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QUENCHED EXIT ESTIMATES AND BALLISTICITY CONDITIONS FOR HIGHER-DIMENSIONAL RANDOM WALK IN RANDOM ENVIRONMENT

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Consider a random walk in an i.i.d. uniformly elliptic environment in dimensions larger than one. In 2002, Sznitman introduced for each $\gamma \in (0,1)$ the ballisticity condition $(T)_\gamma$ and the condition (T') defined as the fulfillment of $(T)_\gamma$ for each $\gamma \in (0,1)$. Sznitman proved that (T') implies a ballistic law of large numbers. Furthermore, he showed that for all $\gamma \in (0.5,1)$, $(T)_\gamma$ is equivalent to (T'). Recently, Berger has proved that in dimensions larger than three, for each $\gamma \in (0,1)$, condition $(T)_\gamma$ implies a ballistic law of large numbers. On the other hand, Drewitz and Ramírez have shown that in dimensions $d \geq 2$ there is a constant $\gamma_d \in (0.366, 0.388)$ such that for each $\gamma \in (\gamma_d, 1)$, condition $(T)_\gamma$ is equivalent to (T'). Here, for dimensions larger than three, we extend the previous range of equivalence to all $\gamma \in (0,1)$. For the proof, the so-called *effective criterion* of Sznitman is established employing a sharp estimate for the probability of atypical quenched exit distributions of the walk leaving certain boxes. In this context, we also obtain an affirmative answer to a conjecture raised by Sznitman in 2004 concerning these probabilities. A key ingredient for our estimates is the multiscale method developed recently by Berger.

1. Introduction and statement of the main results. We continue our investigation of the interrelations between the ballisticity conditions $(T)_\gamma$ and (T') introduced by Sznitman in [8] for random walk in random environment (RWRE). In dimensions larger than or equal to four, the results we establish in this paper amount to a considerable improvement of what has been obtained in our work [4]. To prove the corresponding results, we take advantage of techniques recently developed by Berger in [1]. We derive sharp estimates on the probability of certain quenched exit distributions of the RWRE and thereby provide an affirmative answer to a slightly stronger version of a conjecture announced by Sznitman in [9].

We start by giving an introduction to the model, thereby fixing the notation we employ. Denote by $\mathcal{M}_d$ the space of probability measures on the set

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$\{e \in \mathbb{Z}^d : \|e\|_1 = 1\}$ of canonical unit vectors and set $\Omega := (\mathcal{M}_d)^{\mathbb{Z}^d}$. For each *environment* $\omega = (\omega(x,\cdot))_{x\in\mathbb{Z}^d} \in \Omega$, we consider the Markov chain $(X_n)_{n\in\mathbb{N}}$ with transition probabilities from x to $x+e$ given by $\omega(x,e)$ for $\|e\|_1 = 1$, and 0 otherwise. We denote by $P_{x,\omega}$ the law of this Markov chain conditioned on $\{X_0 = x\}$. Furthermore, let $\mathbb{P}$ be a probability measure on Ω such that the coordinates $(\omega(x,\cdot))_{x\in\mathbb{Z}^d}$ of the environment ω are i.i.d. under $\mathbb{P}$. Then $\mathbb{P}$ is called *elliptic* if $\mathbb{P}(\min_{\|e\|_1=1} \omega(0,e) > 0) = 1$ while it is called *uniformly elliptic* if there is a constant $\kappa > 0$ such that $\mathbb{P}(\min_{\|e\|_1=1} \omega(0,e) \geq \kappa) = 1$. We call $P_{x,\omega}$ the *quenched law* of the RWRE starting from x, and correspondingly we define the *averaged* (or *annealed*) law of the RWRE by $P_x := \int_\Omega P_{x,\omega}\mathbb{P}(d\omega)$.

Given a direction $l \in \mathbb{S}^d$, we say that the RWRE is *transient in the direction* l if

$$P_0\Big(\lim_{n\to\infty} X_n \cdot l = \infty\Big) = 1.$$

Furthermore, we say that the RWRE is *ballistic in the direction* l if P_0-a.s.

$$\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0.$$

It is well known that in dimension one there exists uniformly elliptic RWRE in i.i.d. environments which is transient but not ballistic to the right. It was also recently established that in dimensions larger than one there exists elliptic RWRE in i.i.d. environments which is transient but not ballistic in a given direction see Sabot and Tournier in [6]. Nevertheless, the following fundamental conjecture remains open.

CONJECTURE 1.1. *In dimensions larger than one, every uniformly elliptic RWRE in an i.i.d. environment which is transient in a given direction is necessarily ballistic in the same direction.*

Some partial progress has been made toward the resolution of this conjecture by studying transient RWRE satisfying some additional assumptions introduced in [8], usually called *ballisticity conditions*. For each $l \in \mathbb{S}^{d-1}$ and $L > 0$, let us define

$$T_L^l := \inf\{n \geq 0 : X_n \cdot l > L\}.$$

DEFINITION 1.2. Let $\gamma \in (0,1)$ and $l \in \mathbb{S}^{d-1}$. We say that *condition* $(T)_\gamma$ is satisfied with respect to l [written $(T)_\gamma|l$ or $(T)_\gamma$] if for each l' in a neighborhood of l and each $b > 0$ one has that

$$\limsup_{L\to\infty} L^{-\gamma} \log P_0(T_L^{l'} > T_{bL}^{-l'}) < 0.$$

We say that *condition* (T') is satisfied with respect to l [written $(T')|l$ or (T')], if for each $\gamma \in (0,1)$, condition $(T)_\gamma|l$ is fulfilled.

It is known that in dimensions $d \geq 2$, condition (T') implies the existence of a deterministic $v \in \mathbb{R}^d \setminus \{0\}$ such that P_0-a.s. $\lim_{n\to\infty} \frac{X_n}{n} = v$, as well as a central limit theorem for the RWRE so that under the annealed law P_0,

$$B_\cdot^n := \frac{X_{\lfloor \cdot n \rfloor} - \lfloor \cdot n \rfloor v}{n}$$

converges in distribution to a Brownian motion in the Skorokhod space $D([0,\infty), \mathbb{R}^d)$ as $n \to \infty$; see, for instance, Theorem 4.1 in [9] for further details. Recently, in [1] the author has shown that in dimensions larger than three, the above law of large numbers and central limit theorem remain valid if condition $(T)_\gamma$ is satisfied for some $\gamma \in (0,1)$. In addition, in [9] the author has proven that if $\mathbb{P}$ is uniformly elliptic, then in dimensions $d \geq 2$, for each $\gamma \in (0.5, 1)$ and each $l \in \mathbb{S}^{d-1}$, condition $(T)_\gamma|l$ is equivalent to $(T')|l$. In [4], the authors pushed down this equivalence to each $\gamma \in (\gamma_d, 1)$, where $\gamma_d \in (0.366, 0.388)$ is decreasing with the dimension. The first main result of the present paper is a considerable improvement of these previous results for dimensions larger than three.

THEOREM 1.3. *Let $d \geq 4$ and $\mathbb{P}$ be uniformly elliptic. Then for all $\gamma \in (0,1)$ and $l \in \mathbb{S}^{d-1}$, condition $(T)_\gamma|l$ is equivalent to $(T')|l$.*

The proof of Theorem 1.3 takes advantage of the effective criterion and is therefore closely related to upper bounds for quenched probabilities of atypical exit behavior of the RWRE. To state the corresponding result, denote for any subset $B \subset \mathbb{Z}^d$ its boundary by

$$\partial B := \{x \in \mathbb{Z}^d \setminus B : \exists y \in B \text{ such that } \|x - y\|_1 = 1\}$$

and define the slab

$$U_{\beta,l,L} := \{x \in \mathbb{Z}^d : -L^\beta \leq x \cdot l \leq L\}.$$

Furthermore, for the rest of this paper we let

$$T_B := \inf\{n \in \mathbb{N}_0 : X_n \in B\}$$

denote the first hitting time. For $x \in \mathbb{Z}^d$ set $T_x := T_{\{x\}}$. In terms of this notation, in [9] the author conjectured the following (cf. Figure 1).

CONJECTURE 1.4. *Let $d \geq 2$, $\mathbb{P}$ be uniformly elliptic and assume $(T')|l$ to hold for some $l \in \mathbb{S}^{d-1}$. Fix $c > 0$ and $\beta \in (0,1)$. Then for all $\alpha \in (0, \beta d)$,*

$$\limsup_{L\to\infty} L^{-\alpha} \log \mathbb{P}\big(P_{0,\omega}(X_{T_{\partial U_{\beta,l,L}}} \cdot l > 0) \leq e^{-cL^\beta}\big) < 0.$$

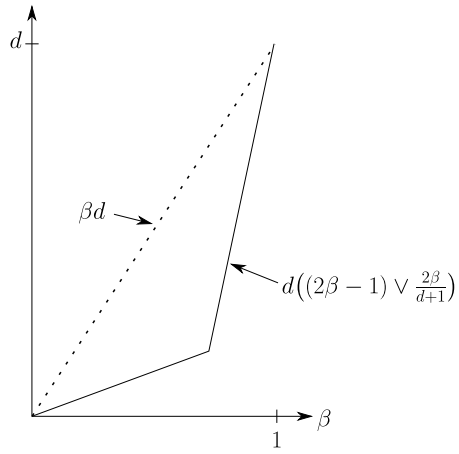

FIG. 1. *Sketch of the known and conjectured bounds for* α.

Theorem 4.4 of [9] states that the above conjecture holds true for all positive α with

$$\alpha < d\Big((2\beta - 1) \vee \frac{2\beta}{d+1}\Big).$$

The second main result of the present paper gives an affirmative answer to a seemingly stronger statement than the one of Conjecture 1.4. For $l \in \mathbb{S}^{d-1}$, denote by

$$\pi_l : \mathbb{R}^d \ni x \quad \mapsto \quad (x \cdot l)l \in \mathbb{R}^d$$

the orthogonal projection on the space $\{\lambda l : \lambda \in \mathbb{R}\}$ as well as by

$$\pi_{l^\perp} : \mathbb{R}^d \ni x \quad \mapsto \quad x - \pi_l(x) \in \mathbb{R}^d$$

the orthogonal projection on the orthogonal complement $\{\lambda l : \lambda \in \mathbb{R}\}^\perp$. Using this notation, for $K > 0$ we define the box

$$B_{L,l,K} := \{x \in \mathbb{Z}^d : 0 \le x \cdot l \le L, \|\pi_{l^\perp}(x)\|_\infty \le KL\}$$

as well as its right boundary part

$$\partial_+ B_{L,l,K} := \{x \in \partial B_{L,l,K} : x \cdot l > L\}, \tag{1.1}$$

see Figure 2.

We can now state the desired result.

THEOREM 1.5. *Let* $d \ge 4$, $\mathbb{P}$ *be uniformly elliptic and assume* $(T)_{\gamma}|l$ *to hold for some* $\gamma \in (0, 1)$, $l \in \mathbb{S}^{d-1}$. *Fix* $c > 0$ *and* $\beta \in (0, 1)$. *Then there exists a constant* $K > 0$ *such that for all* $\alpha \in (0, \beta d)$,

$$\limsup_{L\to\infty} L^{-\alpha} \log \mathbb{P}\big(P_{0,\omega}(T_{\partial B_{L,l,K}} = T_{\partial_+ B_{L,l,K}}) \le e^{-cL^\beta}\big) < 0.$$

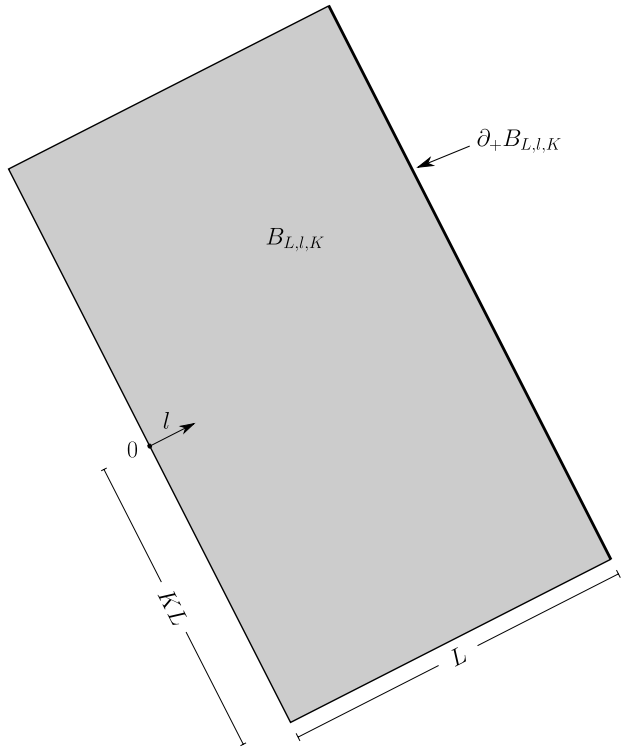

FIG. 2. *The set $B_{L,l,K}$ and its boundary part $\partial_+ B_{L,l,K}$.*

REMARK 1.6.

(a) The result we prove is slightly stronger than the conjecture announced in [9] since we can dispose of the extent of the slab in direction $-l$ as well as restrict the extent in directions orthogonal to l. Scrutinizing the proof it will be clear that one can improve this result replacing the box $B_{L,l,K}$ by a parabola-shaped set which grows in the directions transversal to $\hat{v}$ at least like L^α for some $\alpha > 1/2$.

(b) Note that this theorem is optimal in the sense that its conclusion will not hold in general for $\alpha > \beta d$. In fact, for plain nestling RWRE, this can be shown by the use of so-called *naïve traps* (see [9], page 244).

(c) In both, Theorem 1.3 as well as Theorem 1.5, the restriction to dimensions larger than three is caused by the following: for a very large set of environments we need that the trajectories of two independent d-dimensional random walks in this environment intersect only very rarely; see equations (A.35) and (A.36).

The proof of Theorem 1.5 exploits heavily a recent multiscale technique introduced in [1] to study the slowdown upper bound for RWRE. To explain this in more detail, note that from that source one also infers that every RWRE in a uniformly

elliptic i.i.d. environment which satisfies condition $(T)_\gamma$ for some $\gamma \in (0,1)$, has an asymptotic speed $v \neq 0$. The main result of [1] states that for every RWRE in a uniformly elliptic i.i.d. environment satisfying condition $(T)_\gamma$, some $\gamma \in (0,1)$, the following holds: for each $a \neq v$ in the convex hull of 0 and v as well as $\epsilon > 0$ small enough, and any $\alpha < d$ the inequality

$$P_0\left(\left\|\frac{X_n}{n} - a\right\|_\infty < \epsilon\right) \leq \exp\{-(\log n)^\alpha\}$$

holds for all n large enough. To prove the above result, Berger develops a multiscale technique which describes the behavior of the walk at the scale of the so called naïve traps, which at time n are of radius of order $\log n$. Here, we rely on such a multiscale technique to make explicit the role of the regions of the same scale as the naïve traps to prove Theorem 1.5.

In Section 2, we show how certain exit estimates from boxes imply Theorem 1.5 and how in turn such a result implies Theorem 1.3. In Section 3, we start with giving a heuristic explanation of a modified version of Berger's multiscale technique and of how to deduce the aforementioned exit estimates. We then set up our framework of notation and auxiliary results before making precise the previous heuristics by giving the corresponding proofs. In the Appendix we establish several specific results concerning local limit theorem type results and estimates involving intersections of random walks.

2. Proofs of the main results. The proofs of Theorems 1.3 and 1.5 are based on a multiscale argument and a semi-local limit theorem developed in [1] for RWRE in dimensions larger than or equal to four.

It is well known that if for some $\gamma \in (0,1)$ and $l \in \mathbb{S}^{d-1}$, condition $(T)_\gamma|l$ is fulfilled, then P_0-a.s. the limit

$$\hat{v} := \lim_{n\to\infty} \frac{X_n}{\|X_n\|_2} \in \mathbb{S}^{d-1}$$

exists and is constant (cf., e.g., Theorem 1 in Simenhaus [7]); it is called the *asymptotic direction*.

Define for a vector e_j of the canonical basis of $\mathbb{Z}^d$ and $l \in \mathbb{S}^{d-1}$ such that $l \cdot e_j \neq 0$ the projection $\tilde{\pi}_l^j$ via

$$\tilde{\pi}_l^j : \mathbb{R}^d \ni x \quad \mapsto \quad \frac{x \cdot e_j}{l \cdot e_j} l \in \mathbb{R}^d$$

on the space $\{\lambda l : \lambda \in \mathbb{R}\}$ and by $\tilde{\pi}_{l^\perp}^j$ the projection

$$\tilde{\pi}_{l^\perp}^j : \mathbb{R}^d \ni x \quad \mapsto \quad x - \frac{x \cdot e_j}{l \cdot e_j} l \in \mathbb{R}^d$$

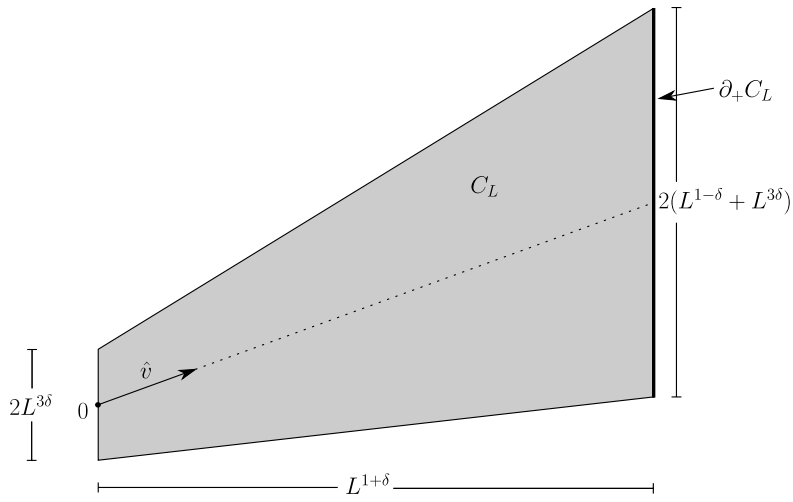

FIG. 3. *The set C_L and its boundary part $\partial_+ C_L$.*

on the space $\{\lambda e_j : \lambda \in \mathbb{R}\}^\perp$. In the case $j = 1$, we will abbreviate this notation by $\tilde{\pi}_l$ and $\tilde{\pi}_{l^\perp}$. For $j \in \{1, \dots, d\}$, $\delta > 0$ and $L > 0$, define the set

$$C_L := \{x \in \mathbb{Z}^d : 0 \le x \cdot e_j \le L^{1+\delta}, \|\tilde{\pi}^j_{\hat{v}^\perp}(x)\|_\infty \le L^{3\delta} + x \cdot e_j L^{-2\delta}\};$$

cf. Figure 3. In analogy to (1.1), we introduce the right boundary parts

$$\partial_+ C_L := \{x \in \partial C_L : x \cdot e_j > L^{1+\delta}\}$$

and $\partial_+(x + C_L) := x + \partial_+ C_L$ for $x \in \mathbb{Z}^d$.

The proof of the following proposition will be deferred to Section 3.

PROPOSITION 2.1. *Let $d \ge 4$, $\mathbb{P}$ be uniformly elliptic and assume $(T)_\gamma|l$ to hold for some $\gamma \in (0, 1)$, $l \in \mathbb{S}^{d-1}$. Without loss of generality, let e_j be a vector of the canonical basis such that $\hat{v} \cdot e_j > 0$ and fix $\beta \in (0, 1)$ as well as $\alpha \in (0, \beta d)$.*

Then for all $\delta > 0$ small enough there exists a sequence of events $(\Xi_L)_{L \in \mathbb{N}}$ such that for all L large enough we have

$$\inf_{\omega \in \Xi_L} P_{0,\omega}(T_{\partial C_L} = T_{\partial_+ C_L}) \ge e^{-L^{\beta-\delta}}$$

and

$$\mathbb{P}(\Xi_L^c) \le e^{-L^\alpha}.$$

For the sake of notational simplicity and without loss of generality, we assume $j = 1$ from now on.

2.1. *Proof of Theorem 1.5.* We will show that Theorem 1.5 is a consequence of Proposition 2.1. For this purpose, let the assumptions of Theorem 1.5 be fulfilled.

In particular, let $(T)_\gamma|l$ be fulfilled (which implies $l\cdot\hat{v}>0$; cf. Theorem 1.1 of [8]) and fix $c>0$, $\beta\in(0,1)$ as well as $\alpha\in(0,\beta d)$. Let $\delta>0$ small enough such that the implication of Proposition 2.1 holds and $3\delta<\beta-\delta$. Choose $\beta'\in(\beta-\delta,\beta)$ and define $x\in\mathbb{Z}^d$ to be one of the (possibly several) sites closest to $L^{\beta'}l$. Then the following property of the displaced set $x+C_L$ will be used:

(Exit) Let K be large enough and $\delta>0$ small enough. Then for L large enough, if the walk starting in x leaves $x+C_L$ through $\partial_+(x+C_L)$, it also leaves the box $B_{L,l,K}$ through $\partial_+B_{L,l,K}$.

Now since the measure $\mathbb{P}$ is uniformly elliptic, we know that there exists a constant C depending on the dimension d, such that for all L large enough and for $\mathbb{P}$-a.a. ω the inequality

$$P_{0,\omega}(T_{\partial B_{L,l,K}}>T_x)\geq e^{-CL^{\beta'}} \tag{2.1}$$

holds true. By Proposition 2.1, for $\alpha\in(0,\beta d)$ fixed, there are subsets $\Xi_L\subset\Omega$ such that for L large enough, $\mathbb{P}(\Xi_L)\geq 1-e^{-L^\alpha}$ and such that for $\omega\in\Xi_L$ one has

$$\begin{aligned}
&P_{0,\omega}(T_{\partial B_{L,l,K}}=T_{\partial_+B_{L,l,K}})\\
&\quad\geq P_{0,\omega}\big(T_{\partial B_{L,l,K}}>T_x,\,T_{\partial(x+C_L)}(\theta_{T_x}(X_\cdot))=T_{\partial_+(x+C_L)}(\theta_{T_x}(X_\cdot))\big)\\
&\quad\geq P_{0,\omega}(T_{\partial B_{L,l,K}}>T_x)P_{x,\omega}\big(T_{\partial(x+C_L)}=T_{\partial_+(x+C_L)}\big)\\
&\quad\geq e^{-CL^{\beta'}}e^{-L^{\beta-\delta}}=e^{-CL^\beta}
\end{aligned}$$

for L large enough, where $\theta_n:(\mathbb{Z}^d)^{\mathbb{N}_0}\to(\mathbb{Z}^d)^{\mathbb{N}_0}$ denotes the canonical n-fold left shift and to obtain the first inequality we used property (Exit). In the second inequality, we have used the strong Markov property and in the third one we employed inequality (2.1) as well as Proposition 2.1 in combination with the translation invariance of the measure $\mathbb{P}$. This finishes the proof of the theorem.

2.2. *Proof of Theorem* 1.3. In [8], the author introduces the so called *effective criterion*, which is a ballisticity condition equivalent to condition (T') and which facilitates the explicit verification of condition (T'). The proof of Theorem 1.3 will rest on the fact that the effective criterion implies condition (T'). Indeed, we will prove that $(T)_\gamma$ implies the effective criterion, the main ingredient being Theorem 1.5.

For the sake of convenience, we recall here the effective criterion and its features. For positive numbers L, L' and $\tilde{L}$ as well as a space rotation R around the origin we define the *box specification* $\mathcal{B}(R,L,L',\tilde{L})$ as the box $B:=\{x\in\mathbb{Z}^d:x\in R((-L,L')\times(-\tilde{L},\tilde{L})^{d-1})\}$. Furthermore, let

$$\rho_{\mathcal{B}}(\omega):=\frac{P_{0,\omega}(X_{T_{\partial B}}\notin\partial_+B)}{P_{0,\omega}(X_{T_{\partial B}}\in\partial_+B)}.$$

Here, $\partial_+ B := \{x \in \partial B : R(e_1)\cdot x \geq L', |R(e_j)\cdot x| < \tilde{L}\ \forall j \in \{2,\dots,d\}\}$. We will sometimes write ρ instead of $\rho_{\mathcal{B}}$ if the box we refer to is clear from the context and use $\hat{R}$ to label any rotation mapping e_1 to $\hat{v}$. Note that due to the uniform ellipticity assumption, $\mathbb{P}$-a.s. we have $\rho \in (0,\infty)$. Given $l \in \mathbb{S}^{d-1}$, we say that the *effective criterion with respect to* l is satisfied if

$$\inf_{\mathcal{B},a}\left\{c_1(d)\left(\log\frac{1}{\kappa}\right)^{3(d-1)}\tilde{L}^{d-1}L^{3(d-1)+1}\mathbb{E}\rho_{\mathcal{B}}^a\right\} < 1. \tag{2.2}$$

Here, when taking the infimum, a runs over $[0,1]$ while $\mathcal{B}$ runs over the box-specifications $\mathcal{B}(R, L-2, L+2, \tilde{L})$ with R a rotation such that $R(e_1) = l$, $L \geq c_2(d)$, $3\sqrt{d} \leq \tilde{L} < L^3$. Furthermore, $c_1(d)$ and $c_2(d)$ are dimension dependent constants.

The following result was proven in [8].

THEOREM 2.2. *For each $l \in \mathbb{S}^{d-1}$, the following conditions are equivalent*:

(a) *The effective criterion with respect to l is satisfied.*
(b) *$(T')|l$ is satisfied.*

Due to this result, we can check condition (T'), which by nature of its definition is asymptotic, by investigating the local behavior of the walk only; indeed, to have the infimum on the left-hand side of (2.2) smaller than 1, it is sufficient to find one box $\mathcal{B}$ and $a \in [0,1]$ such that the corresponding inequality holds.

Recall that from Theorem 1.1 of [8] we infer that for l such that $l\cdot\hat{v} > 0$, we have that $(T)_\gamma|l$ implies $(T)_\gamma|\hat{v}$, and $(T)'|\hat{v}$ implies $(T)'|l$. Thus, because of (2.2) and Theorem 2.2, in order to prove Theorem 1.3 it is then sufficient to show that $(T)_\gamma|\hat{v}$ implies that

(D) for every natural $n \in \mathbb{N}$, one has that $\mathbb{E}\rho^a = o(L^n)$ as $L \to \infty$;

here, ρ corresponds to a box specification $\mathcal{B}(\hat{R}, L-2, L+2, L^2)$.

To show the desired decay, we split $\mathbb{E}\rho^a$ according to

$$\mathbb{E}\rho^a = \mathcal{E}_0 + \sum_{j=1}^{n-1}\mathcal{E}_j + \mathcal{E}_n, \tag{2.3}$$

where $n = n(\gamma)$ is a natural number the choice of which will depend on γ,

$$\mathcal{E}_0 := \mathbb{E}\big(\rho^a, P_{0,\omega}(X_{T_{\partial B}} \in \partial_+ B) > e^{-k_1 L^{\beta_1}}\big),$$
$$\mathcal{E}_j := \mathbb{E}\big(\rho^a, e^{-k_{j+1}L^{\beta_{j+1}}} < P_{0,\omega}(X_{T_{\partial B}} \in \partial_+ B) \leq e^{-k_j L^{\beta_j}}\big)$$

for $j \in \{1,\dots,n-1\}$ and

$$\mathcal{E}_n := \mathbb{E}\big(\rho^a, P_{0,\omega}(X_{T_{\partial B}} \in \partial_+ B) \leq e^{-k_n L^{\beta_n}}\big)$$

with parameters

$$\gamma =: \beta_1 < \beta_2 < \cdots < \beta_n := 1,$$

$a = L^{-\varepsilon}$, $\varepsilon \in (0,1)$, as well as k_n large enough and arbitrary positive constants $k_1, k_2, \ldots, k_{n-1}$. To bound $\mathcal{E}_0$, we employ the following lemma, which has been proven in [4].

LEMMA 2.3. *For all $L > 0$,*

$$\mathcal{E}_0 \leq e^{k_1 L^{\gamma-\varepsilon} - \delta_1 L^{\gamma-\varepsilon} + o(L^{\gamma-\varepsilon})},$$

where

$$\delta_1 := -\limsup_{L\to\infty} L^{-\gamma} \log P_0(X_{T_{\partial B}} \notin \partial_+ B) > 0.$$

To deal with the middle summand in the right-hand side of (2.3), we use the following lemma.

LEMMA 2.4. *For all $L > 0$, $j \in \{1, \ldots, n\}$ and $\varepsilon > 0$, we have that*

$$\mathcal{E}_j \leq e^{k_{j+1} L^{\beta_{j+1}-\varepsilon} - L^{\beta_j d-\varepsilon} + o(L^{\beta_j d-\varepsilon})}.$$

PROOF. Using Markov's inequality, for $j \in \{1, \ldots, n-1\}$ we obtain the estimate

$$\mathcal{E}_j \leq e^{k_{j+1} L^{\beta_{j+1}-\varepsilon}} \mathbb{P}\big(P_{0,\omega}(X_{T_{\partial B}} \in \partial_+ B) \leq e^{-k_j L^{\beta_j}}\big). \tag{2.4}$$

Due to Theorem 1.5, for $\varepsilon > 0$ fixed, the outer probability on the right-hand side of (2.4) can be estimated from above by $e^{-L^{\beta_j d-\varepsilon} + o(L^{\beta_j d-\varepsilon})}$. □

For the term $\mathcal{E}_n$ in (2.3), we have the following estimate.

LEMMA 2.5. *There exists a constant $C > 0$ such that for any $\varepsilon > 0$,*

$$\mathcal{E}_n \leq e^{CL^{1-\varepsilon} - L^{\beta_n d-\varepsilon} + o(L^{\beta_n d-\varepsilon})}.$$

PROOF. Using the uniform ellipticity assumption, we see that there is a constant $C > 0$ such that

$$\mathcal{E}_n \leq e^{CL^{1-\varepsilon}} \mathbb{P}\big(P_{0,\omega}(X_{T_{\partial B}} \in \partial_+ B) \leq e^{-k_n L^{\beta_n}}\big). \tag{2.5}$$

An application of Theorem 1.5 to estimate the second factor of the right-hand side of inequality (2.5) establishes the proof. □

From Lemmas 2.3, 2.4 and 2.5, we deduce that for $k_1 < \delta_1$, $n = n(\gamma)$ large enough, arbitrarily chosen positive constants $k_2, \dots, k_n$ as well as ε and $\beta_1, \dots, \beta_n$ satisfying

$$\beta_1 = \gamma, \qquad \varepsilon < \gamma,$$
$$\beta_{j+1} < \beta_j d$$

for $j \in \{1, \dots, n-1\}$, and

$$1 < \beta_n d,$$

all the terms $\mathcal{E}_0, \dots, \mathcal{E}_n$ on the right-hand side of (2.3) decay stretched exponentially. It is easily observed that the above choice of parameters is feasible, which establishes the desired decay in (D) and thus finishes the proof of Theorem 1.3.

3. Proof of Proposition 2.1 and auxiliary results. The proof of Proposition 2.1 is based on a modified version of the multiscale argument developed in [1]. In general, in our construction, we will name the corresponding results of the construction in [1] in brackets in the corresponding places.

We start with giving a heuristic (and cursory) idea of the proof. Afterward, we will set up all the necessary notation and auxiliary results before providing a rigorous proof of Proposition 2.1.

3.1. *Heuristics leading to Proposition 2.1.* The basic strategy of the proof is to construct, for $\beta \in (0, 1)$ and $\alpha < \beta d$ given, a sequence of events $(G_L)_{L \in \mathbb{N}}$, each a subset of Ω, such that for L large enough one has

$$\mathbb{P}(G_L^c) \le e^{-L^\alpha} \tag{3.1}$$

and at the same time

$$\inf_{\omega \in G_L} P_{0,\omega}(T_{\partial_+ C_L} = T_{\partial C_L}) \ge e^{-cL^\beta}, \tag{3.2}$$

where c is a constant that changes values various times throughout this subsection. In order to define G_L, for each of finitely many scales, we cover the box C_L with boxes of that certain scale. Boxes of the first scale have extent roughly $L^{2\psi}$ in direction $\hat{v}$, and extent marginally larger than L^ψ in directions orthogonal to $\hat{v}$. Here, $\psi > 0$ is much smaller than β. The boxes of larger scale more or less have ψ replaced by larger numbers [see (3.4), (3.7) and (3.8)]. Given an environment, we declare a box to be *good* if within this box and with respect to the given environment, the quenched random walk behaves very much like the annealed one. Otherwise, it is called *bad*.

