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RANDOM CONTINUED FRACTIONS WITH BETA-HYPERGEOMETRIC DISTRIBUTION

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In a recent paper [*Statist. Probab. Lett.* **78** (2008) 1711–1721] it has been shown that certain random continued fractions have a density which is a product of a beta density and a hypergeometric function ${}_2F_1$. In the present paper we fully exploit a formula due to Thomae [*J. Reine Angew. Math.* **87** (1879) 26–73] in order to generalize substantially the class of random continuous fractions with a density of the above form. This involves the design of seven particular graphs. Infinite paths on them lead to random continued fractions with an explicit distribution. A careful study about the set of five real parameters leading to a beta-hypergeometric distribution is required, relying on almost forgotten results mainly due to Felix Klein.

1. Introduction. Recall that ${}_2F_1$ is the hypergeometric function defined as follows: the sequence of Pochhammer's symbols $\{(t)_n\}_{n=0}^{\infty}$ is given for any real number t by $(t)_0 = 1$ and $(t)_{n+1} = (t+n)(t)_n$. For real numbers p, q, r such that $-r \notin \mathbb{N} = \{0, 1, 2, \dots\}$ and for $0 < x < 1$, the number ${}_2F_1(p, q; r; x)$ is the sum of the power series

$$\sum_{n=0}^{\infty} \frac{(p)_n(q)_n}{n!(r)_n} x^n. \tag{1}$$

Let $v = (a, b, p, q, r)^*$ be in $\mathbb{R}^5$, where * means transposition. Consider the function

$$h(a, b, p, q, r; x) = x^{a-1}(1-x)^{b-1} {}_2F_1(p, q; r; x), \qquad 0 < x < 1, \tag{2}$$

and suppose that it is nonnegative and integrable on $(0, 1)$. The distribution $\mathrm{BH}(v)$ of $X \in (0, 1)$ with density proportional to $h(a, b, p, q, r; x)$ is called a beta hypergeometric distribution. An important example is

$$\mathrm{BH}(1, 1, 1, 1, 2)(dx) = \frac{6}{\pi^2 x} \log \frac{1}{1-x} \mathbf{1}_{(0,1)}(x)\, dx. \tag{3}$$

If $c, d > 0$ and $B(c, d) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}$, the probability measure

$$\beta_{c,d}^{(2)}(dw) = \frac{1}{B(c, d)} \frac{w^{c-1}}{(1+w)^{c+d}} \mathbf{1}_{(0,\infty)}(w)\, dw$$

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is called a beta distribution of the second kind. For instance, if $U \sim \gamma_c$ and $V \sim \gamma_d$, where

$$\gamma_c(du) = \frac{1}{\Gamma(c)} w^{c-1} e^{-u} \mathbf{1}_{(0,\infty)}(w)\, dw$$

is the gamma distribution with shape parameter c and scale parameter 1, we have $U/V \sim \beta^{(2)}_{c,d}$. Next define the matrices

$$M = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \end{bmatrix}, \qquad \Pi = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

We have the following

Basic identity: Let $X \sim \mathrm{BH}(v)$ and $W \sim \beta^{(2)}_{\Pi v}$ be independent. Then

$$\frac{1}{1+WX} \sim \mathrm{BH}(Mv). \tag{5}$$

If $X \sim \mathrm{BH}(v)$, $U \sim \gamma_{(\Pi v)_1}$ and $V \sim \gamma_{(\Pi v)_2}$ are independent, another presentation of the result is saying that $\frac{V}{V+UX} \sim \mathrm{BH}(Mv)$.

In the paper by Asci, Letac and Piccioni (2008) a particular case of the basic identity is proved (viz., for $a = p$ and $b = q$). The proof presented there, directly dealing with densities, could be adapted here in the general case. On the other hand, the algebraic structure of the problem is more clearly outlined by means of an alternative proof which will be given later, using a remarkable formula obtained in 1879 by Thomae, which is an identity concerning the generalized hypergeometric function

$${}_3F_2(A, B, C; D, E; x) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n (C)_n}{n!(D)_n (E)_n} x^n. \tag{6}$$

The approach by the Thomae formula, using exactly five parameters like the basic identity, is illuminating while compared to the long proof by densities. Around this basic identity, various considerations arise:

What are the beta hypergeometric distributions? The exact knowledge of the admissible set of parameters $v = (a, b, p, q, r)^*$ such that the distribution $\mathrm{BH}(v)$ exists needs a careful study of the positivity of ${}_2F_1$ on (0, 1). We rely, for this problem, on a paper by [Klein (1890)], studying the number of zeros of ${}_2F_1$. Section 2 is devoted to a detailed description of a set P of parameters $v = (a, b, p, q, r)^*$ such that the distribution $\mathrm{BH}(v)$ exists. Actually the $\mathrm{BH}(v)$s exist on a set of vs which is slightly larger than P; this larger set has an involved description and is not really useful to our purposes.

Identifiability of the beta hypergeometric distributions. Things are complicated by the fact that the same beta hypergeometric distribution can be generically represented in four ways, first because of the symmetry (p,q), and more importantly by the existence of the Euler formula for ${}_2F_1$,

$$ {}_2F_1(p,q;r;x) = (1-x)^{r-p-q}{}_2F_1(r-p,r-q;r;x) \tag{7}$$

[see Rainville (1960), page 60]. In other terms consider the two 5×5 matrices T and S defined respectively by $T(a,b,p,q,r)^* = (a,b,q,p,r)^*$ and $S(a,b,p,q,r)^* = (a,b+r-p-q,r-p,r-q,r)^*$. The symmetry between p and q implies $\mathrm{BH}(v) = \mathrm{BH}(Tv)$ and the Euler formula implies $\mathrm{BH}(v) = \mathrm{BH}(Sv)$. Actually the group generated by T and S has four elements $\{I_5, T, S, TS\}$ and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Therefore the same beta hypergeometric distribution $\mathrm{BH}(v)$ of X can be generically represented in four ways by v, Tv, Sv and TSv. On the other hand we have $\Pi = \Pi TS$, $\Pi T = \Pi S$. This implies that given v there are only two ways, and not four, to use the basic identity:

Basic identity, revisited. Let $X \sim \mathrm{BH}(v)$ and $W \sim \beta^{(2)}_{\Pi v} = \beta^{(2)}_{\Pi TSv}$ be independent. Then

$$\frac{1}{1+WX} \sim \mathrm{BH}(Mv) = \mathrm{BH}(MTSv).$$

If $X \sim \mathrm{BH}(v)$ and $W \sim \beta^{(2)}_{\Pi Tv} = \beta^{(2)}_{\Pi Sv}$ are independent, then

$$\frac{1}{1+WX} \sim \mathrm{BH}(MTv) = \mathrm{BH}(MSv).$$

The reader can immediately verify that $MTS = TM$ and $MS = TMT$.

Explicit distributions of some random continued fractions. The iteration of this basic identity leads to various types of random continued fractions which are described in Section 4. They are split into seven categories corresponding to the seven partitions of the number 5, which are

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1. \tag{8}$$

For instance, as an application of Theorem 4.2 we see that since $v = (1,1,1,1,2)^*$ is an eigenvector of M for the eigenvalue 1 and that $\Pi v = (1,1)$, if the W_ns with $n = 1,2\ldots$ are i.i.d. such that $\Pr(0 < W_n < w) = \frac{w}{1+w}$ (i.e., if $W_n \sim \beta^{(2)}_{1,1}$); then

$$X = \cfrac{1}{1+\cfrac{W_1}{1+\cfrac{W_2}{\cdots}}}$$

has distribution $\mathrm{BH}(v)$ given by (3). This example belongs to partition 5.

To sum up, Section 2 is a thorough study of BH distributions, Section 3 proves the basic identity and introduces a crucial reparameterization of the BH laws for which the particular symmetry due to the Euler formula appears clearly. Section 4 applies the previous material to random continued fractions after introducing the graphs associated to a BH law. This section contains Theorem 4.2, the main result of the paper. Finally Section 5 is a detailed study of the cycles of the graphs defined in Section 4, which makes simple the application of Theorems 4.2 and 4.3.

2. Beta-hypergeometric distributions. This long section (which can be skipped in first reading) describes the class of probability densities on (0, 1) with five real parameters (a, b, p, q, r) called beta-hypergeometric densities. They have the form $x \mapsto Ch(a, b, p, q, r; x)$ where h is defined by (2) and where C is a suitable constant. To achieve this description of the beta-hypergeometric densities we have to answer to the following questions:

1. *Positivity problem*. When ${}_2F_1(p, q; r; x) \geq 0$ on (0, 1)? The set of admissible (p, q, r) will be denoted by $\mathbb{P}$.
2. *Integrability problem*. If $h = h(a, b, p, q, r; x) \geq 0$, when do we have $\int_0^1 h dx < \infty$? We shall define at the end of Section 2.2 below a set P of admissible (a, b, p, q, r) such that this is fulfilled, with (p, q, r) in $\mathbb{P}$.
3. *Identifiability problem*. If $h(a^*, b^*, p^*, q^*, r^*; x)$ and $h(a, b, p, q, r; x)$ are proportional, what are the relations between $(a^*, b^*, p^*, q^*, r^*)$ and (a, b, p, q, r)?

For doing so we recall a few classical facts about hypergeometric functions. It is clear that if p (or q) is a negative integer, then the power series (1) happens to be a polynomial. In particular, if $p = 0$, then

$$ {}_2F_1(0, q; r; x) \equiv 1 \tag{9} $$

for any value of q and r, whereas if $p = -1$ then

$$ {}_2F_1(-1, q; r; x) = 1 - \frac{q}{r}x, \tag{10} $$

meaning that for $q = sr$, the function (10) is always equal to $1 - sx$. Next recall that, provided $-t \notin \mathbb{N}$, we can express $(t)_n$ as $\Gamma(t+n)/\Gamma(t)$. Such a formula gives meaning to $\Gamma(t)$ when $t < 0$ and $t \notin -\mathbb{N}$.

When p and q are not negative integers (this condition is always assumed on r) we can apply Stirling's approximation to the Gamma function to evaluate the order of the general term of the power series (1). It turns out that it is equivalent (up to a multiplicative constant) to $n^{-1-(r-p-q)}$. This implies that its radius of convergence is equal to 1; therefore the series is certainly well defined for $x \in (0, 1)$. It is convergent in $x = 1$ if and only if $r - p - q > 0$, in which case [see Rainville (1960), page 49]

$$ \lim_{x \uparrow 1} {}_2F_1(p, q; r; x) = \frac{\Gamma(r)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}. \tag{11} $$

This makes sense also if $r - p \in -\mathbb{N}$ and/or $r - q \in -\mathbb{N}$, in which case the r.h.s. is equal to zero. For $r - p - q = 0$ we have

$$ \lim_{x \uparrow 1} \frac{{}_2F_1(p, q; p+q; x)}{-\log(1-x)} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}. \tag{12} $$

2.1. *Positivity.* The well-known fact that ${}_2F_1(p,q;r;x) = z(x)$ is an analytic solution of the second order differential equation

$$x(1-x)z'' + [r-(p+q+1)x]z' - pqz = 0 \tag{13}$$

shows that the zeros of ${}_2F_1(p,q;r;x)$ in the unit disk, and in particular on (0,1), are simple. For if $z(x_0) = z'(x_0) = 0$ we easily see by induction using (13) that $z^{(k)}(x_0) = 0$ for all $k \in \mathbb{N}$ and thus $z \equiv 0$ which contradicts $z(0) = 1$. For this reason ${}_2F_1(p,q;r;x) \geq 0$ for $x \in (0,1)$ if and only if ${}_2F_1$ has no zeros in $(0,1)$. Denote

$$\mathbb{P} = \{(p,q,r) : \in \mathbb{R}^3; r \notin -\mathbb{N}, F(p,q;r;x) > 0, \forall x \in (0,1)\}. \tag{14}$$

The aim of this subsection is to describe $\mathbb{P}$.

