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On the invariant measure of the random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case[1]

Sara Brofferio[a], Dariusz Buraczewski[b] and Ewa Damek[b]

[a]*Laboratoire de Mathématiques et IUT de Sceaux, Université Paris-Sud, 91405 Orsay Cedex, France. E-mail: sara.brofferio@math.u-psud.fr*
[b]*Instytut Matematyczny, Uniwersytet Wroclawski, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland.*
E-mail: dbura@math.uni.wroc.pl; edamek@math.uni.wroc.pl

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Abstract. We consider the autoregressive model on $\mathbb{R}^d$ defined by the stochastic recursion $X_n = A_n X_{n-1} + B_n$, where $\{(B_n, A_n)\}$ are i.i.d. random variables valued in $\mathbb{R}^d \times \mathbb{R}^+$. The critical case, when $\mathbb{E}[\log A_1] = 0$, was studied by Babillot, Bougerol and Elie, who proved that there exists a unique invariant Radon measure ν for the Markov chain $\{X_n\}$. In the present paper we prove that the weak limit of properly dilated measure ν exists and defines a homogeneous measure on $\mathbb{R}^d \setminus \{0\}$.

Résumé. Nous considérons le modèle autorégressif sur $\mathbb{R}^d$ défini par récurrence par l'équation stochastique $X_n = A_n X_{n-1} + B_n$, où $\{(B_n, A_n)\}$ sont des variables aléatoires à valeurs dans $\mathbb{R}^d \times \mathbb{R}^+$, indépendantes et de même loi. Le cas critique, c'est-à-dire lorsque $\mathbb{E}[\log A_1] = 0$, a été étudié par Babillot, Bougerol et Elie, qui ont montré qu'il existe une et une seule mesure de Radon ν invariante pour la chaîne de Markov $\{X_n\}$. Dans ce papier nous démontrons que la mesure ν, convenablement dilatée, converge faiblement vers une mesure homogène sur $\mathbb{R}^d \setminus \{0\}$.

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1. Introduction and the main result

We consider the autoregressive process on $\mathbb{R}^d$:

$$\begin{aligned} X_0^x &= x, \\ X_n^x &= A_n X_{n-1}^x + B_n, \end{aligned} \tag{1.1}$$

where the random pairs $\{(B_n, A_n)\}_{n\in\mathbb{N}}$ valued in $\mathbb{R}^d \times \mathbb{R}^+$ are independent, identically distributed (i.i.d.) according to a given probability measure μ. The Markov chain $\{X_n^x\}$ occurs in various applications e.g. in biology and economics, see [1] and the comprehensive bibliography there.

It is convenient to define X_n in the group language. Let G be the "$ax+b$" group, i.e. $G = \mathbb{R}^d \rtimes \mathbb{R}^+$, with multiplication by $(b, a) \cdot (b', a') = (b + ab', aa')$. The group G acts on $\mathbb{R}^d$ by $(b, a) \cdot x = ax + b$, for $(b, a) \in G$ and

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$x \in \mathbb{R}^d$. For each n, we sample the random variables $(B_n, A_n) \in G$ independently with respect to the measure μ, then $X_n^x = (B_n, A_n) \cdot \cdots \cdot (B_1, A_1) \cdot x$.

The Markov chain X_n^x is usually studied under the assumption $\mathbb{E}[\log A_1] < 0$. Then, if additionally $\mathbb{E}[\log^+ |B_1|] < \infty$, there is a unique stationary measure ν [16], i.e. the probability measure ν on $\mathbb{R}^d$ satisfying

$$\mu *_G \nu(f) = \nu(f) \tag{1.2}$$

for every positive measurable function f. Here

$$\mu *_G \nu(f) = \int_G \int_{\mathbb{R}^d} f(ax+b)\nu(\mathrm{d}x)\mu(\mathrm{d}b\,\mathrm{d}a).$$

One of the main results concerning the stationary measure ν is Kesten's theorem [16] (see also [13,15]) saying that if $\mathbb{E}[A_1^\alpha] = 1$ (and some other assumptions are satisfied), then

$$\nu\big(u\colon |u| > z\big) \sim Cz^{-\alpha} \quad \text{as } z \to +\infty$$

for a positive constant C.

Here we study the critical case, when $\mathbb{E}[\log A_1] = 0$. Then X_n has no invariant probability measure. However, it was proved by Babillot, Bougerol and Elie [1] that if

- $\mathbb{P}[A_1 = 1] < 1$ and $\mathbb{P}[A_1 x + B_1 = x] < 1$ for all $x \in \mathbb{R}^d$,
- $\mathbb{E}[(|\log A_1| + \log^+ |B_1|)^{2+\varepsilon}] < \infty$, for some $\varepsilon > 0$,
- $\mathbb{E}[\log A_1] = 0$.

Then there exists a unique (up to a constant factor) invariant Radon measure ν, i.e. a measure satisfying (1.2) (see also [2,3]). We will say that μ satisfies hypothesis (**H**) if all the assumptions above are satisfied. For our purpose we will need an additional assumption, which will be called hypothesis $\mathbf{M}(\delta)$:

- there exists $\delta > 0$ such that $\mathbb{E}[A_1^\delta + A_1^{-\delta} + |B_1|^\delta] < \infty$.