We then define G_L as the event that there are not significantly more than L^α bad boxes of each scale contained in C_L. Using Proposition 3.4, which states that the probability of a box being bad decays faster than polynomially as a function

in L, by large deviations for binomially distributed variables one shows that the probability of the complement of this event is smaller than e^{-L^α}, so that (3.1) is satisfied (cf. Lemma 3.6).

It remains to show that on G_L, inequality (3.2) is satisfied. For this purpose, we associate to the walk a "current scale" that slowly increases as the e_1-coordinate of the walk increases. We will then require the walk to essentially leave in e_1-direction (i.e., through their right boundary parts) all the boxes of its current scale it traverses; this ensures that it leaves C_L through $\partial_+ C_L$. Since the probability that the random walk exits a good box through the right boundary part is relatively large, one can essentially bound the probability of leaving C_L through $\partial_+ C_L$ from below by the cost the walk incurs when traversing bad boxes.

Now each time the walk finds itself in a bad box of its current scale, it will instead move in boxes of smaller scale that contain its current position, and leave these boxes through their right boundary parts. Each time this happens, it has to "correct" the errors incurred by moving in such boxes through some deterministic steps, the cost of which will not exceed $e^{-cL^{2\psi}}$; in a certain way, these corrections make the walk look as if it has been leaving a box of its current scale through its right boundary part. Thus, we can roughly bound the probability of leaving C_L through $\partial_+ C_L$ by

$$e^{-cNL^{2\psi}}, \tag{3.3}$$

where N is the number of bad boxes that the walk visits.

Now in order to obtain a useful upper bound for N, we can force the random walk to have CLT-type fluctuations in directions transversal to $\hat{v}$ at constant cost in each box (see *random direction event*, Section 3.6). By means of this random direction event, one can then infer the existence of a direction (depending on the environment) such that, if the CLT-type fluctuations of the walk essentially center around this direction, then the walk encounters a little less than L^β bad boxes of each scale on its way through C_L. From (3.3), we deduce that the probability for the walker to leave C_L through $\partial_+ C_L$ can then be bounded from below by e^{-cL^β}. This suggests that (3.2) holds.

3.2. *Preliminaries*. We first recall an equivalent formulation of condition $(T)_\gamma$ and introduce the basic notation that will be used throughout the rest of this paper.

We will use C to denote a generic constant that may change from one side to the other of the same inequality. This constant may usually depend on various parameters, but in particular does not depend on the variable L nor N (recall that L is the variable which makes the slabs and boxes grow, and N will play a similar role in general results). In "general lemmas," we will usually denote the corresponding probability measure and expectation by P and E, respectively. Furthermore, when considering stopping times without mentioning the process they apply to, then they will usually refer to the RWRE X.

Not all auxiliary results will appear in the order in which they are employed. In fact, in order to improve readability, we defer the majority of them to the Appendix.

In addition, we assume the conditions of Proposition 2.1 to be fulfilled for the rest of this paper without further mentioning.

We first introduce the regeneration times in direction e_1. Setting $\tau_0 := 0$, we define the *first regeneration time* τ_1 as the first time $X_n \cdot e_1$ obtains a new maximum and never falls below that maximum again, that is,

$$\tau_1 := \inf\Big\{n \in \mathbb{N} : \sup_{0\le k\le n-1} X_k \cdot e_1 < X_n \cdot e_1 \text{ and } \inf_{k\ge n} X_k \cdot e_1 \ge X_n \cdot e_1\Big\}.$$

Now define recursively in n the $(n+1)$*st regeneration time* τ_{n+1} as the first time after τ_n that $X_n \cdot l$ obtains a new maximum and never goes below that maximum again, that is, $\tau_{n+1} := \tau_1(X_{\tau_n+\cdot})$. For $n \in \mathbb{N}$, we define the *radius* of the nth regeneration as

$$X^{*(n)} := \sup_{\tau_{n-1}\le k\le \tau_n} \|X_k - X_{\tau_{n-1}}\|_1.$$

This notation gives rise to the following equivalent formulation of $(T)_\gamma$ proven in [8], Corollary 1.5.

THEOREM 3.1. *Let $\gamma \in (0,1)$ and $l \in \mathbb{S}^{d-1}$. Then the following are equivalent*:

(i) *Condition $(T)_\gamma|l$ is satisfied.*
(ii) *$P_0(\lim_{n\to\infty} X_n \cdot l = \infty) = 1$ and $E_0 \exp\{c(X^{*(1)})^\gamma\} < \infty$ for some $c > 0$.*

REMARK 3.2. Note in particular that, similarly to Proposition 1.3 of Sznitman and Zerner [10], condition (ii) implies $E_0 \exp\{c(X^{*(n)})^\gamma\} < \infty$ for any $n \ge 2$.

We will repeatedly use the above equivalence. Now for each natural k and N we define the scales

$$R_k(N) := \lceil \exp\{(\log\log N)^{k+1}\}\rceil.$$

Note that for every natural n, N and k one has that

$$R_k^n(N) \in o(R_{k+1}(N)) \quad \text{and} \quad R_k(N) \in o(N).$$

Define for each natural N the sublattice

$$\mathcal{L}_N := N^2\mathbb{Z} \times \left\lfloor \frac{R_6(N)N}{4} \right\rfloor \mathbb{Z}^{d-1}$$

of $\mathbb{Z}^d$. Furthermore, for each N and $x \in \mathbb{Z}^d$ we define the blocks

$$\mathcal{P}(0,N) := \{y \in \mathbb{Z}^d : -N^2 < y \cdot e_1 < N^2, \|\tilde{\pi}_{\hat{v}^\perp}(y)\|_\infty < R_6(N)N\} \tag{3.4}$$

and

$$\mathcal{P}(x, N) := x + \mathcal{P}(0, N)$$

as well as their middle thirds

$$\tilde{\mathcal{P}}(0, N) := \{y \in \mathbb{Z}^d : -N^2/3 < y \cdot e_1 < N^2/3, \|\tilde{\pi}_{\hat{v}^\perp}(y)\|_\infty < R_6(N)N/3\}$$

and

$$\tilde{\mathcal{P}}(x, N) := x + \tilde{\mathcal{P}}(0, N).$$

Note that this construction ensures that for each $x \in N^2\mathbb{Z} \times \mathbb{Z}^{d-1}$ there exists a $z \in \mathcal{L}_N$ such that $x \in \tilde{\mathcal{P}}(z, N)$. Furthermore, define its right boundary part

$$\partial_+\mathcal{P}(x, N) := \{y \in \partial\mathcal{P}(x, N) : (y - x) \cdot e_1 = N^2\}.$$

See Figure 4 for an illustration.

For $N \geq 1$, define the event

$$A_N(X) := \{X^{*(n)} < R_2(N)\ \forall n \in \{1, \ldots, 2N^2\}\},$$

where at times we write A_N instead of $A_N(X)$ if the corresponding process X is clear from the context. Using Markov's inequality, the following lemma is a consequence of Theorem 3.1.

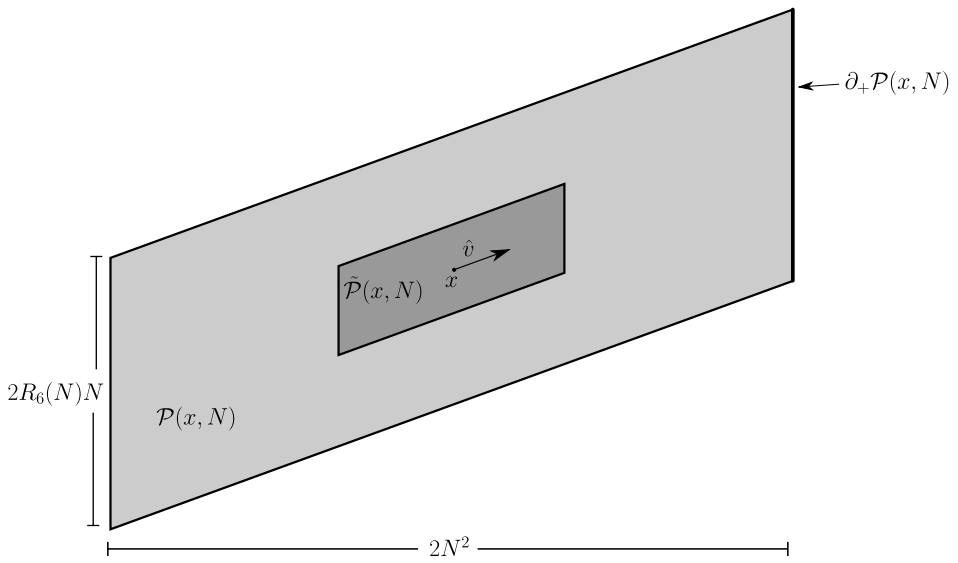

FIG. 4. *The set $\mathcal{P}(x, N)$ and its right boundary part $\partial_+\mathcal{P}(x, N)$ as well as its middle third $\tilde{\mathcal{P}}(x, N)$.*

LEMMA 3.3. *There exists a constant $C > 0$ such that for each $N \geq 1$,*

$$P_0(A_N^c) \leq Ce^{-C^{-1}R_2(N)^\gamma} \tag{3.5}$$

and, defining the event

$$\mathcal{A}_N := \bigcap_{x \in \tilde{\mathcal{P}}(0,N)} \{P_{x,\omega}(A_N^c) \leq e^{-R_1(N)^\gamma}\},$$

which is contained in the Borel-σ-algebra of Ω, one has

$$\mathbb{P}(\mathcal{A}_N^c) \leq Ce^{-C^{-1}R_2(N)^\gamma}.$$

We define the set of rapidly decreasing sequences as

$$\mathcal{S}(\mathbb{N}) := \left\{(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sup_{n\in\mathbb{N}} |n^k a_n| < \infty \ \forall k \in \mathbb{N}\right\}$$

and note that due to Lemma 3.3 we have that $N \mapsto P_0(A_N^c)$ and $N \mapsto \mathbb{P}(\mathcal{A}_N^c)$ are contained in $\mathcal{S}(\mathbb{N})$.

3.3. *Berger's semi-local limit theorem and scaling.* As a first step in the scaling, we introduce a classification of blocks. We need to define some parameters which will remain fixed throughout this paper. For β and α as in the assumptions of Proposition 2.1, choose δ such that

$$0 < \delta < \frac{\beta d - \alpha}{12d}.$$

Furthermore, fix

$$\psi \in \left(2\delta, \frac{20\delta}{9}\right) \tag{3.6}$$

and χ such that

$$0 < \chi < (\beta - 6\delta)/2 \wedge \psi/4 \wedge 6/(d-1).$$

From now on let $L \in \mathbb{N}$, define $L_1 := \lfloor L^\psi \rfloor$ and recursively in k the scales

$$L_{k+1} := L_k \lfloor L^\chi \rfloor. \tag{3.7}$$

Define ι to be the smallest k such that $L_k^2 > L^{1+\delta}$. For $k \in \{1, \ldots, \iota\}$ and $x \in C_L \cap \mathcal{L}_{L_k}$, we call a block

$$\mathcal{P}(x, L_k) \tag{3.8}$$

good with respect to the environment ω if the following three properties are satisfied for $\vartheta := \chi$ and all $z \in \tilde{\mathcal{P}}(x, L_k)$:

(i)

$$P_{z,\omega}\big(T_{\partial\mathcal{P}(x,L_k)} \neq T_{\partial_+\mathcal{P}(x,L_k)}\big) \leq e^{-R_1(L_k)^\gamma}. \tag{3.9}$$

(ii)

$$\begin{aligned}&\big\| E_{z,\omega}\big(X_{T_{\partial\mathcal{P}(x,L_k)}}|T_{\partial\mathcal{P}(x,L_k)}=T_{\partial_+\mathcal{P}(x,L_k)}\big)\\&\qquad - E_z\big(X_{T_{\partial\mathcal{P}(x,L_k)}}|T_{\partial\mathcal{P}(x,L_k)}=T_{\partial_+\mathcal{P}(x,L_k)}\big)\big\|_1\le R_4(L_k).\end{aligned}\tag{3.10}$$

(iii)

$$\begin{aligned}&\max_Q\big|P_{z,\omega}\big(X_{T_{\partial\mathcal{P}(x,L_k)}}\in Q|T_{\partial\mathcal{P}(x,L_k)}=T_{\partial_+\mathcal{P}(x,L_k)}\big)\\&\qquad - P_z\big(X_{T_{\partial\mathcal{P}(x,L_k)}}\in Q|T_{\partial\mathcal{P}(x,L_k)}=T_{\partial_+\mathcal{P}(x,L_k)}\big)\big|\\&\qquad < L_k^{(\vartheta-1)(d-1)-\vartheta(d-1)/(d+1)},\end{aligned}\tag{3.11}$$

where the maximum in Q is taken over all $(d-1)$-dimensional hypercubes $Q\subset\partial_+\mathcal{P}(x,L_k)$ of side length $\lceil L_k^{\vartheta}\rceil$.

Otherwise, we say that the block $\mathcal{P}(x,L_k)$ is *bad*. For $k\in\{1,\dots,\iota\}$ we will usually refer to boxes of the form $\mathcal{P}(x,L_k)$ as a *box of scale k*.

The following result is essentially Proposition 4.5 of [1], which can be understood as a semi-local central limit theorem for RWRE. For the sake of completeness, we will give its proof in the Appendix.

PROPOSITION 3.4 (Proposition 4.5 of [1]). *Assume that* $(T)_\gamma|l$ *is satisfied and fix* $\vartheta\in(0,\frac{6}{d-1}\wedge 1)$. *Then there exists a sequence of events* $(G_L)_{L\in\mathbb{N}}\subset\Omega$ *such that* $\mathbb{P}(G_\cdot^c)\in\mathcal{S}(\mathbb{N})$ *and for all* $\omega\in G_L$ *and* $k\in\{1,\dots,\iota\}$:

(i) *display* (3.9),
(ii) *display* (3.10) *and*
(iii) *display* (3.11)

are satisfied for all $x\in C_L\cap\mathcal{L}_{L_k}$, $z\in\tilde{\mathcal{P}}(x,L_k)$ *and the chosen* ϑ.

In particular, due to the translation invariance of the environment, we have that $\mathbb{P}(\mathcal{P}(x,\cdot)$ *is bad*$)\in\mathcal{S}(\mathbb{N})$ *for any* $x\in\mathbb{Z}^d$.

REMARK 3.5. For the sake of notational simplicity, we will prove the proposition by showing that there exist sequences $G_L^{(\mathrm{i})}$, $G_L^{(\mathrm{ii})}$ and $G_L^{(\mathrm{iii})}$, $L\in\mathbb{N}$, of subsets of Ω such that

$$\mathbb{P}\big(G_L^{(\mathrm{i})^c}\big),\qquad \mathbb{P}\big(G_L^{(\mathrm{ii})^c}\big)\quad\text{and}\quad\mathbb{P}\big(G_L^{(\mathrm{iii})^c}\big)$$

are contained in $\mathcal{S}(\mathbb{N})$ as functions in L and such that for ω contained in these sets, $x=0$, and $z\in\tilde{\mathcal{P}}(0,L)$, displays (3.9), (3.10) and (3.11), respectively, are fulfilled for L instead of L_k. The required result then follows by observing that $\mathbb{P}$ is translation invariant and using $|C_L|\le CL^{2d}$ in combination with a standard union bound.

We next give an upper bound on the probability that an environment has many bad blocks. For this purpose, set

$$\Theta_L := \big\{\omega \in \Omega : |\{x \in C_L \cap \mathcal{L}_{L_k} : \mathcal{P}(x, L_k) \text{ is bad with respect to } \omega\}| \le L^{\alpha+\delta}\ \forall k \in \{1, \dots, \iota\}\big\}. \tag{3.12}$$

Furthermore, observe that $\mathcal{L}_L$ can be represented as the disjoint union of $2 \cdot 8^{d-1}$ (translated) sublattices of $\mathbb{Z}^d$ such that for any sublattice $\mathcal{L}$ of these and $z_1, z_2 \in \mathcal{L}$, we have $\mathcal{P}(z_1, L) \cap \mathcal{P}(z_2, L) = \varnothing$.

LEMMA 3.6 (Lemma 5.1 of [1]). *For L large enough,*

$$\mathbb{P}(\Theta_L^c) \le e^{-L^\alpha}.$$

PROOF. For $k \in \{1, \dots, \iota\}$, set

$$J_{L_k}(\omega) := |\{z \in C_L \cap \mathcal{L}_{L_k} : \mathcal{P}(z, L_k) \text{ is bad with respect to } \omega\}|$$

and note that

$$\mathbb{P}(\Theta_L^c) \le \sum_{k=1}^{\iota} \mathbb{P}(J_{L_k} > L^{\alpha+\delta}). \tag{3.13}$$

As in [1] we can write $J_{L_k} = J_{L_k}^{(1)} + \cdots + J_{L_k}^{(2\cdot 8^{d-1})}$ with $J_{L_k}^{(m)}$ distributed binomially with parameters $D(L_k)$ and $p(L_k)$ for $m \in \{1, \dots, 2 \cdot 8^{d-1}\}$. Here, $p(L) := \mathbb{P}(\mathcal{P}(0, L) \text{ is bad})$, that is, in particular, due to Proposition 3.4,

$$p \in \mathcal{S}(\mathbb{N}), \tag{3.14}$$

and $D(L_k)$ is the maximal number of intersection points any of the above-mentioned translated sublattices has with C_L, that is, in particular

$$D(L_k) \le C L^{2d} \tag{3.15}$$

for some constant C and all L. Now for $m \in \{1, \dots, 2 \cdot 8^{d-1}\}$, we have

$$\mathbb{P}\Big(J_{L_k}^{(m)} > \frac{L^{\alpha+\delta}}{2 \cdot 8^{d-1}}\Big) \le \exp\Big\{-\frac{L^{\alpha+\delta}}{2 \cdot 8^{d-1}}\Big\} \mathbb{E}\exp\{J_{L_k}^{(m)}\} \tag{3.16}$$

with

$$\mathbb{E}\exp\{J_{L_k}^{(m)}\} \le \sum_{j=0}^{D(L_k)} \binom{D(L_k)}{j} (ep(L_k))^j (1 - ep(L_k))^{D(L_k)-j} \times \Big(\frac{1 - p(L_k)}{1 - ep(L_k)}\Big)^{D(L_k)-j}, \tag{3.17}$$

and from (3.14) and (3.15) we conclude that

$$\lim_{L\to\infty}\left(\frac{1-p(L_k)}{1-ep(L_k)}\right)^{D(L_k)-j}=1$$

uniformly in $j\in\{0,\dots,D(L_k)\}$. Substituting this back into displays (3.17), (3.16) and (3.13), we conclude the proof. □

We now need to recall the concept of *closeness* between two probability measures introduced in [1]. Here and in the following, if Z is a d-dimensional random variable defined on a probability space with probability measure μ, we write $E_\mu Z := \int Z\,d\mu$ and if μ is a measure on $\mathbb{R}^d$, then we write $E_\mu := \int x\,d\mu$, whenever the integrals are well defined. Furthermore, we define its variance via $\operatorname{Var} Z := E\|Z-EZ\|_1^2$ whenever this expression is well defined and correspondingly for a probability measure μ on $\mathbb{R}^d$ with appropriate integrability conditions we write Var_μ.

DEFINITION 3.7. Let μ_1 and μ_2 be two probability measures on $\mathbb{Z}^d$. Let $\lambda\in[0,1)$ and K be a natural number. We say that μ_2 is (λ,K)-*close* to μ_1 if there exists a coupling μ of three random variables Z_1, Z_2 and Z_0 such that:

(a) $\mu\circ Z_j^{-1}=\mu_j$ for $j\in\{1,2\}$,
(b) $\mu(Z_1\neq Z_0)\le\lambda$,
(c) $\mu(\|Z_0-Z_2\|_1\le K)=1$,
(d) $E_\mu Z_1=E_\mu Z_0$,
(e) $\sum_x \|x-E_\mu Z_1\|_1^2\cdot|\mu(Z_1=x)-\mu(Z_0=x)|\le\lambda\operatorname{Var} Z_1$.

REMARK 3.8. Assume given a random variable X that is distributed according to some distribution which is (λ,K)-close to some other distribution. Then the corresponding coupling which establishes this closeness can be defined on an extension of the probability space X is defined on, with X playing the role of Z_2. We will therefore assume this property to be fulfilled from now on without further mentioning when dealing with such couplings.

3.4. *General auxiliary results*. The following lemma is a sort of remedy for the fact that

$$\left(\tilde\pi_{\hat v^\perp}\left(\sum_{j=1}^{n}(X_{\tau_j}-X_{\tau_{j-1}})\right)\right)_{n\in\{2,\dots,2L^2\}}$$

with respect to $P_0(\cdot|A_L)$, due to the conditioning on A_L, is not a martingale. To state the result, set

$$\hat v_L := \frac{E_0(X_{\tau_2}-X_{\tau_1})\mathbb{1}_{A_L}}{\|E_0(X_{\tau_2}-X_{\tau_1})\mathbb{1}_{A_L}\|_2}. \tag{3.18}$$

We start with showing that for L large, $\hat v_L$ hardly deviates from the asymptotic direction $\hat v$.

LEMMA 3.9.

$$\|\hat{v} - \hat{v}_\cdot\|_2 \in \mathcal{S}(\mathbb{N}).$$

PROOF. Note that

$$\begin{aligned}\|\hat{v} - \hat{v}_L\|_2 &= \left\| \frac{E_0(X_{\tau_2} - X_{\tau_1})}{\|E_0(X_{\tau_2} - X_{\tau_2})\|_2} - \frac{E_0(X_{\tau_2} - X_{\tau_1}, A_L)}{\|E_0(X_{\tau_2} - X_{\tau_1}, A_L)\|_2} \right\|_2 \\ &= \big\| E_0(X_{\tau_2} - X_{\tau_1}) \|E_0(X_{\tau_2} - X_{\tau_1}, A_L)\|_2 \\ &\qquad - E_0(X_{\tau_2} - X_{\tau_1}, A_L) \|E_0(X_{\tau_2} - X_{\tau_1})\|_2 \big\|_2 \\ &\qquad \times \big(\|E_0(X_{\tau_2} - X_{\tau_1})\|_2 \|E_0(X_{\tau_2} - X_{\tau_1}, A_L)\|_2\big)^{-1}.\end{aligned} \tag{3.19}$$

Inserting a productive 0, the numerator evaluates to

$$\begin{aligned}&\big\| E_0(X_{\tau_2} - X_{\tau_1}) \|E_0(X_{\tau_2} - X_{\tau_1}, A_L)\|_2 - E_0(X_{\tau_2} - X_{\tau_1}) \|E_0(X_{\tau_2} - X_{\tau_1})\|_2 \\ &\quad + E_0(X_{\tau_2} - X_{\tau_1}) \|E_0(X_{\tau_2} - X_{\tau_1})\|_2 - E_0(X_{\tau_2} - X_{\tau_1}, A_L) \|E_0(X_{\tau_2} - X_{\tau_1})\|_2 \big\|_2 \\ &\quad \le \|E_0(X_{\tau_2} - X_{\tau_1})\|_2 \big| \|E_0(X_{\tau_2} - X_{\tau_1}, A_L)\|_2 - \|E_0(X_{\tau_2} - X_{\tau_1})\|_2 \big| \\ &\qquad + \|E_0(X_{\tau_2} - X_{\tau_1}, A_L^c)\|_2 \|E_0(X_{\tau_2} - X_{\tau_1})\|_2 \\ &\quad \le 2 \|E_0(X_{\tau_2} - X_{\tau_1})\|_2 \|E_0(X_{\tau_2} - X_{\tau_1}, A_L^c)\|_2,\end{aligned}$$

where the last inequality follows from the reverse triangle inequality. But Cauchy–Schwarz's inequality and Lemma 3.3 yield

$$\begin{aligned}\|E_0(X_{\tau_2} - X_{\tau_1}, A_L^c)\|_2 &\le E_0(\|X_{\tau_2} - X_{\tau_1}\|_2, A_L^c) \\ &\le E_0(\|X_{\tau_2} - X_{\tau_1}\|_2^2)^{1/2} P_0(A_L^c)^{1/2} \\ &\le C e^{-C^{-1} R_2(L)^\gamma / 2},\end{aligned}$$

whence (3.19) is contained in $\mathcal{S}(\mathbb{N})$ as a function in L. □

Therefore, $(\tilde{\pi}_{\hat{v}^\perp}(\sum_{j=1}^n (X_{\tau_j} - X_{\tau_{j-1}})))_{n \in \{2, \dots, 2L^2\}}$ is nearly a mean-zero martingale with respect to $P_0(\cdot | A_L)$ and this is what we will exploit in the proof of the next lemma.

LEMMA 3.10. *For L and $x \in \tilde{\mathcal{P}}(0, L)$, define the event*

$$\begin{aligned}F_{x,L} := \big\{ \exists n \in \{0, \dots, T_{L^2}\} : &\|\tilde{\pi}_{\hat{v}^\perp}(X_n - x)\|_\infty \ge R_3(L) L \\ &\text{or } (X_n - x) \cdot e_1 < -R_2(L) \big\}.\end{aligned}$$

Then there exists a constant $C > 0$ such that for all L,

$$\max_{x \in \tilde{\mathcal{P}}(0,L)} P_x(F_{x,L}) \le C e^{-C^{-1} R_2(L)^\gamma}.$$

In particular,

$$\max_{x\in\tilde{\mathcal{P}}(0,L)} P_x\big(X_{T_{\partial\mathcal{P}(0,L)}} \notin \partial_+\mathcal{P}(0,L)\big) \le Ce^{-C^{-1}R_2(L)^\gamma}. \tag{3.20}$$

PROOF. Setting $F'_{x,L} := \{\exists n \in \{0,\dots,T_{L^2}\} : \|\tilde{\pi}_{\hat{v}^\perp}(X_n - x)\|_\infty \ge R_3(L)L\}$, we have

$$P_x(F_{x,L}) \le P_x(F'_{x,L}, A_L) + P_x(A_L^c). \tag{3.21}$$

Note that $(\tilde{\pi}_{\hat{v}_L^\perp}(\sum_{j=1}^n (X_{\tau_j} - X_{\tau_{j-1}})))_{n\in\{2,\dots,2L^2\}}$ is a $(d-1)$-dimensional martingale with respect to $P_x(\cdot|A_L)$. Furthermore, observe that due to Lemma 3.9, in particular we have

$$\sup_{y\in\mathcal{P}(0,L)} \|\tilde{\pi}_{\hat{v}_L^\perp}(y) - \tilde{\pi}_{\hat{v}^\perp}(y)\|_\infty \le R_2(L)$$

for L large enough. Therefore, Azuma's inequality applied to the coordinates yields

$$\begin{aligned}
&P_x(F_{x,L}, A_L)\\
&\quad\le P_x\big(\exists n \in \{\tau_1,\dots,\tau_{2L^2}\} : \|\tilde{\pi}_{\hat{v}^\perp}(X_n - X_{\tau_1})\|_\infty \ge R_3(L)L - 2R_2(L)|A_L\big)\\
&\quad\le P_x\big(\exists n \in \{\tau_1,\dots,\tau_{2L^2}\} : \|\tilde{\pi}_{\hat{v}_L^\perp}(X_n - X_{\tau_1})\|_\infty \ge R_3(L)L - 3R_2(L)|A_L\big)\\
&\quad\le P_x\big(\exists n \in \{1,\dots,2L^2\} : \|\tilde{\pi}_{\hat{v}_L^\perp}(X_{\tau_n} - X_{\tau_1})\|_\infty \ge R_3(L)L - 4R_2(L)|A_L\big)\\
&\quad\le 2(d-1)\sum_{j=1}^{2L^2} \exp\left\{-\frac{(R_3(L)L/2)^2}{2jR_2(L)^2}\right\}\\
&\quad\le 4(d-1)L^2 \exp\left\{-\frac{(R_3(L)L/2)^2}{4L^2R_2(L)^2}\right\}\\
&\quad\le \exp\{-R_3(L)\}
\end{aligned}$$

for L large enough. In particular, in combination with (3.21) and (3.5) this reasoning finishes the proof of the first part. Equality (3.20) is an immediate consequence. □

The following lemma, which we will prove in Section A.6 (see page 533), provides lower bounds on certain exit probabilities.

LEMMA 3.11. *Let C' be a positive constant. Then there exists a positive constant c such that for all L large enough, and all $x \in \tilde{\mathcal{P}}(0,L)$, $y \in \partial_+\mathcal{P}(0,L)$ with $\|\tilde{\pi}_{\hat{v}^\perp}(y-x)\|_1 < C'L$, we have*

$$P_x(X_{T_{\partial\mathcal{P}(0,L)}} = y) \ge cL^{1-d}.$$

Let now $x \in \mathcal{L}_L$ and $z \in \tilde{\mathcal{P}}(x, L)$. Then, following [1], for $\omega \in \Omega$ we define $\mu^L_{z,x,\omega}$ to be the distribution of $X_{T_{\partial\mathcal{P}(x,L)}}$ with respect to $P_{z,\omega}(\cdot | T_{\partial\mathcal{P}(x,L)} = T_{\partial_+\mathcal{P}(x,L)})$. Similarly, we define $\mu^L_{z,x}$ to be the distribution of $X_{T_{\partial\mathcal{P}(x,L)}}$ with respect to $P_z(\cdot | T_{\partial\mathcal{P}(x,L)} = T_{\partial_+\mathcal{P}(x,L)})$.

We now get the following bounds for $\mathrm{Var}_{\mu^L_{x,0}}$, which will turn out to be useful in the proof of Corollary 3.13 below.

LEMMA 3.12. *There exists a constant C such that for all $x \in \tilde{\mathcal{P}}(0, L)$ and all L,*

$$C^{-1}L^2 \le \mathrm{Var}_{\mu^L_{x,0}} \le CL^2.$$

PROOF. The lower bound is a consequence of Lemma 3.11.

To prove the upper bound, note that $S_n := \sum_{k=1}^n X_{\tau_k} - X_{\tau_{k-1}} - E_0(X_{\tau_k} - X_{\tau_{k-1}})$ is a martingale in n with respect to P_0. We define the stopping time

$$T := \inf\left\{ n \in \mathbb{N} : \left(S_n + \sum_{k=1}^n E_0(X_{\tau_k} - X_{\tau_{k-1}}) \right) \cdot e_1 \ge L^2 \right\}$$

and note that in particular $(S_{n\wedge T} \cdot e_j)_{n\in\mathbb{N}}$ is a martingale for any $j \in \{2, \dots, d\}$. The independence of the increments yields that so is

$$\big((S_{n\wedge T} \cdot e_j)^2 - \big(E(S_m \cdot e_j)^2\big)_{m=n\wedge T}\big)_{n\in\mathbb{N}}.$$

Since for $n = 0$ the martingale equals 0, we have, noting that

$$E(S_m \cdot e_j)^2 = \sum_{k=1}^m E\big(\big(X_{\tau_k} - X_{\tau_{k-1}} - E_0(X_{\tau_k} - X_{\tau_{k-1}})\big) \cdot e_j\big)^2$$

as well as $T \le L^2$, that

$$E(S_T \cdot e_j)^2 \le CL^2.$$

Taking into consideration Lemma 3.3 and Lemma 3.10, this implies the upper bound. □

For $x \in \mathbb{Z}^d$ and $k \in \mathbb{Z}$, we will use the

$$H_k := \{x \in \mathbb{Z}^d : x \cdot e_1 = k\} \tag{3.22}$$

from the following proof onward.

In [1], the author derived a result similar to the following corollary of Proposition 3.4.