The determination of the number of zeros of ${}_2F_1$ is a difficult question which has been investigated in a number of papers, including [Klein (1890), Hurwitz (1891), Van Vleck (1902)]. To this aim Klein introduces the function $x \mapsto E(x)$ on the real line defined by $E(x) = 0$ if $x \leq 1$ and $E(x) = N$ if N is the positive integer such that $N < x \leq N+1$. Klein finally introduces the nonnegative integer

$$X = X(p,q,r) = E\big(\tfrac{1}{2}(|p-q| - |r-1| - |r-p-q| + 1)\big) \tag{15}$$

and proves the following theorem:

THEOREM 2.1. *Suppose that $r \notin -\mathbb{N}$. Then the number $Z = Z(p,q,r)$ of zeros of $x \mapsto F(p,q;r;x)$ in $(0,1)$ is either X or $X+1$. Therefore, a necessary condition for $Z = 0$ is $X = 0$.*

According to the definition given before, $X = 0$ if and only if the argument of the function E at the r.h.s. of (15) does not exceed 1. By a simple calculation it is realized that this happens if and only if:

1. *either $r \geq 1$, $r-p-q \geq 0$ and $r-p, r-q \geq 0$;*
2. *or $r \geq 1$, $r-p-q < 0$ and $p,q \geq 0$;*
3. *or $r < 1$, $r-p-q \geq 0$, $p,q \leq 1$;*
4. *or $r < 1$, $r-p-q < 0$, $r-p, r-q \leq 1$.*

It is immediately verified that either of the first two conditions is also sufficient for $(p,q,r) \in \mathbb{P}$. But this is not true in the remaining cases. Luckily, the question is settled by adding a further necessary condition which allows us to control the positivity of the hypergeometric function near 1. To this aim we introduce the following notation: for $x \in \mathbb{R}$ the number $s(x) \in \{-1, 0, 1\}$ is the sign of $1/\Gamma(x)$ with the convention $\operatorname{sign}(0) = 0$. Therefore $s(x) = 1$ for $x > 0$ or $-2k-2 < x < -2k-1$ with $k \in \mathbb{N}$, $s(x) = -1$ for $-2k-1 < x < -2k$ with $k \in \mathbb{N}$. We need to define the set

$$S = \{(x,y,z) \in \mathbb{R}^3; s(x)s(y)s(y) \geq 0\}. \tag{16}$$

Therefore the part of S which is in the octant $x, y, z \leq 0$ is a union of unit cubes; in the octant $x, y \leq 0$, $z \geq 0$ the set S is a union of columns with unit square section; in the octant $x \leq 0$, $y, z \geq 0$ the set S is the union of slices of unit height; in the octant $x, y, z \geq 0$ S is the octant itself.

THEOREM 2.2. *Suppose that $r \notin -\mathbb{N}$. Then $(p, q, r) \in \mathbb{P}$ if and only if*:

- *either* 1 *holds*;
- *or* 2 *holds*; *or*
- 3 *and* $(r - p, r - q, r) \in S$ *hold*;
- *or* 4 *and* $(p, q, r) \in S$ *hold*.

PROOF. *The case* 3 *and* $(r - p, r - q, r) \in S$, *with* $r - p - q > 0$.

Since $r - p - q > 0$, formula (11) holds. Moreover $\Gamma(r - p - q) > 0$, thus $(r-p, r-q, r) \in S$ guarantees that ${}_2F_1(p, q; r; 1)$ is positive as well, provided $r - p, r - q \notin -\mathbb{N}$. This, together with $X = 0$ implies $Z = 0$, therefore $(p, q, r) \in \mathbb{P}$. When $r - p$ or $r - q \in -\mathbb{N}$ this argument does not work since ${}_2F_1(p, q; r; 1) = 0$. Suppose, for example, $r - p = -n$, where $n \in \mathbb{N}$. Then by Euler's identity (7) and the fact that $X(p, q, r) = X(r - p, r - q, r)$ it is enough to apply the previous argument to ${}_2F_1(r - p, r - q; r; x)$, which is a polynomial of degree at most n, whose value in 1 is given by

$$\begin{aligned} {}_2F_1(-n, r-q; r; 1) &= \frac{1}{(1-n-r)_n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (r-q)_k (1-n-r)_{n-k} \\ &= \frac{(1-n-q)_n}{(1-n-r)_n} > 0 \end{aligned} \tag{17}$$

given that $-n - q = r - p - q > 0$ (thus the numerator is positive) and $-n - r = -p > -1$ (thus the denominator is positive as well).

Case 3, *with $r = p + q$, and $(p, q, p + q) \in S$.*

This time we need to use (12) to prove the positivity of ${}_2F_1(p, q; r; x)$ in a left neighborhood of 1 in order to conclude as before.

Case 4 *and* $(p, q, r) \in S$.

Again using Euler's identity we reduce to apply the previous argument to ${}_2F_1(r - p, r - q; r; x)$, since $p + q - r > 0$. $\square$

2.2. *Integrability.*

PROPOSITION 2.3. *Let $(a, b, p, q, r) \in \mathbb{R}^5$ such that $r \notin -\mathbb{N}$. Assume that $h(a, b, p, q, r; x) > 0$ on $(0, 1)$, that is, $(p, q, r) \in \mathbb{P}$ defined by* (14). *The condition*

$$a, b, r + b - p - q > 0 \tag{18}$$

is sufficient for having $I = \int_0^1 h(a,b,p,q,r;x)\,dx$ *finite. Under these circumstances*

$$I = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \sum_{n=0}^{\infty} \frac{(p)_n(q)_n(a)_n}{n!(r)_n(a+b)_n} = B(a,b)\,{}_3F_2(p,q,a;r,a+b;1). \tag{19}$$

Conversely, if:

1. *either* $r-p-q \geq 0$, $r-p \notin -\mathbb{N}$ *and* $r-q \notin -\mathbb{N}$;
2. *or* $r-p-q<0$, $p \notin -\mathbb{N}$ *and* $q \notin -\mathbb{N}$.

(18) *is also necessary to have I finite.*

PROOF. The first part can be found in Erdelyi et al. (1954), page 399. Conversely, if $r-p-q>0$ and none of $r-p$ and $r-q$ is in $-\mathbb{N}$, we can claim that the limit in (11) exists and is not zero. If, furthermore, h is integrable, clearly a and b are positive, and a fortiori $r+b-p-q>0$. The case $r-p-q<0$ is dealt similarly by means of the Euler identity (7). For $r-p-q=0$ we use expression (12). If $(p,q,p+q)$ is in $\mathbb{P}$, Theorem 2.2 implies that for $p+q \geq 1$, then p and q are positive, and for $p+q<1$ we must have $(p,q,p+q)$ in S; thus $B(p,q)>0$, anyway. Clearly from (12), if h is integrable, then $a,b>0$. □

For the sake of completeness in the following proposition we consider the cases where conditions 1 and 2 of the previous proposition are violated.

PROPOSITION 2.4. *Let* $(a,b,p,q,r) \in \mathbb{R}^5$ *such that* $r \notin -\mathbb{N}$. *Assume that* $h = h(a,b,p,q,r;x) > 0$ *on* $(0,1)$ *or equivalently that* (p,q,r) *is in* $\mathbb{P}$.

1. *If* $r-p-q \geq 0$ *and* $r-p$ *or* $r-q \in -\mathbb{N}$, *then h is integrable in* $(0,1)$ *if and only if* $a, r+b-p-q>0$ (*but* $b \leq 0$ *is allowed*).
2. *If* $r-p-q<0$ *and* p *or* $q \in -\mathbb{N}$, *then h is integrable in* $(0,1)$ *if and only if* $a,b>0$ (*but* $r+b-p-q \leq 0$ *is allowed*).

PROOF. To show the first part, recall that if $(r,p,q) \in \mathbb{P}$, $r-p-q \geq 0$ (case 3) and $r-p=-n$, then, by Euler's identity (7),

$$h(a,b,p,q,r;x) = x^{a-1}(1-x)^{r+b-p-q-1}{}_2F_1(-n,r-q;r;x),$$

the hypergeometric function at the r.h.s. being a polynomial which is positive in $x=1$ [see (17)]. Thus h is integrable if and only if $a, r+b-p-q>0$. The second part is proved in the same way, without the need to use Euler's identity (7). □

REMARK. From the two previous results the cases where $\int_0^1 h\,dx < \infty$ does not imply $\min(b, r+b-p-q)>0$ are truly exceptional. An example is obtained from Proposition 2.4, part 1, with $r=p$ and $q=-2$; in this case $\int_0^1 h\,dx<\infty$ if and only if $a, b+2>0$. Thus, for instance, $b=-1$ is allowed. For this reason we coin the following definition:

DEFINITION. The beta hyperbolic distribution

$$\mathrm{BH}(a,b,p,q,r)(dx)=\frac{x^{a-1}(1-x)^{b-1}{}_2F_1(p,q;r;x)}{B(a,b){}_3F_2(a,p,q;r;a+b;1)}\mathbf{1}_{(0,1)}(x)\,dx$$

[where ${}_3F_2$ has been introduced in (6)] is defined if and only if:

1. $r\notin-\mathbb{N}$;
2. (p,q,r) is in $\mathbb{P}$ described in Theorem 2.2;
3. $a>0$, $b>0$ and $r+b-p-q>0$.

In the sequel we denote by P the set of (a,b,p,q,r) satisfying the three conditions above.

REMARK. It is immediately verified that the set P is invariant under both T and S. It is also immediately seen that the positivity of all the components of the vector $v=(a,b,p,q,r)^*$ (or Sv) together with the condition $r+b-p-q>0$ (or $b>0$) ensure that $v\in P$.

2.3. *Identifiability.* This subsection addresses to the problem of the identifiability. We already know that $\mathrm{BH}(v)=\mathrm{BH}(Sv)=\mathrm{BH}(Tv)=\mathrm{BH}(TSv)$. Therefore BH distributions could have four different representations. On the other hand, formulas (9) and (10) show that the number of representations of the same BH distribution can even be infinite. More generally we will discuss when a function h defined by (2) can be represented with different values of the parameters a,b,p,q and r. The following result says essentially that, aside from the symmetry of ${}_2F_1$ in p and q and Euler's identity (7), the only lack of a unique representation is due to both relations (9) and (10). The theorem does not use the results of Sections 2.1 and 2.2.

THEOREM 2.5. *Suppose that*

$$h(a^*,b^*,p^*,q^*,r^*;x)=Ch(a,b,p,q,r;x),\qquad 0<x<1.$$

Then $C=1$, $a^*=a$ *and*:

(a) *either* $r^*=r$, *and*:

1. *either* $b^*=b$, $\{p^*,q^*\}=\{p,q\}$,
2. *or* $b^*=b+r-p-q$, $\{p^*,q^*\}=\{r-p,r-q\}$;

(b) *or the function has the form*

$$h(a,b,p,q,r;x)=x^{a-1}(1-x)^{u-1}(1-vx). \tag{20}$$

1. *If* $v=0$ *either one between* p *and* q *is zero and* $u=b$, *or one between* p *and* q *is equal to* r, *say* p, *and* $u=b-q$ (*the same obviously holds for* b^*, p^*, q^* *and* r^*).