The measure ν appears in a natural way when problems related to the process X_n^x or to random walks on the group G in the critical case are investigated. Let us mention two examples. Le Page and Peigné [17] proved the local limit theorem for X_n^x, saying that under some further assumptions, $\sqrt{n}\mathbb{E}[f(X_n^x)]$ converges to $\nu(f)$ for any compactly supported function f. Elie [10] described the Martin boundary for the left random walk on the affine group with the measure ν playing the central role. Therefore, it is natural to ask about a quantified description of the measure ν and the aim of this paper is to answer this question. Our main result is an analogue of Kesten's theorem in the critical case.

Theorem 1.1. *Assume that hypotheses* (**H**), $\mathbf{M}(\delta)$ *are satisfied and the law* μ_A *of* A_1 *is aperiodic. Then there exists a probability measure* Σ *on the unit ball* $S^{d-1} \subset \mathbb{R}^d$ *and a strictly positive number* C_+ *such that the measures* $\delta_{(0,z^{-1})} *_G \nu$ *converge weakly on* $\mathbb{R}^d \setminus \{0\}$ *to* $C_+ \Sigma \otimes \frac{\mathrm{d}a}{a}$ *as* $z \to +\infty$, *that is*

$$\lim_{z\to+\infty} \int_{\mathbb{R}^d} \phi\big(z^{-1}u\big)\nu(\mathrm{d}u) = C_+ \int_{\mathbb{R}^+} \int_{S^{d-1}} \phi(aw)\Sigma(\mathrm{d}w)\frac{\mathrm{d}a}{a}$$

for every function $\phi \in C_c(\mathbb{R}^d \setminus \{0\})$.

In particular, for every $\alpha < \beta$

$$\lim_{z\to\infty} \nu\big(u\colon \alpha z < |u| < \beta z\big) = C_+ \log\frac{\beta}{\alpha}. \tag{1.3}$$

The first estimate of the behavior of the measure ν at infinity was given by Babillot, Bougerol and Elie [1], who proved, for $d = 1$ and under some nondegeneracy hypotheses, that for every $\alpha < \beta$

$$\nu\big((\alpha z, \beta z]\big) \sim \log(\beta/\alpha) \cdot L(z) \quad \text{as } z \to +\infty,$$

where L is a slowly varying function.

The second author recently proved [4] that the function $L(z)$ is in fact constant, but in a more restrictive setting: besides the hypotheses stated above, one assumes in [4] that $d = 1$, the closed semigroup generated by the support of μ is the whole group G and the measure μ_A is spread-out. Moreover nondegeneracy of the limiting constant C_+ was proved there only in the particular case when $B_1 \ge \varepsilon$ a.s.

When the measure μ is related to a differential operator, stronger results have been obtained recently in [6,8]. Namely, let $\{\mu_t\}$ be the one parameter semigroup of probability measures, whose infinitesimal generator is a second-order elliptic differential operator on $\mathbb{R}^d \times \mathbb{R}^+$. Then there exists a unique Radon measure ν that is μ_t-invariant, for any t. Moreover, ν has a smooth density m such that

$$m(zu) \sim C(u) z^{-d} \quad \text{as } z \to +\infty$$

for some continuous nonzero function C on $\mathbb{R}^d \setminus \{0\}$.

In this paper we also describe the behavior of the measure ν in the case when the measure μ_A is periodic. This situation has been quite neglected up to now, also in the contracting case. Even if we cannot obtain the convergence of the measure ν at infinity, we still have a good estimation of the asymptotic of the measure of the ball of radius z.

Theorem 1.2. *Suppose that hypotheses* (**H**) *and* **M**(δ) *are satisfied. If the measure μ_A is periodic of period p, i.e. $\langle \mathrm{supp}\mu_A \rangle = \{\mathrm{e}^{np}\}_{n\in\mathbb{Z}}$, then the family of measures $\delta_{(0,z^{-1})} *_G \nu$ is weakly compact and there exists a positive constant C_+ such that*

$$\lim_{z\to\infty} \int_{\mathbb{R}^d} \phi\big(z^{-1}u\big)\nu(\mathrm{d}u) = C_+ \sum_{k\in\mathbb{Z}} \phi\big(\mathrm{e}^{pk}\big)$$

for any function ϕ belonging to $\mathcal{T}$, the subset of $C_C(\mathbb{R}^d \setminus \{0\})$ consisting of radial functions such that $\sum_{k\in\mathbb{Z}} \phi(a\mathrm{e}^{pk}) = \sum_{k\in\mathbb{Z}} \phi(\mathrm{e}^{pk})$ for all $a \in \mathbb{R}^+$. In particular

$$\nu\big(u\colon |u| \le z\big) \sim \frac{C_+}{p} \log z \quad \textit{as } z \to +\infty.$$

The case when B_1 is positive is also of a particular interest in applications and generally allows to use more powerful techniques. It will be the subject of a forthcoming paper, where we prove that the moment hypothesis of Theorem 1.1 can be weakened.