COROLLARY 3.13. *Fix $\vartheta \in (0, 5/8]$ and let L be large enough. Furthermore, let $k \in \{1, \dots, \iota\}$, $x \in C_L \cap \mathcal{L}_{L_k}$ and $\omega \in \Omega$ such that (3.10) and (3.11) are fulfilled for this choice of ϑ and all $z \in \tilde{\mathcal{P}}(x, L_k)$.*

Then $\mu^{L_k}_{z,x,\omega}$ is $(L_k^{-\vartheta(d-1)/(2(d+1))}, 2dL_k^{\vartheta})$-close to $\mu^{L_k}_{z,x}$.

PROOF. For fixed k, x, ω and z as in the assumptions, we will show the desired closeness for L large enough. Observing that this lower bound on L holds uniformly in the admissible choices of k, x, ω and z then finishes the proof.

We will construct the coupling of Definition 3.7 in the case $x = 0$, the remaining cases being handled in exactly the same manner. Cover $\partial_+\mathcal{P}(0, L)$ by at most $n = \lceil 2R_6(L_k)L_k^{1-\vartheta}\rceil^{d-1}$ disjoint cubes $Q_1, Q_2, \ldots, Q_n$ of side length $\lceil L_k^{\vartheta}\rceil$. Consider an i.i.d. sequence $(Y_j)_{j\in\mathbb{N}}$ of random variables defined on a probability space with probability measure P^* (the space should also be large enough to accommodate the random variables we will define in the remaining part of this proof) such that

$$P^*(Y_j = x) = \frac{\mu_{z,0}^{L_k}(\{x\})}{\mu_{z,0}^{L_k}(Q_j)}, \qquad x \in Q_j,$$

and $P^*(Y_j = x) = 0$ if $x \notin Q_j$; set

$$Y := \sum_{j=1}^{n} Y_j \mathbb{1}_{\{X_{T_{\partial P(0,L_k)}} \in Q_j\}}$$

and

$$\mathbf{P}_{z,\omega} := P_{z,\omega}\big(\cdot | T_{\partial_+\mathcal{P}(0,L_k)} = T_{\partial\mathcal{P}(0,L_k)}\big) \otimes P^*.$$

Clearly, $\mathbf{P}_{z,\omega}$-a.s., $\|X_{T_{\partial P(0,L_k)}} - Y\|_1 \le (d-1)\lceil L_k^{\vartheta}\rceil$ and consequently we have

$$\|\mathbf{E}_{z,\omega}Y - \mathbf{E}_{z,\omega}X_{T_{\partial P(0,L_k)}}\|_1 < (d-1)\lceil L_k^{\vartheta}\rceil.$$

Display (3.10) yields

$$\big\|\mathbf{E}_{z,\omega}X_{T_{\partial P(0,L_k)}} - E_z\big(X_{T_{\partial P(0,L_k)}} | T_{\partial_+\mathcal{P}(0,L_k)} = T_{\partial\mathcal{P}(0,L_k)}\big)\big\|_1 \le R_4(L_k)$$

and thus

$$\big\|\mathbf{E}_{z,\omega}Y - E_z\big(X_{T_{\partial P(0,L_k)}} | T_{\partial_+\mathcal{P}(0,L_k)} = T_{\partial\mathcal{P}(0,L_k)}\big)\big\|_1 < dL_k^{\vartheta}$$

for L large enough. Let now U be an H_0-valued random variable defined on the same probability space as the sequence $(Y_j)_{j\in\mathbb{N}}$ (and choose U to be independent of everything else) such that $\mathbf{P}_{z,\omega}$-a.s. we have $\|U\|_1 \le dL_k^{\vartheta}$ as well as

$$\mathbf{E}_{z,\omega}U = E_z\big(X_{T_{\partial P(0,L_k)}} | T_{\partial_+\mathcal{P}(0,L_k)} = T_{\partial\mathcal{P}(0,L_k)}\big) - \mathbf{E}_{z,\omega}Y.$$

Define

$$Z_0 := Y + U$$

and

$$Z_2 := X_{T_{\partial P(0,L_k)}}.$$

Then taking $\mathbf{P}_{z,\omega}$ as the μ of Definition 3.7, part (c) of that definition is fulfilled for $K = 2dL_k^{\vartheta}$ and L large enough. To show the remaining parts, we first note that for $y \in \mathbb{Z}^d$ we have

$$\mathbf{P}_{z,\omega}(Z_0 = y) = \sum_{u\,:\,\|u\|_1 \le dL_k^{\vartheta}} \mathbf{P}_{z,\omega}(U = u)\mathbf{P}_{z,\omega}(Y = y - u).$$

Since furthermore $\mathbf{P}_{z,\omega} \circ Y^{-1}$ is supported on $\partial_+\mathcal{P}(0, L_k)$, we get

$$\begin{aligned}
&\sum_{y\in\mathbb{Z}^d} |\mathbf{P}_{z,\omega}(Z_0 = y) - \mu_{z,0}^{L_k}(y)| \\
&\le 2 \sum_{u\,:\,\|u\|_1 \le dL_k^{\vartheta}} \mathbf{P}_{z,\omega}(U = u) \\
&\qquad \times \sum_{y\,:\,y-u\in\partial_+\mathcal{P}(0,L_k)} |\mathbf{P}_{z,\omega}(Y = y - u) - \mu_{z,0}^{L_k}(y)|. \qquad (3.23)
\end{aligned}$$

By heat-kernel estimates to be proven later [cf. part (b) of Lemma A.2], for each $y \in H_{L_k^2}$ and every u such that $\|u\|_1 \le dL_k^{\vartheta}$,

$$|\mu_{z,0}^{L_k}(y - u) - \mu_{z,0}^{L_k}(y)| \le CL_k^{\vartheta} \cdot L_k^{-d} = CL_k^{\vartheta-d}.$$

In combination with (3.23) and the validity of (3.11), this yields

$$\begin{aligned}
&\sum_{y\in\mathbb{Z}^d} |\mathbf{P}_{z,\omega}(Z_0 = y) - \mu_{z,0}^{L_k}(y)| \\
&\le 2 \sum_{y\in\partial_+\mathcal{P}(0,L_k)} \big(|\mathbf{P}_{z,\omega}(Y = y) - \mu_{z,0}^{L_k}(y)| + CL_k^{\vartheta-d}\big) \\
&\le 2\left[\left(\sum_{j=1}^{n} |\mu_{z,0,\omega}^{L_k}(Q_j) - \mu_{z,0}^{L_k}(Q_j)|\right) + (2R_6(L_k))^{d-1}L_k^{d-1} \cdot CL_k^{\vartheta-d}\right] \qquad (3.24) \\
&\le CR_6(L_k)^{d-1}L_k^{\vartheta-1} + \lceil 2R_6(L_k)L_k^{1-\vartheta}\rceil^{d-1} \cdot L_k^{(\vartheta-1)(d-1)-\vartheta(d-1)/(d+1)} \\
&\le CR_6(L_k)^{d-1}\big(L_k^{\vartheta-1} + L_k^{-\vartheta(d-1)/(d+1)}\big) \le R_7(L_k)L_k^{-\vartheta(d-1)/(d+1)} \\
&< L_k^{-\vartheta(d-1)/(2(d+1))},
\end{aligned}$$

L large enough; here, the second inequality is obtained by noting that the sign of $|\mathbf{P}_{z,\omega}(Y = y) - \mu_{0,z}^{L_k}(y)|$ is constant as y varies over Q_j for fixed j, while the penultimate inequality takes advantage of $\vartheta \le 5/8$ and $d \ge 4$. Thus, due to (3.24), there exists a random variable Z_1 defined on the probability space with probability measure $\mathbf{P}_{z,\omega}$ such that $\mathbf{P}_{z,\omega} \circ Z_1^{-1} = \mu_{z,0}$ and $\mathbf{P}_{z,\omega}(Z_1 \ne Z_0) \le L_k^{-\vartheta(d-1)/(2(d+1))}$. This establishes (a), (b) and (d) of Definition 3.7 for $\lambda = L_k^{-\vartheta(d-1)/(2(d+1))}$.

To see (e), observe that

$$\mathbf{Var}_{x,\omega} Z_1 = \mathrm{Var}_x\big(X_{T_{\partial\mathcal{P}(0,L)}} | T_{\partial\mathcal{P}(0,L)} = T_{\partial_+\mathcal{P}(0,L)}\big).$$

Now note that the support of $\mu^{L_k}_{0,z}(\cdot) - \mathbf{P}_{z,\omega}(Z_0 = \cdot)$ is contained in

$$\{y \in H_{L_k^2} : \exists z \in \partial_+\mathcal{P}(0, L_k) \text{ such that } \|y - z\|_1 \le dL_k^{\vartheta}\}.$$

Thus, for any y in the support of $\mu^{L_k}_{z,0}(\cdot) - \mathbf{P}_{z,\omega}(Z_0 = \cdot)$ we get as a consequence of (b) in combination with the penultimate line of (3.24) that

$$\begin{aligned}
&\sum_x \big\|x - E_z\big(X_{T_{\partial P(0,L_k)}} | T_{\partial_+\mathcal{P}(0,L_k)} = T_{\partial\mathcal{P}(0,L_k)}\big)\big\|_1^2 \\
&\quad \times |\mathbf{P}_{z,\omega}(Z_1 = x) - \mathbf{P}_{z,\omega}(Z_0 = x)| \\
&\qquad \le 4d^2 (L_k R_6^2(L_k))^2 \sum_x |\mu^{L_k}_{z,0}(x) - \mathbf{P}_{z,\omega}(Z_0 = x)| \\
&\qquad \le 4d^2 R_6^2(L_k) L_k^2 \cdot R_7(L_k) L_k^{-\vartheta(d-1)/(d+1)},
\end{aligned}$$

where the last inequality holds for L large enough. In combination with Lemma 3.12, we deduce that the right-hand side is bounded from above by $\lambda \,\mathrm{Var}\, Z_1$ for L large enough, which finishes the proof. □

3.5. *Auxiliary walk.* As a preparation to prove Proposition 2.1, for each environment, we introduce a refinement (Y_n) of the finite-time auxiliary random walk defined in [1]. In blocks $\mathcal{P}(x, L_k)$ where the environment is such that the quenched RWRE (X_n) behaves similarly to the annealed one, the quenched walk (Y_n) will behave quite like (X_n). In blocks where the quenched and annealed behavior of (X_n) differ significantly, the quenched walk (Y_n) will make up for this deviation by corrections, in order to more or less mimick the annealed behavior of (X_n). As a consequence, the quenched walk (Y_n) starting in 0 will leave C_L through $\partial_+ C_L$ with a probability not too small, with respect to sufficiently many environments. Note that its construction will depend on a couple of parameters and in particular will be done for each $L > 0$ separately. For the sake of notational simplicity, we do not explicitly name these dependencies in the notation (Y_n). In order to facilitate understanding for the reader familiar with [1], we stick to the notation of that paper wherever appropriate.

On a heuristic level, the construction of the auxiliary walk (Y_n) can be described as follows. Let L and ω be given. In order to leave C_L through $\partial_+ C_L$, the walker starts with performing a few deterministic steps in positive e_1-direction.

Then, starting a recursive step, there is associated a natural scale $k' \in \{1, \dots, \iota\}$ to the current position of the walker (this scale is roughly given by the largest $k \in \{1, \dots, \iota\}$ for which L_k^2 divides the current e_1-coordinate of the walker); the walker then looks for good boxes of the form $\mathcal{P}(x, L_k)$, such that $k \in \{1, \dots, k'\}$, $x \in \mathcal{L}_{L_k}$ and such that his current position is contained in $\tilde{\mathcal{P}}(x, L_k)$. We now distinguish cases:

- If such a box exists, then the walker picks the largest of these boxes and moves according to a random walk in the corresponding environment, conditioned on leaving this box through its right boundary part. If this box is of the form $\mathcal{P}(x, L_k)$ for some $k < k'$, then before starting the recursion step from a position with natural scale k' again, the walker will perform a correction, making up for having moved in boxes smaller than the ones corresponding to its natural scale.
- If no such good box exists, the walker performs some deterministic steps in positive e_1-direction again and then returns to the start of the recursive step.

To formally construct our process, we need some auxiliary results. The following lemma will be proved in Section A.6 (see page 523).

LEMMA 3.14. *There exists a finite constant C such that for all L and $x \in \tilde{\mathcal{P}}(0, L)$,*

$$\left\| E_{\mu^L_{x,0}} - x - \frac{L^2 - x \cdot e_1}{\hat{v} \cdot e_1} \hat{v} \right\|_1 \leq C R_2(L)$$

and

$$\left\| E_x X_{T_{\partial \mathcal{P}(0,L)}} - x - \frac{L^2 - x \cdot e_1}{\hat{v} \cdot e_1} \hat{v} \right\|_1 \leq C R_2(L). \tag{3.25}$$

In order to state further auxiliary results, for $x \in \mathbb{Z}^d$ such that $x \cdot e_1 \in L_k^2 \mathbb{N}$, define $z(x, k)$ to be an element $z \in \mathcal{L}_{L_k}$ such that $x \cdot e_1 = z \cdot e_1$ and $x \in \tilde{\mathcal{P}}(z, L_k)$. Furthermore, for x such that $x \cdot e_1 \notin L_k^2 \mathbb{N}$ set $z(x, k) := 0$. In addition, abbreviate for $j, k \in \mathbb{N}$ the hitting times

$$T_k(j) := \inf\{n \in \mathbb{N} : Y_n \cdot e_1 = j L_k^2\}.$$

LEMMA 3.15. *Let $k \in \{1, \ldots, \iota - 1\}$, $\Delta_0 \in H_0 \cap \tilde{\mathcal{P}}(0, L_{k+1})$ deterministic and $(\Delta_i)_{i \in \{1, \ldots, \lfloor L^\chi \rfloor^2\}}$ be random variables. Set $S_j := \sum_{i=0}^j \Delta_i$ and assume furthermore that for every i, conditioned on $\Delta_1, \ldots, \Delta_{i-1}$, the variable Δ_i takes values in $\partial_+ \mathcal{P}(z(S_{i-1}, k), L_k) - z(S_{i-1}, k)$ only, with*

$$\| E(\Delta_i | \Delta_1, \ldots, \Delta_{i-1}) - (E_{\mu^{L_k}_{S_{i-1}, z(S_{i-1}, k)}} - S_{i-1}) \|_1 \leq R_4(L_k) \tag{3.26}$$

a.s. Then for L large enough and $t \geq R_5(L_k) L_{k+1}$,

$$\begin{aligned} &P\big(\exists j \in \{1, \ldots, \lfloor L^\chi \rfloor^2\} : \| \tilde{\pi}_{\hat{v}^\perp}(S_j - \Delta_0) \|_\infty \geq t\big) \\ &\quad \leq 2(d-1) \lfloor L^\chi \rfloor^2 \exp\left\{ - \frac{t^2}{72 L_{k+1}^2 R_6(L_k)^2} \right\}. \end{aligned} \tag{3.27}$$

PROOF. Noting that $\tilde{\pi}_{\hat{v}^\perp}(\lambda\hat{v}) = 0$ for all $\lambda \in \mathbb{R}$, the triangle inequality yields

$$\|\tilde{\pi}_{\hat{v}^\perp}(S_j - \Delta_0)\|_\infty \le \|Z_j^{(1)}\|_\infty + \|Z_j^{(2)}\|_\infty + \|Z_j^{(3)}\|_\infty, \tag{3.28}$$

where

$$Z_j^{(1)} := \sum_{i=1}^{j} \tilde{\pi}_{\hat{v}^\perp}\big(\Delta_i - E(\Delta_i|\Delta_1, \dots, \Delta_{i-1})\big), \qquad j \in \{1, \dots, \lfloor L^\chi \rfloor^2\},$$

$$Z_0^{(1)} := 0,$$

$$Z_j^{(2)} := \sum_{i=1}^{j} \tilde{\pi}_{\hat{v}^\perp}\big(E(\Delta_i|\Delta_1, \dots, \Delta_{i-1}) - (E_{\mu^{L_k}_{S_{i-1}, z(S_{i-1},k)}} - S_{i-1})\big)$$

and

$$Z_j^{(3)} := \sum_{i=1}^{j} \tilde{\pi}_{\hat{v}^\perp}\bigg((E_{\mu^{L_k}_{S_{i-1}, z(S_{i-1},k)}} - S_{i-1}) - \frac{L_k^2}{\hat{v}\cdot e_1}\hat{v}\bigg).$$

Due to (3.26), a.s.

$$\|Z_j^{(2)}\|_\infty \le j R_4(L_k), \tag{3.29}$$

while Lemma 3.14 results in

$$\|Z_j^{(3)}\|_\infty \le C j R_2(L_k). \tag{3.30}$$

Using (3.28) to (3.30) and because of $t \ge R_5(L_k)L_{k+1}$, for L large enough the probability in (3.27) can be bounded from above by

$$P\big(\exists j \in \{1, \dots, \lfloor L^\chi \rfloor^2\} : \|\tilde{\pi}_{\hat{v}^\perp}(Z_j^{(1)})\|_\infty \ge t/3\big).$$

Now with respect to P, the sequence $(\tilde{\pi}_{\hat{v}^\perp}(Z_j^{(1)}))_{j\in\{0,\dots,\lfloor L^\chi\rfloor^2\}}$ is a $(d-1)$-dimensional mean zero martingale such that $\|\tilde{\pi}_{\hat{v}^\perp}(Z_{j+1}^{(1)}) - \tilde{\pi}_{\hat{v}^\perp}(Z_j^{(1)})\|_\infty \le 2L_k R_6(L_k)$ for all $j \in \{0, \dots, \lfloor L^\chi \rfloor^2\}$. Thus, Azuma's inequality yields

$$\begin{aligned} &P\big(\exists j \in \{1, \dots, \lfloor L^\chi \rfloor^2\} : \|\tilde{\pi}_{\hat{v}^\perp}(Z_j^{(1)})\|_\infty \ge t/3\big) \\ &\quad \le 2(d-1)\lfloor L^\chi \rfloor^2 \exp\bigg\{-\frac{t^2}{72\lfloor L^\chi \rfloor^2 (L_k R_6(L_k))^2}\bigg\} \\ &\quad = 2(d-1)\lfloor L^\chi \rfloor^2 \exp\bigg\{-\frac{t^2}{72 L_{k+1}^2 R_6(L_k)^2}\bigg\}. \end{aligned}$$

□

We now introduce some quantities that will play an important role in the remaining part of this paper. For $k \ge 2$, let

$$\lambda_k := R_{9+k}(L) L_1^{-\vartheta(d-1)/(2(d+1))}$$

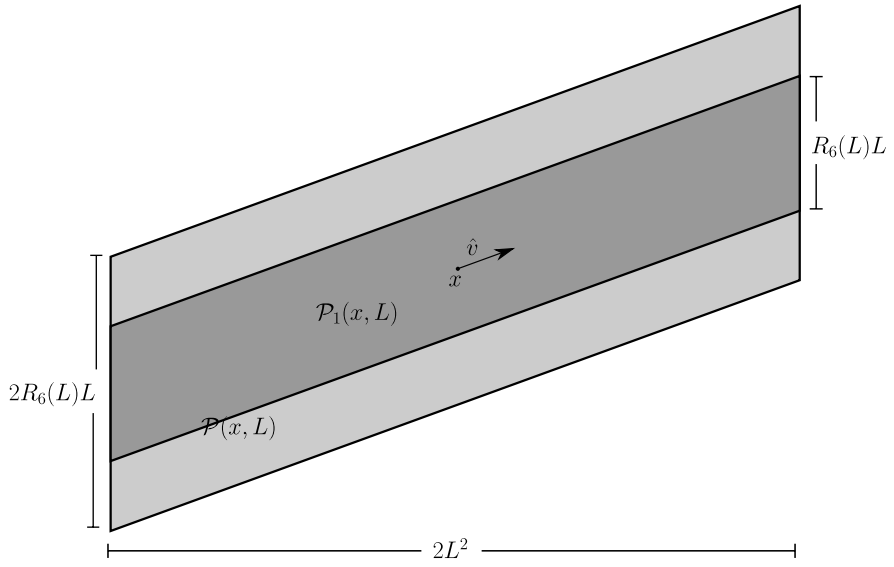

FIG. 5. *The set $\mathcal{P}_1(x,L)$ contained in $\mathcal{P}(x,L)$.*

and

$$K_k := 4\lfloor L^\chi \rfloor^2 d \lceil L_{k-1}^\vartheta \rceil,$$

where $\vartheta := \chi$ as in the definition of good blocks. Furthermore, define the boxes

$$\mathcal{P}_1(0,L) := \{y \in \mathcal{P}(0,L) : \|\tilde{\pi}_{\hat{v}^\perp}(y)\|_\infty \le R_6(L)L/2\}$$

and

$$\mathcal{P}_1(x,L) := x + \mathcal{P}_1(0,L)$$

as well as its right boundary part

$$\partial_+\mathcal{P}_1(x,L) := \{y \in \partial\mathcal{P}_1(x,L) : (y-x)\cdot e_1 = L^2\};$$

cf. Figure 5.

From now on, we will occasionally emphasize the process to which a certain random time refers by writing it as a superscript to the corresponding random time (as, e.g., $T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})}$ in the following lemma).

LEMMA 3.16. *Consider $S_{\lfloor L^\chi \rfloor^2}$ of Lemma 3.15 and assume that the distribution $P \circ S^{-1}_{\lfloor L^\chi \rfloor^2}$ of $S_{\lfloor L^\chi \rfloor^2}$ with respect to P is $(2k\lambda_{k+1}, 2kK_{k+1})$-close to $\mu^{L_{k+1}}_{x,0}$ for some $x \in \widetilde{\mathcal{P}}(0,L_{k+1})$. Then for L large enough, the distribution of $S_{\lfloor L^\chi \rfloor^2}$ with respect to $P(\cdot|T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} = T^S_{\partial\mathcal{P}_1(0,L_{k+1})})$ is $((2k+1)\lambda_{k+1}, (2k+1)K_{k+1})$-close to $\mu^{L_{k+1}}_{x,0}$ for all admissible choices of Δ_0, k and x.*

PROOF. Let μ be the coupling of Z_0, Z_1 and Z_2 as in the definition of $(2k\lambda_{k+1}, 2kK_{k+1})$-closeness, that is, such that:

(a) $\mu \circ Z_1^{-1} = \mu_{x,0}^{L_{k+1}}$ and $\mu \circ Z_2^{-1} = P \circ S_{\lfloor L^\chi \rfloor^2}^{-1}$,
(b) $\mu(Z_1 \neq Z_0) \leq 2k\lambda_{k+1}$,
(c) $\mu(\|Z_0 - Z_2\|_1 \leq 2kK_{k+1}) = 1$,
(d) $E_\mu Z_1 = E_\mu Z_0$,
(e) $\sum_x \|x - E_\mu Z_1\|_1^2 \cdot |\mu(Z_1 = x) - \mu(Z_0 = x)| \leq 2k\lambda_{k+1} \operatorname{Var} Z_1$.

Then we look for Z_0', Z_1' and Z_2' such that:

(a′) $\mu \circ {Z_1'}^{-1} = \mu_{x,0}^{L_{k+1}}$ and $\mu \circ {Z_2'}^{-1} = P(\cdot | T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} = T^S_{\partial\mathcal{P}_1(0,L_{k+1})}) \circ S_{\lfloor L^\chi \rfloor^2}^{-1}$,
(b′) $\mu(Z_1' \neq Z_0') \leq (2k+1)\lambda_{k+1}$,
(c′) $\mu(\|Z_0' - Z_2'\|_1 \leq (2k+1)K_{k+1}) = 1$,
(d′) $E_\mu Z_1' = E_\mu Z_0'$,
(e′) $\sum_x \|x - E_\mu Z_1'\|_1^2 \cdot |\mu(Z_1' = x) - \mu(Z_0' = x)| \leq (2k+1)\lambda_{k+1} \operatorname{Var} Z_1'$.

For this purpose and due to Remark 3.8, we can assume

$$Z_2 = S_{\lfloor L^\chi \rfloor^2}$$

without loss of generality. Set

$$Z_1' := Z_1$$

and

$$Z_2' := Z_2 \mathbb{1}_{T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} = T^S_{\partial\mathcal{P}_1(0,L_{k+1})}} + Z_2^* \mathbb{1}_{T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} \neq T^S_{\partial\mathcal{P}_1(0,L_{k+1})}},$$

where Z_2^* is independent of the remaining random variables and distributed as $S_{\lfloor L^\chi \rfloor^2}$ with respect to

$$P\big(\cdot | T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} = T^S_{\partial\mathcal{P}_1(0,L_{k+1})}\big).$$

Furthermore, set

$$Z_0^* := Z_0 \mathbb{1}_{T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} = T^S_{\partial\mathcal{P}_1(0,L_{k+1})}} + Z_2' \mathbb{1}_{T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} \neq T^S_{\partial\mathcal{P}_1(0,L_{k+1})}}.$$

Now as $EZ_1' = EZ_0$ and since due to Lemma 3.15 we have that

$$\max_{\Delta_0} P\big(T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} \neq T^S_{\partial\mathcal{P}_1(0,L_{k+1})}\big)$$

is contained in $\mathcal{S}(\mathbb{N})$ as a function in L (where the maximum is taken over all admissible choices of Δ_0; see the assumptions of Lemma 3.15), it follows that $\|EZ_0^* - EZ_1'\|_1$ is contained in $\mathcal{S}(\mathbb{N})$ as a function in L. Thus, there exists a random

variable U taking values in H_0 such that $P(\|U\|_1 \le K_{k+1}) = 1$, $P(U \neq 0)$ is contained in $\mathcal{S}(\mathbb{N})$ as a function of L, and such that $EZ_0^* + EU = EZ_1'$. Set

$$Z_0' := Z_0^* + U.$$

Then (a′), (c′) and (d′) are fulfilled. Furthermore,

$$\begin{aligned} P(Z_0' \neq Z_1') &\le 2k\lambda_{k+1} + P\big(T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} \neq T^S_{\partial\mathcal{P}_1(0,L_{k+1})}\big) + P(U \neq 0) \\ &\le (2k+1)\lambda_{k+1} \end{aligned}$$

for L large enough, which establishes (b′). With respect to the variance bound we obtain

$$\begin{aligned} &\sum_y \|y - E_{\mu_{x,0}^{L_{k+1}}}\|_1^2 \cdot |\mu_{x,0}^{L_{k+1}}(y) - P(Z_0' = y)| \\ &\quad= \sum_y \|y - E_{\mu_{x,0}^{L_{k+1}}}\|_1^2 \\ &\qquad\times |\mu_{x,0}^{L_{k+1}}(y) - P(Z_0 \mathbb{1}_{T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} = T^S_{\partial\mathcal{P}_1(0,L_{k+1})}} \\ &\qquad\qquad + Z_2' \mathbb{1}_{T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} \neq T^S_{\partial\mathcal{P}_1(0,L_{k+1})}} + U = y)| \\ &\quad\le (3dL_{k+1}R_6(L_{k+1}))^2 \big(P\big(T^S_{\partial_+\mathcal{P}_1(0,L_{k+1})} \neq T^S_{\partial\mathcal{P}_1(0,L_{k+1})}\big) + P(U \neq 0)\big) \\ &\qquad+ \sum_y \|y - E_{\mu_{x,0}^{L_{k+1}}}\|_1^2 \cdot |\mu_{x,0}^{L_{k+1}}(y) - P(Z_0 = y)| \\ &\quad\le (2k+1)\lambda_{k+1} \operatorname{Var}_{\mu_{x,0}^{L_{k+1}}} \end{aligned}$$

for L large enough. Since the above computations are uniform in the admissible choices of Δ_0, k and x, the result follows. □

LEMMA 3.17. *Let $k \in \{1, \dots, \iota\}$ and $x \in \tilde{\mathcal{P}}(0, L_k) \cap H_0$. Furthermore, let a distribution ν be given which is supported on $\partial_+\mathcal{P}(0, L_k)$ and $((2k-1)\lambda_k, (2k-1)K_k)$-close to $\mu_{x,0}^{L_k}$. Then for L large enough, $\nu(\cdot + x)$ is $(2k\lambda_k, 2kK_k)$-close to $\mu_{0,0}^{L_k}$ for all admissible choices of k and x.*

PROOF. If ν is $((2k-1)\lambda_k, (2k-1)K_k)$-close to $\mu_{x,0}^{L_k}$, then there exist Z_0, Z_1 and Z_2 fulfilling the requirements of Definition 3.7, where we denote the coupling measure by P.

We set $Z_2' := Z_2 - x$ and will construct Z_0' and Z_1' such that the corresponding points of Definition 3.7 are satisfied. First of all, note that (as a consequence of Lemma 3.10 and a decomposition into regenerations) there exist random variables Z_1^* and V taking values in $\partial_+\mathcal{P}(0, L_k)$ and $\{0, 1\}$, respectively, and such that $P(V = 0) \in \mathcal{S}(\mathbb{N})$ as a function in L and

$$Z_1' := (Z_1 - x)\mathbb{1}_{Z_0 \in \partial_+\mathcal{P}_1(x,L_k), V=1} + Z_1^* \mathbb{1}_{\{Z_0 \notin \partial_+\mathcal{P}_1(x,L_k)\} \cup \{V=0\}}$$

is distributed according to $\mu_{0,0}^{L_k}$. Let furthermore

$$Z_0^* := (Z_0 - x)\mathbb{1}_{Z_0 \in \partial_+ \mathcal{P}_1(x, L_k)} + Z_2' \mathbb{1}_{Z_0 \notin \partial_+ \mathcal{P}_1(x, L_k)}.$$

As a consequence, there exists an H_0-valued random variable independent from everything else such that $P(\|U\|_1 \leq K_k) = 1$, $P(U \neq 0)$ is contained in $\mathcal{S}(\mathbb{N})$ as a function in L, and $E(Z_0^* + U) = EZ_1'$. Set

$$Z_0' := Z_0^* + U.$$

Then, since $P(Z_0 \neq Z_1) \leq (2k-1)\lambda_k$ by assumption, we get

$$\begin{aligned} P(Z_0' \neq Z_1') &\leq P(Z_0 \neq Z_1) + P\big(Z_0 \notin \partial_+ \mathcal{P}_1(x, L_k)\big) + P(U \neq 0) + P(V = 0) \\ &\leq 2k\lambda_k \end{aligned}$$

for L large enough. Furthermore, $P(\|Z_0' - Z_2'\|_1 \leq 2kK_k) = 1$. To check the remaining variance condition, note that

$$\begin{aligned} &\sum_y \|y - E_{\mu_{0,0}^{L_k}}\|_1^2 \cdot |P(Z_1' = y) - P(Z_0' = y)| \\ &\quad = \sum_y \|y - E_{\mu_{0,0}^{L_k}}\|_1^2 \\ &\qquad \times \big(\big|P\big((Z_1 - x)\mathbb{1}_{Z_0 \in \partial_+ \mathcal{P}_1(x, L_k), V=1} + Z_1^* \mathbb{1}_{\{Z_0 \notin \partial_+ \mathcal{P}_1(x, L_k)\} \cup \{V=0\}} = y\big) \\ &\qquad\qquad - P\big((Z_0 - x)\mathbb{1}_{Z_0 \in \partial_+ \mathcal{P}_1(x, L_k)} + Z_2' \mathbb{1}_{Z_0 \notin \partial_+ \mathcal{P}_1(x, L_k)} + U = y\big)\big|\big) \\ &\quad \leq (dL_k R_6(L_k))^2 \big(P\big(Z_0 \notin \partial_+ \mathcal{P}_1(x, L_k)\big) + P(U \neq 0) + P(V = 0)\big) \\ &\qquad + \sum_y \|y - E_{\mu_{x,0}^{L_k}}\|_1^2 \cdot |\mu_{x,0}^{L_k}(y) - P(Z_0 = y)| \\ &\quad \leq 2k\lambda_k \operatorname{Var}_{\mu_{0,0}^{L_k}} \end{aligned}$$

for L large enough, where to obtain the last inequality we employed the $((2k-1)\lambda_k, (2k-1)K_k)$-closeness of ν to $\mu_{x,0}^{L_k}$ as well as Lemma 3.12. Again, since the above computations are uniform in the admissible choices of k and x, this yields the result. □

In order to construct the auxiliary walk, we need the following result which guarantees that if boxes on a certain scale are left in some way close to the annealed distribution conditioned on leaving through the right boundary part of the boundary, then the same applies to the containing box on the larger scale as well. Essentially, this is Lemma 4.16 of [1].