2. *If $v \neq 0, 1$ (which can be assumed without lack of generality) either one between p and q is equal to -1, say p, $u = b$ and $v = \frac{q}{r}$, or one between p and q is equal to $r + 1$, say p, $u = b - q - 1$ and $v = \frac{r-q}{r}$ (the same obviously holds for b^*, p^*, q^* and r^*).*

REMARKS. We see that, when $v = 0$, and $a, u > 0$, the r.h.s. of (20) yields the (unnormalized) density of the beta law $\beta_{a,u} = \mathrm{BH}(a, u, p, 0, r)$ where p and r are arbitrary. By a proper choice of p and r we can ensure that $(a, u, p, 0, r) \in P$. This shows that beta distributions are BH distributions in the sense of the last remark in Section 2.2.

Next notice that if the r.h.s. of (20) is positive in the whole interval (0, 1), either it is $0 < v < 1$ or v is negative; if, moreover, $a, b > 0$, then h is integrable and can be normalized to become the density of $\mathrm{BH}(a, u, -1, rv, r)$ where r is arbitrary. If v is negative this is clearly a mixture of two beta distributions. By taking r large enough we can ensure $(a, u, -1, rv, r) \in P$. The corresponding law will be called quasibeta and indicated by $q\beta_{a,u,v}$. Thus beta and quasi-beta distributions are beta hypergeometric according to the definition given at the end of Section 2.2.

PROOF OF THEOREM 2.5. By assumption for $x \in (0, 1)$

$$x^{a-1}(1-x)^{b-1}{}_2F_1(p, q; r; x) = Cx^{a^*-1}(1-x)^{b^*-1}{}_2F_1(p^*, q^*; r^*; x).$$

Dividing the r.h.s. by x^{a-1} and going to the limit as $x \downarrow 0$ we get that the l.h.s. converges to 1. Then the same has to hold for the r.h.s., which implies necessarily that $a^* = a$, and $C = 1$. We thus get

$$(1-x)^d{}_2F_1(p, q; r; x) = {}_2F_1(p^*, q^*; r^*; x),$$

where $d = b - b^*$. Recall that ${}_2F_1(p, q; r; x) = z(x)$ is a solution of the second order differential equation (13). In the sequel we fix the four numbers p, q, r, d and we define the linear differential operator $L_d(y)$ as

$$\begin{aligned} x(1-x)y'' &+ [r + (2d - p - q - 1)x]y' \\ &+ \frac{dr - pq + [d(d-p-q) + pq]x}{1-x}y. \end{aligned} \tag{21}$$

It is easy to see that $y(x) = (1-x)^d{}_2F_1(p, q; r; x)$ satisfies $L_d(y)(x) = 0$. On the other hand $y(x) = {}_2F_1(p^*, q^*; r^*; x)$, hence

$$x(1-x)y'' + [r^* - (p^* + q^* + 1)x]y' - p^*q^*y = 0$$

which subtracted from $L_d(y) = 0$ yields $(Cx + D)y' + \frac{Ax+B}{(1-x)}y = 0$ where

$$A = d(d-p-q) + pq - p^*q^*, \qquad B = dr + p^*q^* - pq,$$
$$C = 2d + p^* + q^* - p - q, \qquad D = r - r^*.$$

Let us now suppose that $C = D = 0$, but A and B are not both zero. Then $y \equiv 0$, which is impossible. Now assume $C = 0$, $D \neq 0$. Thus $y' = -\frac{Ax+B}{D(1-x)} y$ and thus

$$y = (1-x)^d {}_2F_1(p, q; r; x) = e^{A/Dx}(1-x)^{(B-A)/D}.$$

Define $d_1 = d - \frac{B-A}{D}$ and $y_1 = e^{A/Dx}$. Then y_1 satisfies $L_{d_1}(y_1) = 0$ [using notation of (21)]. Since $L_{d_1}(y_1)(x) = \frac{P(x)}{1-x} y_1(x)$, where P is a polynomial of degree 3 with leading coefficient $(\frac{A}{D})^2$ this implies that $A = 0$. Therefore $y = (1-x)^{(B-A)/D}$ and we are in the second case. Next let $C \neq 0$ and $D = 0$, in which case

$$\frac{y'}{y} = -\frac{Ax+B}{Cx(1-x)} = \frac{C_1}{x} - \frac{C_2}{1-x},$$

and therefore $y = C_3 x^{C_1}(1-x)^{C_2}$. Since $y(0) = 1$, this implies that $C_1 = 0$ so we are in the second case again. If $C \neq 0$ and $D \neq 0$, then we can write

$$y' = \frac{\alpha x + \beta}{(1+cx)(1-x)} y,$$

where $c = \frac{C}{D} \neq 0$. We now distinguish between the cases $c \neq -1$ and $c = -1$. If $c \neq -1$, then

$$\frac{y'}{y} = -\frac{cC_1}{1+cx} - \frac{C_2}{1-x}$$

from which $y = (1+cx)^{-C_1}(1-x)^{C_2}$. Define $d_2 = d - C_2$ and $y_2 = (1+cx)^{-C_1}$. Then y_2 satisfies $L_{d_2}(y_2) = 0$. This implies that $C_1(C_1+1) = 0$, therefore either $C_1 = 0$ [thus we are in case (b)] or $C_1 = -1$ [thus we are in case (c)]. Next suppose that $c = -1$. Then

$$\frac{y'}{y} = \frac{\alpha x + \beta}{(1-x)^2} = -\frac{C_1}{1-x} - \frac{C_2}{(1-x)^2}$$

from which $y = (1-x)^{C_1} \exp\{\frac{C_2}{1-x}\}$. Define $d_3 = d - C_1$ and $y_3 = \exp\{\frac{C_2}{1-x}\}$. Then y_3 satisfies $L_{d_3}(y_3) = 0$ [using notation of (21)]. Since $L_{d_3}(y_3)(x) = \frac{P(x)}{(1-x)^3} y_3(x)$, where P is a polynomial such that $P(1) = C_2^2$, which implies that $C_2 = 0$, which falls into case (b). Finally suppose that $A = B = C = D = 0$. From $D = 0$ we get $r^* = r$. Furthermore

$$A = B \quad \Longleftrightarrow \quad d(r + d - p - q) = 0.$$

Thus either $d = 0$: as a consequence $b = b^*$, moreover $p + q = p^* + q^*$ (from $C = 0$) and $pq = p^*q^*$ (from $A = 0$), which means $\{p, q\} = \{p^*, q^*\}$. Or $d = p + q - r$: as a consequence $b^* = b + r - p - q$, moreover $2r - p - q = p^* + q^*$ (from $C = 0$) and $(r-p)(r-q) = p^*q^*$ (from $A = 0$), which means $\{r-p, r-q\} =$

$\{p^*, q^*\}$. Thus the condition $A = B = C = D = 0$ yields case (a) of the theorem. To complete part (b), if $f_v(x) = x^{a-1}(1-x)^{c-1}$, we have

$$(1-x)^{b-c}{}_2F_1(p, q; r; x) \equiv 1.$$

Thus define $d_4 = b - c$ and $y_4 \equiv 1$. Then y_4 satisfies $L_{d_4}(y_4) = 0$, which implies $d_4 r - pq = 0$ and $d_4(d_4 - p - q) + pq = 0$. Summing the two equalities we get $d_4(d_4 + r - p - q) = 0$. Thus either $d_4 = 0$ (and $c = b$) and $pq = 0$; or $d_4 + r - p - q = 0$ (and $c = b + r - p - q$) and $(r - p)(r - q) = 0$. Finally, to complete part (c), if $f_v(x) = x^{a-1}(1-x)^{b_1-1}(1+cx)$, then

$$(1-x)^{b_1}(1+cx) = (1-x)^b{}_2F_1(p, q; r; x).$$

Define $d_5 = b - b_1$ and $y_5(x) \equiv 1 + cx$. Then y_5 satisfes $L_{d_5}(y_5) = 0$. This means

$$c(r + (2d_5 - p - q - 1)x) + \frac{d_5 r - pq + (d_5(d_5 - p - q) + pq)x}{1-x}(1+cx) = 0.$$

Since $c \neq -1{,}0$, this implies $d_5(d_5 + r - p - q) = 0$. As a consequence the fractional term is the constant $d_5 r - pq$, so by equating the coefficients of the polynomial at the l.h.s. to zero we get the two equations

$$\begin{cases} c(2+r)d_5 = (p+1)(q+1), \\ (c + d_5)r = pq. \end{cases}$$

Thus either $d_5 = 0$, in which case $b_1 = b$, and $(p+1)(q+1) = 0$ and $cr = pq$. Thus if, say, $p = -1$, we get $q = -cr$ as stated in the theorem. Or $d_5 = p + q - r$, in which case $b_1 = b + r - p - q$, and we get similarly $(r - p + 1)(r - q + 1) = 0$ and $cr = (r - p)(r - q)$. Thus if, say, $r - p = -1$, we get $r - q = -cr$ as stated in the theorem. □

2.4. *Examples*. For illustrating the above results, recall that a few classical equalities can be found, for instance, in Abramovitz and Stegun (1965), pages 556–557. The identities describing a hypergeometric function in terms of $x = \sin^2\theta$ are sometimes useful for describing the density of the image of $\mathrm{BH}(a, b, p, q, r)$ under $x \mapsto \theta = \arcsin\sqrt{x}$ which has a density proportional to $\sin^{2a-1}\theta\cos^{2b-1}\theta\,{}_2F_1(p, q; r; \sin^2\theta)$ on the interval $(0, \pi/2)$. Let us comment also on the case of the distributions

$$\mathrm{BH}(a, a, a, a; 2a)(dx) = C\,{}_2F_1(a, a; 2a; x)\beta_{a,a}(dx).$$

Since the vectors $v = (a, a, a, a, 2a)^*$ satisfy $Mv = v$ where M is defined by (4), the application of (5) to $W \sim \mathrm{BH}(v)$ is specially simple. The case $a = 1$ is (3). It is possible to make the laws explicit when $a = k$ is a positive integer: in this circumstance the normalizing constant $1/C$ is

$$c_k = {}_3F_2(k, k, k; 2k, 2k; 1) = \frac{(2k-1)!)^2}{(k-1)!^3}\sum_{n=0}^{\infty} u_k(n),$$

where

$$u_k(n) = \frac{(n+1)(n+2)\cdots(n+k-1)}{(n+k)^2(n+k+1)^2\cdots(n+2k-1)^2}.$$

The sum of this series can be explicitely computed for fixed k by expansion of the rational function $n \mapsto u_k(n)$ in partial fractions. The case $c_1 = \frac{\pi^2}{6}$ has been used in (3). A hand calculation gives $c_2 = 18(10-\pi^2)$ and $c_3 = -(21/2)^2 \times 127 + 2850\pi^2$. More generally we have $c_k = a_k + b_k\pi^2$ where a_k and b_k are rational numbers. The density uses

$${}_2F_1(k,k;2k;z) = \frac{(2k-1)!}{(k-1)!^2}\sum_{n=0}^{\infty} v_k(n)x^n,$$

where

$$v_k(n) = \frac{(n+1)(n+2)\cdots(n+k-1)}{(n+k)(n+k+1)\cdots(n+2k-1)}.$$

Again, by expansion of the rational function $n \mapsto v_k(n)$ in partial fractions we get that

$${}_2F_1(k,k;2k;x) = A_k(1/x) + B_k(1/x)\log\frac{1}{1-x},$$

where

$$B_k(x) = \frac{(2k-1)!}{(k-1)!^2}\sum_{j=1}^{k}(-1)^{k-j}\frac{(j)_{k-1}}{(j-1)!(k-j)!}x^{k+j-1},$$

and where $A_k(x)$ the unique polynomial of degree $\leq 2k-2$, with $A_k(0) = 0$, such that $A_k(1/x) + B_k(1/x)\log(\frac{1}{1-x})$ is analytic on zero. In particular $A_1(x) = 0$, $B_1(x) = x$, $A_2(x) = -6x^2$, $B_2(x) = -3x^2 + 6x^3$ and $A_3(x) = 90x^3(1-2x)$, $B_3(x) = 30x^3(1-6x+6x^2)$. Therefore the densities on $(0,1)$ of $\mathrm{BH}(2,2,2,2,4)$ and $\mathrm{BH}(3,3,3,3,6)$ are respectively

$$\frac{1}{10-\pi^2}\left[-\frac{2}{x} + \frac{2-x}{x^2}\log\frac{1}{1-x}\right](1-x),$$

$$\frac{900}{c_3}\left[\frac{6-3x}{x^2} + \frac{6-6x+x^2}{x^3}\log\frac{1}{1-x}\right](1-x)^2.$$

These remarks are a significant specialization of 15.4.7 in [Abramovitz and Stegun (1965)] where ${}_2F_1(a,b;2b;x)$ is rather expressed in terms of Legendre functions.