Let us mention that Theorems 1.1 and 1.2 have been applied recently to study tails of fixed points of the so-called smoothing transform in a boundary case. However in this context ‘boundary case’ concerns probability measures having infinite mean. See [5] for more details.

The structure of the paper is the following. First we estimate the behavior of ν at infinity in Section 2 under the very mild hypothesis (**H**). In Theorem 2.1, we show that $\delta_{(0,z^{-1})} *_G \nu(K)$ is smaller than $C_K L(z)$, for all compact sets K and a slowly varying function L, i.e. the family of measures $\delta_{(0,z^{-1})} *_G \nu/L(z)$ is weakly compact. We also prove that $\int_{\mathbb{R}^d} (1+|u|)^{-\gamma}\nu(\mathrm{d}u) < \infty$ for any $\gamma > 0$ and we obtain some invariance properties of the accumulation points of $\delta_{(0,z^{-1})} *_G \nu/L(z)$.

Next, as in [4], we reduce the problem to study asymptotic behavior of positive solutions of the Poisson equation. More precisely, let $\overline{\mu}$ be the law of $-\log A_1$. The mean of $\overline{\mu}$ is equal to 0 and given a positive $\phi \in C_c(\mathbb{R}^d \setminus \{0\})$ we define the function on $\mathbb{R}$:

$$f_\phi(x) = \delta_{(0,\mathrm{e}^{-x})} *_G \nu(\phi) = \int_{\mathbb{R}^d} \phi\big(\mathrm{e}^{-x}u\big)\nu(\mathrm{d}u). \tag{1.4}$$

Then f_ϕ can be considered as a solution of the Poisson equation

$$\overline{\mu} *_{\mathbb{R}} f_\phi(x) = f_\phi(x) + \psi_\phi(x), \quad x \in \mathbb{R}, \tag{1.5}$$

for a specific function ψ_ϕ. The function ψ_ϕ possesses some regularity properties and it is easier to study than f_ϕ. The main problem can be formulated as follows: given a function ψ_ϕ describe the behavior at infinity of positive solutions of the Poisson equation. An answer to this rather classical question was given by Port and Stone [19], under the hypothesis that $\overline{\mu}$ is spread out. However, their methods, slightly developed, work for general centered measure $\overline{\mu}$ on $\mathbb{R}$. Namely we can construct a class $\mathcal{F}(\overline{\mu})$ of functions ψ, with well defined potential that can be used to describe solutions of the corresponding Poisson equation. All the details will be figured out in Section 3.

The next step is to prove that the function ψ_ϕ belongs to $\mathcal{F}(\overline{\mu})$. A priori, this is not true for all function ϕ. However we are able to construct special functions that have the good properties and allow to deduce our main results (Section 4).

Finally in Section 5 we prove that the limit of $\nu(\alpha z < |u| < \beta z)$ is strictly positive and the only hypothesis needed for this result is condition **(H)**.

2. The upper bound

The goal of this section is to prove a preliminary estimate of the measure ν at infinity. We prove that, under the very mild hypothesis (**H**) on the measure μ, the tail measure of a compact set $\delta_{(0,z^{-1})} * \nu(K)$ is bounded by a slowly varying function $L(z)$, that is a function on $\mathbb{R}^+$ such that $\lim_{z\to+\infty} L(az)/L(z) = 1$ for all $a > 0$. Such functions grow very slowly, namely they are smaller than z^γ for any $\gamma > 0$, in a neighborhood of $+\infty$.

Theorem 2.1. *If hypothesis* (**H**) *is fulfilled, there exists a positive slowly varying function* L *on* $\mathbb{R}_+^*$ *such that the normalized family of measures on* $\mathbb{R}^d \setminus \{0\}$

$$\frac{\delta_{(0,z^{-1})} *_G \nu}{L(z)} \tag{2.1}$$

is weakly compact for $z \ge 1$. *Thus* $(1 + |x|)^{-\gamma} \in L^1(\nu)$ *for any* $\gamma > 0$. *Furthermore, all limit measures* η *are nonnull and invariant under the action of* $G(\mu_A)$, *the closed sub-group of* $\mathbb{R}_+^*$ *generated by the support of* μ_A, *that is*

$$\delta_{(0,a)} *_G \eta = \eta \quad \forall a \in G(\mu_A).$$

This theorem is a partial generalization of Proposition 5.2 in [1].

We first prove that the μ-invariance of ν implies that the accumulation points of the tail are invariant under the action of $G(\mu_A)$, namely we have

Lemma 2.2. *Suppose that there exists a function* $L(z)$ *such that the family* (2.1) *is weakly compact when* z *goes to* $+\infty$, *then the accumulation points* η *are invariant under the action of* $G(\mu_A)$.