LEMMA 3.18. *Let $\lambda \in (0, 1)$, L be large enough and $n \in \mathbb{N}$ such that $n \leq \lambda L$. Furthermore, let $(\Delta_i)_{i=1}^n$ be random variables such that for every i, the variable*

Δ_i takes values in $\partial_+\mathcal{P}(0,L)$ only, and, conditioned on $\Delta_1,\dots,\Delta_{i-1}$, the distribution of Δ_i is (λ,K)-close to $\mu_{0,0}^L$. In addition, assume $R_3(L)\le K\le L$.

Then for $S_n:=\sum_{i=1}^n\Delta_i$, the distribution of S_n is $(R_9(L)\lambda,4nK)$-close to $\mu_{0,0}^{\sqrt{n}L}$.

The proof of this crucial lemma can be found from page 527 onward.

Now we rigorously construct the auxiliary walk (Y_n) in environment ω starting in 0, and denote the corresponding probability measure by $P_{0,\omega}$ also. For $k\in\{1,\dots,\iota-1\}$ we recursively define

$$\begin{aligned}&M_k \text{ as the smallest integer larger than or equal to}\\ &L^{\beta-6\delta}+L^{2\chi} \text{ such that } L_{k+1}^2 \text{ divides } \sum_{j=1}^{k} M_jL_j^2.\end{aligned}\tag{3.31}$$

Note that $L_{k+1}=L_k\lfloor L^\chi\rfloor$ implies that $M_k\le\lceil L^{\beta-6\delta}\rceil+2\lceil L^{2\chi}\rceil$, and that for every $k\in\{2,\dots,\iota\}$, from $x\cdot e_1-\sum_{j=1}^{k-1}M_jL_j^2\in L_k^2\mathbb{N}_0$ we can infer that $x\cdot e_1\in L_k^2\mathbb{N}$. Define $\mathcal{P}^{(k)}(x):=\mathcal{P}(z(x,k),L_k)$,

$$\begin{aligned}k(x):=\max\Bigg\{k\in\{1,\dots,\iota\}:x\cdot e_1-\sum_{j=1}^{k-1}M_jL_j^2\in L_k^2\mathbb{N}_0\\ \text{and } \mathcal{P}^{(k)}(x) \text{ is good}\Bigg\}\end{aligned}\tag{3.32}$$

and

$$k'(x):=\max\Bigg\{k\in\{1,\dots,\iota\}:x\cdot e_1-\sum_{j=1}^{k-1}M_jL_j^2\in L_k^2\mathbb{N}\Bigg\}\tag{3.33}$$

with the maximum of the empty set defined to be 0. We now define the auxiliary random walk (Y_n) and a corresponding sequence of stopping times (ζ_n) recursively. For $z\in\partial_+\mathcal{P}(0,L_1)$ chosen according to $\mu_{0,0}^L$, fix $Y_0,\dots,Y_{l_1}$ to be an arbitrary nearest-neighbor path (independent of ω) of shortest length connecting 0 with z such that $\{Y_0,\dots,Y_{l_1-1}\}\subset\mathcal{P}(0,L_1)$. Furthermore, set $\zeta_1:=\zeta_1':=T^Y_{\partial_+\mathcal{P}(0,L_1)}=l_1$. Next, we define the recursive step of the construction.

(R) Assume that the walk is defined up to time ζ_n' and set $x:=Y_{\zeta_n'}$.

- If $k(x)>0$, then choose $Y_{\zeta_n'+\cdot}$ according to the law of $X_\cdot$ with respect to

$$P_{x,\omega}\big(\cdot|T_{\partial\mathcal{P}^{(k(x))}(x)}=T_{\partial_+\mathcal{P}^{(k(x))}(x)}\big),$$

up to time $\zeta_n'+l_n$, where

$$l_n:=T^{Y_{\zeta_n'+\cdot}}_{\partial\mathcal{P}^{(k(x))}(x)}.$$

- Otherwise, if $k(x) = 0$, then similarly to the start of the construction, we choose

$$\{Y_{\zeta_n'}, \dots, Y_{\zeta_n'+l_n}\}$$

to be a nearest-neighbor path of shortest length connecting x with z, where z is chosen according to $\mu^{L_1}_{x,z(x,1)}$ and such that this path leaves $\mathcal{P}^{(1)}(x)$ in its last step only.

In both cases, set $\zeta_{n+1} := \zeta_n' + l_n$. If $Y_{\zeta_{n+1}} \cdot e_1 > L^{1+\delta}$, then we stop the construction of Y.

If $1 \vee k(x) = k'(Y_{\zeta_{n+1}})$, then set $Y_{\zeta_{n+1}+1} := Y_{\zeta_{n+1}} + e_1$, $Y_{\zeta_{n+1}+2} := Y_{\zeta_n}$ as well as $\zeta'_{n+1} := \zeta_{n+1} + 2$ and repeat step (R).

Otherwise, if $1 \vee k(x) < k'(Y_{\zeta_{n+1}})$, given $\zeta_1, \dots, \zeta_{n+1}$ and $(Y_i)_{i\in\{0,\dots,\zeta_{n+1}\}}$, define for each $k \in \{(1 \vee k(x)) + 1, \dots, k'(Y_{\zeta_{n+1}})\}$ the number $j(k) := Y_{\zeta_{n+1}} \cdot e_1/L_k^2$. Furthermore, define for $j, k \in \mathbb{N}$ the stopping time $T'_k(j)$ equal to ζ'_m if there exists $m \le n+1$ such that $\zeta_m = T_k(j)$, and equal to $T_k(j)$ otherwise.

Now for $k \in \{(1 \vee k(x)) + 1, \dots, k'(Y_{\zeta_{n+1}})\}$ with increasing order we iteratively perform the following step, where $\zeta_{n+1}^{(1\vee k(x))} := \zeta_{n+1}$:

(B) Conditioned on $Y_{T'_k(j(k)-1)}$, by construction (and as a consequence of Corollary 3.13 and Lemma 3.18), the distribution of the variable

$$Y_{\zeta_{n+1}^{(k-1)}} - z\big(Y_{T'_k(j(k)-1)}, k\big)$$

is $(2(k-1)\lambda_k, 2(k-1)K_k)$-close to

$$\mu^{L_k}_{Y_{T'_k(j(k)-1)}-z(Y_{T'_k(j(k)-1)},k),0}.$$

We now condition the variable

$$Y_{\zeta_{n+1}^{(k-1)}} - z\big(Y_{T'_k(j(k)-1)}, k\big)$$

on the event

$$D_k := \big\{T^{Y_{T'_k(j(k)-1)+\cdot}}_{\partial_+\mathcal{P}_1(z(Y_{T'_k(j(k)-1)},k),L_k)} = T^{Y_{T'_k(j(k)-1)+\cdot}}_{\partial\mathcal{P}_1(z(Y_{T'_k(j(k)-1)},k),L_k)}\big\}. \tag{3.34}$$

In combination with Lemma 3.16 we may infer that for L large enough, the distribution of this conditioned random variable still is $((2k-1)\lambda_k, (2k-1)K_k)$-close to

$$\mu^{L_k}_{Y_{T'_k(j(k)-1)}-z(Y_{T'_k(j(k)-1)},k),0}.$$

Thus, Lemma 3.17 implies that

$$\begin{aligned}&\text{the distribution of the variable } Y_{\zeta_{n+1}^{(k-1)}} - Y_{T'_k(j(k)-1)}\\ &\text{is } (2k\lambda_k, 2kK_k)\text{-close to } \mu^{L_k}_{0,0}.\end{aligned} \tag{3.35}$$

Set $\zeta_{n+1}^{(k)} := \zeta_{n+1}^{(k-1)} + \|\beta_{k,j(k)}\|_1$, where the $\beta_{k,j(k)}$ defined below take values in H_0 and play a correcting role. Furthermore, let $Y_{\zeta_{n+1}^{(k-1)}}, \dots, Y_{\zeta_{n+1}^{(k)}}$ be a nearest-neighbor path of shortest length from $Y_{\zeta_{n+1}^{(k-1)}}$ to $Y_{\zeta_{n+1}^{(k-1)}} + \beta_{k,j(k)}$. Note that from the conditioning in (3.34) in combination with Remark 3.19 below, we may infer that $Y_{\zeta_{n+1}^{(k)}} - Y_{T'_k(j(k)-1)}$ takes values in $\partial_+ \mathcal{P}^{(k)}(0)$ only. If $k < k'(Y_{\zeta_{n+1}})$, then repeat step (B) for $k+1$; if $k = k'(Y_{\zeta_{n+1}})$, continue below.

Set $Y_{\zeta_{n+1}^{(k)}+1} := Y_{\zeta_{n+1}^{(k)}} + e_1$ as well as $Y_{\zeta_{n+1}^{(k)}+2} := Y_{\zeta_{n+1}^{(k)}}$ and $\zeta'_{n+1} := \zeta_{n+1}^{(k)} + 2$. Now we continue the construction at the recursion step (R).

It remains to define the variables $\beta_{k,j}$. Set $\beta_{1,j} = 0$ for all j. For any $n \in \mathbb{N}$, we will define those $\beta_{k,j}$, $k \in \{2, \dots, \iota\}$, for which $Y_{\zeta_n} \in H_{jL_k^2}$, using only the environment ω, the auxiliary walk Y up to time ζ_n as well as the values of $\{\beta_{\hat{k},\hat{j}} : \hat{k} \in \{2, \dots, k-1\}$ and $\hat{j} L_{\hat{k}}^2 = jL_k^2\}$. We define $\beta_{k,j}$ to be 0 in the following cases:

- If there is no $n \in \mathbb{N}$ such that $\zeta_n = T_k(j-1)$, then $\beta_{k,j} = 0$.
- Otherwise, let n be such that $\zeta_n = T_k(j-1)$. If $\mathcal{P}^{(k)}(Y_{\zeta'_n})$ is good, then $\beta_{k,j} = 0$.

Thus, assume now that $\zeta_n = T_k(j-1)$ such that $\mathcal{P}^{(k)}(Y_{\zeta'_n})$ is bad. Let $x := Y_{\zeta'_n}$ and let $\mu_{x,\omega}^{k,j,Y}$ be the distribution of the variable $Y_{\zeta_{n+1}^{(k-1)}} - x$, which due to (3.35) is $(2k\lambda_k, 2kK_k)$-close to $\mu_{0,0}^{L_k}$. Thus, we find (Z_0, Z_1, Z_2) defined on the same probability space as (Y_n) (which without loss of generality is assumed to be large enough) such that Z_2 equals $Y_{\zeta_{n+1}^{(k-1)}} - x$ (cf. Remark 3.8), such that $Z_1 \sim \mu_{0,0}^{L_k}$, and such that furthermore the requirements of $(2k\lambda_k, 2kK_k)$-closeness (cf. Definition 3.7) are satisfied. Now define

$$\beta_{k,j} := Z_0 - Z_2, \tag{3.36}$$

and note that $\beta_{k,j} \in H_0$ a.s. This completes the definition of $\beta_{k,j}$.

The deterministic corrections caused by the variables $\beta_{k,j}$ are not too big, that is, not too expensive in terms of probability. This is made precise in the following remark.

REMARK 3.19. By construction of (Y_n), for every $k \in \{2, \dots, \iota\}$ and $j \in \mathbb{N}$ such that $\beta_{k,j}$ has been defined above, with probability 1,

$$\beta_{k,j} < 2\iota K_\iota \le L^{4\chi}$$

for L large.

REMARK 3.20. Observe that by construction we infer that $T^Y_{\partial C_L} = T^Y_{\partial_+ C_L}$ a.s., with C_L denoting the set of Proposition 2.1.

3.6. *Random direction event.* As in [1], we will introduce a so-called *random direction event* in order to ensure that, in most environments, the walker does not hit too many bad boxes. For this purpose, for $k \in \{1, \dots, \iota\}$ set

$$B_k := \frac{\sum_{j=1}^{k-1} M_j L_j^2}{L_k^2}. \tag{3.37}$$

For $w \in [-1,1]^{d-1}$, $k \in \{2, \dots, \iota\}$ and $j \in \{B_k + 1, \dots, M_k\}$, define

$$W_k^{(w)}(j) := \{\|Y_{T'_k(j)} - Y_{T'_k(B_k)} - (j - B_k)(E_{\mu_{0,0}^{L_k}} - L_k(0,w))\|_\infty < L_k\},$$

where in a slight abuse of notation we write $L_k(0,w)$ to denote the vector $(0, L_k w_1, \dots, L_k w_{d-1}) \in \mathbb{R}^d$. Furthermore, define

$$W_k^{(w)} := \bigcap_{j=B_k+1}^{B_k+M_k} W_k^{(w)}(j)$$

as well as the *random direction event*

$$W^{(w)} := \bigcap_{k=1}^{\iota} W_k^{(w)}.$$

To obtain a lower bound for the probability of this event, we have to establish some auxiliary results first.

CLAIM 3.21. *For all L large enough and all $k \in \{1, \dots, \iota\}$, $j \in \{B_k + 1, \dots, B_k + M_k\}$ and $\omega \in \Omega$, one has that $P_{0,\omega}(\cdot | Y_1, \dots, Y_{T'_k(j-1)})$-a.s. the distribution of $Y_{T'_k(j)} - Y_{T'_k(j-1)}$ is $(2k\lambda_k, 2kK_k)$-close to $\mu_{0,0}^{L_k}$.*

PROOF. Similarly to Lemma 6.6 of [1], this result is a consequence of the construction of the auxiliary walk. In fact, if $\mathcal{P}^{(k)}(Y_{T'_k(j-1)})$ is good, then the statement follows from the first part of step (R) in the construction of the auxiliary walk in combination with Corollary 3.13.

Otherwise, if $\mathcal{P}^{(k)}(Y_{T'_k(j-1)})$ is bad, it follows from step (B) of that construction. □

We now get the following corollary.

COROLLARY 3.22 (Corollary 6.7 of [1]). *There exists a constant $\rho > 0$ such that for all L large enough, $\omega \in \Omega$, all k, j as in Claim 3.21, $\overline{Y} := Y_{T'_k(j-1)} + E_{\mu_{0,0}^{L_k}}$, and for all $x \in H_{jL_k^2}$ such that $\|\overline{Y} - x\|_1 < 4L_k$, one has*

$$P_{0,\omega}(\|Y_{T'_k(j)} - x\|_1 < L_k | Y_1, \dots, Y_{T'_k(j-1)}) > \rho. \tag{3.38}$$

PROOF. This follows from Claim 3.21 in combination with Lemmas 3.14 and 3.11. □

LEMMA 3.23 (Lemma 7.1 of [1]). *There exists $\rho > 0$ such that for all L large enough, $\omega \in \Omega$, all $w \in [-1,1]^{d-1}$, as well as k, j as in Claim 3.21, one has*

$$P_{0,\omega}\big(W_k^{(w)}(j) | W_1^{(w)}, \dots, W_{k-1}^{(w)}, W_k^{(w)}(B_k+1), \dots, W_k^{(w)}(j-1)\big) > \rho$$

[with $W_k^{(w)}(B_k) := \Omega$].

PROOF. On the event

$$W_1^{(w)} \cap \cdots \cap W_{k-1}^{(w)} \cap W_k^{(w)}(B_k+1) \cap \cdots \cap W_k^{(w)}(j-1)$$

one has

$$\big\| Y_{T_k'(j-1)} - Y_{T_k'(B_k)} - (j-1-B_k)\big(E_{\mu_{0,0}^{L_k}} - L_k(0,w)\big)\big\|_\infty < L_k$$

and thus

$$\big\| \underbrace{Y_{T_k'(B_k)} + (j-B_k)\big(E_{\mu_{0,0}^{L_k}} + L_k(0,w)\big)}_{=:x} - \big(Y_{T_k'(j-1)} + E_{\mu_{0,0}^{L_k}}\big)\big\|_\infty$$
$$< 2L_k.$$

Corollary 3.22 now yields the desired result. □

Departing from this result we obtain the desired lower bound on the probability of the random direction event.

LEMMA 3.24. *There exists a constant $C > 0$ such that for all L large enough as well as all $\omega \in \Omega$ and $w \in [-1,1]^{d-1}$,*

$$P_{0,\omega}\big(W^{(w)}\big) \geq e^{-CL^{\beta-6\delta}}.$$

PROOF. We compute

$$\begin{aligned} &P_{0,\omega}\big(W^{(w)}\big) \\ &\quad = \prod_{k=1}^{\iota} \prod_{j=B_k+1}^{B_k+M_k} P_{0,\omega}\big(W_k^{(w)}(j) | W_1^{(w)}, \dots, W_{k-1}^{(w)}, \\ &\qquad\qquad W_k^{(w)}(B_k+1), \dots, W_k^{(w)}(j-1)\big) \\ &\quad \geq \rho^{\sum_{k=1}^{\iota} M_k} \geq e^{-CL^{\beta-6\delta}} \end{aligned}$$

for $C > 0$ large enough, where the first inequality is a consequence of Lemma 3.23 while the second follows from the bound $M_k \leq 2\lceil L^{\beta-6\delta}\rceil$ for L large enough; see directly after (3.31). □

We now want to bound from above the probability that the auxiliary walk hits too many bad boxes. For this purpose, we start with the following auxiliary result.

LEMMA 3.25 (Lemma 7.4 of [1]). *For all L large enough, $\omega \in \Omega$, $k \in \{1, \dots, \iota - 1\}$, $j \in \{B_{k+1}\lfloor L^{\chi} \rfloor^2, \dots, \lfloor L^{1+\delta}/L_k^2 \rfloor\}$ and $z \in \mathcal{L}_{L_k} \cap H_{jL_k^2}$ one has*

$$\begin{aligned} &\int_{[-1,1]^{d-1}} P_{0,\omega}\big(\{Y_n : n \in \{1, \dots, T^Y_{L^{1+\delta}}\}\} \cap \mathcal{P}(z, L_k) \neq \varnothing | W^{(w)}\big)\, dw \\ &\qquad \leq L^{(-\beta+6\delta+2\chi)(d-1)}. \end{aligned} \tag{3.39}$$

PROOF. Choose k' to be the number out of $\{k, \dots, \iota - 1\}$ such that $B_{k'}L_{k'}^2 \leq jL_k^2 < B_{k'+1}L_{k'+1}^2$. We start with noting that for fixed $w \in [-1, 1]^{d-1}$, with probability 1 with respect to $P_0(\cdot | W^{(w)})$, the walk Y is located in a $(d-1)$-dimensional hypercube of side length $\sum_{j=1}^{k'-1} L_j \leq \iota L_{k'-1}$ at time $T'_{k'}(B_{k'})$. Letting w vary over $[-1, 1]^{d-1}$, the union of all appearing hypercubes covers a hypercube of side length at least $M_{k'-1}L_{k'-1} \geq \lceil L^{\beta-6\delta} \rceil L_{k'-1}$.

Now let $\{y_1, \dots, y_r\} \subset \mathcal{L}_{L_{k'}}$ be the set of all elements $y_j \in \mathcal{L}_{L_{k'}}$ such that $\mathcal{P}(z, L_k) \cap \mathcal{P}(y_j, L_{k'}) \neq \varnothing$ for all $j \in \{1, \dots, r\}$, and note that, due to a reasoning similar to the observation just before Lemma 3.6, r is bounded from above by $3 \cdot 15^{d-1}$.

From steps (R) and (B) in the construction of the auxiliary walk Y, it follows that if there exists ζ'_n such that $z(Y_{\zeta'_n}, k') = y_j$, then Y leaves $\mathcal{P}(y_j, L_{k'})$ through $\partial_+\mathcal{P}(y_j, L_{k'})$. Therefore, we conclude that

$$\{\{Y_n : n \in \mathbb{N}\} \cap \mathcal{P}(z, L_k) \neq \varnothing\} \subset \bigcup_{j=1}^{r} \{\{Y_n : n \in \mathbb{N}\} \cap \mathcal{P}(y_j, L_{k'}) \neq \varnothing\}.$$

But due to the above reasoning, there exists a constant C such that the right-hand side can have positive probability with respect to $P_{0,\omega}(\cdot | W^{(w)})$ only if w lies in a certain $(d-1)$-dimensional hypercube of side length

$$\frac{CR_6(L_{k'})L_{k'}}{L^{\beta-6\delta}L_{k'-1}} \leq CL^{-\beta+6\delta+3\chi/2}.$$

This establishes (3.39). □

Now adopt the notation

$$\mathcal{D}_{k,\omega} := \{x \in C_L \cap \mathcal{L}_{L_k} : x \cdot e_1 \geq B_k L_k^2 \text{ and } \mathcal{P}(x, L_k) \text{ is bad with respect to } \omega\}$$

and

$$\mathcal{B}_{k,\omega} := \big|\{x \in \mathcal{D}_{k,\omega} : \{Y_n : n \in \{1, \dots, T^Y_{L^{1+\delta}}\}\} \cap \mathcal{P}(x, L_k) \neq \varnothing\}\big|. \tag{3.40}$$

We are interested in the distribution of the variable $\mathcal{B}_{k,\omega}$. Recall that Θ_L has been defined in (3.12).

LEMMA 3.26 (Lemma 7.5 of [1]). *For all L large enough and all $k \in \{1, \dots, \iota - 1\}$ as well as $\omega \in \Theta_L$,*

$$\int_{[-1,1]^{d-1}} E_{0,\omega}\big(\mathcal{B}_{k,\omega} | W^{(w)}\big)\, dw \le 15^d L^{\beta - 6\delta}.$$

PROOF. With the same reasoning as in the proof of Lemma 3.25, steps (R) and (B) of the construction of the auxiliary walk Y imply that $P_0(\cdot | W^{(w)})$-a.s. we have

$$\begin{aligned} &\big|\{x \in C_L \cap \mathcal{L}_{L_k} : B_k L_k^2 \le x \cdot e_1 < B_{k+1} L_{k+1}^2, \\ &\qquad \mathcal{P}(x, L_k) \cap \{Y_n : n \in \{1, \dots, T^Y_{L^{1+\delta}}\}\} \ne \varnothing\}\big| \\ &\qquad \le 3 \cdot 15^{d-1} M_k \le (15^d - 1) L^{\beta - 6\delta}. \end{aligned} \tag{3.41}$$

Now consider $x \in \mathcal{D}_{k,\omega}$ with $x \cdot e_1 \ge B_{k+1} L_{k+1}^2$. Then by Lemma 3.25,

$$\begin{aligned} &\int_{[-1,1]^{d-1}} P_{0,\omega}\big(\{Y_n : n \in \{1, \dots, T^Y_{L^{1+\delta}}\}\} \cap \mathcal{P}(x, L_k) \ne \varnothing | W^{(w)}\big)\, dw \\ &\qquad \le L^{(-\beta + 6\delta + 2\chi)(d-1)} \end{aligned} \tag{3.42}$$

for L large enough. Therefore, (3.41) and (3.42) in combination with (3.12) yield

$$\begin{aligned} \int_{[-1,1]^{d-1}} E_{0,\omega}\big(\mathcal{B}_{k,\omega} | W^{(w)}\big)\, dw &\le (15^d - 1) L^{\beta - 6\delta} + L^{(-\beta + 6\delta + 2\chi)(d-1)} L^{\alpha + \delta} \\ &\le 15^d L^{\beta - 6\delta}, \end{aligned}$$

due to our choice of δ. □

Because of the modifications in our construction of the auxiliary walk in comparison to the one in [1], we give here a modified result concerning the density of the path measures of X with respect to Y.

LEMMA 3.27 (Lemma 6.5 of [1]). *Let $(v_n) = (v_1, \dots, v_{T^v_{L^{1+\delta}}})$ be a finite nearest-neighbor path in $\mathbb{Z}^d$ starting in 0 such that $T^v_{L^{1+\delta}} = \inf\{n \in \mathbb{N} : v_n \cdot e_1 > L^{1+\delta}\}$. Furthermore, for $k \in \{1, \dots, \iota\}$ and $\omega \in \Omega$, let*

$$Q_{k,\omega}(v) := \big|\{z \in \mathcal{D}_{k,\omega} : \{v_n : n \in \{1, \dots, T^v_{L^{1+\delta}}\}\} \cap \mathcal{P}(z, L_k) \ne \varnothing\}\big|$$

and set $Q_\omega(v) := \sum_{k=1}^{\iota} Q_{k,\omega}(v)$.

Then for all L large enough and all $\omega \in \Omega$ we have

$$\frac{P_{0,\omega}(X_j = v_j \ \forall j \in \{1, \dots, T^v_{L^{1+\delta}}\})}{P_{0,\omega}(Y_j = v_j \ \forall j \in \{1, \dots, T^v_{L^{1+\delta}}\})} \ge \frac{1}{2} \kappa^{3 Q_\omega(v) \iota L^{9\psi/4} + 4\iota L^{\beta - 6\delta}} \tag{3.43}$$

for all admissible choices of (v_n).

PROOF. Due to ellipticity, the numerator in (3.43) is positive; therefore, it is sufficient to consider those trajectories (v_n) only for which the probability in the denominator is positive as well.

To any such (v_n) and environment ω, there belong sequences (ζ_n) and (ζ'_n) as in the definition of Y. In fact, set $\zeta_0 := \zeta'_0 := 0$ and $\zeta_1 := \zeta'_1 := T^v_{\partial\mathcal{P}(0,L_1)}$. Given $\zeta_0, \dots, \zeta_n$ and $\zeta'_0, \dots, \zeta'_{n-1}$, define $x_n := v_{\zeta_n}$, $\zeta'_n := \min\{l > \zeta_n : v_{l-1} \cdot e_1 > x_n \cdot e_1\}$ (only if $n > 1$) as well as $x'_n := v_{\zeta'_n}$. For $k(x'_n)$ as in (3.32), if $k(x'_n) > 0$, set $\zeta_{n+1} := T^v_{\partial_+\mathcal{P}^{(k)}(x'_n)}$, otherwise set $\zeta_{n+1} := T^v_{\partial_+\mathcal{P}^{(1)}(x'_n)}$.

Now to estimate the probability in the denominator from above, we only consider the contributions coming from Y moving in good boxes in which it behaves like the quenched walk X conditioned on leaving the box through its right boundary part:

$$\begin{aligned}
&P_{0,\omega}(Y_j = v_j \ \forall j \in \{1, \dots, T^v_{L^{1+\delta}}\}) \\
&\quad \le \prod_{n\,:\,k(x'_n)>0} P_{x'_n,\omega}\big(X_l = v_{\zeta'_n+l} \\
&\qquad\qquad \forall l \in \{1, \dots, \zeta_{n+1} - \zeta'_n\} | T_{\partial\mathcal{P}^{(k(x'_n))}(x'_n)} = T_{\partial_+\mathcal{P}^{(k(x'_n))}(x'_n)}\big).
\end{aligned}$$

To obtain a lower bound for the numerator, as a consequence of the strong Markov property we may decompose it into movements within the corresponding boxes as follows:

$$\begin{aligned}
&P_{0,\omega}(X_j = v_j \ \forall j \in \{1, \dots, T^v_{L^{1+\delta}}\}) \\
&\quad \ge \prod_{n\,:\,k(v_{\zeta'_n})>0} P_{v_{\zeta'_n},\omega}\big(X_l = v_{\zeta'_n+l} \\
&\qquad\qquad \forall l \in \{1, \dots, \zeta_{n+1} - \zeta'_n\} | T_{\partial\mathcal{P}^{(k(v_{\zeta'_n}))}(v_{\zeta'_n})} = T_{\partial_+\mathcal{P}^{(k(v_{\zeta'_n}))}(v_{\zeta'_n})}\big) \\
&\qquad \times \prod_{n\,:\,k(v_{\zeta'_n})>0} P_{v_{\zeta'_n},\omega}\big(T_{\partial\mathcal{P}^{(k(v_{\zeta'_n}))}(v_{\zeta'_n})} = T_{\partial_+\mathcal{P}^{(k(v_{\zeta'_n}))}(v_{\zeta'_n})}\big) \\
&\qquad \times (\kappa^{\iota L^{3\chi}})^{Q_\omega(v)} (2^{-\iota})^{Q_\omega(v)} \\
&\qquad \times \prod_{n\,:\,\zeta'_n < T^v_{L^{1+\delta}}} \kappa^2 \kappa^{CL^{2\psi}} \prod_{n\,:\,k(v_{\zeta'_n})=0} \kappa^{CL^{2\psi}}
\end{aligned}$$

for L large enough as well as $k(v_{\zeta'_n})$ and $k'(v_{\zeta'_n})$ as defined in (3.32) and (3.33). Here, the first and second product on the right-hand side come from X moving in good boxes. The third and fourth factor on the right-hand side originate from the corrections in the case of moving in bad boxes. In this case, Remark 3.19 tells us that each of the correcting variables $\beta_{k,j}$ is bounded from above by $L^{3\chi}$. Since each time such a correction occurs, the number of influencing correcting variables $\beta_{k,j}$ is bounded from above by ι, we obtain the third factor. The fourth factor

originates from the conditioning on D_k in (3.34), the probability of which can be estimated using Lemma 3.15. The fifth factor follows from the fact that directly before each time ζ_n' we force the walk to do one step in the direction of e_1 and one step back, while the last factor originates from the deterministic moves performed within bad boxes of scale one. Consequently, we obtain

$$\begin{aligned}\frac{P_{0,\omega}(X_j = v_j\ \forall j \in \{1,\dots,T^v_{L^{1+\delta}}\})}{P_{0,\omega}(Y_j = v_j\ \forall j \in \{1,\dots,T^v_{L^{1+\delta}}\})}&\\ \geq \prod_{n\,:\,k(x_n')>0} P_{x_n',\omega}\big(T_{\partial \mathcal{P}^{(k(x_n'))}(x_n')} = T_{\partial_+ \mathcal{P}^{(k(x_n'))}(x_n')}\big)\big(\kappa^{3Q_\omega(v)\iota L^{3\chi}}\big)&\\ \times \prod_{n\,:\,\zeta_n'<T^v_{L^{1+\delta}}} \kappa^2\kappa^{CL^{2\psi}} \prod_{n\,:\,k(v_{\zeta_n'})=0} \kappa^{CL^{2\psi}}&\end{aligned} \tag{3.44}$$

for L large enough. Since $k(v_{\zeta_n'}) > 0$ implies that $\mathcal{P}^{(k(v_{\zeta_n'}))}(v_{\zeta_n'})$ is good, from (3.9) we infer that the value of the first product on the right-hand side is bigger than $1/2$ uniformly in all (v_n) we consider, for all L large enough. Due to the construction of the auxiliary walk Y, there are at most $\sum_{k=1}^{\iota} M_k \leq 2\iota L^{\beta-6\delta}$ stopping times ζ_n' such that $\zeta_n' < T^v_{L^{1+\delta}}$. Therefore, and due to the choice of δ and ψ, for L large enough, the total expression on the right-hand side is bounded from below by $\kappa^{3Q_\omega(v)\iota L^{9\psi/4}+4\iota L^{\beta-6\delta}}/2$, which finishes the proof. □

3.7. *Proof of Proposition 2.1.* With $\mathcal{B}_{k,\omega}$ as defined in (3.40) and for L large enough, Lemma 3.26 yields

$$\int_{[-1,1]^{d-1}} E_{0,\omega}\Big(\sum_{k=1}^{\iota} \mathcal{B}_{k,\omega} \big| W^{(w)}\Big)\, dw \leq 15^d \iota L^{\beta-6\delta}$$

for $\omega \in \Theta_L$. Hence, for such ω and L fixed, we can find $w \in [-1,1]^{d-1}$ such that

$$E_{0,\omega}\Big(\sum_{k=1}^{\iota} \mathcal{B}_{k,\omega} \big| W^{(w)}\Big) \leq 15^d \iota L^{\beta-6\delta}. \tag{3.45}$$

Fix such w and define

$$\overline{W} := \Big\{\sum_{k=1}^{\iota} \mathcal{B}_{k,\omega} \leq 2\cdot 15^d \iota L^{\beta-6\delta}\Big\} \cap W^{(w)}.$$

Using (3.45), Markov's inequality yields

$$P_{0,\omega}\Big(\Big\{\sum_{k=1}^{\iota} \mathcal{B}_{k,\omega} \geq 2\cdot 15^d \iota L^{\beta-6\delta}\Big\}\Big| W^{(w)}\Big) \leq \frac{1}{2},$$

whence we obtain

$$\begin{aligned} P_{0,\omega}(\overline{W}) &= P_{0,\omega}\left(\left\{\sum_{k=1}^{\iota}\mathcal{B}_{k,\omega}\le 2\cdot 15^d \iota L^{\beta-6\delta}\right\}\middle| W^{(w)}\right)P_{0,\omega}\big(W^{(w)}\big)\\ &\ge \frac{1}{2}P_{0,\omega}\big(W^{(w)}\big)\ge e^{-CL^{\beta-6\delta}} \end{aligned} \tag{3.46}$$

for L large enough and where the last inequality follows from Lemma 3.24.