3. Proof of the basic identity. The θ parameterization. *Mellin-like transforms of* $\mathrm{BH}(v)$. Assume now that $v = (a,b,p,q,r)^*$ is in P. Recall that this set P has been defined at the very end on Section 2.2 and is

$$P = \{(a,b,p,q,r); (p,q,r) \in \mathbb{P}, r \notin -\mathbb{N}, a, b, r+b-p-q > 0\}. \tag{22}$$

(The set $\mathbb{P}$ is described in the statement of Theorem 2.2.) Let us fix (s,t) such that $(a+s, b+t, p, q, r)$ is in P or equivalently such that $s > -a$ and

$$t > -b - \min\{0, r-p-q\}.$$

We get from (19) the important formula

$$\begin{aligned}
(23)\qquad &\int_0^1 x^s(1-x)^t\,\mathrm{BH}(v)(dx)\\
&\quad= \frac{B(a+s, b+t)_3F_2(p, q, a+s; r, a+b+s+t; 1)}{B(a,b)_3F_2(p,q,a;r,a+b;1)}.
\end{aligned}$$

The Thomae formula. It is a fundamental relation between Euler's gamma function Γ and the generalized hypergeometric function ${}_3F_2$ defined by (6) and evaluated at $x = 1$.

LEMMA 3.1 (Thomae's formula). *The function*

$$\mathcal{T}(A, B, C, D, E) = \frac{\Gamma(C)_3F_2(A, B, C; D, E; 1)}{\Gamma(D)\Gamma(E)}$$

has the invariance property

$$\begin{aligned}
&\mathcal{T}(A, B, C, D, E)\\
&\quad= \mathcal{T}(D-C, E-C, D+E-A-B-C,\\
&\qquad\qquad D+E-A-C, D+E-B-C).
\end{aligned}$$

REMARK. This relation was originally obtained by Thomae (1879). In the papers by Maier (2006) and Beyer, Louck and Stein (1987) such a relation is reformulated as the invariance of a suitably defined function with respect to the symmetric group $\mathcal{S}_5$. See Bailey (1935) and Andrews, Askey and Roy (1999) for two different proofs and Asci, Letac and Piccioni (2008) for a probabilistic one based on the following idea: if U and V are two arbitrary beta random variables on $(0,1)$ and if s is real, compute $\mathbb{E}((1-UV)^s)$ in two ways: expansion in a series of powers of UV or computation of the density of UV by multiplicative convolution. We obtain in this way an identity involving the parameter s and the four parameters of the beta distributions. This identity is equivalent to the Thomae's formula. It should be emphasized that Thomae's formula is an equality between analytic functions in their whole domain of analyticity, so it certainly holds true when all the arguments $A, B, C, D-C, E-C$ and $D+E-A-B-C$ are positive.

Proof of the basic identity (5).

THEOREM 3.2. *Let $v = (a, b, p, q, r)^*$ be in P, and assume $a + b - p > 0$ and $r - a > 0$. Let M and Π be the matrices defined in* (4). *Then Mv is in P and if $X \sim \mathrm{BH}(v)$ and $W \sim \beta^{(2)}_{\Pi v}$ are independent we have*

$$\frac{1}{1+XW} \sim \mathrm{BH}(Mv).$$

PROOF. Denote $Mv = (a', b', p', q', r')^*$. The facts that v is in P, $b' = a + b - p > 0$ and $a' = r - a > 0$ imply that $p' = r + b - p - q > 0$ and $q' = b > 0$. It is more delicate to prove that $r' = r + b - q > 0$. This certainly holds when $q \le 0$ and, from $r + b - p - q > 0$, it holds when $p \ge 0$ as well. Next we assume both $p < 0$ and $q > 0$. Since $(p, q, r) \in \mathbb{P}$ we make the following deductions. Case 1 is impossible. In case 2, from $r - q \ge 0$, we get $r + b - q > 0$. In case 3 we have $(p, q, r) \in S$, and since $q > 0$ and $r > a > 0$, it has to be $1/\Gamma(p) \ge 0$ which is fulfilled with $p < 0$ only when $p = -n$, n being a positive integer. This is excluded since it would imply $r - p > n$, a contradiction with $r - p \le 1$. Finally, in case 4 we have $0 < r < 1$, $0 < q \le 1$ and $1/\gamma(r-q) \ge 0$. Thus either $r - q > 0$, which implies $r + b - q > 0$, or $-1 < r - q \le 0$. The only possibility is thus $r = q$, in which $r + b - q > 0$ as well. Finally, the inequality $r' + b' - p' - q' = a > 0$ together with the previous ones implies $Mv \in P$ [see the remark at the end of Section 2.2].

Now denote $Y = \frac{1}{1+XW}$. This implies $\frac{1-Y}{Y} = XW$ and therefore for $t \in (-b, a)$ we can write $\mathbb{E}((1-Y)^t Y^{-t}) = \mathbb{E}(X^t)\mathbb{E}(W^t)$. Because in the following calculation the constants are quite long to write, let us adopt the following convention: we say that two positive functions f and g of $t \in (-b, a)$ are equivalent if $t \mapsto f(t)/g(t)$ is a constant with respect to t. This fact is denoted $f \equiv g$ or (with some abuse of notation) $f(t) \equiv g(t)$. With this convention we get by replacing (s, t) in (23) by $(t, 0)$

$$\mathbb{E}(X^t) \equiv \frac{\Gamma(a+t)}{\Gamma(a+b+t)}\, {}_3F_2(p, q, a+t; r, a+b+t; 1)$$

as well as

$$\mathbb{E}(W^t) \equiv \Gamma(a+b-p+t)\Gamma(r-a-t),$$

since the Mellin transform of the law $\beta^{(2)}_{c,d}$ is $t \mapsto \Gamma(c+t)\Gamma(d-t)$. This implies

$$\mathbb{E}\big((1-Y)^t Y^{-t}\big) \equiv \frac{\Gamma(a+b-p+t)\Gamma(r-a-t)\Gamma(a+t)}{\Gamma(a+b+t)} \times {}_3F_2(p, q, a+t; r, a+b+t; 1). \tag{24}$$

From formula (23) note that

$$
\begin{aligned}
&\int_0^1 y^t(1-y)^{-t}\,\mathrm{BH}(Mv)(dy)\\
&\qquad \equiv \Gamma(a'+t)\Gamma(b'-t)\\
&\qquad\qquad \times {}_3F_2(p',q',a'+t;r',a'+b';1)\\
&\qquad \equiv \Gamma(r-a+t)\Gamma(a+b-p-t)\\
&\qquad\qquad \times {}_3F_2(r+b-p-q,b,r-a+t;r+b-q,r+b-p;1),
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
&a' = r-a, \qquad b' = a+b-p, \qquad p' = r+b-p-q,\\
&q' = b, \qquad r' = r+b-q.
\end{aligned}
\tag{26}
$$

The knowledge of the function $t \mapsto \mathbb{E}((1-Y)^tY^{-t})$ over a nonempty open interval determines the distribution of Y. More specifically, this function is the Laplace transform of $f(Y) = \log(Y^{-1}-1)$, and f is injective on $(0,1)$. Therefore enough is to show that the r.h.s. of (24) and (25) are equivalent. To see this we simply apply the Thomae formula (Lemma 3.1) to

$$
A = p, \qquad B = q, \qquad C = a+t, \qquad D = r, \qquad E = a+b+t,
$$

and this concludes the proof of Theorem 3.2. $\square$

The θ parameterization. Up to now, the BH distributions have been parametrized by $v = (a,b,p,q,r)$ belonging to the subset P of $\mathbb{R}^5$ described in (22). One defect of this parameterization is the fact that

$$
\mathrm{BH}(v) = \mathrm{BH}(Sv)
$$

(as implied by the Euler formula) is not apparent. A second defect of the parameterization is that it makes complicated the application of the basic identity. For these reasons we choose to make a linear transformation of v as follows. Introduce a 5-tuple of parameters $\theta = (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) \in \mathbb{R}^5$. For notational convenience, we set

$$
\begin{aligned}
&a = \theta_4+\theta_5, \qquad b = \theta_1+\theta_3, \qquad p = \theta_3+\theta_5,\\
&q = \theta_3+\theta_4, \qquad r = \theta_2+\theta_3+\theta_4+\theta_5.
\end{aligned}
\tag{27}
$$

This can be inverted as

$$
\begin{aligned}
&2\theta_1 = a+2b-p-q, \qquad 2\theta_2 = -a-p-q+2r,\\
&2\theta_3 = -a+p+q, \qquad 2\theta_4 = a-p+q, \qquad 2\theta_5 = a+p-q.
\end{aligned}
\tag{28}
$$

For the reader's convenience these are compactly written as $v = Q\theta$ and $\theta = Q^{-1}v$, where

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad Q^{-1} = \begin{bmatrix} 1/2 & 1 & -1/2 & -1/2 & 0 \\ -1/2 & 0 & -1/2 & -1/2 & 1 \\ -1/2 & 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & -1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & -1/2 & 0 \end{bmatrix}. \tag{29}$$

From now on we denote $\mathrm{BH}_\theta = \mathrm{BH}(a, b, p, q, r)$ where (a, b, p, q, r) is given by (27). For example, important particular cases like BH(1, 1, 1, 1, 2) and BH(k, k, k, k, $2k$) considered in (3) and in Section 2.4 are rewritten as $\mathrm{BH}_{k/2,k/2,k/2,k/2,k/2}$. We say that BH_θ exists when $(a, b, p, q, r) \in P$. Necessary and sufficient conditions for this are given in Proposition 3.4 below. This new parameterization has many advantages. We see immediately that exchange of p with q is equivalent to the exchange of θ_4 with θ_5, whereas, since

$$r + b - p - q = \theta_1 + \theta_2, \qquad r - p = \theta_2 + \theta_4, \qquad r - q = \theta_2 + \theta_5$$

whereas Euler's identity corresponds to exchange θ_2 with θ_3 and θ_4 with θ_5. Therefore actually the distribution

$$\mathrm{BH}_{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5} = \mathrm{BH}_{\theta_1,\{\theta_2,\theta_3\},\{\theta_4,\theta_5\}} \tag{30}$$

being symmetric in (θ_2, θ_3) and (θ_4, θ_5) has rather to be considered as depending on θ_1 and on the two sets $\{\theta_2, \theta_3\}$, $\{\theta_4, \theta_5\}$. This notation $\mathrm{BH}_{\theta_1,\{\theta_2,\theta_3\},\{\theta_4,\theta_5\}}$ is, however, a slight abuse of language since the set $\{\theta_2, \theta_3\}$ could be reduced at one point if $\theta_2 = \theta_3$, and the language of multisets (sets with entire positive weights) could be more adapted. Up to this, we consider that the notations (30) are sufficiently informative. The revisited basic identity can be reformulated in this new notation as follows:

THEOREM 3.3. *Let $X \sim \mathrm{BH}_{\theta_1,\{\theta_2,\theta_3\},\{\theta_4,\theta_5\}}$, $W \sim \beta^{(2)}_{\theta_1+\theta_5,\theta_2+\theta_3}$ and $W' \sim \beta^{(2)}_{\theta_1+\theta_4,\theta_2+\theta_3}$ such that (W, W') are independent of X. Then*

$$\frac{1}{1 + WX} \sim \mathrm{BH}_{\theta_5,\{\theta_1,\theta_4\},\{\theta_2,\theta_3\}}, \qquad \frac{1}{1 + W'X} \sim \mathrm{BH}_{\theta_4,\{\theta_1,\theta_5\},\{\theta_2,\theta_3\}}.$$

Theorem 3.3 shows again that there are two ways to apply the basic identity. It shows also that the matrix M appearing in (4) is similar to a permutation matrix of order 5.