Proof. Let η be a limit measure along a sequence $\{z_n\}$ and fix a function $\phi \in C_c^1(\mathbb{R}^d \setminus \{0\})$. We claim that the function

$$h(y) = \delta_{(0,y)} *_G \eta(\phi) = \lim_{n\to\infty} \frac{\delta_{(0,z_n^{-1}y)} *_G \nu(\phi)}{L(z_n)}$$

on $\mathbb{R}^+$ is μ_A-superharmonic. Indeed, observe that for all $(b,a) \in G$ there is a compact set $K = K(b)$ and a constant C such that

$$\bigl|\phi\bigl(z^{-1}(au+b)\bigr) - \phi\bigl(z^{-1}(au)\bigr)\bigr| < C\bigl|z^{-1}b\bigr|\mathbf{1}_K\bigl(z^{-1}(au)\bigr)$$

for all $z > 1$ and $u \in \mathbb{R}^d$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{|\delta_{(0,z_n^{-1})} *_G \delta_{(b,a)} *_G \nu(\phi) - \delta_{(0,z_n^{-1})} *_G \delta_{(0,a)} *_G \nu(\phi)|}{L(z_n)} &\le \lim_{n\to\infty} \frac{C|z_n^{-1}b|\nu(a^{-1}z_n K)}{L(z_n)} \\ &\le C\eta\bigl(a^{-1}K\bigr) \cdot \lim_{n\to\infty}\bigl|z_n^{-1}b\bigr| = 0,\end{aligned}$$

hence

$$\begin{aligned}\int_G h(ay)\mu_A(\mathrm{d}a) &= \int_G \lim_{n\to\infty} \frac{\delta_{(0,z_n^{-1}y)} *_G \delta_{(0,a)} *_G \nu(\phi)}{L(z_n)}\mu(\mathrm{d}b\,\mathrm{d}a)\\ &= \int_G \lim_{n\to\infty} \frac{\delta_{(0,z_n^{-1}y)} *_G \delta_{(b,a)} *_G \nu(\phi)}{L(z_n)}\mu(\mathrm{d}b\,\mathrm{d}a)\\ &\le \lim_{n\to\infty} \frac{\delta_{(0,z_n^{-1}y)} *_G \mu *_G \nu(\phi)}{L(z_n)} \quad \text{by Fatou's Lemma}\\ &= \lim_{n\to\infty} \frac{\delta_{(0,z_n^{-1}y)} *_G \nu(\phi)}{L(z_n)} = h(y).\end{aligned}$$

Since h is positive and continuous, then by the Choquet–Deny theorem $h(ay) = h(y)$ for every $a \in G(\mu_A)$, that is $\delta_{(0,a)} *_G \eta(\phi) = \eta(\phi)$. □

Proof of Theorem 2.1. *Step 1.* The first step is to prove that the tail of the measure ν satisfies a quotient theorem. Namely that there exists a family of bounded compactly supported functions s such that $\delta_{(0,z^{-1})} *_G \nu(s)$ is strictly positive for all $z \ge 1$ and that for every compact set K there is a positive constant C_K such that

$$\delta_{(0,z^{-1})} *_G \nu(K) \le C_K \delta_{(0,z^{-1})} *_G \nu(s) \quad \forall z \ge 1. \tag{2.2}$$

In other words we show the quotient family $\frac{\delta_{(0,z^{-1})} *_G \nu}{\delta_{(0,z^{-1})} *_G \nu(s)}$ is weakly compact.

The proof of this property relies only on the fact that, by hypothesis (**H**), the support of μ contains at least two elements, one contracting and the other delating $\mathbb{R}^d$. Let call them $g_+ = (b_+, a_+)$ and $g_- = (b_-, a_-)$ with $a_+ > 1 > a_-$.

Given two real numbers α and β we consider the annulus

$$C(\alpha, \beta) = \{u \in \mathbb{R}^d \mid \alpha \le |u| \le \beta\}.$$

Observe that for all $(b, a) \in G$ the following implication holds

$$u \in C\left(\frac{\alpha + |b|}{a}, \frac{\beta - |b|}{a}\right) \quad \Rightarrow \quad au + b \in C(\alpha, \beta).$$

Using this remark and the fact that ν is invariant with respect to μ^{*n}, one can verify that

$$\delta_{(0,z^{-1})} *_G \nu\big(C(\alpha, \beta)\big) \ge \mu^{*n}(U)\nu\left(C\left(\max_{(b,a)\in U} \frac{\alpha z + |b|}{a}, \min_{(b,a)\in U} \frac{\beta z - |b|}{a}\right)\right) \tag{2.3}$$

for any U subset of G and $n \in \mathbb{N}$.

First we prove that there exists a sufficiently large $R > 0$ such that $\delta_{(0,z^{-1})} *_G \nu(C(1/R, R))$ is strictly positive for all $z \ge 1$.

