m}$. Therefore it suffices to show that (A.1) implies $\rho_C(K) = 0$. Since $\rho_C(K) \le \rho_X(K)$ (cf. [19], page 321), it is even sufficient to show that $\rho_X(K) = 0$.

It will be convenient to use the notion of the Hausdorff measure of noncompactness χ which is defined for a bounded subset S of a Banach space X by

$$\chi(S) = \inf\{d > 0 : S \text{ has a finite } d\text{-net in } X\}.$$

By finite d-net in X we mean a finite subset $\{x_1, \ldots, x_n\}$ of X such that for any $y \in \mathcal{S}$ there exists an index j s.t. $\|y - x_j\| < d$, where $\|\cdot\|$ is the norm on X.

Let

$$\chi(K) := \inf\{b \geq 0 : \chi(K(S)) \leq b\chi(S) \text{ for bounded subsets } S \text{ of } X\},$$

and $\sigma(K) := \lim_{n\to\infty} \sqrt[n]{\chi(K^n)}$. The Kuratowski and Hausdorff measures of noncompactness are equivalent in the following sense (cf. [1], page 4): $\chi(S) \leq \alpha(S) \leq 2\chi(S)$ for every bounded subset S of X. Thus, it suffices to show that $\sigma(K) = 0$ when (A.1) holds. The latter assertion follows from the following lemma.

LEMMA B.2. *Let X be a Banach space and K be a bounded linear operator in X. Further, let $\varepsilon > 0$ be a positive constant and assume that there is a compact operator Q in X such that $\|Q - K\| < \varepsilon$. Then, $\chi(K) \leq 2\varepsilon\|K\|$.*

PROOF. Fix a bounded set $S \subseteq X$. Let $\{x_1, x_2, \ldots, x_n\} \subseteq X$ be a finite d-net of S for some $d > 0$. It suffices to show that the set $K(S)$ has a finite η_d-net in X, where we denote $\eta_d := 2\varepsilon d\|K\|$. Let B_i, $i = 1, \ldots, n$, be the balls in X of radius d and centered in x_i. Then, $S \subseteq \bigcup_{i=1}^n B_i$ and $K(S) \subseteq \bigcup_{i=1}^n K(B_i)$. Therefore, it is sufficient to show that each set $K(B_i)$, $i = 1, 2, \ldots, n$, has a finite η_d-net in X.

Fix any $\delta > 0$. By the semi-homogeneity property of the measures of noncompactness and their invariance under translations (cf. [1], page 4) we can assume without loss of generality that $d = 1$ and consider only the unit ball B_0 centered at $0 \in X$. Let $Z := \{z_1, z_2, \ldots, z_m\}$ be a finite δ-net of the totally bounded set $Q(B_0)$. Then, the balls of radius $\delta + \|K\| \cdot \|K - Q\|$ with centers in $z_1, z_2, \ldots, z_m$ cover the set $K(B_0)$. Indeed, for a point $x \in K(B_0)$, let $z(x) \in Z$ be such that $\|Qx - z(x)\| \leq \delta$. Then,

$$\begin{aligned}\|Kx - z(x)\| &\leq \|Kx - Qx\| + \|Qx - z(x)\| \leq \|x\| \cdot \|K - Q\| + \delta \\ &\leq \|K\| \cdot \|K - Q\| + \delta \leq \|K\| \cdot \|K - Q\| + \delta.\end{aligned}$$

This completes the proof of the lemma since $\delta > 0$ is arbitrary. □

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