We now observe that there is a set $V_{L,\omega}$ of paths such that $\overline{W}=\{(Y_n)\in V_{L,\omega}\}$ and in particular, for $(v_n)\in V_{L,\omega}$ we have $Q_\omega(v)\le 2\cdot 15^d\iota L^{\beta-6\delta}$. Thus, as a consequence of (3.6) and Lemma 3.27,

$$\begin{aligned} P_{0,\omega}\big((X_n)\in V_{L,\omega}\big) &\ge e^{-L^{\beta-\delta}/2}P_{0,\omega}\big((Y_n)\in V_{L,\omega}\big)\\ &= e^{-L^{\beta-\delta}/2}P_{0,\omega}(\overline{W})\ge e^{-L^{\beta-\delta}} \end{aligned} \tag{3.47}$$

for L large enough, where the first inequality follows from the fact that $\omega\in\Theta_L$ in combination with Lemma 3.27 and our choices of δ and ψ, while the last estimate follows from (3.46). Due to Remark 3.20, we may and do choose $V_{L,\omega}$ in such a way that it only contain paths that start in 0 and leave C_L through ∂_+C_L. We take the required family of events in Proposition 2.1 as $\Xi_L:=\Theta_L$, and observe that from (3.47) and Lemma 3.6 we can infer that Ξ_L has the desired properties.

APPENDIX: AUXILIARY RESULTS AND PROOF OF PROPOSITION 3.4

This section contains slight modifications of auxiliary results proven in [1] as well as some further lemmas. With respect to results to which the first point applies, this section is very much based on [1].

In order to prove Proposition 3.4, we will proceed as outlined in Remark 3.5.

A.1. Proof of Proposition 3.4(i).

Set

$$G_L^{(\mathrm{i})}:=\Big\{\omega\in\Omega\colon \max_{z\in\tilde{\mathcal{P}}(0,L)}P_{z,\omega}\big(T_{\partial\mathcal{P}(0,L)}\ne T_{\partial_+\mathcal{P}(0,L)}\big)\le e^{-R_1(L)^\gamma}\Big\}.$$

Then Markov's inequality in combination with Lemma 3.10 yields

$$\begin{aligned} \mathbb{P}\big(G_L^{(\mathrm{i})^C}\big) &\le e^{R_1(L)^\gamma}\mathbb{E}\max_{z\in\tilde{\mathcal{P}}(0,L)}P_{z,\omega}\big(T_{\partial\mathcal{P}(0,L)}\ne T_{\partial_+\mathcal{P}(0,L)}\big)\\ &\le e^{R_1(L)^\gamma}\sum_{z\in\tilde{\mathcal{P}}(0,L)}P_z\big(T_{\partial\mathcal{P}(0,L)}\ne T_{\partial_+\mathcal{P}(0,L)}\big)\\ &\le e^{R_1(L)^\gamma}Ce^{-C^{-1}R_2(L)^\gamma}\le Ce^{-C^{-1}R_2(L)^\gamma}. \end{aligned}$$

In combination with Remark 3.5, this finishes the proof.

A.2. Auxiliary results for the proof of Proposition 3.4(ii) and (iii). We need the following local CLT-type results.

CLAIM A.1. *Let $(Y_i)_{i\in\mathbb{N}}$ be $\mathbb{Z}^d$-valued, independent random variables with finite $(m+1)$st moments for some $m \geq 3$. Furthermore, assume that $(Y_i)_{i\geq 2}$ are identically distributed and that there exists $v \in \mathbb{Z}^d$ such that $P(Y_2 = v) > 0$ and $P(Y_2 = v + e_j) > 0$ for all $j \in \{1, \ldots, d\}$. Let Γ denote the covariance matrix of Y_2 and $S_n = \sum_{i=1}^n (Y_i - EY_i)$. Then there exists a constant C which is determined by the distributions of Y_1 and Y_2 such that for all $n \in \mathbb{N}$ and all x, y and $z \in \mathbb{Z}^d$ with $\|x - y\|_1 = 1$ and $z - y = y - x$:*

(a)

$$P(S_n = x) \leq Cn^{-d/2}, \tag{A.1}$$

(b)

$$|P(S_n = x) - P(S_n = y)| \leq Cn^{-(d+1)/2}, \tag{A.2}$$

(c)

$$|P(S_n = x) - 2P(S_n = y) + P(S_n = z)| \leq Cn^{-(d+2)/2}. \tag{A.3}$$

(d) *In addition, for all w, x, y and z such that there exist $i \neq j$ with $x - y = w - z = e_i$ and $x - w = y - z = e_j$,*

$$|P(S_n = x) + P(S_n = z) - P(S_n = y) - P(S_n = w)| < Cn^{-(d+2)/2}. \tag{A.4}$$

PROOF. Display (A.1) is essentially a consequence of the local limit theorem, see, for example, Theorem 2.3.8 in Lawler and Limic [5]. Indeed, if $EY_2 \in \mathbb{Z}^d$, that source yields that for $S'_n := \sum_{k=2}^{n+1}(Y_k - EY_k)$ and Γ the covariance matrix of Y_2, there exists a constant C such that

$$\begin{aligned} &|P(S'_n = x) - p_n(x)| \\ &\quad \leq Cn^{-(d+1)/2}\big((\|x\|_1^m n^{-m/2} + 1)e^{-(x^T\Gamma^{-1}x)/(2n)} + n^{-(m-2)/2}\big) \end{aligned} \tag{A.5}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, where

$$p_n(x) := \frac{1}{(2\pi n)^{d/2}\sqrt{\det\Gamma}} e^{-(x^T\Gamma^{-1}x)/(2n)}$$

denotes the heat-kernel. Equality (A.5) in particular implies $P(S'_n = x) \leq Cn^{-d/2}$, which entails (A.1). If $EY_2 \notin \mathbb{Z}^d$, then as one may check by redoing the proof, (A.5) holds true for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d - nEY_2$, with $P(S'_n = x)$ replaced by $P(S'_n = x + nEY_2)$, which again implies (A.1).

Now in order to prove (A.2), note that the triangle inequality yields

$$\begin{aligned}
&|P(S_{n+1}=x)-P(S_{n+1}=y)|\\
&\quad\le \max_{z_1,z_2:\,\|z_1-z_2\|_1=1} |P(S'_n=z_1)-P(S'_n=z_2)|\\
&\quad\le \max_{z_1,z_2:\,\|z_1-z_2\|_1=1} |P(S'_n=z_1)-p_n(z_1)|+|p_n(z_1)-p_n(z_2)|\\
&\qquad +|p_n(z_2)-P(S'_n=z_2)|.
\end{aligned}$$

Then (A.5) in combination with standard heat kernel estimates yields the desired result.

In a similar manner, (A.3) and (A.4) can be deduced from Theorem 2.3.8 of [5], which we will omit for the sake of conciseness. □

Using a decomposition according to regeneration times, the previous claim can be employed to prove the following lemma.

LEMMA A.2. *For L and $x\in\tilde{\mathcal{P}}(0,L)$, let $\nu_{x,L}$ denote either $P_x(X_{T_{L^2}}\in\cdot)$, $P_x(X_{T_{\partial\mathcal{P}(0,L)}}\in\cdot)$, $\mu^L_{x,0}$ or $P_x(X_{T_{L^2}}\in\cdot|(X_n-x)\cdot e_1\ge 0\ \forall n\in\mathbb{N})$.*

(a) *There exists a constant C such that for all L, $x\in\tilde{\mathcal{P}}(0,L)$ and $y\in H_{L^2}$,*

$$\nu_{x,L}(y)\le CL^{-d+1}. \tag{A.6}$$

(b) *There exists a constant C such that for all L, $x\in\tilde{\mathcal{P}}(0,L)$, $y\in H_{L^2}$ and $j\in\{2,\dots,d\}$,*

$$|\nu_{x,L}(y)-\nu_{x,L}(y\pm e_j)|<CL^{-d}.$$

(c) *There exists a constant C such that for all L and $x,y\in\tilde{\mathcal{P}}(0,L)$ with $\|x-y\|_1$ as well as $z\in H_{L^2}$,*

$$|\nu_{x,L}(z)-\nu_{y,L}(z)|<CL^{-d}.$$

(d) *There exists a constant C such that for all L, $x\in\tilde{\mathcal{P}}(0,L)$ and $w,y,z\in H_{L^2}$ such that $\|w-y\|_1=1$ and $w-y=y-z$,*

$$|\nu_{x,L}(w)-2\nu_{x,L}(y)+\nu_{x,L}(z)|\le CL^{-d-1}.$$

(e) *There exists a constant C such that for all L, $x\in\tilde{\mathcal{P}}(0,L)$ and $v,w,y,z\in H_{L^2}$ such that $\|v-w\|_1=1$, $z-y=w-v$ and $z-w=y-v$,*

$$\big|\nu_{x,L}(z)-\nu_{x,L}(y)-\big(\nu_{x,L}(w)-\nu_{x,L}(v)\big)\big|\le CL^{-d-1}.$$

PROOF. The fact that the particular choice among the first three possibilities for $\nu_{x,L}$ is irrelevant, is a direct consequence of Lemma 3.10. With respect to the

case that $\nu_{x,L} = P_x(X_{T_{L^2}} \in \cdot | (X_n - x)\cdot e_1 \geq 0\ \forall n \in \mathbb{N})$, the desired result follows analogously from what comes below in combination with Corollary 1.5 of [10].

We will give the proof for $\nu_{x,L} = P_x(X_{T_{L^2}} \in \cdot)$.

For $k, l \in \mathbb{N}$ we define the event $B(l,k) := \{X_{\tau_k}\cdot e_1 = l\}$ as well as $B(l) := \bigcup_{k=1}^{l} B(l,k)$ and

$$\hat{B}(l) := B(l) \cap \bigcap_{j=l+1}^{L^2-1} B^c(j);$$

that is, for $l < L^2$ and on $\hat{B}(l)$, one has that l is the e_1-coordinate at which the last renewal before reaching the e_1-coordinate L^2 occurs.

(a) We have

$$P_x(X_{T_{L^2}} = y) \leq P_x(A_L^c) + \sum_{l=L^2-R_2(L)}^{L^2} F_l \tag{A.7}$$

with $F_l := P_x(X_{T_{L^2}} = y, \hat{B}(l))$, and furthermore

$$\begin{aligned} F_l &= \sum_{k=0}^{L^2} \sum_{z\in H_l} P_x\big(X_{\tau_k} = z, X_{T_{L^2}} = y, \hat{B}(l)\big) \\ &= \sum_{k=0}^{L^2} \sum_{z\in H_l} P_x(X_{\tau_k} = z) P_x\big(X_{T_{L^2}} = y, \hat{B}(l) | X_{\tau_k} = z\big) \qquad \text{(A.8)} \\ &= \sum_{z\in H_l} P_x\big(X_{T_{L^2}} = y, \hat{B}(l) | X_{\tau_1} = z\big) \sum_{k=0}^{L^2} P_x(X_{\tau_k} = z). \end{aligned}$$

In order to estimate the inner sum of (A.8), set $m := E_0(X_{\tau_2} - X_{\tau_1})$ and for $l \in \{L^2 - R_2(L), \ldots, L^2\}$ fixed, define $l^* := \lfloor \frac{l}{m\cdot e_1} \rfloor$. We now distinguish cases.

First, assume $k \geq l^*$. Then $\{X_{\tau_k}\cdot e_1 = l\} \subset H^1 \cup H^2$, where $H^1 := \{X_{\tau_{\lfloor k/2\rfloor}}\cdot e_1 \leq l/2\}$ and $H^2 := \{(X_{\tau_k} - X_{\tau_{\lfloor k/2\rfloor}})\cdot e_1 \leq l/2\}$. We get

$$P_x(X_{\tau_k} = z, H^1) = \sum_{y\,:\, y\cdot e_1 \leq l/2} P_x(X_{\tau_k} = z | X_{\tau_{\lfloor k/2\rfloor}} = y) P_x(X_{\tau_{\lfloor k/2\rfloor}} = y),$$

and uniformly in y and z we have $P_x(X_{\tau_k} = z | X_{\tau_{\lfloor k/2\rfloor}} = y) \leq Ck^{-d/2}$ due to the independence of the renewals (cf. Corollary 1.5 in [10]) and (A.1). Now observe that using standard estimates for centred random variables with finite $2d$th moment [note that $(X_{\tau_2} - X_{\tau_1})\cdot e_1$ has finite $2d$th moment as a consequence of the assumption $(T)_\gamma$], there exists a constant C such that uniformly in k and L we have

$$P_x(X_{\tau_{\lfloor k/2\rfloor}}\cdot e_1 \leq l/2) \leq 1 \wedge Ck^d(k - l^*)^{-2d}. \tag{A.9}$$

We therefore get

$$\sum_{k=l^*}^{L^2} P_x(X_{\tau_k} = z, H^1) \le Cl^{-d/2}\left(\sqrt{l^*} + \sum_{j=1}^{\infty}\left(l^* + (j+1)\sqrt{l^*}\right)^d \left(j\sqrt{l^*}\right)^{-2d}\right)$$
$$\le Cl^{(-d+1)/2}$$

and analogously for H^2, whence

$$\sum_{k=l^*}^{L^2} P_x(X_{\tau_k} = z) \le Cl^{(-d+1)/2}. \tag{A.10}$$

Now assume $k < l^*$. Then in the same manner as above we obtain

$$\sum_{k=l^*/2}^{l^*} P_x(X_{\tau_k} = z) \le Cl^{(-d+1)/2} \tag{A.11}$$

and furthermore (A.9) supplies us with

$$\sum_{k=0}^{l^*/2} P_x(X_{\tau_k} = z) \le C\sum_{k=0}^{l^*/2} k^d (l^* - k)^{-2d} \le Cl^{-d+1}. \tag{A.12}$$

In order to deal with the outer sum of (A.8), note that for fixed l as well as $y^* \in H_{L^2}$ and $z^* \in H_l$ we have

$$\begin{aligned}\sum_{z\in H_l} P_x\big(X_{T_{L^2}} = y^*, \hat{B}(l) | X_{\tau_1} = z\big) &= \sum_{y\in H_{L^2}} P_x\big(X_{T_{L^2}} = y, \hat{B}(l) | X_{\tau_1} = z^*\big) \\ &= P_x\big(\hat{B}(l) | X_{\tau_1} = z^*\big).\end{aligned} \tag{A.13}$$

Using (A.10) to (A.12) in combination with (A.8), we therefore deduce that for all $l \in \{L^2 - R_2(L), \dots, L^2\}$,

$$F_l \le C P_x(\hat{B}(l)) L^{-d+1}.$$

Thus, in combination with (A.7) and Lemma 3.3 we get

$$P_x(X_{T_{L^2}} = y) \le C L^{-d+1},$$

which finishes the proof.

(b) We have

$$|P_x(X_{T_{L^2}} = y) - P_x(X_{T_{L^2}} = y \pm e_j)| \le 2P_x(A_L^c) + \sum_{l=L^2-R_2(L)}^{L^2} F_l \tag{A.14}$$

with $\hat{B}(l)$ as defined before and

$$F_l := P_x\big(X_{T_{L^2}} = y, \hat{B}(l)\big) - P_x\big(X_{T_{L^2}} = y \pm e_j, \hat{B}(l)\big).$$

We compute

$$
\begin{aligned}
F_l &= \sum_{k=0}^{L^2} \sum_{z\in H_l} \big(P_x\big(X_{\tau_k}=z, X_{T_{L^2}}=y, \hat{B}(l)\big) \\
&\qquad\qquad - P_x\big(X_{\tau_k}=z\pm e_j, X_{T_{L^2}}=y\pm e_j, \hat{B}(l)\big)\big) \\
&= \sum_{k=0}^{L^2} \sum_{z\in H_l} \big(P_x(X_{\tau_k}=z)P_x\big(X_{T_{L^2}}=y, \hat{B}(l)|X_{\tau_k}=z\big) \\
&\qquad\qquad - P_x(X_{\tau_k}=z\pm e_j)P_x\big(X_{T_{L^2}}=y\pm e_j, \hat{B}(l)|X_{\tau_k}=z\pm e_j\big)\big) \\
&= \sum_{z\in H_l} P_x\big(X_{T_{L^2}}=y, \hat{B}(l)|X_{\tau_1}=z\big) \\
&\qquad\qquad \times \sum_{k=0}^{L^2} |P_x(X_{\tau_k}=z) - P_x(X_{\tau_k}=z\pm e_j)|,
\end{aligned}
\tag{A.15}
$$

where to obtain the last line we used the translation invariance of $\mathbb{P}$. Fix $l\in\{L^2 - R_2(L),\dots,L^2\}$ and let m and l^* as before. Again we distinguish cases.

First, assume $k\geq l^*$. Then $\{X_{\tau_k}\cdot e_1 = l\}\subset H^1\cup H^2$, where H^1 and H^2 as before. Then

$$
\begin{aligned}
&|P_x(X_{\tau_k}=z, H^1) - P_x(X_{\tau_k}=z\pm e_j, H^1)| \\
&\quad = \sum_{y\,:\,y\cdot e_1\leq l/2} \big|P_x(X_{\tau_k}=z|X_{\tau_{\lfloor k/2\rfloor}}=y) - P_x(X_{\tau_k}=z\pm e_j|X_{\tau_{\lfloor k/2\rfloor}}=y)\big| \\
&\qquad\qquad \times P_x(X_{\tau_{\lfloor k/2\rfloor}}=y),
\end{aligned}
$$

and uniformly in y, we have

$$
\big|P_x(X_{\tau_k}=z|X_{\tau_{\lfloor k/2\rfloor}}=y) - P_x(X_{\tau_k}=z\pm e_j|X_{\tau_{\lfloor k/2\rfloor}}=y)\big| \leq Ck^{(-d-1)/2}
$$

due to the independence of the renewals and (A.2). Using (A.9), we get

$$
\begin{aligned}
&\sum_{k=l^*}^{L^2} |P_x(X_{\tau_k}=z, H^1) - P_x(X_{\tau_k}=z\pm e_j, H^1)| \\
&\qquad \leq Cl^{(-d-1)/2}\left(\sqrt{l^*} + \sum_{j=1}^{\infty}\big(l^* + (j+1)\sqrt{l^*}\big)^d\big(j\sqrt{l^*}\big)^{-2d}\right) \leq Cl^{-d/2}
\end{aligned}
$$

and analogously for H^2, whence

$$
\sum_{k=l^*}^{L^2} |P_x(X_{\tau_k}=z) - P_x(X_{\tau_k}=z\pm e_j)| \leq Cl^{-d/2}. \tag{A.16}
$$

Now assume $k < l^*$. Then in the same manner we obtain

$$\sum_{k=L^2/2}^{l^*} |P_x(X_{\tau_k} = z) - P_x(X_{\tau_k} = z \pm e_j)| \le C l^{-d/2} \tag{A.17}$$

and furthermore (A.9) supplies us with

$$\begin{aligned}\sum_{k=0}^{l^*/2} |P_x(X_{\tau_k} = z) - P_x(X_{\tau_k} = z \pm e_j)| &\le C \sum_{k=0}^{l^*/2} k^d (l^* - k)^{-2d} \\ &\le C l^{-d+1}.\end{aligned} \tag{A.18}$$

Using (A.13) and (A.15) to (A.18), we deduce that there exists C such that for all $l \in \{L^2 - R_2(L), \dots, L^2\}$,

$$F_l \le C P_x(\hat{B}(l)) L^{-d}.$$

In combination with (A.14) and Lemma 3.3, we get

$$|P_x(X_{T_{L^2}} = y) - P_x(X_{T_{L^2}} = y \pm e_j)| \le C L^{-d},$$

which finishes the proof.

Parts (c), (d) and (e) follow from analogous calculations using (A.2), (A.3) and (A.4), respectively. For the sake of conciseness, we omit giving the corresponding proofs here. □

To prove parts (ii) and (iii) of Proposition 3.4, we quote and reprove a conditional Azuma-type inequality appearing in [1].

In this context, denote by $(M_k)_{k\in\mathbb{N}_0}$ a one-dimensional martingale on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_k)_{k\in\mathbb{N}_0}$ and $M_0 = 0$. Set $\Delta_k := M_k - M_{k-1}$ and assume that the $|\Delta_k|$ are uniformly bounded from above by a finite constant. Define for any nonnegative random variable X its *conditional essential supremum* with respect to $\mathcal{F}_k$ as $\operatorname{ess\,sup}(X|\mathcal{F}_{k-1}) := \lim_{n\to\infty} E(X^n|\mathcal{F}_{k-1})^{1/n}$, where the right-hand side exists due to Jensen's inequality. Set

$$\sigma_k := \operatorname{ess\,sup}(|\Delta_k||\mathcal{F}_{k-1}).$$

Then the *essential variance* of the martingale is defined as

$$V_k := \operatorname{ess\,sup}\left(\sum_{j=1}^{k} \sigma_j^2\right).$$

LEMMA A.3. *If the Δ_k are uniformly bounded, then for all $n \in \mathbb{N}$ and $t > 0$,*

$$P(|M_n| > t) \le 2e^{-t^2/(2V_n)}.$$

Furthermore, if $M_n = (M_n^{(1)}, \dots, M_n^{(d)})$ *with* $M_n^{(j)}$ *being one-dimensional martingales such that the differences* Δ_k *are uniformly bounded and with* $V_n^{(j)}$ *as essential variance, then writing* $V_n^{\max} := \max_{j\in\{1,\dots,d\}} V_n^{(j)}$ *one has*

$$P(\|M_n\|_\infty > t) \le 2de^{-t^2/(2V_n^{\max})}.$$

PROOF. First, observe that the d-dimensional case is a direct consequence of the one-dimensional case by considering its components and a standard union bound. It is therefore sufficient to prove the one-dimensional case.

We start with showing that for each $k \in \{1, \dots, n\}$,

$$E(e^{\sum_{j=k}^n \Delta_j}|\mathcal{F}_{k-1}) \le e^{(1/2)\operatorname{ess\,sup}(\sum_{j=k}^n \sigma_j^2|\mathcal{F}_{k-1})}. \tag{A.19}$$

To establish this inequality in the case $k = n$, we first of all note that

$$\lim_{m\to\infty} E(|\Delta_n|^m|\mathcal{F}_{n-1})^{1/m}\mathbb{1}_A \ge \operatorname{ess\,sup}(|\Delta_n|\mathbb{1}_A)\mathbb{1}_A - \varepsilon \tag{A.20}$$

for all $x \in [0, \infty)$, $\varepsilon > 0$ and

$$A := A_{x,\varepsilon} := \Big\{\lim_{m\to\infty} E(|\Delta_n|^m|\mathcal{F}_{n-1})^{1/m} \in (x, x+\varepsilon]\Big\} \in \mathcal{F}_{n-1}.$$

We then observe that for such A and with $C_A := \operatorname{ess\,sup}(|\Delta_n|\mathbb{1}_A)$ as well as

$$h_A : [-C_A, C_A] \ni s \mapsto \frac{e^{C_A} + e^{-C_A}}{2} + \frac{e^{C_A} - e^{-C_A}}{2}\frac{s}{C_A},$$

we obtain

$$\begin{aligned} E(e^{\Delta_n}|\mathcal{F}_{n-1}) &\le E(h_A(\Delta_n)|\mathcal{F}_{n-1})\mathbb{1}_A \\ &= h_A(E(\Delta_n\mathbb{1}_A|\mathcal{F}_{n-1}))\mathbb{1}_A \\ &= h_A(0)\mathbb{1}_A = \frac{e^{C_A} + e^{-C_A}}{2}\mathbb{1}_A \\ &= \cosh(C_A)\mathbb{1}_A. \end{aligned}$$

Since by comparison of the corresponding power series one has $\cosh(x) \le e^{x^2/2}$, we obtain with (A.20) that

$$\begin{aligned} &E(e^{\Delta_n}\mathbb{1}_A|\mathcal{F}_{n-1}) \\ &\quad\le e^{C_A^2/2}\mathbb{1}_A \le \exp\Big\{\frac{1}{2}\Big(\lim_{m\to\infty} E(|\Delta_n|^m|\mathcal{F}_{n-1})^{1/m} + \varepsilon\Big)^2\Big\}\mathbb{1}_A. \end{aligned} \tag{A.21}$$

Summing (A.21) over all $A := A_{x,\varepsilon}$ for $x = j\varepsilon$, $j \in \mathbb{N}_0$ we get

$$\begin{aligned} E(e^{\Delta_n}|\mathcal{F}_{n-1}) \le \exp\Big\{\frac{1}{2}\lim_{m\to\infty} E(|\Delta_n|^m|\mathcal{F}_{n-1})^{2/m}\Big\} \\ \times \exp\{\operatorname{ess\,sup}|\Delta_n|^2\varepsilon + \varepsilon^2/2\}. \end{aligned}$$

Since Δ_n was assumed to be bounded, taking $\varepsilon \downarrow 0$ yields (A.19) for $k = n$.

Now we assume (A.19) to hold true for $k+1$ and deduce its validity for k:

$$\begin{aligned}
E(e^{\sum_{j=k}^{n}\Delta_j}|\mathcal{F}_{k-1}) &= E(e^{\Delta_k}E(e^{\sum_{j=k+1}^{n}\Delta_j}|\mathcal{F}_k)|\mathcal{F}_{k-1}) \\
&\le E\big(e^{\Delta_k}e^{(1/2)\operatorname{ess\,sup}(\sum_{j=k+1}^{n}\sigma_j^2|\mathcal{F}_k)}|\mathcal{F}_{k-1}\big) \\
&\le E\big(e^{\Delta_k}e^{(1/2)\operatorname{ess\,sup}(\sum_{j=k+1}^{n}\sigma_j^2|\mathcal{F}_{k-1})}|\mathcal{F}_{k-1}\big) \\
&= e^{(1/2)\operatorname{ess\,sup}(\sum_{j=k+1}^{n}\sigma_j^2|\mathcal{F}_{k-1})}E(e^{\Delta_k}|\mathcal{F}_{k-1}) \\
&\le e^{(1/2)\operatorname{ess\,sup}(\sum_{j=k+1}^{n}\sigma_j^2|\mathcal{F}_{k-1})}e^{(1/2)\sigma_k^2} \\
&= e^{(1/2)\operatorname{ess\,sup}(\sum_{j=k}^{n}\sigma_j^2|\mathcal{F}_{k-1})},
\end{aligned}$$

where to obtain the second inequality we used that for any nonnegative random variable X we have

$$\operatorname{ess\,sup}(X|\mathcal{F}_k) \le \operatorname{ess\,sup}(X|\mathcal{F}_{k-1}).$$

Altogether, this establishes (A.19).

Inserting $k=1$ in (A.19), we deduce $Ee^{\lambda M_n} \le e^{(1/2)\lambda^2 V_n}$ for any real λ. This estimate in combination with the exponential Chebyshev inequality yields

$$\begin{aligned}
P(|M_n|>t) &= P(M_n>t)+P(M_n<-t) \\
&\le e^{-\lambda t}(Ee^{\lambda M_n}+Ee^{-\lambda M_n}) \\
&\le 2e^{-\lambda t}e^{(1/2)\lambda^2 V_n}
\end{aligned}$$

for $\lambda>0$. Setting $\lambda := t/V_n$, this finishes the proof. □

The following result appears as Lemma 3.3 in Berger and Zeitouni [2] and will prove helpful in the following.

LEMMA A.4. *Let $d\ge 3$ and let $(v_n)_{n\in\mathbb{N}}$ be i.i.d., $\mathbb{Z}^d$-valued random variables such that P-a.s. we have $v_1\cdot e_1\ge 1$ as well as $E\|v_1\|^r<\infty$ for some $r\in[2,d-1]$. Furthermore, assume that for some $\delta>0$,*

$$P(v_1\cdot e_1=1)>\delta,$$

and that for all $z\in\mathbb{Z}^d$ of the form $z=e_1\pm e_j$, $j\in\{2,\dots,d\}$, one has

$$P(v_1=z|v_1\cdot e_1=1)>\delta.$$

Set $S_n:=\sum_{i=1}^{n}v_i$. Then there exists a constant $K>0$ such that for all $z\in\mathbb{Z}^d$,

$$P(\exists n\in\mathbb{N}: S_n=z)\le K|z\cdot e_1|^{-r(d-1)/(r+d-1)}.$$

Furthermore, for all $l\in\mathbb{N}$,

$$\sum_{z\in H_l}P(\exists n\in\mathbb{N}: S_n=z)\le 1.$$

The following result guarantees that with positive probability with respect to the annealed measure, the trajectories of two independent RWRE do never intersect.

LEMMA A.5. *Let $d \geq 4$. Then there exists $M \in (0, \infty)$ such that for $x_1, x_2 \in \mathbb{Z}^d$ with $(x_1 - x_2) \cdot e_1 = 0$ and $\|x_1 - x_2\|_\infty > M$ we have*

$$\mathbf{P}_{x_1,x_2}\big(\{X_n^{(1)} : n \in \mathbb{N}\} \cap \{X_n^{(2)} : n \in \mathbb{N}\} = \varnothing\big) > 0, \tag{A.22}$$

where

$$\mathbf{P}_{x_1,x_2} := P_{x_1}\big(\cdot | X_n^{(1)} \cdot e_1 \geq x_1 \cdot e_1 \ \forall n \in \mathbb{N}\big) \otimes P_{x_2}\big(\cdot | X_n^{(2)} \cdot e_2 \geq x_2 \cdot e_2 \ \forall n \in \mathbb{N}\big)$$

and $X^{(1)}$ and $X^{(2)}$ denote copies of the RWRE X "driven" by P_{x_1} and P_{x_2}, respectively.

In particular, for all $l \in \mathbb{N}$,

$$\inf_{x_1,x_2 \in H_l, x_1 \neq x_2} P_{x_1} \otimes P_{x_2}\big(\{X_n^{(1)} : n \in \mathbb{N}\} \cap \{X_n^{(2)} : n \in \mathbb{N}\} = \varnothing\big) > 0$$

also.

PROOF. Due to uniform ellipticity, the last statement is a direct consequence of (A.22). Thus, we prove (A.22) now.