Fix $z \ge 1$ and take $n \in \mathbb{N}$ such that $a_+^{n-1} \le z \le a_+^n$. Clearly, if $g^n = (b(g^n), a(g^n))$ is the nth power of an element $g = (b, a) \in G$ then

$$a\big(g^n\big) = a^n \quad \text{and} \quad b\big(g^n\big) = \sum_{i=0}^{n-1} a^i b = \frac{a^n - 1}{a - 1} b.$$

Consider the δ-neighborhood of g^n

$$U_\delta\big(g^n\big) = \big\{(b, a) \in G \mid \mathrm{e}^{-\delta} < a^{-1}a\big(g^n\big) < \mathrm{e}^\delta \text{ and } \big|b - b\big(g^n\big)\big| < \delta\big\}.$$

Observe that $\mu^{*n}(U_\delta(g_+^n))>0$ for all $\delta>0$ and for $(b,a)\in U_\delta(g_+^n)$

$$\frac{z/R+|b|}{a}\le\frac{a_+^n/R+|b(g_+^n)|+\delta}{\mathrm{e}^{-\delta}a_+^n}\le\mathrm{e}^{\delta}\left(\frac{1}{R}+\frac{|b_+|}{a_+-1}+\delta\right)=:\alpha_R,$$

$$\frac{Rz-|b|}{a}\ge\frac{Ra_+^{n-1}-|b(g_+^n)|-\delta}{\mathrm{e}^{\delta}a_+^n}\ge\mathrm{e}^{-\delta}\left(\frac{R}{a_+}-\frac{|b_+|}{a_+-1}-\delta\right)=:\beta_R.$$

Since ν is a Radon measure with the infinite mass, its support cannot be compact. Thus, for a fixed δ, there exits a sufficiently large R such that: $\nu(C(\alpha_R,\beta_R))>0$. Then by (2.3):

$$\delta_{(0,z^{-1})}*_G\nu\bigl(C(1/R,R)\bigr)\ge\mu^{*n}\bigl(U_\delta\bigl(g_+^n\bigr)\bigr)\nu\bigl(C(\alpha_R,\beta_R)\bigr)>0\tag{2.4}$$

for all $z\ge1$.

For $R>2$ consider the compact sets $K_\pm^n=C(2a_\pm^{-n}/R,a_\pm^{-n}R/2)$. Notice that for $\delta<\log(4/3)$, $(b,a)\in U_\delta(g_\pm^n)$ and $z>z_\pm^n:=2R(|b(g_\pm^n)|+\delta)$:

$$\frac{z/R+|b|}{a}\le z\mathrm{e}^{\delta}\frac{1/R+z^{-1}(|b(g_\pm^n)|+\delta)}{a_\pm^n}\le z\frac{2a_\pm^{-n}}{R}\left(\mathrm{e}^{\delta}\frac{1+z^{-1}R(|b(g_\pm^n)|+\delta)}{2}\right)\le z\frac{2a_\pm^{-n}}{R},$$

$$\frac{zR-|b|}{a}\ge z\mathrm{e}^{-\delta}\frac{R-z^{-1}(|b(g_\pm^n)|+\delta)}{a_\pm^n}\ge z\frac{a_\pm^{-n}R}{2}\cdot2\mathrm{e}^{-\delta}\left(1-z^{-1}\frac{|b(g_\pm^n)|+\delta}{R}\right)\ge z\frac{a_\pm^{-n}R}{2}.$$

Thus by (2.3):

$$\delta_{(0,z^{-1})}*_G\nu\bigl(C(1/R,R)\bigr)\ge\mu^{*n}\bigl(U_\delta\bigl(g_\pm^n\bigr)\bigr)\nu\bigl(C\bigl(z2a_\pm^{-n}/R,za_\pm^{-n}R/2\bigr)\bigr)=C_{K_\pm^n}^{-1}\delta_{(0,z^{-1})}*_G\nu\bigl(K_\pm^n\bigr)$$

for all $z>z_\pm^n$. Since $\delta_{(0,z^{-1})}*_G\nu(C(1/R,R))>0$, the above inequality holds in fact for all $z\ge1$, possibly with a bigger constant C_K and sufficiently large R. We may assume that $R>2\max\{a_+,1/a_-\}$, then the family of sets $K_\pm^n$ covers $\mathbb{R}^d\setminus\{0\}$.

Finally notice, that every function $s\in C_c(\mathbb{R}^d\setminus\{0\})$ and such that $s(u)\ge\mathbf{1}_{C(1/R,R)}(u)$ satisfies (2.2). Indeed, let K be a generic compact set in $\mathbb{R}^d\setminus\{0\}$ covered by a finite number of compacts $\{K_i\}_{i\in I}$ of the type $K_\pm^n$, then

$$\delta_{(0,z^{-1})}*_G\nu(K)\le\sum_{i\in I}\delta_{(0,z^{-1})}*\nu(K_i)\le\Bigl(|I|\max_{i\in I}C_{K_i}\Bigr)\delta_{(0,z^{-1})}*_G\nu(s)$$

for all $z\ge1$.

Step 2. Let $L(z)=\delta_{(0,z^{-1})}*_G\nu(s)$, so that $\delta_{(0,z^{-1})}*_G\nu/L(z)$ is weakly compact when z goes to $+\infty$. It remains to prove that L is a slowly varying function. Fix $a\in G(\mu_A)$ and observe that

$$\frac{L(az)}{L(z)}=\frac{\delta_{(0,a^{-1})}*_G\delta_{(0,z^{-1})}*_G\nu(s)}{L(z)}.$$

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence such that $\delta(0,z_n^{-1})*_G\nu/L(z_n)$ converges to some limit measure η. Then by invariance of the limit measure

$$\lim_{n\to\infty}\frac{L(az_n)}{L(z_n)}=\delta_{(0,a^{-1})}*_G\eta(s)=\eta(s)=1.$$

Since for any sequence, there exists a subsequence such that the conclusion above holds, if μ_A aperiodic, L is slowly varying and the proof is completed.