The existence of BH_θ. In order to check whether $\mathrm{BH}_{\theta_1,\{\theta_2,\theta_3\},\{\theta_4,\theta_5\}}$ does exist we dissymetrize $\{\theta_2, \theta_3\}$ and $\{\theta_4, \theta_5\}$ by assuming $\theta_2 \le \theta_3$ and $\theta_4 \le \theta_5$. Recall that the subset S of $\mathbb{R}^3$ has been defined in (16) and is the set of (x, y, z) such that $1/\Gamma(x)\Gamma(y)\Gamma(z) \ge 0$. The condition $(a, b, p, q, r) \in P$ where P is given by (22) gives the following:

PROPOSITION 3.4. *The distribution* $\mathrm{BH}_{\theta_1,\{\theta_2,\theta_3\},\{\theta_4,\theta_5\}}$, *where* $\theta_2 \le \theta_3$ *and* $\theta_4 \le \theta_5$ *exists if and only if*:

- $\theta_1 + \theta_2 > 0$ *and* $\theta_4 + \theta_5 > 0$, *and*
- *either* $r = \theta_2 + \theta_3 + \theta_4 + \theta_5 \ge 1$ *and* $\theta_3 + \theta_4 > 0$,
- *or* $r = \theta_2 + \theta_3 + \theta_4 + \theta_5 < 1$, $\theta_2 + \theta_5 \le 1$ *and* $(\theta_3 + \theta_4, \theta_3 + \theta_5, r) \in S$.

DEFINITION. We will call Θ the set of parameters $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \mathbb{R}^5$ which is the image of P by the linear map $(a, b, p, q, r) \mapsto (\theta_1, \dots, \theta_5)$ described by (28). The part of the set Θ such that $\theta_2 \le \theta_3$ and $\theta_4 \le \theta_5$ is also described by Proposition 3.4.

REMARK. From (27) one can observe that $a, b, p, q, r, r + b - p - q > 0$ is equivalent to

$$\theta_1 + \theta_2, \theta_1 + \theta_4, \theta_3 + \theta_4, \theta_4 + \theta_5, \theta_2 + \theta_3 + \theta_4 + \theta_5 > 0. \tag{31}$$

Thus the vectors θ satisfying the inequalities (31) belong necessarily to Θ. This certainly happens if θ is such that $\theta_i + \theta_j > 0$ for all $1 \le i < j \le 5$ except for $(i, j) = (2, 4)$ (here $\theta_2 \le \theta_3$ and $\theta_4 \le \theta_5$). Thus in this case BH_θ exists (this observation will turn out to be useful for Theorem 5.1 below). Moreover, recalling the first part of the proof of Theorem 3.2 we see that the application of the basic identity always yields beta hypergeometric distributions with the vector of parameters θ satisfying the inequalities (31).

4. Random continued fractions with a beta hypergeometric distribution. It is clear that the iteration of the random transformations appearing in Theorem 3.3, applied to $X \sim \mathrm{BH}_\theta$ with $\theta \in \Theta$, yields random variables whose distribution is of the form $\mathrm{BH}_{\theta'}$ where $\theta' \in \Theta$ is obtained from a permutation of the components of θ.

For this reason, for any $\theta \in \mathbb{R}^5$ we define the finite subset $V_\theta \subset \mathbb{R}^5_s$ of vectors θ^* which can be obtained in this way. Motivated by Theorem 3.3, we are going to define a directed graph structure on V_θ. The possible forms of these graphs will be quite limited.

4.1. *The graphs* G_θ *and their subgraphs.*

The role of the seven partitions of 5. There are seven partitions of 5 enumerated in (8). To each point $\theta \in \mathbb{R}^5$ we attach the discrete measure on $\mathbb{R}$ which is $\sum_{j=1}^5 \delta_{\theta_j} = \sum_{k=1}^n m_k \delta_{x_k}$ where $\{x_1, \dots, x_n\}$ is the image of $j \mapsto \theta_j$ and where m_k is the positive integer which is the number of $j = 1, \dots, 5$ such that $\theta_j = x_k$. Thus $n \le 5$, and $m_1 + \dots + m_n = 5$ defines the partition of 5 induced by $\theta \in \mathbb{R}^5$. For convenience in the sequel we take $m_1 \ge m_2 \ge \dots \ge m_n$, and we write

$$x = x_1, \qquad y = x_2, \qquad z = x_3, \qquad u = x_4, \qquad v = x_5$$

when these x_k do exist. Suppose, for instance, that the partition attached to θ is $3+2$. Therefore we shall use three times the letter x and two times the letter y; the 5 elements of V_θ will be

$$(x,\{x,x\},\{y,y\}),(y,\{x,y\},\{x,x\}),(x,\{y,y\},\{x,x\}),$$
$$(y,\{x,x\},\{x,y\}),(x,\{x,y\},\{x,y\}),$$

that we quickly code

$$x|x^2|y^2,\ y|xy|x^2,\ x|y^2|x^2,\ y|x^2|xy,\ x|xy|xy. \tag{32}$$

The directed graph G_θ. According to Theorem 3.3, given $\theta\in\mathbb{R}^5$, if $\theta\in\Theta$ (which means that BH_θ exists) and if $\theta_1+\theta_5$, $\theta_2+\theta_3$ and $\theta_1+\theta_4$ are positive we can move to two new beta hypergeometric distributions. Ignore for a while the constraints linked to inequalities. Let us extend the process to the whole θs in $\mathbb{R}^5$ or rather to the elements of the quotient described by $(\theta_1,\{\theta_2,\theta_3\},\{\theta_4,\theta_5\})$: from this element we move either to $(\theta_5,\{\theta_1,\theta_4\},\{\theta_2,\theta_3\})$ or to $(\theta_4,\{\theta_1,\theta_5\},\{\theta_2,\theta_3\})$. These two elements may not be distinct.

We need to introduce in (32) the arrows

$$\begin{aligned}
x|x^2|y^2 &\to y|xy|x^2,\\
y|xy|x^2 &\to x|xy|xy,\\
x|y^2|x^2 &\to x|x^2|y^2,\\
y|x^2|xy &\to x|y^2|x^2,\ y|xy|x^2,\\
x|xy|xy &\to y|x^2|xy,\ x|xy|xy
\end{aligned}$$

getting the following directed graph, called the graph G_θ:

$$\begin{tikzcd}
x|x^2|y^2 \arrow[r] & y|xy|x^2 \arrow[dr] & \\
x|y^2|x^2 \arrow[u] & y|x^2|xy \arrow[l] \arrow[u] & x|xy|xy. \arrow[l] \arrow[loop above]
\end{tikzcd} \tag{33}$$

It is clear that the graph G_θ is the same for all $\theta\in\mathbb{R}^5$ sharing the same partition of 5 given by the numbers $m_1\ge m_2\ge\cdots\ge m_n$. Thus we explore the 7 possible forms of this graph. We are particularly interested in determining the cycles in these graphs. A cycle of order $k\ge 2$ in a directed graph is a sequence $v_0,\dots,v_{k-1}$ of *distinct* vertices such that (v_{k-1},v_0) and (v_i,v_{i+1}) are oriented edges of the graph for $i=0,\dots,k-2$. A cycle of order 1 is a vertex v such that (v,v) is an edge).

Description of the seven graphs G_θ:

- *Case* 5. Here $\theta = (x, x, x, x, x)$, and the graph G_θ is rather trivial with one point and one cycle of order 1

$$\begin{tikzcd} x|x^2|x^2 \arrow[loop above] \end{tikzcd}\ . \tag{34}$$

- *Case* $4+1$. Here the graph has three vertices and is

$$\begin{tikzcd} x|x^2|xy \arrow[r, leftrightarrow] \arrow[d] & x|xy|x^2 \\ y|x^2|x^2. \arrow[ur] & \end{tikzcd} \tag{35}$$

  It has two cycles of order 2 and 3.
- *Case* $3+2$. The graph has already been drawn in (33). It has three cycles of order 1, 3 and 5.
- *Case* $3+1+1$. Here the graph has eight vertices and is

$$\begin{tikzcd} & z|x^2|xy \arrow[r] \arrow[d] & x|yz|x^2 \arrow[d] & y|x^2|xz \arrow[l] \arrow[d] & \\ & y|xz|x^2 \arrow[dl] & x|x^2|yz \arrow[l] \arrow[r] & z|xy|x^2 \arrow[dr] & \\ x|xz|xy \arrow[uur] \arrow[rrrr, leftrightarrow] & & & & x|xz|xy. \arrow[uul] \end{tikzcd}$$

  There is one cycle of order 2, two of orders 3, 5 and 6.
- *Case* $2+2+1$. Here the graph has 11 vertices and is

$$\begin{tikzcd} & & z|y^2|x^2 \arrow[dll] & & \\ x|xz|y^2 \arrow[r] & y|xy|xz \arrow[rr] \arrow[dr] & & x|yz|xy \arrow[r] \arrow[dd] & y|x^2|yz \arrow[ull] \arrow[dd] \\ & & z|xy|xy \arrow[ur] \arrow[dl] & & \\ x|y^2|xz \arrow[uu] \arrow[drr] & y|xz|xy \arrow[l] \arrow[uu] & & x|xy|yz \arrow[ll] \arrow[ul] & y|yz|x^2 \arrow[l] \\ & & z|x^2|y^2. \arrow[urr] & & \end{tikzcd} \tag{36}$$

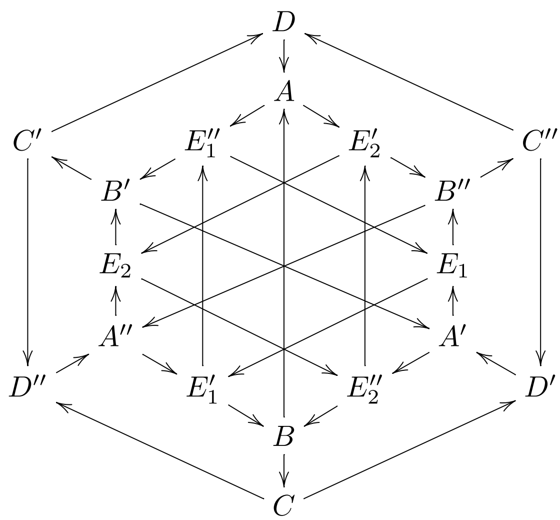
  There are two cycles of order 3, one of order 4, six of order 5, four of order 6, two of order 7 and 9 and one of order 8.
- *Case* $2+1+1+1$. The 18 vertices are

$$\begin{aligned} &A = x|xu|yz, && A' = x|xy|uz, && A'' = x|xz|uy,\\ &B = x|yz|xu, && B' = x|uz|xy, && B'' = x|uy|xz,\\ &C = u|x^2|yz, && C' = y|x^2|yz, && C'' = z|x^2|uy, \end{aligned}$$

$$D = u|yz|x^2, \qquad D' = y|uz|x^2, \qquad D'' = z|uy|x^2,$$
$$E_1 = u|xz|xy, \qquad E_1' = y|ux|xz, \qquad E_1'' = z|xy|ux,$$
$$E_2 = u|xy|xz, \qquad E_2' = y|xz|ux, \qquad E_2'' = z|ux|xy.$$

Here is the graph.