The proof is inspired by the proof of Proposition 3.4 in [2]. The translation invariance of $\mathbb{P}$ implies that we can assume $x \cdot e_1 = x \cdot e_2 = 0$ without loss of generality. Denote by $m := E_{x_1}(X^{(1)})^{*(2)}$ the expectation of the second renewal radius and for $N \in \mathbb{N}$ set

$$B_N^{(j)} := \left\{ \sum_{k=1}^{N/(4m)} \big\|\big(X^{(j)}\big)^{*(k)}\big\|_1 \leq N/2 \right\}.$$

For $j \in \{1, 2\}$, with respect to $\mathbf{P}_{x_1,x_2}(\cdot | A_N(X^{(j)}))$,

$$\left(\sum_{k=1}^{n} \big\|\big(X^{(j)}\big)^{*(k)}\big\|_1 - \mathbf{E}_{x_1,x_2}\big(\big\|\big(X^{(j)}\big)^{*(k)}\big\|_1 | A_N\big(X^{(j)}\big)\big) \right)_{n \in \{0,\ldots,2N^2\}}$$

is a martingale with bounded increments. Therefore, applying Azuma's inequality for $N \in 4m\mathbb{N}$ large enough results in

$$\begin{aligned}
&\mathbf{P}_{x_1,x_2}\big((B_N^{(j)})^c\big) \\
&\quad = \mathbf{P}_{x_1,x_2}\left(\sum_{k=1}^{N/(4m)} \big\|\big(X^{(j)}\big)^{*(k)}\big\|_1 > N/2 \right) \\
&\quad \leq \mathbf{P}_{x_1,x_2}\left(\left[\sum_{k=1}^{N/(4m)} \big\|\big(X^{(j)}\big)^{*(k)}\big\|_1 \right. \right. \\
&\qquad\qquad \left. \left. - \mathbf{E}_{x_1,x_2}\big(\big\|\big(X^{(j)}\big)^{*(k)}\big\|_1 | A_N\big(X^{(j)}\big)\big) > N/4 \right] \middle| A_N\big(X^{(j)}\big) \right) \qquad \text{(A.23)} \\
&\qquad + \mathbf{P}_{x_1,x_2}\big(A_N\big(X^{(j)}\big)^c\big) \\
&\quad \leq 2\exp\left\{ -\frac{(N/4)^2}{2NR_2^2(N)/4} \right\} + \mathbf{P}_{x_1,x_2}\big(A_N\big(X^{(j)}\big)^c\big); \qquad \text{(A.24)}
\end{aligned}$$

here, we took advantage of

$$m \geq \mathbf{E}_{x_1,x_2}\big(\big\|\big(X^{(j)}\big)^{*(k)}\big\|_1 | A_N\big(X^{(j)}\big)\big)$$

for all k and $N \in \mathbb{N}$.

Furthermore, for $j \in \{1,2\}$, $n \in \mathbb{N}$ and $\nu \in (0,1)$ define the random times $h_{j,n} := \max\{k \in \mathbb{N}_0 : X^{(j)}_{\tau_k} \cdot e_1 \leq n\}$ as well as the event

$$T^{(j)}_{\nu,N} := \bigcap_{n \geq N/(4m)} \big\{\big(X^{(j)}\big)^{*(h_{j,n+1})} \leq (2mn)^{\nu}\big\}.$$

Then $(T)_\gamma$ implies that for any $\nu > 0$ and $K > 0$ there exists a constant $C > 0$ such that for all N we have

$$\mathbf{P}_{x_1,x_2}\big(\big(T^{(j)}_{\nu,N}\big)^c\big) \leq CN^{-K}. \tag{A.25}$$

Now we distinguish the situations in which the trajectories of the two walks could intersect in order to explain the decomposition in (A.27) and (A.28) below; for this purpose, assume that x_1 and x_2 from the assumptions satisfy

$$\|x_1 - x_2\|_\infty \geq N^4. \tag{A.26}$$

(a) If the walks intersect within the first $N/(4m)$ renewal times of both walks, then due to (A.26) this event is a subset of $(B^{(1)}_N)^c \cup (B^{(2)}_N)^c$. This yields the first summand in (A.27).

(b) Otherwise, the intersection may occur on $(T^{(1)}_{\nu,N})^c \cup (T^{(2)}_{\nu,N})^c$, which yields the second summand in (A.27).

(c) It remains to consider intersections after $N/(4m)$ renewal times for at least one walk on $B^{(1)}_N \cap B^{(2)}_N \cap T^{(1)}_{\nu,N} \cap T^{(2)}_{\nu,N}$; note that due to the restriction to $B^{(1)}_N \cap B^{(2)}_N$ and (A.26), the intersection can take place in H_n with $n \geq N/(4m)$ only. In this case, since we restrict to $T^{(1)}_{\nu,N} \cap T^{(2)}_{\nu,N}$, if the trajectories intersect in the hyperplane H_n, there must have occurred a renewal for each of the walks in distance at most $(2mn)^\nu$ from the point of intersection which implies that the two renewals must occur at sites that have distance $2(2mn)^\nu$ at most from each other. Thus, (A.28) corresponds to an intersection after at least $N/(4m)$ renewals for at least one walk, on $B^{(1)}_N \cap B^{(2)}_N \cap T^{(1)}_{\nu,N} \cap T^{(2)}_{\nu,N}$.

Consequently, choosing $\nu > 0$ small enough, we obtain using Lemma A.4 with $r = 2$ as well as (A.24) and (A.25), that

$$\begin{aligned}
&\mathbf{P}_{x_1,x_2}\big(\big\{X^{(1)}_n : n \in \mathbb{N}\big\} \cap \big\{X^{(2)}_n : n \in \mathbb{N}\big\} \neq \varnothing\big) \\
&\leq 2\mathbf{P}_{x_1,x_2}\big(\big(B^{(1)}_N\big)^c\big) + 2\mathbf{P}_{x_1,x_2}\big(\big(T^{(1)}_\nu\big)^c\big) \qquad \text{(A.27)} \\
&\quad + \sum_{j \geq N/(4m)} \sum_{z \in H_j} \sum_{z' : \|z - z'\|_1 \leq 2(2mj)^\nu} \mathbf{P}_{x_1,x_2}\big(\exists i : X^{(1)}_{\tau_i} = z\big) \\
&\qquad\qquad \times \mathbf{P}_{x_1,x_2}\big(\exists k : X^{(2)}_{\tau_k} = z'\big) \qquad \text{(A.28)}
\end{aligned}$$

$$\leq C\Bigg(N^{-K} + \sum_{j\geq N/(4m)} \sum_{z\in H_j} \mathbf{P}_{x_1,x_2}\big(\exists i : X^{(1)}_{\tau_i} = z\big) \times \sum_{z' :\, \|z-z'\|_1 \leq 2(2mj)^\nu} \big(j - 2(2mj)^\nu\big)^{-2(d-1)/(d+1)}\Bigg) \tag{A.29}$$

$$\leq C\big(N^{-K} + N^{\nu d} N^{-2(d-1)/(d+1)+1}\big) \to 0 \tag{A.30}$$

as $N \to \infty$, and where for ease of notation we omitted to emphasize that the renewal times τ_i refer to the process that is evaluated at these times. Choosing $M = N^4$ for some N such that the term in (A.30) is smaller than 1, this establishes the lemma. □

For $\omega \in \Omega$ and $z \in \mathbb{Z}^d$ we set $\mathbf{P}_{z,\omega} := P_{z,\omega} \otimes P_{z,\omega}$ as well as $\mathbf{P}_z := \int_\Omega P_{z,\omega} \otimes P_{z,\omega} \mathbb{P}(d\omega)$, where the RWRE "driven" by the first factor is denoted by $X^{(1)}$ and the one driven by the second factor is denoted by $X^{(2)}$. Using the previous lemma, we can bound the number of intersections of two independent RWREs as follows.

LEMMA A.6. *There exists a positive constant C such that for all L large enough as well as $z \in \tilde{\mathcal{P}}(0, L)$ and $m \in \mathbb{N}$,*

$$\mathbf{P}_z\big(\big|\{X^{(1)}_n : n \in \mathbb{N}\} \cap \{X^{(2)}_n : n \in \mathbb{N}\} \cap \mathcal{P}(0, L)\big| > m R_2^{d+1}(L) | A_L(X^{(1)}), A_L(X^{(2)})\big) < e^{-Cm}.$$

PROOF. For L large enough, any k such that $k + R_2(L) < L$ and $j \in \{1, 2\}$ we have

$$\mathbb{1}_{A_L(X^{(j)})} \cdot \big|\{x \in \{X^{(j)}_n : n \in \mathbb{N}\} : k < x \cdot e_1 < k + R_2(L)\}\big| < R_2^{d+1}(L) \tag{A.31}$$

as well as

$$\mathbb{1}_{A_L(X^{(j)})} \cdot \big|\{x \in \{X^{(j)}_n : n \in \mathbb{N}\} : x \cdot e_1 \leq 0\}\big| < R_2^{d+1}(L). \tag{A.32}$$

For every k, let $Q_k^- := \mathcal{P}(0, L) \cap \{x : x \cdot e_1 < k R_2(L)\}$ and $Q_k^+ := \mathcal{P}(0, L) \cap \{x : x \cdot e_1 \geq k R_2(L)\}$. Due to Lemma A.5, we can infer that there exists $\rho > 0$ such that for every k and uniformly in $z \in \tilde{\mathcal{P}}(0, L)$,

$$\mathbf{P}_z\big(\big|\{X^{(1)}_n : n \in \mathbb{N}\} \cap \{X^{(2)}_n : n \in \mathbb{N}\} \cap Q_{k+1}^+ = \varnothing\big| \, A_L(X^{(1)}), A_L(X^{(2)}), \{X^{(1)}_n : n \in \mathbb{N}\} \cap Q_k^-, \{X^{(2)}_n : n \in \mathbb{N}\} \cap Q_k^-\big) > \rho. \tag{A.33}$$

Let

$$J^{(\text{even})} := \big\{k \in 2\mathbb{N}_0 : \{X^{(1)}_n : n \in \mathbb{N}\} \cap \{X^{(2)}_n : n \in \mathbb{N}\} \cap Q_k^+ \cap Q_{k+1}^- \neq \varnothing\big\}$$

and

$$J^{(\mathrm{odd})} := \{k \in 2\mathbb{N}_0 + 1 : \{X_n^{(1)} : n \in \mathbb{N}\} \cap \{X_n^{(2)} : n \in \mathbb{N}\} \cap Q_k^+ \cap Q_{k+1}^- \neq \varnothing\}.$$

Then by (A.33), conditioned on $A_L(X^{(1)}) \cap A_L(X^{(2)})$, both $J^{(\mathrm{even})}$ and $J^{(\mathrm{odd})}$ are stochastically dominated by a geometric variable with parameter ρ.

The lemma now follows when we remember that by (A.31) and (A.32),

$$\begin{aligned}&\mathbb{1}_{A_L(X^{(1)})}\mathbb{1}_{A_L(X^{(2)})} \cdot \big|\{X_n^{(1)} : n \in \mathbb{N}\} \cap \{X_n^{(2)} : n \in \mathbb{N}\} \cap \mathcal{P}(0, L)\big| \\ &\quad \le R_2^{d+1}(L)\big(J^{(\mathrm{even})} + J^{(\mathrm{odd})}\big).\end{aligned}$$

□

As a corollary of Lemma A.6, we obtain the following estimate.

COROLLARY A.7. *With the same notation as in Lemma A.6,*

$$\begin{aligned}&\mathbb{P}\big(\exists z \in \tilde{\mathcal{P}}(0, L) : \\ &\quad \mathbf{E}_{z,\omega}\big(\big|\{X_n^{(1)} : n \in \mathbb{N}\} \\ &\qquad \cap \{X_n^{(2)} : n \in \mathbb{N}\} \cap \mathcal{P}(0, L)\big| \big| A_L\big(X^{(1)}\big), A_L\big(X^{(2)}\big)\big) \ge R_3(L)\big)\end{aligned} \tag{A.34}$$

is contained in $\mathcal{S}(\mathbb{N})$ as a function in L.

PROOF. Set $Z := |\{X_n^{(1)} : n \in \mathbb{N}\} \cap \{X_n^{(2)} : n \in \mathbb{N}\} \cap \mathcal{P}(0, L)|$ and note that on $A_L(X^{(1)}) \cap A_L(X^{(2)})$, the variable Z is bounded from above by $|\mathcal{P}(0, L)| \le (2L^2)^d$. Thus,

$$\begin{aligned}&\mathbb{P}\big(\mathbf{E}_{z,\omega} Z \ge R_3(L) | A_L\big(X^{(1)}\big), A_L\big(X^{(2)}\big)\big) \\ &\quad \le \mathbb{P}\big(\mathbf{E}_{z,\omega}\big(Z, Z \le nR_2^{d+1}(L) | A_L\big(X^{(1)}\big), A_L\big(X^{(2)}\big)\big) \ge R_3(L)/2\big) \\ &\qquad + \underbrace{\mathbb{P}\big(\mathbf{E}_{z,\omega}\big(Z, Z \le nR_2^{d+1}(L) | A_L\big(X^{(1)}\big), A_L\big(X^{(2)}\big)\big) \ge R_3(L)/2\big)}_{=0 \text{ for } n = R_2(L)} \\ &\quad \le (2L^2)^d \mathbf{P}_z\big(Z \ge nR_2^{d+1}(L) | A_L\big(X^{(1)}\big), A_L\big(X^{(2)}\big)\big) \le e^{-CR_2(L)}\end{aligned}$$

for $n = R_2(L)$ and L large enough due to Lemma A.6. Taking the union bound for $z \in \tilde{\mathcal{P}}(0, L)$ finishes the proof. □

We define $J(L) \subset \Omega$ to be the set of all ω such that for every $z \in \tilde{\mathcal{P}}(0, L)$,

$$\mathbf{E}_{z,\omega}\big(\big|\{X_n^{(1)} : n \in \mathbb{N}\} \cap \{X_n^{(2)} : n \in \mathbb{N}\} \cap \mathcal{P}(0, L)\big| \big| A_L\big(X^{(1)}\big), A_L\big(X^{(2)}\big)\big) \le R_3(L).$$

Then by Corollary A.7,

$$\mathbb{P}(J(\cdot)^c) \in \mathcal{S}(\mathbb{N}), \tag{A.35}$$

and for $\omega \in J(L)$ and $z \in \tilde{\mathcal{P}}(0, L)$,

$$\sum_{x \in \mathcal{P}(0,L)} P_{z,\omega}(T_x < \infty)^2 < R_3(L). \tag{A.36}$$

A.3. Proof of Proposition 3.4(ii). The following lemma will yield part (ii) of Proposition 3.4.

LEMMA A.8. *There exists a sequence of events $G_L^{(\mathrm{ii})} \subset \Omega$ such that*

$$\mathbb{P}\big(G_\cdot^{(\mathrm{ii})^c}\big) \in \mathcal{S}(\mathbb{N})$$

and for every $\omega \in G_L^{(\mathrm{ii})}$ and $z \in \tilde{\mathcal{P}}(0, L)$,

$$\big\| E_{z,\omega}\big(X_{T_{\partial\mathcal{P}(0,L)}} | T_{\partial\mathcal{P}(0,L)} = T_{\partial_+\mathcal{P}(0,L)}\big) - E_z\big(X_{T_{\partial\mathcal{P}(0,L)}} | T_{\partial\mathcal{P}(0,L)} = T_{\partial_+\mathcal{P}(0,L)}\big)\big\|_1 \leq R_4(L).$$

PROOF. As a consequence of Lemma 3.3, Proposition 3.4(i) and (A.35), it is sufficient to show that denoting

$$U(\omega, z) := \|E_{z,\omega}(X_{T_{L^2}}, A_L, J(L)) - E_z(X_{T_{L^2}}, A_L, J(L))\|_1,$$

one has that

$$\mathbb{P}\Big(\bigcup_{z \in \tilde{\mathcal{P}}(0,L)} \{\omega : U(\omega, z) > R_4(L)/2\}\Big) \tag{A.37}$$

is contained in $\mathcal{S}(\mathbb{N})$ as a function in L.

To this end, note that on A_L the walk starting in $\tilde{\mathcal{P}}(0, L)$ can visit sites in

$$S_L := \{x \in \mathbb{Z}^d : -R_2(L) - L^2/3 \leq x \cdot e_1 < L^2, \ \|\pi_{e_1^\perp}(x)\|_\infty \leq 2L^2 R_2(L)\} \tag{A.38}$$

only before hitting H_{L^2}. Order the vertices contained in S_L lexicographically, that is in increasing order of their first differing coordinate, as $x_1, x_2, \ldots, x_n$. Let $\mathcal{G}_0 := \{\Omega, \varnothing\}$ and for $k \in \{1, \ldots, n\}$, let $\mathcal{G}_k$ be the σ-algebra on Ω that is generated by $(\omega(x_j))_{j \in \{1,\ldots,k\}}$. Furthermore, define the martingale

$$M_k := E_z(X_{T_{L^2}}, A_L, J(L) | \mathcal{G}_k).$$

Note that due to the independence structure of $\mathbb{P}$, taking the conditional expectation with respect to $\mathcal{G}_k$ is nothing else than taking the expectation with re-

spect to the process as well as over all those $\omega(x)$ for which $x \notin \{x_1, \dots, x_k\}$. Thus, $E_{z,\omega}(X_{T_{L^2}}, A_L, J(L)) = E_z(X_{T_{L^2}}, A_L, J(L)|\mathcal{G}_n)(\omega)$ for $\mathbb{P}$-a.a. ω as well as $E_z(X_{T_{L^2}}, A_L, J(L)) = E_z(X_{T_{L^2}}, A_L, J(L)|\mathcal{G}_0)$.

Next, we estimate $\operatorname{ess\,sup}(\|M_k - M_{k-1}\|_1|\mathcal{G}_{k-1})$ similarly to [1], which again is based on ideas from Bolthausen and Sznitman [3]. For $x \in \mathbb{Z}^d$, let

$$B(x) := \{y \in H_{x\cdot e_1 - 1} : \|x - y\|_1 \le R_2(L) + 1\}.$$

Note that if x is visited, then on A_L the first visit to the affine hyperplane $H_{x\cdot e_1 - 1}$ will occur at a point contained in $B(x)$. Therefore,

$$\begin{aligned}
U_k &:= \operatorname{ess\,sup}(\|M_k - M_{k-1}\|_1|\mathcal{G}_{k-1}) \\
&= \operatorname{ess\,sup}\Big(\Big\| E_z\big(X_{T_{L^2}}, A_L, J(L), T_{x_k} < \infty|\mathcal{G}_k\big) \\
&\qquad\qquad - E_z\big(X_{T_{L^2}}, A_L, J(L), T_{x_k} < \infty|\mathcal{G}_{k-1}\big)\Big\|_1|\mathcal{G}_{k-1}\Big) \\
&\le R_2^2(L) P_z\big(T_{x_k} < \infty, A_L, J(L)|\mathcal{G}_{k-1}\big) \\
&\le R_2^2(L) \sum_{y \in B(x_k)\cap S_L} P_z\big(X_{T_{y\cdot e_1}} = y, J(L)|\mathcal{G}_{k-1}\big) \\
&= R_2^2(L) \sum_{y \in B(x_k)\cap S_L} P_{z,\omega}\big(X_{T_{y\cdot e_1}} = y, J(L)\big) \\
&\le R_2^2(L)\Bigg(\sum_{y \in B(x_k)\cap \mathcal{P}(0,L)} P_{z,\omega}\big(T_y < \infty, J(L)\big) \\
&\qquad\qquad + P_{z,\omega}\big(T_{\partial_+\mathcal{P}(0,L)} \ne T_{\partial\mathcal{P}(0,L)}\big)\Bigg).
\end{aligned} \tag{A.39}$$

Here, the first equality follows since

$$E_z\big(X_{T_{L^2}}, A_L, J(L), T_{x_k} = \infty|\mathcal{G}_k\big) - E_z\big(X_{T_{L^2}}, A_L, J(L), T_{x_k} = \infty|\mathcal{G}_{k-1}\big) = 0,$$

which is due to the fact that the restriction to $T_{x_k} = \infty$ makes the inner random variables independent of the realization of $\omega(x_k)$. The first inequality follows since, if the walker hits x_k, then on A_L the site of the subsequent renewal has distance at most $R_2(L)$ to x_k and consequently, using standard coupling arguments, one obtains that the values of

$$E_z\big(X_{T_{L^2}}, A_L, J(L), T_{x_k} < \infty|\mathcal{G}_k\big)$$

as a function in $\omega(x_k)$ (and all other coordinates fixed) lie within distance of $R_2^2(L)$ of each other.

Now for $\omega \in G_L^{(\mathrm{i})} \cap J(L)$, remembering that $|B(x_k)| \le (3R_2(L))^d$ and that every y is in $B(x)$ for at most $(3R_2(L))^d$ different points x, using (A.39) we infer

$$\begin{aligned}
\sum_{k=1}^{n} U_k^2 &\le \sum_{k=1}^{n} R_2^4(L)\Big(\sum_{y\in B(x_k)\cap\mathcal{P}(0,L)} P_{z,\omega}\big(T_y<\infty, J(L)\big) \\
&\qquad + P_{z,\omega}\big(T_{\partial_+\mathcal{P}(0,L)} \ne T_{\partial\mathcal{P}(0,L)}\big)\Big)^2 \\
&\le 2(3R_2(L))^{2d} R_2^4(L) \sum_{k=1}^{n}\Big(\sum_{y\in B(x_k)\cap\mathcal{P}(0,L)} P_{z,\omega}\big(T_y<\infty, J(L)\big)^2 \\
&\qquad + P_{z,\omega}\big(T_{\partial_+\mathcal{P}(0,L)} \ne T_{\partial\mathcal{P}(0,L)}\big)^2\Big) \\
&\le 2(3R_2(L))^{4d} R_2^4(L)\Big(\sum_{y\in \mathcal{P}(0,L)} P_{z,\omega}\big(T_y<\infty, J(L)\big)^2 \\
&\qquad + P_{z,\omega}\big(T_{\partial_+\mathcal{P}(0,L)} \ne T_{\partial\mathcal{P}(0,L)}\big)^2\Big) \\
&\le 4(3R_2(L))^{4d} R_2^4(L) R_3(L) \le R_3^2(L)
\end{aligned}$$

for L large enough, where the fourth inequality is a consequence of (A.36) and part (i) of Proposition 3.4.

Therefore, by Lemma A.3 applied to the $(d-1)$-dimensional martingale M_k,

$$\mathbb{P}\big(U(\omega,z) > R_4(L)/2\big) < 2de^{-R_4^2(L)/(8R_3^2(L))} + \mathbb{P}(J(L)^c) + \mathbb{P}\big(G_L^{(\mathrm{i})c}\big)$$

and the right-hand side is contained in $\mathcal{S}(\mathbb{N})$ as a function in L due to Proposition 3.4(i), Lemma 3.3 and (A.35). Now since the above estimates and hence the last inclusion were uniform in $z \in \tilde{\mathcal{P}}(0,L)$, we infer that (A.37) holds, which in combination with Remark 3.5 finishes the proof. □

A.4. Auxiliary results for the proof of Proposition 3.4(iii). The following lemma is the basis for proving Proposition 3.4(iii).

LEMMA A.9. *Fix $\vartheta \in (\frac{d-1}{d}, 1]$, let C be a constant and denote by $B^\vartheta(L)$ the set of those ω for which for all $M \in \{\lfloor \frac{2}{5}L^2\rfloor, \dots, L^2\}$, all $z \in \tilde{\mathcal{P}}(0,L)$ and all $(d-1)$-dimensional hypercubes Q of side length $\lceil L^\vartheta\rceil$ that are contained in H_M, one has*

$$|P_{z,\omega}(X_{T_M} \in Q) - P_z(X_{T_M} \in Q)| \le C L^{(\vartheta-1)(d-1)}. \tag{A.40}$$

Then for C large enough, $\mathbb{P}(B^\vartheta(L)^c)$ is contained in $\mathcal{S}(\mathbb{N})$ as a function in L.

PROOF. Choose $\vartheta' \in (\frac{d-1}{d}, \vartheta)$, set $U := \lfloor L^{2\vartheta'} \rfloor$, fix $M \in \{\lfloor \frac{2}{5}L^2 \rfloor, \dots, L^2\}$ and with S_L as in the previous proof set $S_L^M := S_L \cap \{x \in \mathbb{Z}^d : x \cdot e_1 \le M\}$. Similarly to the proof of Lemma A.8, we let $x_1, x_2, \dots, x_n$ be a lexicographic ordering of the vertices in S_L^M and denote by $\mathcal{G}_k$ the σ-algebra on Ω generated by $\omega(x_1), \dots, \omega(x_k)$. For $v \in H_{M+U}$, we start with estimating

$$\big|P_z\big(X_{T_{M+U}} = v, A_L, J(L)|\mathcal{G}_n\big) - P_z\big(X_{T_{M+U}} = v, A_L, J(L)\big)\big|,$$

and for this purpose consider the martingale

$$M_k := P_z\big(X_{T_{M+U}} = v, A_L, J(L)|\mathcal{G}_k\big).$$

As in the proof of Lemma A.8, from which we borrow the notation $B(x_k)$, we are going to take advantage of Lemma A.3, whence we will need to bound $\Delta_k := \operatorname{ess\,sup}(|M_k - M_{k-1}||\mathcal{G}_{k-1})$. By part (c) of Lemma A.2, again in combination with standard coupling arguments, we obtain

$$\begin{aligned}
\Delta_k &\le C R_2^2(L) P_z\big(T_{x_k} < \infty, A_L, J(L)|\mathcal{G}_{k-1}\big) \cdot U^{-d/2} \\
&\le C U^{-d/2} R_2^2(L) \sum_{y \in B(x_k) \cap S_L^M} P_{z,\omega}\big(X_{T_{y \cdot e_1}} = y, A_L, J(L)\big) \\
&\le C U^{-d/2} R_2^2(L) \Bigg(\sum_{y \in B(x_k) \cap \mathcal{P}(0,L)} P_{z,\omega}\big(T_y < \infty, J(L)\big) \\
&\qquad + P_{z,\omega}\big(T_{\partial_+ \mathcal{P}(0,L)} \ne T_{\partial \mathcal{P}(0,L)}\big)\Bigg).
\end{aligned}$$

Therefore, for $\omega \in J(L) \cap G_L^{(\mathrm{i})}$, and based on the same calculation as in the proof of Lemma A.8,

$$\operatorname{ess\,sup}\Bigg(\sum_{k=1}^n \Delta_k^2\Bigg) \le R_4(L) U^{-d}. \tag{A.41}$$

Indeed, continuing the previous chain, for $\omega \in J(L) \cap G_L^{(\mathrm{i})}$ we have

$$\begin{aligned}
\sum_{k=1}^n \Delta_k^2 &\le C U^{-d} R_2^4(L) \sum_{k=1}^n \Bigg(\sum_{y \in B(x_k) \cap \mathcal{P}(0,L)} P_{z,\omega}\big(T_y < \infty, J(L)\big) \\
&\qquad + P_{z,\omega}\big(T_{\partial_+ \mathcal{P}(0,L)} \ne T_{\partial \mathcal{P}(0,L)}\big)\Bigg)^2 \\
&\le C U^{-d} R_2^4(L) (3R_2(L))^d \sum_{k=1}^n \Bigg(\sum_{y \in B(x_k) \cap \mathcal{P}(0,L)} P_{z,\omega}\big(T_y < \infty, J(L)\big)^2 \\
&\qquad + P_{z,\omega}\big(T_{\partial_+ \mathcal{P}(0,L)} \ne T_{\partial \mathcal{P}(0,L)}\big)^2\Bigg)
\end{aligned}$$

$$\le CU^{-d}R_2^4(L)(3R_2(L))^{2d}\Big(\sum_{y\in\mathcal{P}(0,L)} P_{z,\omega}\big(T_y<\infty, J(L)\big)^2 + nP_{z,\omega}\big(T_{\partial\mathcal{P}(0,L)}\neq T_{\partial_+\mathcal{P}(0,L)}\big)^2\Big)$$
$$\le R_3^2(L)U^{-d}$$

for L large enough and where to obtain the second line we applied the Cauchy–Schwarz inequality in combination with $|B(x_k)|\le (3R_2(L))^d$. Therefore, using (A.41) and Lemma A.3, for each $v\in H_{M+U}$ we have

$$\mathbb{P}\big(\big|P_z(X_{T_{M+U}}=v, A_L, J(L)|\mathcal{G}_n\big)-P_z\big(X_{T_{M+U}}=v, A_L, J(L)\big)\big|>L^{1-d}/4\big) \le 2e^{-U^{2\eta}/(32R_4(L))}$$

with $\eta:=\frac{d-(d-1)/\vartheta'}{2}>0$ [here we use the assumption $\vartheta'>(d-1)/d$ to guarantee the positivity of η]. We define the subset

$$T(L):=\bigcap_{\substack{M\in\{\lfloor(2/5)L^2\rfloor,\dots,L^2\},\\ v\in H_{M+U},\\ z\in\tilde{\mathcal{P}}(0,L)}}\big\{\big|P_z(X_{T_{M+U}}=v|\mathcal{G}_n)-P_z(X_{T_{M+U}}=v)\big|\le L^{1-d}/2\big\}$$

of Ω. Now for any of these choices of M, v and z, we obtain

$$\begin{aligned}\mathbb{P}\big(\big|P_z(X_{T_{M+U}}=v|\mathcal{G}_n)-P_z(X_{T_{M+U}}=v)\big|>L^{1-d}/2\big)&\le \mathbb{P}\big(\big|P_z\big(X_{T_{M+U}}=v, A_L, J(L)|\mathcal{G}_n\big)\\ &\quad -P_z\big(X_{T_{M+U}}=v, A_L, J(L)\big)\big|>L^{1-d}/4\big)\\ &\quad +\mathbb{P}\big(P_z\big(A_L^c\cup J(L)^c|\mathcal{G}_n\big)>L^{1-d}/8\big)\\ &\quad +\mathbb{P}\big(P_z\big(A_L^c\cup J(L)^c\big)>L^{1-d}/8\big).\end{aligned}$$

Thus, in combination with Lemma 3.3 and Proposition 3.4(i), and since the previous bounds were uniform in the (at most polynomially many) admissible choices of M, v and z,[4] we get that

$$\mathbb{P}(T(\cdot)^c)\in\mathcal{S}(\mathbb{N}). \tag{A.42}$$

Now in order to estimate

$$|P_{z,\omega}(X_{T_M}\in Q)-P_z(X_{T_M}\in Q)|,$$

[4]More precisely, in order to have only polynomially many choices for v, we restrict v to be contained in the union of all admissible hypercubes $Q^{(2)}$ appearing in (A.43).

we denote by $c(Q)$ the centre of the cube Q and set $c'(Q) := c(Q) + \frac{U}{\hat{v}\cdot e_1}\hat{v}$. Furthermore, set

$$Q^{(1)} := \{y \in H_{M+U} : \|y - c'(Q)\|_\infty < (0.9)^{1/(d-1)} L^\vartheta/2\}$$

and

$$Q^{(2)} := \{y \in H_{M+U} : \|y - c'(Q)\|_\infty < (1.1)^{1/(d-1)} L^\vartheta/2\}. \tag{A.43}$$

Then by standard annealed estimates there exists $\varphi \in \mathcal{S}(\mathbb{N})$ such that for all $z \in \tilde{\mathcal{P}}(0, L)$,

$$P_z(X_{T_{M+U}} \in Q^{(1)}) < P_z(X_{T_M} \in Q) + \varphi(L), \tag{A.44}$$

$$P_z(X_{T_{M+U}} \in Q^{(2)}) > P_z(X_{T_M} \in Q) - \varphi(L), \tag{A.45}$$

$$P_z(X_{T_{M+U}} \in Q^{(1)} | \mathcal{G}_n) < P_{z,\omega}(X_{T_M} \in Q) + \varphi(L) \tag{A.46}$$

and

$$P_z(X_{T_{M+U}} \in Q^{(2)} | \mathcal{G}_n) > P_{z,\omega}(X_{T_M} \in Q) - \varphi(L) \tag{A.47}$$

for $\omega \in \mathcal{A}_L$. Indeed, in order to prove equation (A.44) note that

$$\begin{aligned} P_z(X_{T_{M+U}} \in Q^{(1)}) \le{}& P_z(X_{T_M} \in Q, X_{T_{M+U}} \in Q^{(1)}) \\ &+ P_z(X_{T_M} \notin Q, X_{T_{M+U}} \in Q^{(1)}). \end{aligned}$$

By Lemma 3.3 and restricting to A_L, using Azuma's inequality one can show that

$$\sup_{z \in \tilde{\mathcal{P}}(0,L)} P_z(X_{T_M} \notin Q, X_{T_{M+U}} \in Q^{(1)})$$

is contained in $\mathcal{S}(\mathbb{N})$ as a function in L; this then implies (A.44). The remaining inequalities are shown in similar ways.