If μ_A is periodic, that is $G(\mu_A)=\langle\mathrm{e}^p\rangle$, take any continuous compactly supported function $s_0\ge\mathbf{1}_{C(1/R,R)}$, i.e. a function satisfying (2.2), and define

$$s(u)=\int_{\mathbb{R}_+^*}\mathbf{1}_{[\mathrm{e}^{-p},\mathrm{e}^{p})}(t)s_0(u/t)\frac{\mathrm{d}t}{t}.\tag{2.5}$$

An easy argument shows that also s is in $C_c(\mathbb{R}^d \setminus \{0\})$ and it is bigger than some multiple of $\mathbf{1}_{C(1/R,R)}$. We claim that $\delta_{(0,a^{-1})} *_G \eta(s) = \eta(s)$ for all $a \in \mathbb{R}_+^*$ and not only for $a \in G(\mu_A)$ (and thus $L(z)$ is slowly varying). In fact, let $\mathrm{e}^{Kp} \in G(\mu_A)$ such that $\mathrm{e}^{Kp} > a\mathrm{e}^{p}$ then

$$\begin{aligned}
\delta_{(0,a^{-1})} *_G \eta(s) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} \mathbf{1}_{[a\mathrm{e}^{-p}, a\mathrm{e}^{p})}(t) s_0(u/t) \frac{\mathrm{d}t}{t} \eta(\mathrm{d}u) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} \bigl(\mathbf{1}_{[a\mathrm{e}^{-p}, \mathrm{e}^{Kp})}(t) - \mathbf{1}_{[a\mathrm{e}^{p}, \mathrm{e}^{Kp})}(t)\bigr) s_0(u/t) \frac{\mathrm{d}t}{t} \eta(\mathrm{d}u) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} \bigl(\mathbf{1}_{[a\mathrm{e}^{-Kp}, \mathrm{e}^{p})}(t) s_0(u/t) - \mathbf{1}_{[a\mathrm{e}^{-Kp}, \mathrm{e}^{-p})}(t) s_0(u/t)\bigr) \eta(\mathrm{d}u) \frac{\mathrm{d}t}{t} \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} \mathbf{1}_{[\mathrm{e}^{-p}, \mathrm{e}^{p})}(t) s_0(u/t) \frac{\mathrm{d}t}{t} \eta(\mathrm{d}u) = \eta(s),
\end{aligned}$$

since η is $G(\mu_A)$-invariant. □

3. Recurrent potential kernel and solutions of the Poisson equation for general probability measures

As it has been observed in the Introduction, to understand the asymptotic behavior of the measure ν one has to consider the function

$$f_\phi(x) = \int_{\mathbb{R}^d} \phi\bigl(u\mathrm{e}^{-x}\bigr) \nu(\mathrm{d}u)$$

that is a solution of the Poisson equation

$$\overline{\mu} *_{\mathbb{R}} f = f + \psi \tag{3.1}$$

for a peculiar choice of the function $\psi = \psi_\phi = \overline{\mu} *_{\mathbb{R}} f_\phi - f_\phi$.

Studying solutions of such equation for a centered probability measure on $\mathbb{R}$ is a classical problem. Port and Stone in their papers [18,19] give an explicit formula describing all bounded from below solutions of (3.1) in terms of the recurrent potential kernel A of the function ψ. However, they suppose either that the measure is spread-out or, if not, that the Fourier transform of ψ is compactly supported. This second condition is too restrictive in our setting: such functions decay too slowly and the corresponding function ϕ would not be a ν-integrable. For this reason, the results of the previous paper [4] on the decay of the measure ν were obtained under the hypothesis that $\overline{\mu}$ is spread out. To avoid this restriction we need to generalize the technics used by Port and Stone [18] to a larger class of functions $\mathcal{F}(\overline{\mu})$ associated to an arbitrary measure $\overline{\mu}$.

Let $\overline{\mu}$ be a centered probability measure on $\mathbb{R}$ with finite second moment $\sigma^2 = \int_{\mathbb{R}} x^2 \overline{\mu}(\mathrm{d}x)$. We denote by $\widehat{\overline{\mu}}(\theta) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\theta} \overline{\mu}(\mathrm{d}x)$ its Fourier transform and given a function $\psi \in L^1(\mathbb{R})$ we define its Fourier transform by $\widehat{\psi}(\theta) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\theta} \psi(x)\,\mathrm{d}x$.

Let $\mathcal{F}(\overline{\mu})$ be the class of functions ψ, such that

(1) ψ, $x^2\psi$ and $\widehat{\psi}$ are elements of $L^1(\mathbb{R})$,
(2) the function $\frac{\widehat{\psi}(-\theta)}{1-\widehat{\overline{\mu}}(\theta)}$ is $\mathrm{d}\theta$-integrable outside any neighborhood of zero.