(37)



There are two cycles of order 3, twelve of order 5, nine of order 6, three of order 7, nine of order 8, eight of order 9, three of order 10, three of order 12, six of order 13, six of order 14, two of order 15, nine of order 16.

- *Case* $1+1+1+1+1$. The graph has 30 vertices and is too complicated to be drawn here. The two edges issued from $u|vx|yz$ are given by $u|vx|yz \to y|uz|vx, z|uy|vx$. There are exactly two incoming edges, coming from $x|yz|uv$ and $v|yz|ux$. There is a large number of cycles in this graph; the following remark helps in their determination.

A remark about the automorphisms of the graphs and their cycles. The graphs $3+1+1, 2+2+1, 2+1+1+1+1$ and $1+1+1+1+1$ have automorphisms induced by the permutations of the letters. For instance, the vertices of the graph $2+1+1+1$ are coded by letters x^2yzu and the 6 permutations of yzu induce a group G of automorphisms of the graph. Clearly, G transforms a cycle of size k into a cycle of size k. Therefore the set of cycles of size k is split into orbits. For the simpler graphs $3+1+1, 2+2+1$, and $2+1+1+1$ the number of orbits can be easily found by hand. We indicate below the number of orbits of size k for the graph $1+1+1+1+1$, which have been determined by computer. We have not displayed the sometimes quite large number of cycles of each order as we did for the six others. We get the following results:

$$\begin{aligned}
3+1+1 &: 2(1), 3(1), 5(1), 6(1);\\
2+2+1 &: 3(1), 4(1), 5(3), 6(2), 7(1), 8(1), 9(1);\\
2+1+1+1 &: 3(1), 5(2), 6(2), 7(1), 8(2), 9(2),\\
&\quad 10(1), 12(1), 13(1), 14(1), 15(1), 16(3);\\
1+1+1+1+1 &: 5(1), 6(1), 8(1), 9(1), 12(2), 13(1),\\
&\quad 14(3), 15(4), 16(7), 17(3), 18(4), 19(8),\\
&\quad 20(7), 22(7), 23(10), 24(2), 26(15), 30(4).
\end{aligned}$$

To understand this array, 30(4) on the last line means that the graph $1+1+1+1+1$ has 4 different orbits on the set of cycles of order 30 (the existence of cycles of order 30 implies that the graph is Hamiltonian).

The two subgraphs G^*_θ and G^{**}_θ of G_θ. Consider such a graph $G = G_\theta$ that we have just defined. Denote

$$v_\theta = (\theta_1, \{\theta_2, \theta_3\}, \{\theta_4, \theta_5\}) \tag{38}$$

and write $v_0 = v_\theta$. If (v_0, v) is an edge of the graph recall that either $v = v_1 = (\theta_5, \{\theta_1, \theta_4\}, \{\theta_2, \theta_3\})$ or $v = v_2 = (\theta_4, \{\theta_1, \theta_5\}, \{\theta_2, \theta_3\})$. We say that that the edge (v_0, v) is admissible if $\theta_2 + \theta_3 > 0$ and either $\theta_1 + \theta_5 > 0$ when $v = v_1$ or $\theta_1 + \theta_4 > 0$ when $v = v_2$. We denote by G^*_θ the subgraph of G when we remove the nonadmissible edges. Finally, in the graph G^*_θ let us remove the vertices v such that BH_v does not exist. We also remove the edges of G^*_θ which are adjacent to these erased vertices. The remaining graph is denoted by G^{**}_θ. A detailed example is in order: we start from $\theta = z|y^2|x^2$ where x, y, z are distinct real numbers. Therefore G_θ is of the $2+2+1$ type, and it is graph (36). We now assume $x + y > 0$, $y + z > 0$, $y < 0$. This leads to the following graph G^*_θ:

$$\begin{tikzcd}
 & & z|y^2|x^2 & & \\
x|xz|y^2 \arrow[r] & y|xy|xz \arrow[rr] \arrow[dr] & & x|yz|xy \arrow[r] \arrow[d] & y|x^2|yz \arrow[ull] \\
 & & z|xy|xy \arrow[ur] \arrow[dl] & & \\
x|y^2|xz & y|xz|xy \arrow[l] & & x|xy|yz \arrow[ll] \arrow[ul] & y|yz|x^2 \arrow[l] \\
 & & z|x^2|y^2. \arrow[urr] & &
\end{tikzcd} \tag{39}$$

We assume furthermore $x+\min(x,z)+2y\geq 1$, and we get G_θ^{**}

$$\begin{tikzcd}
& & z|y^2|x^2 & & \\
& & & x|yz|xy \arrow[r] \arrow[dd] & y|x^2|yz \arrow[llu] \\
& & z|xy|xy \arrow[ur] \arrow[dl] & & \\
x|y^2|xz & y|xz|xy \arrow[l] & & x|xy|yz. \arrow[ul] \arrow[ll] &
\end{tikzcd}$$

Here is a second example with $\theta = x|x^2|y^2$ where $x+y\geq 1/2$ and $y<0$. Here G_θ is of the $3+2$ type and the subgraphs G_θ^* and G_θ^{**} are respectively

$$\begin{tikzcd}
x|x^2|y^2 \arrow[r] & y|xy|x^2 \arrow[dr] & \\
x|y^2|x^2 & y|x^2|xy \arrow[l] & x|xy|xy \arrow[l] \arrow[loop above]
\end{tikzcd}
\qquad
\begin{tikzcd}
& y|xy|x^2 \arrow[dr] & \\
x|y^2|x^2 & y|x^2|xy \arrow[l] & x|xy|xy. \arrow[l] \arrow[loop above]
\end{tikzcd}$$

The third example with $\theta = z|x^2|y^2$ with $y>x>0$, $y>1$ and $z=-x$ gives the two graphs G_θ^* and G_θ^{**}. The graph G_θ^{**} has no cycle at all.

$$\begin{tikzcd}
& & z|y^2|x^2 & & \\
x|xz|y^2 & y|xy|xz \arrow[rr] \arrow[dr] & & x|yz|xy \arrow[r] \arrow[dd] & y|x^2|yz \arrow[llu] \arrow[dd] \\
& & z|xy|xy \arrow[dl] & & \\
x|y^2|xz \arrow[uu] & y|xz|xy & & x|xy|yz \arrow[ll] & y|yz|x^2 \arrow[l] \\
& & z|x^2|y^2. \arrow[urr] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& z|y^2|x^2 & & \\
& & & y|x^2|yz \arrow[llu] \arrow[d] \\
y|xz|xy & & x|xy|yz \arrow[ll] & y|yz|x^2 \arrow[l]
\end{tikzcd}$$

REMARK. If $\theta_i+\theta_j>0$ for all $1\leq i<j\leq 5$, then $G_\theta = G_\theta^{**}$.

4.2. *Random continued fractions attached to a path in G_θ^{**}.*

The basic identity and the graphs. Let us fix $\theta\in\mathbb{R}^5$ and consider the directed graph $G_\theta^{**}=(V,E)$. To each vertex $v\in V$ is attached a distribution BH_v. To each edge $(v,v')\in E$ is attached a pair of positive numbers corresponding to a $\beta^{(2)}$ distribution that we denote by $\beta^{(v,v')}$. The basic identity (Theorem 3.3) says that if $X\sim \mathrm{BH}_v$ and $W\sim\beta^{(v,v')}$ are independent, then $\frac{1}{1+XW}\sim \mathrm{BH}_{v'}$.

In the sequel, for $w>0$, we denote by H_w the Moebius transformation

$$H_w(x)=\frac{1}{1+wx}.$$

PROPOSITION 4.1. *Let $v_0\to v_1\to\cdots\to v_n$ be a path in G_θ^{**} of nonnecessarily distinct vertices. Let $X_0, W_1,\ldots,W_n$ be independent random variables such* *that* $X_0 \sim \mathrm{BH}_{v_0}$ *and* $W_j \sim \beta^{(v_{j-1},v_j)}$ *for* $j = 1, \dots, n$. *Define the random Moebius transformations* $F_j = H_{W_j}$. *Then*

$$F_n \circ F_{n-1} \circ \cdots \circ F_1(X_0) = \frac{1}{1 + \frac{W_n}{\cdots + \frac{W_2}{1+W_1 X_0}}} \sim \mathrm{BH}_{v_n}.$$

PROOF. We proceed by induction on n. This is trivially true for $n = 0$. If it is true for $n - 1$ we apply the basic identity (Theorem 3.3) to the pair (v_{n-1}, v_n). $\square$

Here is the simple theorem which can be considered as the main result of the present paper:

THEOREM 4.2. *Let*

$$\cdots v_n \to v_{n-1} \to \cdots \to v_2 \to v_1 \to v_0 \tag{40}$$

be an infinite path in G_θ^{**}, *and let* $W_1, \dots, W_n, \dots$ *be independent random variables such that* $W_n \sim \beta^{(v_n,v_{n-1})}$. *Define the random Moebius transformations* $F_j = H_{W_j}$ *and* $Z_n(x) = F_1 \circ F_2 \circ \cdots \circ F_n(x)$. *Then the random continued fraction*

$$Z = \lim_{n\to\infty} Z_n(x) = \frac{1}{1 + \frac{W_1}{1 + \frac{W_2}{1+W_3\cdots}}}$$

associated to the infinite path almost surely exists and is independent of $x \geq 0$. *Its distribution is* $Z \sim \mathrm{BH}_{v_0}$.

PROOF. Recall that if $r > 0$ and if $V_r(u) = 1/(r + u)$, then the limit of the continued fraction $V_{r_1} \circ \cdots \circ V_{r_n}(x)$ is finite if $\sum_{n=1}^{\infty} r_n = \infty$ [Henrici (1977)]. For proving the convergence of $Z_n(x) = H_{W_1} \circ \cdots \circ H_{W_n}(x)$ the trick is to observe by induction on n that

$$Z_n(x) = V_{r_1} \circ \cdots \circ V_{r_n}(x/r_{n+1})$$

for the sequence (r_n) defined by $r_1 = 1$ and $r_n r_{n+1} = 1/W_n$. Now

$$\sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} (r_{2n-1} + r_{2n}) \geq 2 \sum_{n=1}^{\infty} (r_{2n-1} r_{2n})^{1/2} = 2 \sum_{n=1}^{\infty} W_{2n-1}^{-1/2}.$$

Since the W_{2n-1} are independent and since their distributions belong to a finite set, trivially the last series diverges almost surely as well as the sequence of continued fractions $(Z_n(x))$. Thus $Z = \lim_{n\to\infty} Z_n(x)$ exists and does not depend on x. Since the graph G_θ^{**} is finite, there exists an edge (v, v') such that the set $N = \{n; (v_n, v_{n-1}) = (v, v')\}$ is infinite. Let now X be independent of $W_1, \dots, W_n, \dots$ such that $X \sim \mathrm{BH}_v$. Let us apply Proposition 4.1 to $Z_n(X)$

when $n \in N$: we get that $Z_n(X) \sim \mathrm{BH}_{v_0}$ for all $n \in N$. We deduce from this that $Z \sim \mathrm{BH}_{v_0}$. $\square$

Comments and examples. In Theorem 4.2 we have called sequence (40) an infinite path in the graph G_θ^{**} which can also be written as

$$v_0 \leftarrow v_1 \leftarrow v_2 \leftarrow \cdots \leftarrow v_{n-1} \leftarrow v_n \leftarrow \cdots$$

If one insists that an infinite path should be a map from $\mathbb{N}$ to the set of vertices of the graph and not a map on $-\mathbb{N}$ it would have been be more correct to say that we work with an infinite path in the graph where all arrows have been inverted.