In order to make use of (A.44) to (A.47), note that for $\omega \in T(L) \cap \mathcal{A}_L$ we get with Lemma A.2(a) that

$$\begin{aligned} &|P_z(X_{T_{M+U}} \in Q^{(1)} | \mathcal{G}_n) - P_z(X_{T_{M+U}} \in Q^{(2)})| \\ &\quad \le |P_z(X_{T_{M+U}} \in Q^{(1)} | \mathcal{G}_n) - P_z(X_{T_{M+U}} \in Q^{(1)})| \\ &\qquad + \Big| \sum_{v \in Q^{(2)} \setminus Q^{(1)}} P_z(X_{T_{M+U}} = v) \Big| \\ &\quad \le |P_z(X_{T_{M+U}} \in Q^{(1)} | \mathcal{G}_n) - P_z(X_{T_{M+U}} \in Q^{(1)})| + |Q^{(2)} \setminus Q^{(1)}| C L^{1-d} \\ &\quad \le C L^{(\vartheta-1)(d-1)} \end{aligned}$$

for some constant C. If $P_{z,\omega}(X_{T_M} \in Q) \le P_z(X_{T_M} \in Q)$, then this estimate in combination with (A.45) and (A.46) yields (A.40). Otherwise, again for $\omega \in$

$T(L) \cap \mathcal{A}_L$ we compute

$$\begin{aligned}
&\big|P_{z,\omega}\big(X_{T_{M+U}} \in Q^{(2)}\big) - P_z\big(X_{T_{M+U}} \in Q^{(1)}\big)\big| \\
&\quad \le \big|P_z\big(X_{T_{M+U}} \in Q^{(2)} | \mathcal{G}_n\big) - P_z\big(X_{T_{M+U}} \in Q^{(2)}\big)\big| + \big|Q^{(2)} \setminus Q^{(1)}\big| C L^{1-d} \\
&\quad \le C L^{(\vartheta-1)(d-1)},
\end{aligned}$$

which in combination with (A.44) and (A.47) again implies (A.40).

Thus, for C large enough, and since the bounds we derived so far were uniform in the admissible choices of M, z and Q, it follows that $T(L) \cap \mathcal{A}_L \subseteq B^{\vartheta}(L)$. Therefore, employing (A.42), we get $\mathbb{P}(B^{\vartheta}(\cdot)^c) \in \mathcal{S}(\mathbb{N})$. □

Departing from Lemma A.9, due to the following result, for a large set of environments we can bound from above the quenched probability of hitting a hyperplane in a hypercube of side length $\lceil L^{\vartheta} \rceil$ for any $\vartheta \in (0, 1]$.

LEMMA A.10. *For $\vartheta \in (0, 1]$ and $h \in \mathbb{N}$, denote by $\overline{B}_h^{\vartheta}(L)$ the set of those ω for which for all $z \in \tilde{\mathcal{P}}(0, L)$, all $M \in \{\lfloor \frac{1}{2} L^2 \rfloor, \dots, L^2\}$ and all $(d-1)$-dimensional hypercubes Q of side length $\lceil L^{\vartheta} \rceil$ which are contained in H_M,*

$$P_{z,\omega}(X_{T_M} \in Q) \le R_h(L) L^{(\vartheta-1)(d-1)}. \tag{A.48}$$

Then there exists $h(\vartheta) \in \mathbb{N}$ such that $\mathbb{P}(\overline{B}_{h(\vartheta)}^{\vartheta}(L)^c)$ is contained in $\mathcal{S}(\mathbb{N})$ as a function in L.

PROOF. We prove the lemma by descending induction on ϑ. Lemma A.9 in combination with Lemma A.2(a) implies that $\mathbb{P}(\overline{B}_1^{\vartheta}(\cdot)^c) \in \mathcal{S}(\mathbb{N})$ for each $\vartheta \in (\frac{d-1}{d}, 1]$. For the induction step, assume that the statement of the lemma holds for some ϑ' and choose ϑ such that $\rho := \frac{\vartheta}{\vartheta'} \in (\frac{d-1}{d}, 1]$. Set $h' := h(\vartheta')$. For $z \in \mathbb{Z}^d$, define the canonical shift on $\mathbb{Z}^d$ via $\sigma_z : \mathbb{Z}^d \ni x \mapsto x + z \in \mathbb{Z}^d$ and let

$$G := B^{\rho}(L) \cap G^*_{\lfloor L^{\rho} \rfloor} \cap \bigcap_{z \in \mathcal{P}(0,L)} \sigma_z(\overline{B}_{h'}^{\vartheta'}(\lfloor L^{\rho} \rfloor)),$$

where $B^{\rho}(L)$ as in Lemma A.9 and

$$G^*_{\lfloor L^{\rho} \rfloor} := \bigcap_{x \in \mathcal{P}(0,L)} \bigcap_{y \in \tilde{\mathcal{P}}(x, \lfloor L^{\rho} \rfloor)} \big\{P_{y,\omega}\big(T_{\partial \mathcal{P}(x, \lfloor L^{\rho} \rfloor)} \ne T_{\partial_+ \mathcal{P}(x, \lfloor L^{\rho} \rfloor)}\big) < e^{-R_2(\lfloor L^{\rho} \rfloor)^{\gamma}}\big\}.$$

The translation invariance of $\mathbb{P}$ implies that

$$\mathbb{P}(\sigma_z(\overline{B}_{h'}^{\vartheta'}(\lfloor L^{\rho} \rfloor))) = \mathbb{P}(\overline{B}_{h'}^{\vartheta'}(\lfloor L^{\rho} \rfloor)),$$

and therefore, as a consequence of Proposition 3.4(i), $\mathbb{P}(G^c)$ is contained in $\mathcal{S}(\mathbb{N})$ as a function in L. Thus, it is sufficient to show that for some h and all L large enough, we have that $G \subseteq \overline{B}_h^{\vartheta}(L)$. To this end we fix $\omega \in G$, $z \in \tilde{\mathcal{P}}(0, L)$, $M \in$

$\{\lfloor \frac{1}{2}L^2 \rfloor, \dots, L^2\}$ and a cube Q of side length $\lceil L^\vartheta \rceil$ in $\mathcal{P}(0, L) \cap H_M$. Let $c(Q)$ be the centre of Q and x' be an element of $\mathbb{Z}^d$ closest to $c(Q) - \frac{\lfloor L^\rho \rfloor^2}{\hat{v}\cdot e_1}\hat{v}$. Due to the strong Markov property and the fact that $\omega \in G^*_{\lfloor L^\rho \rfloor}$,

$$\left| P_{z,\omega}(X_{T_M} \in Q) - \sum_{v \in H_{M-\lfloor L^\rho \rfloor^2} \cap \mathcal{P}(x', \lfloor L^\rho \rfloor)} P_{z,\omega}(X_{T_{M-\lfloor L^\rho \rfloor^2}} = v) \times P_{v,\omega}(X_{T_M} \in Q) \right| \tag{A.49}$$

is contained in $\mathcal{S}(\mathbb{N})$ as a function in L. To estimate the second factor of the sum, observe that since

$$\omega \in \bigcap_{z \in \mathcal{P}(0,L)} \sigma_z(\overline{B}^{\vartheta'}_{h'}(\lfloor L^\rho \rfloor)),$$

we get that for every $v \in H_{M-\lfloor L^\rho \rfloor^2}$,

$$P_{v,\omega}(X_{T_M} \in Q) < R_{h'}(L) L^{\rho(\vartheta'-1)(d-1)} = R_{h'}(L) L^{(\vartheta-\rho)(d-1)}. \tag{A.50}$$

With respect to the first factor of the sum, for L large enough, $H_{M-\lfloor L^\rho \rfloor^2} \cap \mathcal{P}(x', \lfloor L^\rho \rfloor)$ is the union of less than $R_7(L)$ cubes of side length $\lfloor L^\rho \rfloor$. Since $\omega \in B^\rho(L)$, we deduce that for every cube Q' of side length $\lfloor L^\rho \rfloor$ that is contained in $H_{M-\lfloor L^\rho \rfloor^2} \cap \mathcal{P}(0, L)$, one has

$$P_{z,\omega}(X_{T_{M-\lfloor L^\rho \rfloor^2}} \in Q') < R_2(L) L^{(\rho-1)(d-1)} \tag{A.51}$$

for L large enough. Combining (A.49), (A.50) and (A.51), we infer that

$$\begin{aligned} P_{z,\omega}(X_{T_M} \in Q) &\le R_7(L) R_{h'}(L) L^{(\vartheta-\rho)(d-1)} \cdot R_2(L) L^{(\rho-1)(d-1)} \\ &\le R_h(L) L^{(\vartheta-1)(d-1)} \end{aligned}$$

for $h = \max\{7, h'\} + 1$ and L large enough.

Noting that the above estimates are uniform in the (at most polynomially many) admissible choices of z, M and Q, this finishes the proof. □

The next result employs the previous lemmas to yield bounds on the difference of certain annealed and semi-annealed hitting probabilities.

LEMMA A.11. *Let $\mathcal{G}$ be the σ-algebra in Ω generated by the functions $\{\Omega \ni \omega \mapsto \omega(z) : z \cdot e_1 \le L^2\}$. Let $\eta \in (0, \frac{6}{d-1} \wedge 1)$, $U := \lfloor L^\eta \rfloor$ and denote by $B(L, \eta)$ the set of those ω for which for all $z \in \tilde{\mathcal{P}}(0, L)$ and all $v \in H_{L^2+U}$, one has*

$$|P_z(X_{T_{L^2+U}} = v | \mathcal{G}) - P_z(X_{T_{L^2+U}} = v)| \le L^{1-d} U^{(1-d)/3}.$$

Then $\mathbb{P}(B(L, \eta)^c)$ is contained in $\mathcal{S}(\mathbb{N})$ as a function in L.

PROOF. Let $v \in H_{L^2+U}$ and let $\vartheta > 0$ be such that $\vartheta < \frac{1}{12}\eta$. Define K_L to be the natural number such that $2^{-K_L} L^2 > U \ge 2^{-K_L - 1} L^2$, and for $k \in \{1, \dots, K_L -$

1} we set

$$\mathcal{P}^{(k)} := \mathcal{P}(0, L) \cap \{x : L^2 - 2^{-k}L^2 < x \cdot e_1 \le L^2 - 2^{-k-1}L^2\}.$$

In addition, we take

$$\mathcal{P}^{(K_L)} := \mathcal{P}(0, L) \cap \{x : L^2 - 2^{-K_L}L^2 < x \cdot e_1 \le L^2\},$$
$$\mathcal{P}^{(0)} := \mathcal{P}(0, L) \cap \{x : x \cdot e_1 \le L^2/2\}$$

and

$$F(v) := \{x \in \mathcal{P}(0, L) : \|x - u(v, x)\|_1 \le R_7(L)\|(v - x) \cdot e_1\|_1^{1/2}\},$$

where $u(v, x) := v + \frac{(x-v)\cdot e_1}{\hat{v}\cdot e_1}\hat{v}$. Then for $k \in \{0, \dots, K_L\}$ we define

$$\mathcal{P}^{(k)}(v) := \mathcal{P}^{(k)} \cap F(v)$$

and

$$\hat{\mathcal{P}}^{(k)}(v) := \big\{y \in \mathbb{Z}^d : \exists x \in \mathcal{P}^{(k)}(v) \text{ such that } \|x - y\|_1 < R_2(L)\big\}.$$

See Figure 6 for an illustration.

Similarly to the previous, we use a lexicographic enumeration $x_1, x_2, \dots, x_n$ of

$$\hat{\mathcal{P}} := \bigcup_{k=0}^{K_L} \hat{\mathcal{P}}^{(k)} \tag{A.52}$$

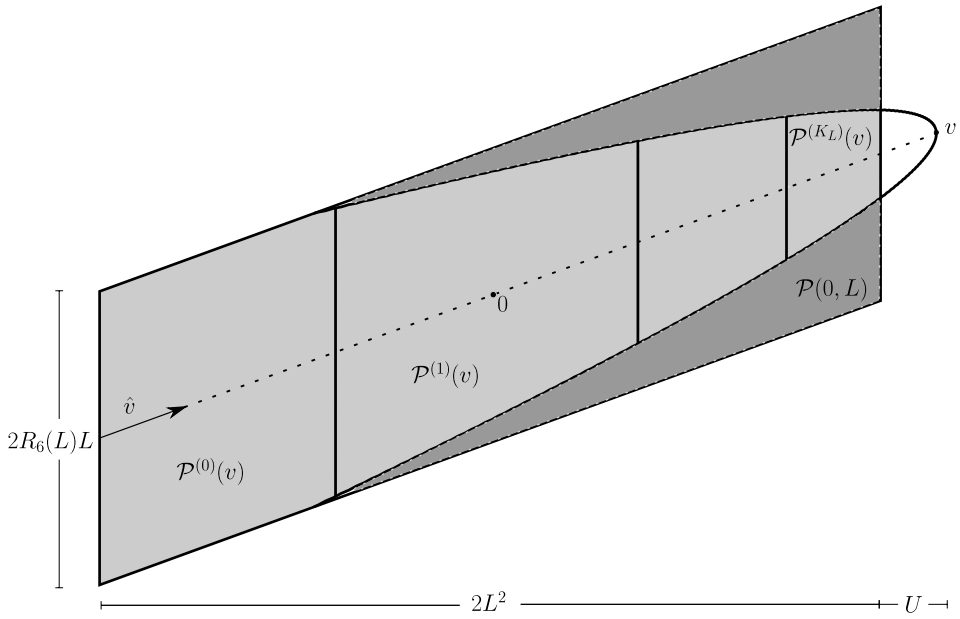

FIG. 6. *The sets $\mathcal{P}^{(k)}(v)$ contained in $\mathcal{P}(0, L)$.*

and the corresponding filtration $\{\mathcal{G}_i\}_{i\in\{0,\dots,n\}}$. We consider the martingale $M_i := P_z(X_{T_{L^2+U}} = v, A_L, J(L)|\mathcal{G}_i)$. Again, in order to use Lemma A.3, we need to bound $U_i := \operatorname{ess\,sup}(|M_i - M_{i-1}||\mathcal{G}_{i-1})$. With the same reasoning as in the proof of Lemma A.9 and with Lemma A.2(c), we obtain for i such that $x_i \in \mathcal{P}^{(k)}(v)$:

$$U_i \le C R_2(L) P_z\big(T_{x_i} < \infty, A_L, J(L)|\mathcal{G}_{i-1}\big) \cdot L^{-d} 2^{(k+1)(d/2)}.$$

To obtain a useful upper bound for U_i with $k \in \{0,\dots,K_L\}$ and $\omega \in \overline{B}^{\vartheta}_{h(\vartheta)} \cap J(L)$, we will estimate

$$V_\omega(k) = \sum_{x\in\mathcal{P}^{(k)}(v)} P_{z,\omega}\big(T_x < \infty, A_L, J(L)\big)^2.$$

Using (A.36), we get for $\omega \in J(L)$ that

$$V_\omega(0) \le R_3(L).$$

Now choose $h(\vartheta) \ge 8$ such that the implication of Lemma A.10 holds true; then for $k > 0$ as well as $B(x)$ as in the proof of Lemma A.9,

$$\begin{aligned}
V_\omega(k) &= \sum_{x\in\mathcal{P}^{(k)}(v)} P_{z,\omega}(T_x < \infty, A_L)^2 \\
&\le \sum_{x\in\mathcal{P}^{(k)}(v)} \Bigg(\sum_{y\in B(x)} P_{z,\omega}(X_{T_{y\cdot e_1}} = y)\Bigg)^2 \\
&\le (3R_2(L))^{d-1} \sum_{y\in\hat{\mathcal{P}}^{(k)}(v)} P_{z,\omega}(X_{T_{y\cdot e_1}} = y)^2 \qquad \text{(A.53)} \\
&\le (3R_2(L))^{d-1} \sum_{y\in\hat{\mathcal{P}}^{(k)}(v)} R^2_{h(\vartheta)}(L) L^{2(\vartheta-1)(d-1)} \\
&\le R_{h(\vartheta)+1}(L) L^{2((d+1)/2+(\vartheta-1)(d-1))} 2^{-k\lfloor (d+1)/2\rfloor}
\end{aligned}$$

for L large enough, where inequality (A.53) follows from the fact that $\omega \in \overline{B}^{\vartheta}_{h(\vartheta)}(L)$.

Therefore, we get that for $\omega \in J(L) \cap \overline{B}^{\vartheta}_{h(\vartheta)}$ we have

$$\begin{aligned}
\operatorname{ess\,sup}\Bigg(\sum_{i=1}^n U_i^2\Bigg) &\le C R_2^2(L) \sum_{k=0}^{K_L} V_\omega(k) L^{-2d} 2^{kd} \\
&\le C R_{h(\vartheta)+1}(L) L^{-2d} \\
&\quad + C R_{h+1}(L) L^{2((d+1)/2+(\vartheta-1)(d-1))-2d} \sum_{k=1}^{K_L} 2^{kd-k(d+1)/2} \\
&\le C R_{h(\vartheta)+1}(L)\big(L^{-2d} + L^{3-3d+2(d-1)\vartheta} 2^{K_L(d-1)/2}\big)
\end{aligned}$$

$$\leq C R_{h(\vartheta)+1}(L)\big(L^{-2d} + L^{2-2d+2(d-1)\vartheta} U^{-(d-1)/2}\big)$$
$$\leq L^{2-2d} U^{(1-d)/3}/2$$

for L large enough and where the penultimate inequality follows from the definition of K_L, while the last inequality is due to the choice of ϑ. Thus, Lemma A.3 yields that

$$\mathbb{P}\big(\big|P_z(X_{T_{L^2+U}} = v, A_L|\mathcal{G}_n) - P_z(X_{T_{L^2+U}} = v, A_L)\big| > L^{1-d}U^{(1-d)/3}/2\big) \tag{A.54}$$

is contained in $\mathcal{S}(\mathbb{N})$ as a function in L, uniformly in the admissible choices of z and v.

Observe furthermore that with $\hat{\mathcal{P}}$ defined in (A.52), for example, by Azuma's inequality,

$$P_z(X_{T_{L^2+U}} = v, A_L, T_{\partial\hat{\mathcal{P}}} < T_v)$$

is contained in $\mathcal{S}(\mathbb{N})$, and thus, due to Markov's inequality, so is

$$\mathbb{P}\big(P_z(X_{T_{L^2+U}} = v, A_L, T_{\partial\hat{\mathcal{P}}} < T_v|\mathcal{G}) \geq L^{1-d}U^{(1-d)/3}/2\big).$$

In combination with (A.54) and the fact that

$$\big\{\omega : \big|P_z(X_{T_{L^2+U}} = v, A_L|\mathcal{G}) - P_z(X_{T_{L^2+U}} = v, A_L|\mathcal{G}_n)\big| \geq L^{1-d}U^{(1-d)/3}/2\big\}$$
$$\subset \big\{\omega : P_z(X_{T_{L^2+U}} = v, A_L, T_{\partial\hat{\mathcal{P}}} < T_v|\mathcal{G}) \geq L^{1-d}U^{(1-d)/3}/2\big\},$$

this supplies us with the fact that

$$\mathbb{P}\big(\big|P_z(X_{T_{L^2+U}} = v, A_L|\mathcal{G}) - P_z(X_{T_{L^2+U}} = v, A_L)\big| > L^{1-d}U^{(1-d)/3}\big)$$

is contained in $\mathcal{S}(\mathbb{N})$ also.

A union bound in combination with Lemma 3.3 and Lemma 3.10 completes the proof of the lemma. □

LEMMA A.12. *For any $\vartheta \in (0, \frac{6}{d-1} \wedge 1)$ denote by $D^{\vartheta}(L)$ the set of those ω for which for all $z \in \tilde{\mathcal{P}}(0, L)$ and all $(d-1)$-dimensional hypercubes Q of side length $\lceil L^{\vartheta} \rceil$ that are contained in $\partial_+\mathcal{P}(0, L)$,*

$$\big|P_{z,\omega}\big(X_{T_{\partial\mathcal{P}(0,L)}} \in Q|T_{\partial\mathcal{P}(0,L)} = T_{\partial_+\mathcal{P}(0,L)}\big)$$
$$- P_z\big(X_{T_{\partial\mathcal{P}(0,L)}} \in Q|T_{\partial\mathcal{P}(0,L)} = T_{\partial_+\mathcal{P}(0,L)}\big)\big| \tag{A.55}$$
$$\leq L^{(\vartheta-1)(d-1)-\vartheta(d-1)/(d+1)}.$$

Then $\mathbb{P}(D^{\vartheta}(L)^c)$ is contained in $\mathcal{S}(\mathbb{N})$ as a function in L.

PROOF. Choose $\vartheta' \in (\frac{3}{4}\vartheta, \vartheta)$ and $U := \lfloor L^{4\vartheta'/(d+1)} \rfloor$. Then by Lemma A.11 and Proposition 3.4(i), we know that $\mathbb{P}(B(\cdot, \frac{4\vartheta'}{d+1})^c \cup G_{\cdot}^{(\mathrm{i})c}) \in \mathcal{S}(\mathbb{N})$ whence it is

sufficient to show that $B(L, \frac{4\vartheta'}{d+1}) \cap G_L^{(\mathrm{i})} \subset D^{\vartheta}(L)$; this we will do similarly to the last step of the proof of Lemma A.9. We denote by $c(Q)$ one of those elements of $\mathbb{Z}^d$ closest to the centre of Q and let $x' \in H_{L^2+U}$ be one of the lattice points closest to $c(Q) + \frac{U}{\hat{v}\cdot e_1}\hat{v}$. Furthermore, let $Q^{(1)}$ and $Q^{(2)}$ be $(d-1)$-dimensional hypercubes that are contained in H_{L^2+U} and are centred in x', such that the side length of $Q^{(1)}$ is $\lfloor L^{\vartheta} - R_6(L)\sqrt{U} \rfloor$ and the side length of $Q^{(2)}$ is $\lceil L^{\vartheta} + R_6(L)\sqrt{U} \rceil$. Then due to Lemma A.11, on $B(L, \frac{4\vartheta'}{d+1})$ for $i \in \{1, 2\}$,

$$\big|P_z\big(X_{T_{L^2+U}} \in Q^{(i)} | \mathcal{G}\big) - P_z\big(X_{T_{L^2+U}} \in Q^{(i)}\big)\big| \le \big|Q^{(i)}\big| L^{1-d} U^{(1-d)/3} \tag{A.56}$$

for all corresponding z and Q. Now similarly to (A.44) to (A.47), there exists $\varphi \in \mathcal{S}(\mathbb{N})$ such that for all such z and Q,

$$\begin{aligned} &P_z\big(X_{T_{L^2+U}} \in Q^{(1)}\big) - \varphi(L) \\ &\qquad < P_z(X_{T_{L^2}} \in Q) < P_z\big(X_{T_{L^2+U}} \in Q^{(2)}\big) + \varphi(L) \end{aligned} \tag{A.57}$$

as well as

$$\begin{aligned} &P_z\big(X_{T_{L^2+U}} \in Q^{(1)} | \mathcal{G}\big) - \varphi(L) \\ &\qquad < P_{z,\omega}(X_{T_{L^2}} \in Q) < P_z\big(X_{T_{L^2+U}} \in Q^{(2)} | \mathcal{G}\big) + \varphi(L). \end{aligned} \tag{A.58}$$

Proposition 3.4(i) and Lemma 3.10 imply that for $\omega \in G_L^{(\mathrm{i})}$,

$$\big|P_{z,\omega}\big(X_{T_{\partial\mathcal{P}(0,L)}} \in Q | T_{\partial\mathcal{P}(0,L)} = T_{\partial_+\mathcal{P}(0,L)}\big) - P_{z,\omega}(X_{T_{L^2}} \in Q)\big|$$

and

$$\big|P_z\big(X_{T_{\partial\mathcal{P}(0,L)}} \in Q | T_{\partial\mathcal{P}(0,L)} = T_{\partial_+\mathcal{P}(0,L)}\big) - P_z(X_{T_{L^2}} \in Q)\big|$$

are both contained in $\mathcal{S}(\mathbb{N})$ as functions in L. Therefore, for $\omega \in B(L, \frac{4\vartheta'}{d+1}) \cap G_L^{(\mathrm{i})}$, using (A.56) to (A.58) and as a consequence of Lemma A.2(a),

$$\begin{aligned} &\big|P_{z,\omega}\big(X_{T_{\partial\mathcal{P}(0,L)}} \in Q | T_{\partial\mathcal{P}(0,L)} = T_{\partial_+\mathcal{P}(0,L)}\big) \\ &\quad - P_z\big(X_{T_{\partial\mathcal{P}(0,L)}} \in Q | T_{\partial\mathcal{P}(0,L)} = T_{\partial_+\mathcal{P}(0,L)}\big)\big| \\ &\qquad \le |P_{z,\omega}(X_{T_{L^2}} \in Q) - P_z(X_{T_{L^2}} \in Q)| + \varphi(L) \\ &\qquad \le \big|Q^{(2)}\big| L^{1-d} U^{(1-d)/3} + C\big(\big|Q^{(2)}\big| - \big|Q^{(1)}\big|\big) L^{1-d} + \varphi(L) \\ &\qquad \le C\big(L^{(\vartheta-1)(d-1)} U^{(1-d)/3} + R_6(L)\sqrt{U}\lceil L^{\vartheta}\rceil^{d-2} L^{1-d}\big) \\ &\qquad \le L^{(\vartheta-1)(d-1) - \vartheta(d-1)/(d+1)} \end{aligned}$$

for L large enough and some $\varphi \in \mathcal{S}(\mathbb{N})$. Here, we used that $U = \lfloor L^{4\vartheta'/(d+1)} \rfloor$ and $\vartheta' \in (\frac{3}{4}\vartheta, \vartheta)$ to obtain the last line. □

A.5. Proof of Proposition 3.4(iii). Denote by $D^{\vartheta}(L)$ the set of all ω such that

$$\max_{z\in\tilde{\mathcal{P}}(0,L)}\max_{Q}\big|P_{z,\omega}\big(X_{T_{\partial\mathcal{P}(0,L)}}\in Q\,|\,T_{\partial\mathcal{P}(0,L)}=T_{\partial_+\mathcal{P}(0,L)}\big) - P_z\big(X_{T_{\partial\mathcal{P}(0,L)}}\in Q\,|\,T_{\partial\mathcal{P}(0,L)}=T_{\partial_+\mathcal{P}(0,L)}\big)\big| < L^{(\vartheta-1)(d-1)-\vartheta(d-1)/(d+1)}$$

holds, where the maximum in Q is taken over all $(d-1)$-dimensional hypercubes $Q\subset\partial_+\mathcal{P}(0,L)$ of side length $\lceil L^{\vartheta}\rceil$. Then for $\vartheta\in(0,\frac{6}{d-1}\wedge 1)$, Lemma A.12 is applicable and yields that $\mathbb{P}(D^{\vartheta}(L)^c)$ is contained in $\mathcal{S}(\mathbb{N})$ as a function in L. In combination with Remark 3.5, this finishes the proof.

A.6. Further auxiliary results. The principal purpose of this subsection is to prove Lemma 3.18 that has been employed in step (B) in the construction of the auxiliary random walk on page 490.

We start with proving some further auxiliary results, parts of which have been stated and employed above already.

PROOF OF LEMMA 3.14. We observe that due to Lemma 3.10, it is sufficient to establish (3.25). With $\hat{v}_L$ as defined in (3.18), we obtain

$$\begin{aligned}&\Big\|E_xX_{T_{\partial\mathcal{P}(0,L)}}-x-\frac{L^2-x\cdot e_1}{\hat{v}\cdot e_1}\hat{v}\Big\|_1\\ &\le\Big\|E_xX_{T_{\partial\mathcal{P}(0,L)}}-x-\frac{L^2-x\cdot e_1}{\hat{v}_L\cdot e_1}\hat{v}_L\Big\|_1\\ &\quad+\Big\|\frac{L^2-x\cdot e_1}{\hat{v}_L\cdot e_1}\hat{v}_L-\frac{L^2-x\cdot e_1}{\hat{v}\cdot e_1}\hat{v}\Big\|_1.\end{aligned}\tag{A.59}$$

To estimate the first summand on the right-hand side of (A.59), note that for $H:=\inf\{n\in\mathbb{N}:\sum_{j=1}^n(X_{\tau_j}-X_{\tau_{j-1}})\cdot e_1\ge L^2\}$, we can infer from Lemma 3.10 and Lemma 3.3 that

$$\|E_xX_{T_{\partial\mathcal{P}(0,L)}}-E_x(X_{\tau_H},A_L)\|_1\le 2R_2(L)\tag{A.60}$$

for L large enough. Now $(\sum_{j=1}^n X_{\tau_j}-X_{\tau_{j-1}}-E_x(X_{\tau_j}-X_{\tau_{j-1}}|A_L))_{n\in\{1,\dots,2L^2\}}$ is a zero-mean martingale with respect to $P_x(\cdot|A_L)$, whence the optimal stopping theorem implies

$$\begin{aligned}E_x(X_{\tau_H}-x|A_L)&=\big(E_x(H|A_L)-1\big)\cdot E_x(X_{\tau_2}-X_{\tau_1}|A_L)\\&\quad+E_x(X_{\tau_1}-x|A_L).\end{aligned}\tag{A.61}$$

But as a consequence of the conditioning on A_L, we have

$$\big\|E_x(X_{\tau_H}-x|A_L)\cdot e_1-(L^2-x\cdot e_1)\big\|_1\le R_2(L).\tag{A.62}$$

Since furthermore

$$\|E_x(X_{\tau_2} - X_{\tau_1}|A_L) - E_x(X_{\tau_1} - x|A_L)\|_1 \le 2R_2(L) \tag{A.63}$$

using (A.61) to (A.63), we get

$$\left\| E_x(X_{\tau_H} - x|A_L) - \frac{L^2 - x\cdot e_1}{\hat{v}_L \cdot e_1}\hat{v}_L \right\|_1 \le 3R_2(L).$$

Combining this with (A.60) we obtain that the first summand on the right-hand side of (A.59) is bounded from above by $5R_2(L)$.

Furthermore, the second summand on the right-hand side of (A.59) is contained in $\mathcal{S}(\mathbb{N})$ as a function in L due to Lemma 3.9. This finishes the proof. □

In the following, we will sometimes consider distributions $\mu_{0,0}^{\sqrt{j}L}$ for $j \in \mathbb{N}$, and in particular, $\sqrt{j}L$ is not necessarily a natural number anymore. However, as one may check, this does not lead to any complications.

CLAIM A.13. *For $j \in \{1, \dots, \lfloor L^\chi \rfloor^2\}$, let U be distributed according to the convolution $\mu_{0,0}^{L} * \mu_{0,0}^{\sqrt{j-1}L}$. Then U can be represented as $U = \hat{U} + U'$ such that $\hat{U} \sim \mu_{0,0}^{\sqrt{j}L}$ and*

$$P\big(\|U'\|_1 > 2R_2(L)\big) \le Ce^{-C^{-1}R_2(L)^\gamma}$$

for some constant C independent of j and L.