The second condition is satisfied e.g. when the measure $\overline{\mu}$ is aperiodic and $\widehat{\psi}$ has a compact support or when the measure $\overline{\mu}$ is spread-out (since is this case $\sup_{|\theta|>a} |\widehat{\overline{\mu}}(\theta)| < 1$). Thus, the set $\mathcal{F}(\overline{\mu})$ contains the set of functions on which Port and Stone define the recurrent potential but it is, in many cases, bigger. In particular we will see in Lemma 3.3, that if the measure has an exponential moment, then $\mathcal{F}(\overline{\mu})$ always contains some functions that decay exponentially. That will be sufficient to prove our main theorem in the next section.

Let $J(\psi) = \int_{\mathbb{R}} \psi(x)\,\mathrm{d}x$ and $K(\psi) = \int_{\mathbb{R}} x\psi(x)\,\mathrm{d}x$, then we have:

Theorem 3.1. *Assume that* $\psi, g \in \mathcal{F}(\overline{\mu})$, g *is positive and such that* $J(g) = 1$. *Then*:

- *The recurrent potential*

$$A\psi(x) := \lim_{\lambda \nearrow 1}\left[J(\psi)\sum_{n=0}^{\infty}\lambda^n \overline{\mu}^{*n} * g(0) - \sum_{n=0}^{\infty}\lambda^n \overline{\mu}^{*n} * \psi(x)\right]$$

  is a well defined continuous function.
- $A\psi$ *is a solution of the Poisson equation* (3.1).
- *If* $J(\psi) \geq 0$, *then* $A\psi$ *is bounded from below and*

$$\lim_{x\to\pm\infty}\frac{A\psi(x)}{x} = \pm\sigma^{-2}J(\psi). \tag{3.2}$$

- *If* $J(\psi) = 0$, *then* $A\psi$ *is bounded and has a limit at infinity*

$$\lim_{x\to\pm\infty} A\psi(x) = \mp\sigma^{-2}K(\psi). \tag{3.3}$$

The proof of this result is rather technical and follows the ideas of [18] and [19]. A sketch of the proof is proposed in the Appendix for reader convenience.

A direct consequence of the previous theorem, is the following characterization of the bounded solutions of the Poisson equation:

Corollary 3.2. *If* $J(\psi) = 0$, *then every continuous solution of the Poisson equation bounded from below is of the form*

$$f = A\psi + h,$$

where h *is constant if* $\overline{\mu}$ *is aperiodic, and it is periodic of period* p *if the support of* $\overline{\mu}$ *is contained in* $p\mathbb{Z}$. *Thus every continuous solution of the Poisson equation is bounded and the limit of* $f(x)$ *exists when* x *goes to* $+\infty$ *and* $x \in G(\overline{\mu})$.

Conversely if there exists a bounded solution of the Poisson equation, then $A\psi$ *is bounded and* $J(\psi) = 0$. *In particular the first part of corollary is valid.*

Proof. Let $J(\psi) = 0$ and assume that f is a continuous solution of the Poisson equation. Since

$$\overline{\mu} * f = f + \psi \quad \text{and} \quad \overline{\mu} * A\psi = A\psi + \psi,$$

the function $h = f - A\psi$ is $\overline{\mu}$-harmonic. It is bounded from below because both $-A\psi$ and f are bounded from below. Therefore by the Choquet–Deny theorem [9], $h(x + y) = h(x)$ for all y in the closed subgroup generated by the support of $\overline{\mu}$.

Conversely, suppose that there exists a bounded solution f_0 of the Poisson equation. Then $A\psi - f_0$ is $\overline{\mu}$-harmonic and bounded from below, and so the Choquet–Deny theorem implies that $A\psi$ is bounded. Thus

$$\lim_{x\to\infty}\frac{A\psi(x)}{x} = 0$$

and by (3.2), we deduce $J(\psi) = 0$. □

As announced we need to construct a class of functions in $\mathcal{F}(\overline{\mu})$ that will be used later on and that have the same type of decay at infinity as $\overline{\mu}$:

Lemma 3.3. *Let* Y *be a random variable with the law* $\overline{\mu}$, *then the function*

$$r(x) = \mathbb{E}\big[|Y + x| - |x|\big]$$

is nonnegative and

$$\widehat{r}(\theta) = C \cdot \frac{\widehat{\overline{\mu}}(-\theta) - 1}{\theta^2}$$

for $\theta \neq 0$. *Moreover if* $\mathbb{E}[e^{\delta Y} + e^{-\delta Y}] < \infty$, *then* $r(x) \le C e^{-\delta_1 |x|}$ *for* $\delta_1 < \delta$.

Hence r *belongs to* $\mathcal{F}(\overline{\mu})$ *and for every function* $\zeta \in L^1(\mathbb{R})$ *such that* $x^2\zeta$ *is integrable the convolution* $r *_{\mathbb{R}} \zeta$ *belongs to* $\mathcal{F}(\overline{\mu})$.