A simple example of application of Theorem 4.2 is the graph (34) with the sequence (v_n) equal to the constant $x|x^2|x^2$ where $x > 0$. Here the distribution of Z is $\mathrm{BH}_{x,x,x,x,x}(dz)$. These distributions have been studied in Section 2.4 when x is an integer or a half integer. Another example of application of Theorem 4.2 is graph (35) that we present in a simpler way as

$$\begin{tikzcd} a \arrow[r, leftrightarrow] \arrow[d] & b \\ c \arrow[ur] & \end{tikzcd} \tag{41}$$

where $a = x|x^2|xy, b = x|xy|x^2, c = y|x^2|x^2$. It is easily seen that $x + y, x > 0$ implies $G_\theta^{**} = G_\theta$. To any infinite word of $\{0, 1\}$ we associate an infinite path $v_0 \leftarrow v_1 \leftarrow \cdots \leftarrow v_{n-1} \leftarrow v_n \leftarrow \cdots$ in this graph as follows: each one is replaced by $b \leftarrow c \leftarrow a \leftarrow$, and each zero is replaced by $b \leftarrow a \leftarrow$. For instance the word $00101\ldots$ gives an infinite path ending at b as

$$b \leftarrow a \leftarrow b \leftarrow a \leftarrow b \leftarrow c \leftarrow a \leftarrow b \leftarrow a \leftarrow b \leftarrow c \leftarrow a \leftarrow \cdots.$$

Theorem 4.5 says that whatever is the infinite word of $\{0, 1\}$ the distribution of the random continued fraction Z corresponding to the infinite path in graph (41) deduced from this word is

$$\mathrm{BH}_b(dz) = Cz^{2x-1}(1-z)^{2x-1}{}_2F_1(2x, 2x; 3x + y; z)\mathbf{1}_{(0,1)}(z)\,dz.$$

A Cauchy distribution analogy. If $w = a + ib$ with $b > 0$, consider the Cauchy distribution $C_w(dx) = \frac{1}{\pi}\frac{b\,dx}{b^2+(x-a)^2}$. Now let $W_1, \ldots, W_n, \ldots$ be independent random variables such that $W_n \sim C_{w_n}$. Assume that $\sum_{n=1}^\infty |w_n|^{-1} = \infty$. Define

$$Z = W_1 - \cfrac{1}{W_2 - \frac{1}{W_3 - \cdots}}, \qquad z = w_1 - \cfrac{1}{w_2 - \frac{1}{w_3 - \cdots}}.$$

Then $Z \sim C_z$. We find some analogy between this elegant statement [due to Lloyd (1969) in the particular case of a constant sequence $(w_n)_{n=1}^\infty$] and Theorem 4.2: here we consider the Moebius transformations $F_w(x) = w - \frac{1}{x}$, and we are given an arbitrary infinite word $w_1 w_2 \cdots$. It leads to the exact distribution C_z of

$$Z = \lim_{n\to\infty} F_{W_1} \circ F_{W_2} \circ \cdots \circ F_{W_n}(x).$$

4.3. *Markov chains attached to a cycle in G_θ^{**}.* A consequence of Theorem 4.2 is about the stationary distribution of some Markov chains. It shows the importance of cycles in the graphs G_θ^{**} (later in Theorem 5.1 below it will proved that the cycles of G_θ^{*} and G_θ^{**} coincide). It is based on the "coupling from the past" idea for Markov chains which can be found in Letac (1986) and Chamayou and Letac (1991). Later on this idea was developed by Propp and Wilson (1996) in order to design simulation algorithms which are able to sample from the stationary distribution of an ergodic Markov chain without any initialization bias (perfect simulation).

THEOREM 4.3. *Let $v_0 \to v_1 \to \cdots \to v_{k-1} \to v_0$ be a cycle of order k in G_θ^{**}; let $W_j \sim \beta^{(v_{j-1},v_j)}$ with $j = 1, 2, \ldots$ be independent with the convention $v_j = v_{j'}$ if $j \equiv j' \bmod k$. Consider the random Moebius transformation*

$$G_n(x) = H_{W_{nk}} \circ H_{W_{nk-1}} \circ \cdots \circ H_{W_{(n-1)k+1}}(x)$$

and the homogeneous Markov chain $(X_n(x))_{n=0}^{\infty}$ on $(0, 1)$ defined by $X_0(x) = x > 0$ and

$$X_n(x) = G_n(X_{n-1}(x)) = G_n \circ G_{n-1} \circ \cdots \circ G_1(x).$$

Under these circumstances the stationary distribution of the chain is unique and is BH_{v_0}.

PROOF. Consider $Z_n(x) = G_1 \circ G_2 \circ \cdots \circ G_n(x)$. Consider the infinite path

$$v_0 \leftarrow v_k \leftarrow v_{k-1} \leftarrow \cdots \leftarrow v_1 \leftarrow v_0 \leftarrow v_k \leftarrow v_{k-1} \leftarrow \cdots.$$

Theorem 4.2 shows that $Z = \lim_{n\to\infty} Z_n(x)$ exists almost surely and that $Z \sim \mathrm{BH}_{v_0}$. Since convergence almost surely implies convergence in law, in terms of the Markov chain $(X_n(x))$ this implies that it converges in law to BH_{v_0}, for any x. Thus BH_{v_0} has to be the unique stationary law for the chain. □

Comments and examples. For other random continued fractions there are some analogs of Theorem 4.3 in literature with cycles only of size 1 or 2 [see Letac and Seshadri (1983), Lloyd (1969), Dyson (1953), Marklov, Tourigny and Wolovski (2008) and the one by Asci, Letac and Piccioni (2008) that we have mentioned before]. In this last paper two BH distributions $\mathrm{BH}(a, b, a, b, a + a')$ and $\mathrm{BH}(a', b, a', b, a + a')$ are considered. Their θ parameterizations are

$$\mathrm{BH}_{b/2,\{b/2,a'-b/2\},\{b/2,a-b/2\}}, \qquad \mathrm{BH}_{b/2,\{b/2,a-b/2\},\{b/2,a'-b/2\}}.$$

These θs are of $3 + 1 + 1$ type. With $x = b/2$, $y = (2a - b)/2$ and $z = (2a' - b)/2$ the vertices $x|xz|xy$ and $x|xy|xz$ are the vertices of the unique cycle of order 2 of the graph $3 + 1 + 1$. The parameters of the beta type two random variable W, used by the random Moebius transformation H_W, sending the first law into the second, are thus (b, a) (and (b, a') for the opposite).

In Section 4.1 we have mentioned the existence of cycles of all sizes between 1 and 30 (except 11, 21, 25, 27, 28, 29). Each of these cycles is associated to explicit distributions of periodic random continued fractions and stationary measures of Markov chains. In particular we give the example of the homogeneous Markov chain $G_n \circ G_{n-1} \circ \cdots \circ G_1(z)$ with stationary distribution $\mathrm{BH}_{x,y,z,u,v}$, where the random Moebius transformations G_k are i.i.d. with

$$G_1 = H_{W_{30}} \circ \cdots \circ H_{W_1}$$

with the W_is independent with $\beta^{(2)}$ distribution with parameters (increasing from $i = 1$ to $i = 30$)

$$\begin{gathered}(x+y, u+v), (y+u, x+z), (u+x, v+y), (x+v, z+u), (v+z, x+y),\\ (z+x, u+v), (x+u, z+y), (u+z, x+v), (z+x, u+y), (x+u, v+z),\\ (u+v, y+x), (v+y, u+z), (y+u, v+x), (u+v, z+y), (v+z, u+x),\\ (z+u, v+y), (u+v, z+x), (v+z, y+u), (z+y, x+v), (y+x, z+u),\\ (x+z, y+v), (z+y, x+u), (y+x, v+z), (x+v, y+u), (v+y, z+x),\\ (y+z, u+v), (z+u, y+x), (u+y, v+z), (y+v, x+u), (v+x, y+z).\end{gathered}$$

We should also mention here that if

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a nonsingular real matrix, and if $h_M(x) = (ax+b)/(cx+d)$ is the corresponding Moebius transformation, we have necessarily $h_M \circ h_{M'} = h_{MM'}$. Therefore the above results could be interpreted in terms of random walks on the group $GL(2, \mathbb{R})$ suitably quotiented since $h_{\lambda M} = h_M$ for any nonzero scalar λ.

To be more specific, denote by $P^d(\mathbb{R})$ the real projective space of dimension d, namely the set of equivalence classes of $\mathbb{R}^{d+1} \setminus \{0\}$ for the following relation: we say that x and x' are equivalent if there exists λ in $\mathbb{R} \setminus \{0\}$ such that $x' = \lambda x$. The space $P^d(\mathbb{R})$ is compact with the topology inherited from the quotient. We write $\tilde{x}$ for the equivalence class of x. The action of $M \in GL(d+1, \mathbb{R})$ on $P^d(\mathbb{R})$ is defined by $M\tilde{x} = \widetilde{Mx}$. We say that a sequence $(M_n)_{n\geq 0}$ of $GL(d+1, \mathbb{R})$ converges weakly if $\lim_{n\to\infty} M_n\tilde{x}$ exists in $P^d(\mathbb{R})$ for all $\tilde{x} \in P^d(\mathbb{R})$. In particular for $d = 1$ the sequence $(M_n)_{n\geq 0}$ of $(2, 2)$ invertible matrices converges weakly if and only if $\lim_{n\to\infty} h_{M_n}(x)$ exists in the set $\overline{\mathbb{R}}$ of the real numbers compactified by one point ∞ for all $x \in \overline{\mathbb{R}}$ (with the convention $h_M(\infty) = b/d$ and $h_M(-d/c) = \infty$ if $h_M(x) = (ax+b)/(cx+d)$). Note that $\overline{\mathbb{R}}$ can be identified to $P^1(\mathbb{R})$: to $x \in \mathbb{R}$ we associate all elements $(\lambda x, \lambda)^*$, and to ∞ we associate all elements $(\lambda, 0)^*$.

Let us apply this definition to the simple case of random matrices $M_n = A_1 A_2 \cdots A_n$ where

$$A_n = \begin{bmatrix} 0 & 1 \\ W_n & 1 \end{bmatrix}$$

and where $W_1, \dots, W_n$ are i.i.d. with distribution $\beta^{(2)}_{a,a}$. Note that $h_{A_n} = H_{W_n}$ and that $h_{M_n}(x)$ is expressed as a finite continued fraction. Theorem 4.2 says that in this particular case the random walk $(A_1A_2\cdots A_n)_{n\geq 0}$ on $GL(2,\mathbb{R})$ converges weakly and that $\lim A_1A_2\cdots A_n\tilde{x}$ in $\overline{\mathbb{R}}$ exists, does not depend on $\tilde{x}$ and has distribution $\mathrm{BH}(a,a,a,a,2a)$ (Actually we proved it for $x>0$, but it can be extended to all $\tilde{x}$). Warning: $(A_1A_2\cdots A_n)_{n\geq 0}$ does not converge in $GL(2,\mathbb{R})$. On the other hand Theorem 4.3 says that $(A_nA_{n-1}\cdots A_1\tilde{x})_{n\geq 0}$ is simply a stationary Markov chain on $\overline{\mathbb{R}}$ with stationary distribution $\mathrm{BH}(a,a,a,a,2a)$. We have illustrated Theorems 4.2 and 4.3 in the simple case (34). Similar interpretations for the six other graphs are available.