PROOF. Since we assume all appearing probability spaces to be large enough, it is sufficient to construct U, $\hat{U}$ and U' as desired. First, observe that for

$$A_{k,N} := \big\{X^{*(n)} < R_2(N)\ \forall n \in \{1, \dots, k\}\big\},$$

the same reasoning as in the proof of Lemma 3.3 yields that

$$P_0\big(A^c_{(\lfloor L^\chi \rfloor L)^2, L}\big) \le C\exp\{-C^{-1}R_2(L)^\gamma\}.$$

This in combination with Azuma's inequality, Lemma 3.10 and Lemma 3.14, yields that for L large enough we have

$$\begin{aligned} &P_0\big(T_{\partial\mathcal{P}(0,\sqrt{j-1}L)} \ne T_{\partial_+\mathcal{P}(0,\sqrt{j-1}L)}\big) \\ &\quad\le P_0\big(A^c_{(\lfloor L^\chi \rfloor L)^2, L}\big) + 2d(\lfloor L^\chi \rfloor L)^2 \exp\left\{-\frac{R_6(L)^2}{4R_2(L)^2}\right\} \\ &\quad\le C\exp\{-C^{-1}R_2(L)^\gamma\}. \end{aligned}$$

Now for $l \in \mathbb{N}$, let $n(l)$ be the unique natural number such that $\tau_{n(l)-1} < T_l \le \tau_{n(l)}$. Then due to the above, in combination with Lemma 3.3,

$$P_0\big(\|X_{\tau_{n((j-1)L^2)}} - X_{T_{\partial\mathcal{P}(0,\sqrt{j-1}L)}}\|_1 \ge R_2(L)\big) \le C\exp\{-C^{-1}R_2(L)^\gamma\}.$$

Now let Z denote a RWRE coupled to X in such a way that $Z_0 = X_{T_{\partial_+\mathcal{P}(0,\sqrt{j-1}L)}}$ and

$$Z_{\tau_1^Z+\cdot} = X_{\tau_{n((j-1)L^2)}+\cdot} - X_{\tau_{n((j-1)L^2)}},$$

whereas between times 0 and τ_1^Z it evolves independently of X. Then

$$\begin{aligned} &P_0\Big(\{T_{\partial\mathcal{P}(0,\sqrt{j}L)} \neq T_{\partial_+\mathcal{P}(0,\sqrt{j}L)}\} \\ &\qquad \cup \{T_{\partial\mathcal{P}(0,\sqrt{j-1}L)} \neq T_{\partial_+\mathcal{P}(0,\sqrt{j-1}L)}\} \cup \{T^Z_{\partial\mathcal{P}(Z_0,L)} \neq T^Z_{\partial_+\mathcal{P}(Z_0,L)}\} \\ &\qquad\qquad \cup \Big\{\max_{0\le n\le\tau_1^Z} \|Z_n - Z_0\|_1 \ge R_2(L)\Big\} \cup A^c_{(\lfloor L^\chi\rfloor L)^2,L}\Big) \\ &\qquad \le C\exp\{-C^{-1}R_2(L)^\gamma\} \end{aligned} \tag{A.64}$$

for C large enough and all L; restricted to the complement of the event on the left-hand side of (A.64),

$$\|X_{T_{\partial_+\mathcal{P}(0,\sqrt{j-1}L)}} + (Z_{T^Z_{\partial_+\mathcal{P}(Z_0,L)}} - Z_0) - X_{T_{\partial_+\mathcal{P}(0,\sqrt{j}L)}}\|_1 \le 2R_2(L). \tag{A.65}$$

Furthermore, with respect to

$$P_0\big(\cdot|T_{\partial\mathcal{P}(0,\sqrt{j-1}L)} = T_{\partial_+\mathcal{P}(0,\sqrt{j-1}L)}, T^Z_{\partial\mathcal{P}(Z_0,L)} = T^Z_{\partial_+\mathcal{P}(Z_0,L)}\big),$$

the variable

$$U := X_{T_{\partial_+\mathcal{P}(0,\sqrt{j-1}L)}} + Z_{T^Z_{\partial_+\mathcal{P}(Z_0,L)}} - Z_0$$

is distributed according to $\mu_{0,0}^{\sqrt{j-1}L} * \mu_{0,0}^L$, while with respect to

$$P_0\big(\cdot|T_{\partial\mathcal{P}(0,\sqrt{j}L)} = T_{\partial_+\mathcal{P}(0,\sqrt{j}L)}\big),$$

the variable $\hat{U} := X_{T_{\partial\mathcal{P}(0,\sqrt{j}L)}}$ is distributed as $\mu_{0,0}^{\sqrt{j}L}$. Therefore, setting $U' := U - \hat{U}$, in combination with (A.64) and (A.65) we deduce the desired result. □

The following lemma is essentially a discrete second-order Taylor expansion.

LEMMA A.14. *Let μ be a finite signed measure on $\mathbb{Z}^d$ and let $f : \mathbb{Z}^d \to \mathbb{R}$. Choose $m, k, J, N \in \mathbb{N}$ and $\varrho \in \mathbb{Z}^d$ such that*:

(a) *for every $x, y \in \mathbb{Z}^d$ such that $\|x - y\|_1 = 1$, we have $|f(x) - f(y)| < m$;*

(b) *for every $x, y, z, w \in \mathbb{Z}^d$ and $i, j \in \{1, \dots, d\}$ such that $x - y = z - w = e_i$ and $x - z = y - w = e_j$, we have that $|f(x) + f(w) - f(y) - f(z)| < k$ (note that if $i = j$ then $y = z$ and this is the discrete second derivative, while if $i \neq j$ it is a discrete mixed second derivative)*;

(c) $\sum_x \mu(x) = 0$;

(d) $\|\sum_x x\mu(x)\|_1 \le N$;
(e) $\sum_x \|x-\varrho\|_1^2 \cdot |\mu(x)| < J$.

Then

$$\left|\sum_x f(x)\mu(x)\right| \le mN + kJ.$$

PROOF. From (c), we infer that $\sum_x f(x)\mu(x) = \sum_x (f(x)+c)\mu(x)$ for every $c \in \mathbb{R}$. Therefore, without loss of generality, we may assume that $f(\varrho) = 0$. Let $g : \mathbb{Z}^d \to \mathbb{R}$ be the affine function characterized by

$$g(\varrho) = f(\varrho) = 0 \quad \text{and} \quad g(\varrho + e_i) = f(\varrho + e_i) \qquad \forall i \in \{1, \dots, d\}. \tag{A.66}$$

Then for any $x \in \mathbb{Z}^d$,

$$|f(x) - g(x)| < k\|x-\varrho\|_1^2. \tag{A.67}$$

In fact, setting $h := f - g$ we get for $B(x,\varrho) := \{y \in \mathbb{Z}^d : x_i \wedge \varrho_i \le y_i \le x_i \vee \varrho_i \ \forall i \in \{1,\dots,d\}\}$ that

$$|f(x) - g(x)| \le |h(\varrho)| + \max_{\substack{i\in\{1,\dots,d\},\\ y\in B(x,\varrho)}} \left|\frac{\partial}{\partial e_i} h(y)\right| \cdot \|x-\varrho\|_1, \tag{A.68}$$

where $\frac{\partial}{\partial e_i} h(y) := h(y+e_i) - h(y)$.

In addition, for $\frac{\partial^2}{\partial e_j\, \partial e_i} h(y) := h(y+e_i) - h(y) - (h(y+e_i+e_j) - h(y+e_j))$ we get for $y \in B(x,\varrho)$ that

$$\left|\frac{\partial}{\partial e_i} h(y)\right| \le \left|\frac{\partial}{\partial e_i} h(\varrho)\right| + \max_{\substack{j\in\{1,\dots,d\},\\ z\in B(x,\varrho)}} \left|\frac{\partial^2}{\partial e_j\, \partial e_i} h(z)\right| \cdot \|y-\varrho\|_1. \tag{A.69}$$

Noting that $h(\varrho) = \frac{\partial}{\partial e_i} h(\varrho) = 0$ as well as $\frac{\partial^2}{\partial e_j\, \partial e_i} h = \frac{\partial^2}{\partial e_j\, \partial e_i} f$, and plugging (A.69) into (A.68), (b) yields (A.67).

Now (e) in combination with (A.67) results in

$$\left|\sum_x f(x)\mu(x) - \sum_x g(x)\mu(x)\right| \le \sum_x |f(x) - g(x)| \cdot |\mu(x)| \le kJ.$$

In addition, since g is affine, $g - g(0)$ is linear and hence (A.66) in combination with (a) and (d) yields

$$\left|\sum_x g(x)\mu(x)\right| = \left|g\left(\sum_x x\mu(x)\right) - g(0) + \sum_x g(0)\mu(x)\right| \le mN.$$

Due to the triangle inequality, these two estimates imply the statement of the lemma. □

PROOF OF LEMMA 3.18. We will construct a coupling that establishes the desired closeness. For each $k \in \{1, \dots, n\}$, conditioned on $\Delta_1, \dots, \Delta_{k-1}$, the distribution of Δ_k is (λ, K)-close to $\mu_{0,0}^L$ by assumption, whence a coupling as defined in Definition 3.7 exists. As mentioned in Remark 3.8, the coupling can be constructed on the (possibly extended) probability space the variables Δ_k are defined on, with Δ_k playing the role of Z_2 of that definition. We will assume such couplings to be given. Thus, for each such k we still denote the variable corresponding to Z_2 in Definition 3.7 by Δ_k; the variable corresponding to Z_0 will be denoted by Y_k. Without loss of generality, due to the fact that the Δ_k's and $\mu_{0,0}^L$ are supported on $\partial_+\mathcal{P}(0, L)$, we may assume that the Y_k's take values in $\partial_+\mathcal{P}(0, L)$ only. Again, without loss of generality, we assume all these couplings to be defined on one common probability space $(\Omega, \mathcal{F}, P)$. Thus, using the notation $\mathcal{F}_{k-1} := \sigma(\Delta_1, \dots, \Delta_{k-1})$ for $k \in \{2, \dots, n\}$ and $\mathcal{F}_0 := \{\varnothing, \Omega\}$, the following hold P-a.s.:

(a′) $\sum_x |P(Y_k = x|\mathcal{F}_{k-1}) - \mu_{0,0}^L(x)| \le \lambda$;
(b′) $P(\|Y_k - \Delta_k\|_1 \le K|\mathcal{F}_{k-1}) = 1$;
(c′) $E(Y_k|\mathcal{F}_{k-1}) = E_{\mu_{0,0}^L}$;
(d′) $\sum_x \|x - E_{\mu_{0,0}^L}\|_1^2 \cdot |P(Y_k = x|\mathcal{F}_{k-1}) - \mu_{0,0}^L(x)| \le \lambda \operatorname{Var}_{\mu_{0,0}^L}$.

To prove the desired result, it is sufficient to show that there exists a random variable Y' defined on the same probability space such that:

(a″) $\sum_x |P(Y' = x) - \mu_{0,0}^{\sqrt{n}L}(x)| \le \lambda R_9(L)$;
(b″) $P(\|Y' - S_n\|_1 < 4nK) = 1$;
(c″) $EY' = E_{\mu_{0,0}^{\sqrt{n}L}}$;
(d″) $\sum_x \|x - E_{\mu_{0,0}^{\sqrt{n}L}}\|_1^2 \cdot |P(Y' = x) - \mu_{0,0}^{\sqrt{n}L}(x)| \le \lambda R_9(L) \operatorname{Var}_{\mu_{0,0}^{\sqrt{n}L}}$.

To this end, set

$$S^{(j)} := \sum_{k=j}^{n} Y_k. \tag{A.70}$$

Using descending induction, we start with showing that for all $j \in \{1, \dots, n\}$ the following holds:

(IS) Conditioned on $\Delta_1, \dots, \Delta_{j-1}$, we can write $S^{(j)} = Y^{(j)} + Z^{(j)}$ for some $Y^{(j)}$ and $Z^{(j)}$ such that $\|Z^{(j)}\|_1 \le (n - j)R_3(L)$ a.s. and such that with respect to $P(\cdot|\mathcal{F}_{j-1})$, the variable $Y^{(j)}$ is distributed as $\mu_{0,0}^{\sqrt{n-j+1}L} + D_2^{(j)}$, where $D_2^{(j)}$ is a signed measure the variational norm $\|D_2^{(j)}\|_{\mathrm{TV}}$ of which is bounded from above by $\lambda^{(j)}$ with $\lambda^{(n)} = \lambda$ and

$$\lambda^{(j)} := \lambda^{(j+1)} + C\lambda R_6\big(\sqrt{n - j}L\big)(n - j)^{-1}$$

for $j < n$ and some constant C.

For $j = n$, the statement holds true due to the assumptions with $Z^{(n)} = 0$. We now assume that the statement holds for $j+1$ and prove it for j.

Setting $H := Y_j + Y^{(j+1)}$, for each z we have

$$P(H = z|\mathcal{F}_{j-1}) = \sum_x P(Y_j = x|\mathcal{F}_{j-1}) P\big(Y^{(j+1)} = z - x|Y_j = x, \mathcal{F}_{j-1}\big).$$

With $\hat{\mu}^L_{n,j,Y_j}$ defined as the convolution $P(Y_j \in \cdot|\mathcal{F}_{j-1}) * \mu_{0,0}^{\sqrt{n-j}L}$, this yields that

$$\begin{aligned}
&\sum_z \big|P(H = z|\mathcal{F}_{j-1}) - \hat{\mu}^L_{n,j,Y_j}(z)\big| \\
&\quad \le \sum_z \sum_x P(Y_j = x|\mathcal{F}_{j-1}) \\
&\quad\qquad \times \big|P\big(Y^{(j+1)} = z - x|Y_j = x, \mathcal{F}_{j-1}\big) - \mu_{0,0}^{\sqrt{n-j}L}(z - x)\big| \qquad \text{(A.71)} \\
&\quad = \sum_{x,y} P(Y_j = x|\mathcal{F}_{j-1}) \cdot \big|P\big(Y^{(j+1)} = y|Y_j = x, \mathcal{F}_{j-1}\big) - \mu_{0,0}^{\sqrt{n-j}L}(y)\big| \\
&\quad \le \|D_2^{(j+1)}\|_{\mathrm{TV}} \le \lambda^{(j+1)}
\end{aligned}$$

holds a.s.

Next, we set $\hat{\mu}^L_{1,n-j} := \mu^L_{0,0} * \mu_{0,0}^{\sqrt{n-j}L}$ and will bound

$$\begin{aligned}
&|\hat{\mu}^L_{1,n-j}(z) - \hat{\mu}^L_{n,j,Y_j}(z)| \\
&\quad = \Big|\sum_x \mu_{0,0}^{\sqrt{n-j}L}(x)\big(P(Y_j = z - x|\mathcal{F}_{j-1}) - \mu^L_{0,0}(z - x)\big)\Big|
\end{aligned} \qquad \text{(A.72)}$$

from above.

For this purpose, for given z, we will apply Lemma A.14 to the function $\mu_{0,0}^{\sqrt{n-j}L}$ with the corresponding measure μ given by $P(Y_j \in \cdot|\mathcal{F}_{j-1}) - \mu^L_{0,0}$ (note that $\|\mu\|_{\mathrm{TV}} \le \lambda$).

We now determine the parameters k, m, J and N of the assumptions of Lemma A.14. Parts (d) and (e) of Lemma A.2 yield that we can choose $k \le C(\sqrt{n-j}L)^{-d-1}$. Furthermore, as a consequence of (c′) we can choose N equal to 0, whence the exact value of m does not matter ($m = 1$ works). In addition, (d′) yields that J can be chosen equal to $2\lambda \operatorname{Var}_{\mu^L_{0,0}}$, with ϱ equal to one of the elements of $\mathbb{Z}^d$ closest to $E_{\mu^L_{0,0}}$. Thus, Lemmas 3.12 and A.14 in combination with (A.72) yield that a.s.,

$$|\hat{\mu}^L_{1,n-j}(z) - \hat{\mu}^L_{n,j,Y_j}(z)| \le C\lambda L^{1-d}(n-j)^{(-d-1)/2}. \qquad \text{(A.73)}$$

Note that for z such that $\|z - E_{\hat{\mu}^L_{1,n-j}}\|_\infty > 4dR_6(\sqrt{n-j}L)\sqrt{n-j}L$, the terms $\hat{\mu}^L_{1,n-j}(z)$ and $\hat{\mu}^L_{n,j,Y_j}(z)$ vanish. Thus, using (A.71) and (A.73), the triangle in-

equality implies that a.s.,

$$\begin{aligned}
&\sum_z \bigl|P(H=z|\mathcal{F}_{j-1}) - \hat{\mu}^L_{1,n-j}(z)\bigr| \\
&\quad\le \sum_z \bigl|P(H=z|\mathcal{F}_{j-1}) - \hat{\mu}^L_{n,j,Y_j}(z)\bigr| \\
&\qquad + \sum_{\substack{z\in H_{(n-j)L^2},\\ \|z-E_{\hat{\mu}^L_{1,n-j}}\|_1\le 4dR_6(\sqrt{n-j}L)\sqrt{n-j}L}} |\hat{\mu}^L_{n,j,Y_j}(z) - \hat{\mu}^L_{1,n-j}(z)| \\
&\quad\le \lambda^{(j+1)} + C\lambda R_6\bigl(\sqrt{n-j}L\bigr)^{d-1}(n-j)^{-1}.
\end{aligned}$$

Consequently, we get that the distribution of H can be written as $\hat{\mu}^L_{1,n-j} + \overline{D}_2^{(j)}$ for a signed measure $\overline{D}_2^{(j)}$ with

$$\bigl\|\overline{D}_2^{(j)}\bigr\|_{\mathrm{TV}} \le \lambda^{(j+1)} + C\lambda R_6\bigl(\sqrt{n-j}L\bigr)^{d-1}(n-j)^{-1}. \tag{A.74}$$

By Claim A.13, there exists $Z'(j)$ such that

$$P\bigl(\|Z'(j)\|_1 > R_3(L)\bigr) \le Ce^{-C^{-1}R_2(L)^\gamma},$$

and such that the distribution of $H + Z'(j)$ is $\mu_{0,0}^{\sqrt{n-j+1}L} + \overline{D}_2^{(j)}$. Let

$$\overline{H} := H + Z'(j)\cdot \mathbb{1}_{\|Z'(j)\|_1\le R_3(L)}.$$

Then due to (A.74), the distribution of $\overline{H}$ equals $\mu_{0,0}^{\sqrt{n-j+1}L} + \check{D}_2^{(j)}$ for some signed measure $\check{D}_2^{(j)}$ such that

$$\bigl\|\check{D}_2^{(j)}\bigr\|_{\mathrm{TV}} \le \bigl\|\overline{D}_2^{(j)}\bigr\|_{\mathrm{TV}} + Ce^{-C^{-1}R_2(L)^\gamma} \le \lambda^{(j+1)} + C\lambda R_6\bigl(\sqrt{n-j}L\bigr)^{d-1}(n-j)^{-1}.$$

We let

$$Z^{(j)} := Z^{(j+1)} + Z'(j)\cdot \mathbb{1}_{\|Z'(j)\|_1\le R_3(L)}$$

and

$$Y^{(j)} := S^{(j)} - Z^{(j)}.$$

Then we infer that

$$\bigl\|Z^{(j)}\bigr\|_1 \le (n-j)R_3(L) \tag{A.75}$$

and the distribution of $Y^{(j)}$ is $\mu_{0,0}^{\sqrt{n-j+1}L} + D_2^{(j)}$ where $D_2^{(j)}$ is a signed measure such that $\|D_2^{(j)}\|_{\mathrm{TV}} \le \lambda^{(j)}$ with

$$\lambda^{(j)} \le \lambda^{(j+1)} + C\lambda R_6\bigl(\sqrt{n-j}L\bigr)^{d-1}(n-j)^{-1}.$$

This establishes (IS).

Using (c′) and (A.70), the expectation of $Y^{(1)}$ is computed via

$$EY^{(1)} = ES^{(1)} - EZ^{(1)} = nE_{\mu_{0,0}^L} - EZ^{(1)}. \tag{A.76}$$

Therefore, in combination with (A.75), we get

$$\begin{aligned}\big\|EY^{(1)} - E_{\mu_{0,0}^{\sqrt{n}L}}\big\|_1 &\le \|nE_{\mu_{0,0}^{L}} - E_{\mu_{0,0}^{\sqrt{n}L}}\|_1 + \big\|EZ^{(1)}\big\|_1\\ &\le CnR_2\big(\sqrt{n}L\big) + nR_3(L)\\ &\le 2nR_3(L)\end{aligned}$$

for L large enough, since $n \le L$ by assumption; indeed, with the help of Lemma 3.14 one deduces

$$\begin{aligned}\|nE_{\mu_{0,0}^{L}} - E_{\mu_{0,0}^{\sqrt{n}L}}\|_1 &\le n\left\|E_{\mu_{0,0}^{L}} - \frac{L^2}{\hat{v}\cdot e_1}\hat{v}\right\|_1 + \left\|\frac{nL^2}{\hat{v}\cdot e_1}\hat{v} - E_{\mu_{0,0}^{\sqrt{n}L}}\right\|_1\\ &\le C\big(nR_2(L) + R_2\big(\sqrt{n}L\big)\big).\end{aligned}$$

As in the proof of Corollary 3.13, we can find a variable U which is independent of all the variables we have seen so far, and such that $\|U\|_1 \le 2nR_3(L)$, $U \in H_0$ almost surely and

$$EU = E_{\mu_{0,0}^{\sqrt{n}L}} - EY^{(1)}.$$

We define

$$Y' := Y^{(1)} + U,$$

which directly yields that (c″) holds. To check (b″), note that in combination with (IS) and the definition of $S^{(1)}$ we get

$$\begin{aligned}\|S_n - Y'\|_1 &\le \big\|S_n - S^{(1)}\big\|_1 + \big\|S^{(1)} - Y^{(1)}\big\|_1 + \big\|Y^{(1)} - Y'\big\|_1\\ &\le nK + nR_3(L) + 2nR_3(L) \le 4nK\end{aligned} \tag{A.77}$$

since $K \ge R_3(L)$. Now from (IS) it follows that $\lambda^{(1)} \le C\lambda R_6(L^2)^{d-1}\log(n) \le C\lambda R_7(L)$ for L large enough. Thus, (a″) is a consequence of

$$\begin{aligned}&\sum_x \big|P(Y' = x) - \mu_{0,0}^{\sqrt{n}L}(x)\big|\\ &\quad\le 2\sum_{x\in\partial_+\mathcal{P}(0,\sqrt{n}L)}\Bigg|\sum_{\substack{y\in H_0,\\ \|y\|_1\le 2nR_3(L)}} P(U=y)P\big(Y^{(1)} = x-y\big) - \mu_{0,0}^{\sqrt{n}L}(x)\Bigg|\\ &\quad\le 2\sum_{x\in\partial_+\mathcal{P}(0,\sqrt{n}L)}\Bigg(\Bigg|\sum_{\substack{y\in H_0,\\ \|y\|_1\le 2nR_3(L)}} P(U=y)\big(P\big(Y^{(1)} = x-y\big)\\ &\qquad\qquad - \mu_{0,0}^{\sqrt{n}L}(x-y)\big)\Bigg|\\ &\qquad\qquad + CnR_3(L)\big(\sqrt{n}L\big)^{-d}\Bigg)\\ &\quad\le C\lambda R_7(L)\end{aligned} \tag{A.78}$$

for L large enough and where we used Lemma A.2(b) to obtain the second inequality, and also the fact that $\lambda \geq nL^{-1}$.

The remaining part of the proof consists of establishing that Y' also satisfies (d″). Denoting by D_2 be the signed measure such that $Y' \sim \mu_{0,0}^{\sqrt{n}L} + D_2$, this amounts to showing that

$$\sum_x \|x - E_{\mu_{0,0}^{\sqrt{n}L}}\|_1^2 \cdot |D_2(x)| \leq \lambda R_9(L) \operatorname{Var}_{\mu_{0,0}^{\sqrt{n}L}} \tag{A.79}$$

holds.

To start with, note that

$$\begin{aligned} &\sum_x \|x - E_{\mu_{0,0}^{\sqrt{n}L}}\|_1^2 \cdot |D_2(x)| \\ &\leq (d-1) \sum_x \|x - E_{\mu_{0,0}^{\sqrt{n}L}}\|_2^2 \cdot |D_2(x)| \qquad \text{(A.80)} \\ &= (d-1) \sum_{i=2}^{d} \sum_x \big((x - E_{\mu_{0,0}^{\sqrt{n}L}}) \cdot e_i\big)^2 \cdot |D_2(x)|. \end{aligned}$$

To proceed, we write $D_2 = D_2^+ - D_2^-$ for the Jordan decomposition of D_2 and estimate

$$\begin{aligned} &\sum_x \big((x - E_{\mu_{0,0}^{\sqrt{n}L}}) \cdot e_i\big)^2 \cdot |D_2(x)| \\ &\leq 2 \sum_x \big((x - E_{\mu_{0,0}^{\sqrt{n}L}}) \cdot e_i\big)^2 \cdot D_2^-(x) \qquad \text{(A.81)} \\ &\quad + \left| \sum_x \big((x - E_{\mu_{0,0}^{\sqrt{n}L}}) \cdot e_i\big)^2 \cdot D_2(x) \right|. \qquad \text{(A.82)} \end{aligned}$$

To bound (A.81) from above, note that $D_2^-(x) \leq \mu_{0,0}^{\sqrt{n}L}(x)$ for all x. Combined with the fact that $\|D_2^-\|_{\mathrm{TV}} \leq \|D_2\|_{\mathrm{TV}} \leq C\lambda R_7(L)$ [due to (A.78)], we obtain

$$\sum_x \big((x - E_{\mu_{0,0}^{\sqrt{n}L}}) \cdot e_i\big)^2 D_2^-(x) \leq \lambda R_8(L) n L^2 \tag{A.83}$$

for L large enough, since $\mu_{0,0}^{\sqrt{n}L}$ is supported on $\partial_+ \mathcal{P}(0, \sqrt{n}L)$.

In order to estimate (A.82), note that due to (c″) we have $\sum_x x D_2(x) = 0$, and hence (A.82) equals $|\operatorname{Var}(D_2, i)|$ with

$$\operatorname{Var}(D_2, i) := \sum_x (x \cdot e_i)^2 D_2(x) = \operatorname{Var}(Y' \cdot e_i) - \operatorname{Var}(W \cdot e_i), \tag{A.84}$$

where W denotes a random variable distributed according to $\mu_{0,0}^{\sqrt{n}L}$. By Claim A.13, there exists a random variable W' such that $W' \sim (\mu_{0,0}^L)^{*n}$, with $(\mu_{0,0}^L)^{*n}$ denoting

the n-fold convolution of $\mu_{0,0}^L$, and such that

$$P\big(\|W - W'\|_1 > nR_3(L)\big) < Cne^{-C^{-1}R_2(L)^\gamma}$$

for L large enough. Then

$$\begin{aligned}|\mathrm{Var}(D_2, i)| &\le |\mathrm{Var}(W \cdot e_i) - \mathrm{Var}(W' \cdot e_i)| \\ &\quad + \big|\mathrm{Var}(W' \cdot e_i) - \mathrm{Var}\big(S^{(1)} \cdot e_i\big)\big| \\ &\quad + \big|\mathrm{Var}\big(S^{(1)} \cdot e_i\big) - \mathrm{Var}(Y' \cdot e_i)\big|. \end{aligned} \tag{A.85}$$

Now for L large enough,

$$\begin{aligned}&|\mathrm{Var}(W \cdot e_i) - \mathrm{Var}(W' \cdot e_i)| \\ &\quad \le \mathrm{Var}\big((W - W') \cdot e_i\big) \\ &\qquad + 2\big|\mathrm{Cov}\big((W - W') \cdot e_i, W' \cdot e_1\big)\big| \\ &\quad \le \operatorname{ess\,sup}\big((W - W') \cdot e_i\big)^2 P\big(\|W - W'\|_1 > nR_3(L)\big) \\ &\qquad + 2n^2 R_3(L)^2 + 2nR_3(L)\sqrt{\mathrm{Var}(W')} \\ &\quad \le Cn^{3/2} R_3(L) L, \end{aligned} \tag{A.86}$$

where among others we used $\sqrt{n} \le L$ and Lemma 3.12. Furthermore,

$$\begin{aligned}\big|\mathrm{Var}(Y' \cdot e_i) - \mathrm{Var}\big(S^{(1)} \cdot e_i\big)\big| &= \big|\mathrm{Var}\big(\big(S^{(1)} + U'\big) \cdot e_i\big) - \mathrm{Var}\big(S^{(1)} \cdot e_i\big)\big| \\ &\le 2 \operatorname{ess\,sup}(\|U'\|_1)\sqrt{\mathrm{Var}\big(S^{(1)}\big)} + \operatorname{ess\,sup}(\|U'\|_1)^2 \\ &\le Cn^{3/2} R_3(L) L + 4n^2 R_3^2(L) \qquad \text{(A.87)} \\ &\le Cn^{3/2} R_3(L) L, \qquad \text{(A.88)}\end{aligned}$$

where we used $n \le L$ and that by the definition of Y', we know that $U' := Y' - S^{(1)}$ satisfies $\|U'\|_1 \le 3nR_3(L)$.

To estimate the remaining summand, note that $\mathrm{Cov}(Y_j, Y_k) = 0$ for $j \ne k$, and hence

$$\big|\mathrm{Var}\big(S^{(1)} \cdot e_i\big) - \mathrm{Var}(W' \cdot e_i)\big| \le \sum_{j=1}^{n} \big|\mathrm{Var}(Y_j \cdot e_i) - \mathrm{Var}_{\mu_{0,0}^L \circ (\cdot \cdot e_i)^{-1}}\big|. \tag{A.89}$$

Furthermore, since $E_{\mu_{0,0}^L} = EY_j$ for every j, from (d′) we infer that

$$\begin{aligned}&\big|\mathrm{Var}(Y_j \cdot e_i) - \mathrm{Var}_{\mu_{0,0}^L \circ (\cdot \cdot e_i)^{-1}}\big| \\ &\quad = \Big|\sum_x \big(x \cdot e_i - (E_{\mu_{0,0}^L} \cdot e_i)\big)^2 \big(P(Y_j = x) - \mu_{0,0}^L(x)\big)\Big| \\ &\quad \le \sum_x \big(x \cdot e_i - (E_{\mu_{0,0}^L} \cdot e_i)\big)^2 |P(Y_j = x) - \mu_{0,0}^L(x)| \\ &\quad \le \lambda \, \mathrm{Var}_{\mu_{0,0}^L}, \end{aligned}$$

and as a consequence

$$\big|\mathrm{Var}\big(S^{(1)}\cdot e_i\big)-\mathrm{Var}(W'\cdot e_i)\big|\le n\lambda\,\mathrm{Var}_{\mu_{0,0}^L}. \tag{A.90}$$

Using (A.85) to (A.90) in combination with Lemma 3.12, we deduce that

$$|\mathrm{Var}(D_2,i)|\le C\lambda nL^2,$$

whence in combination with (A.81) to (A.84) we have

$$\sum_x (x\cdot e_i - E_{\mu_{0,0}^{\sqrt{n}L}}\cdot e_i)^2|D_2(x)|\le 2R_8(L)\lambda nL^2$$

for L large enough. Therefore, (A.80) yields

$$\sum_x \|x-E_{\mu_{0,0}^{\sqrt{n}L}}\|_1^2\cdot|D_2(x)|\le 2d^2R_8(L)\lambda nL^2$$

for L large enough. In combination with Lemma 3.12, we deduce that (A.79) holds and thus (d″) is fulfilled. □

We now prove the previously employed Lemma 3.11.

PROOF OF LEMMA 3.11. We continue to use the notation $B(l,k)$ and $B(l)$ introduced in the proof of Lemma A.2, from which this proof draws its strategy. Again, denote by Γ the covariance matrix of $X_{\tau_2}-X_{\tau_1}$ with respect to P_0 and set $m:=E_0(X_{\tau_2}-X_{\tau_1})$. Using (A.5) and the fact that, since Γ^{-1} is positive definite, the corresponding quadratic form induces a norm, we infer that for any $C>0$ there exists a constant $c>0$ such that for k large enough and $y\in H_{L^2}$ with $\|y-x-km\|_1\le C\sqrt{k}$, we have

$$P_x\big(X_{T_{L^2}}=y,\,B(L^2,k)\big)\ge ck^{-d/2}. \tag{A.91}$$

Setting $l^*:=\frac{L^2-x\cdot e_1}{m\cdot e_1}$ and $C:=4C'$, for $k\in\{\lfloor l^*-\sqrt{l^*}\rfloor,\dots,\lceil l^*+\sqrt{l^*}\rceil\}$ and x, y as in the assumptions, we have

$$\|y-x-km\|_1\le\|\tilde\pi_{\hat v^\perp}(y-x)\|_1+\|l^*m-km\|_1\le CL$$

for L large enough. Then, using (A.91) with $y\in\partial_+\mathcal{P}(0,L)$, uniformly in $x\in\tilde{\mathcal{P}}(0,L)$ we have

$$\begin{aligned}P_x(X_{T_{L^2}}=y)&\ge P_x\big(X_{T_{L^2}}=y,\,B(L^2)\big)\\&\ge\sum_{k=\lfloor l^*-\sqrt{l^*}\rfloor}^{\lceil l^*+\sqrt{l^*}\rceil}P_x\big(X_{T_{L^2}}=y,\,B(L^2,k)\big)\\&\ge cL^{1-d},\end{aligned}$$

which due to (3.20) finishes the proof. □

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