5. Cycles and positivity. Let $G = G_\theta$ be one of the seven graphs. In this section we show the delicate result that the cycles of G^*_θ are always the cycles of G^{**}_θ. Since the existence of $\beta^{(v,v')}$ is easier to check than the existence of BH_v, Theorem 5.1 happens to be useful and practical. Use again the notation

$$v_\theta = (\theta_1, \{\theta_2,\theta_3\}, \{\theta_4,\theta_5\}),$$

and suppose that $v_0 = v_\theta$ is a vertex of G. If (v_0, v) is an edge of the graph recall that either $v = v_1 = (\theta_5, \{\theta_1,\theta_4\}, \{\theta_2,\theta_3\})$ or $v = v_2 = (\theta_4, \{\theta_1,\theta_5\}, \{\theta_2,\theta_3\})$ and that (v_0, v_1) is said to be admissible if $\theta_2+\theta_3>0$ and $\theta_1+\theta_5>0$.

THEOREM 5.1. *Let $\theta = (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)\in\mathbb{R}^5$, and consider the vertex $v_0 = v_\theta$ of G^*_θ defined by* (38). *We assume without loss of generality $\theta_2\leq\theta_3$ and $\theta_4\leq\theta_5$. Then v_0 belongs to a cycle of G^*_θ if and only if $\theta_i+\theta_j>0$ for all $1\leq i<j\leq 5$, except possibly for $(i,j)=(2,4)$. Furthermore if v_0 belongs to a cycle of G^*_θ, then $v_0\in\Theta$, which means that the distribution BH_{v_0} exists, and v_0 is a vertex of G^{**}_θ. Finally, if $\theta_2+\theta_4\leq 0$, then*

$$v_0\to v_1\to v_2\to v_3\to v_4\to v_5\to v_0,$$

where

$$\begin{aligned} &v_1 = (\theta_5, \{\theta_4,\theta_1\}, \{\theta_2,\theta_3\}), \qquad v_2 = (\theta_3, \{\theta_2,\theta_5\}, \{\theta_4,\theta_1\}),\\ &v_3 = (\theta_1, \{\theta_4,\theta_3\}, \{\theta_2,\theta_5\}),\\ &v_4 = (\theta_5, \{\theta_2,\theta_1\}, \{\theta_4,\theta_3\}), \qquad v_5 = (\theta_3, \{\theta_4,\theta_5\}, \{\theta_2,\theta_1\}) \end{aligned} \tag{42}$$

and $\min\{\theta_1,\theta_3,\theta_5\} > -\max\{\theta_2,\theta_4\}$. *In particular in this case v_0 belongs to a cycle of order* 1, 2, 3 *or* 6.

LEMMA 5.2. *In the graph G, let $v_{-2}\to v_{-1}\to v_0\to v_1\to v_2$ such that the four edges $\{(v_k,v_{k+1}); k=-2,-1,0,1\}$ are admissible. If $v_0 = (\theta_1, \{\theta_2,\theta_3\}, \{\theta_4,\theta_5\})$, then the six following numbers*:

$$\theta_1+\theta_2, \theta_1+\theta_3, \theta_2+\theta_3, \theta_1+\theta_4, \theta_1+\theta_5, \theta_4+\theta_5$$

are positive (*we do not assume $\theta_2\leq\theta_3$ and $\theta_4\leq\theta_5$ here*).

PROOF. The edge (v_0, v_1) being admissible we get $\theta_2 + \theta_3 > 0$. The edge (v_{-1}, v_0) being admissible we get $\theta_4 + \theta_5 > 0$. If $v_1 = (\theta_4, \{\theta_1, \theta_5\}, \{\theta_2, \theta_3\})$ the fact that (v_0, v_1) is an edge implies $\theta_1 + \theta_4 > 0$, and the fact that (v_1, v_2) is an edge implies $\theta_1 + \theta_5 > 0$. Similarly $v_1 = (\theta_5, \{\theta_1, \theta_2\}, \{\theta_2, \theta_3\})$ implies $\theta_1 + \theta_4 > 0$ and $\theta_1 + \theta_5 > 0$ as well. Finally if $v_{-1} = (\theta_3, \{\theta_4, \theta_5\}, \{\theta_1, \theta_2\})$ the fact that (v_{-1}, v_0) is an edge implies $\theta_1 + \theta_2 > 0$ and $\theta_1 + \theta_3 > 0$. Similarly if $v_{-1} = (\theta_2, \{\theta_4, \theta_5\}, \{\theta_1, \theta_3\})$ implies $\theta_1 + \theta_2 > 0$ and $\theta_1 + \theta_3 > 0$, and the lemma is proved. □

PROOF OF THEOREM 5.1. Suppose that v_0 is in a cycle of G^*_θ. Therefore Lemma 5.2 is applicable to v_0. Recall now that in the statement of the theorem we have assumed $\theta_2 \le \theta_3$ and $\theta_4 \le \theta_5$. If $v_1 = (\theta_4, \{\theta_1, \theta_5\}, \{\theta_2, \theta_3\})$, since (v_2, v_3) is an edge we get the new inequality $\theta_2 + \theta_4 > 0$. Similarly if $v_1 = (\theta_5, \{\theta_1, \theta_2\}, \{\theta_2, \theta_3\})$ implies $\theta_2 + \theta_5 > 0$. Since $\theta_4 \le \theta_5$, the inequality $\theta_2 + \theta_5 > 0$ holds for both possible values of v_1.

At this point observe that the only inequality to prove now is $\theta_3 + \theta_4 > 0$, which implies the other one, $\theta_3 + \theta_5 > 0$. We discuss again the two possible values of v_1 for applying Lemma 5.2 to the sequence $v_{-1} \to v_0 \to v_1 \to v_2 \to v_3$. If $v_1 = (\theta_4, \{\theta_1, \theta_5\}, \{\theta_2, \theta_3\})$ we have seen that $\theta_2 + \theta_4 > 0$ which imply $\theta_3 + \theta_4 > 0$. Therefore the result is proved in this case.

Now we assume $v_1 = (\theta_5, \{\theta_1, \theta_2\}, \{\theta_2, \theta_3\})$ and we discuss according to the two possible values of v_2. If $v_2 = (\theta_2, \{\theta_3, \theta_5\}, \{\theta_1, \theta_4\})$ we apply Lemma 5.2 to the sequence $v_0 \to v_1 \to v_2 \to v_3 \to v_4$. Among the six inequalities we find $\theta_2 + \theta_4 > 0$ which implies $\theta_3 + \theta_4 > 0$. The last case is $v_2 = (\theta_3, \{\theta_2, \theta_5\}, \{\theta_1, \theta_4\})$: among the inequalities given by Lemma 5.2 we find the desired one, $\theta_3 + \theta_4 > 0$. Finally from (27) BH_v always exists if v is in a cycle of G^*_θ since in this case all the traditional parameters a, b, p, q, r are positive.

One can observe from the preceeding study that if $\theta_2 + \theta_4 \le 0$ then necessarily $v_1 = (\theta_5, \{\theta_1, \theta_2\}, \{\theta_2, \theta_3\})$ and $v_2 = (\theta_3, \{\theta_2, \theta_5\}, \{\theta_1, \theta_4\})$. Iterating this remark we see that the set $(v_0, v_1, v_2, v_3, v_4, v_5)$ described in (42) is a cycle.

We now discuss the converse. Suppose that $\theta_i + \theta_j > 0$ for all $1 \le i < j \le 5$. In this case $G = G^*_\theta$ and since any vertex of G is the initial vertex and the end vertex of some arrows, any vertex of G belongs to a cycle. Now suppose that $\theta_i + \theta_j > 0$ for all $1 \le i < j \le 5$ except for $(i, j) = (2, 4)$. We have described above the corresponding cycle. For seeing that $\min\{\theta_1, \theta_3, \theta_5\} > \max\{\theta_2, \theta_4\}$ observe that we have seen that in an element of the cycle $\theta_i + \theta_j \le 0$ can happen only for one pair (i, j). For instance, for v_0, we had $(i, j) = (2, 4)$, and therefore $\theta_2 < \theta_3$ and $\theta_4 < \theta_5$ must be strict inequalities. Now making the same remark for the other $(v_1, v_2, v_3, v_4, v_5)$ of (42) shows $\min\{\theta_1, \theta_3, \theta_5\} > \max\{\theta_2, \theta_4\}$. □

Cycles when $\theta_2 + \theta_4 \le 0$. Assume as in Theorem 5.1 that $\theta_2 \le \theta_3$ and $\theta_4 \le \theta_5$ and that $\theta_2 + \theta_4 \le 0$. In this case v_0 as in Theorem 5.1 can belong to a cycle of size 1, 2, 3 or 6. Let us indicate here without proof the necessary and sufficient conditions for this. This is done by analyzing (42).

- v_0 is in a cycle of size 1 if and only if $v_0 = x|xy|xy$ with $x + y > 0$ and $y \leq 0$.
- v_0 is in a cycle of size 2 if and only if $v_0 = x|xy|xz$ with $x + y, x + z > 0$, $y + z \leq 0$ and $y \neq z$.
- v_0 is in a cycle of size 3 if and only if $v_0 = y|xz|xu$ with $x + y, x + z, x + u > 0$, $x \leq 0$ and y, z, u are not all equal.
- v_0 is in a cycle of size 6 if and only if $v_0 = x|yz|uv$ with $y + u \leq 0$, all the other sums of pairs are positive $y \neq u$ and x, z, v are not all equal.

EXAMPLE. Let us use the identity ${}_2F_1(p, -p; 1/2; \sin^2\theta) = \cos 2p\theta$. It provides a hypergeometric function which is certainly positive on (0, 1) if $0 < p \leq 1/2$. Consider the distribution $\mathrm{BH}(2\alpha, b, p, -p, 1/2)$ for $\alpha, b > 0$. From (28) it is equal to BH_θ with

$$\theta_1 = \alpha + b, \qquad \theta_2 = -\alpha, \qquad \theta_3 = -\alpha + \tfrac{1}{2}, \qquad \theta_4 = \alpha - p, \qquad \theta_5 = \alpha + p.$$

Note that $\theta_2 < \theta_3$, $\theta_4 < \theta_5$ and $\theta_2 + \theta_4 = -p < 0$. It is easy to detect with the help of Theorem 5.1 that that v_θ belongs to a cycle of G_θ^* if and only if $1/4 > \alpha$ and $b + 2\alpha > p$, with $0 < p < 1/2$. Since $\theta_2 + \theta_4 < 0$ this cycle is described by (42).

A cycle of order 6 *changed in a cycle of order* 5. The above analysis includes $\theta_2 + \theta_4 = 0$. In this case BH_{v_j} for $j = 0, \dots, 5$ are beta distributions or quasi-beta distributions. Since in this case the same distribution has an infinite number of θ parameters, it makes sense to ask if these different representations could appear within a cycle. This happens only in the following case. Taking $v_0 = (x, y, x + 2y, -y, x)$ with $x > y > 0$ we get $v_1 = (x, -y, x, y, x + 2y)$, $v_2 = (x + 2y, y, x, -y, x)$, $v_3 = (x, -y, x + 2y, y, x)$, $v_4 = (x, y, x, -y, x + 2y)$, $v_5 = (x + 2y, -y, x, y, x)$ and

$$\begin{aligned} &\mathrm{BH}_{v_0} = \beta_{x-y,x+y}, && \mathrm{BH}_{v_1} = \beta_{x+3y,x-y}, && \mathrm{BH}_{v_2} = \beta_{x-y,x+3y}, \\ &\mathrm{BH}_{v_3} = \beta_{x+y,x-y}, && \mathrm{BH}_{v_4} = \beta_{x+y,x+y}, && \mathrm{BH}_{v_5} = \beta_{x+y,x+y}. \end{aligned}$$

We observe that $\mathrm{BH}_{v_4} = \mathrm{BH}_{v_5}$. All the other BH_{v_j} are different. Therefore, starting with a cycle of order 6 we can design a beta distributed random continued fraction of the type of Theorem 4.3 with period $k = 5$. One can prove that this phenomenon appears only for the above choice of parameters.

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