Proof. Observe that, since $\mathbb{E}Y = 0$, for $x \ge 0$ we can write

$$r(x) = \mathbb{E}\big[(Y + x) - 2(Y + x)\mathbf{1}_{Y+x\le 0} - x\big] = -2\mathbb{E}\big[(Y + x)\mathbf{1}_{Y+x\le 0}\big].$$

Proceeding analogously for $x < 0$, we obtain

$$r(x) = \begin{cases} -2\mathbb{E}\big[(Y + x)\mathbf{1}_{Y+x\le 0}\big] & \text{for } x \ge 0, \\ \mathbb{E}\big[(Y + x)\mathbf{1}_{Y+x>0}\big] & \text{for } x < 0. \end{cases} \tag{3.4}$$

Thus the function r is nonnegative.

The Fourier transform of x can be computed in the sense of distributions. Let $a(x) = |x|$ and observe that $r = (\overline{\mu} - \delta_0) * a$. Then $\widehat{a}(\theta) = \frac{C}{\theta^2}$, hence $\widehat{r}(\theta) = C \cdot \frac{\widehat{\overline{\mu}}(-\theta)-1}{\theta^2}$.

To estimate the decay of r we use (3.4). For $x \ge 0$, we write

$$\big|r(x)\big| = 2\mathbb{E}\big[|Y + x|\mathbf{1}_{Y+x\le 0}\big] = 2\int_{x+y\le 0} |x + y|\overline{\mu}(\mathrm{d}y) \le 2\int_{\mathbb{R}} |x + y| e^{-\delta_0(x+y)}\overline{\mu}(\mathrm{d}y) \le C e^{-\delta_1 x}$$

for some constants $\delta_1 < \delta_0 < \delta$.

Finally, if $\psi = r * \zeta$ with ζ and $x^2\zeta$ in $L^1(\mathbb{R})$, then it is easily checked that both ψ and $x^2\psi$ are integrable. Since $\widehat{\psi}(\theta) = \widehat{r}(\theta)\widehat{\zeta}(\theta) = C\frac{\widehat{\overline{\mu}}(-\theta)-1}{\theta^2}\widehat{\zeta}(\theta)$ and $\widehat{\zeta}$ vanishes at infinity, $\psi \in \mathcal{F}(\overline{\mu})$. □

4. Proofs of Theorems 1.1 and 1.2 – Existence of the limit

Our aim is to apply the results of Section 3 and for this purpose we need to show that ψ_ϕ is sufficiently integrable. The upper bound of the tail of ν given in Section 2 will guarantee integrability for positive x. To control the function for x negative we need to perturb slightly the measures μ and ν in order to have more integrability near 0. This is included in the following lemma proved in [4] (Lemma 4.1).

Lemma 4.1. *For all* $x_0 \in \mathbb{R}^d$ *the translated measure* $\nu_0 = \delta_{x_0} *_{\mathbb{R}^d} \nu$ *is the unique invariant measure of* $\mu_0 = \delta_{(x_0,1)} *_G \mu *_G \delta_{(-x_0,1)}$ *and it has the same behavior as* ν *at infinity, that is*:

$$\lim_{x\to+\infty}\left(\int_{\mathbb{R}^d} \phi\big(u e^{-x}\big)\nu(\mathrm{d}u) - \int_{\mathbb{R}^d} \phi\big(u e^{-x}\big)\nu_0(\mathrm{d}u)\right) = 0$$

for every function $\phi \in C_c^1(\mathbb{R}^d \setminus \{0\})$. *Furthermore there is* $x_0 \in \mathbb{R}^d$ *such that the measure* ν_0 *satisfies*

$$\int_{\mathbb{R}^d} \frac{1}{|u|^\gamma}\nu_0(\mathrm{d}u) < \infty \quad \textit{for all } \gamma \in (0, 1). \tag{4.1}$$

Using (4.1) we can guarantee that the function ψ_ϕ decays quickly at infinity, as it is proved in the following lemma.

Lemma 4.2. *Assume that hypotheses* (**H**) *and* $\mathbf{M}(\delta)$ *are satisfied. Furthermore assume that the function* $|u|^{-\gamma}$ *is* $\nu(\mathrm{d}u)$*-integrable for all* $\gamma \in (0,1)$. *Let* ϕ *be a continuous function on* $\mathbb{R}^d$ *such that* $|\phi(u)| \le C(1+|u|)^{-\beta}$ *for some* $\beta, C > 0$. *Then* f_ϕ *and* $\overline{\mu} * f_\phi$ *are well defined and continuous. Furthermore if* ϕ *is Lipschitz, then*

$$\int_{\mathbb{R}} \int_G \int_{\mathbb{R}^d} \bigl|\phi\bigl(\mathrm{e}^{-x}(au+b)\bigr) - \phi\bigl(\mathrm{e}^{-x}au\bigr)\bigr| \nu(\mathrm{d}u)\mu(\mathrm{d}b\,\mathrm{d}a)\,\mathrm{d}x < \infty \tag{4.2}$$

and

$$\bigl|\psi_\phi(x)\bigr| \le C\mathrm{e}^{-\zeta|x|}$$

for $\zeta < \min\{\delta/4, \beta, 1\}$.

Proof. If $\zeta < \min\{\beta, 1\}$, then

$$\bigl|f_\phi(x)\bigr| \le \int_{\mathbb{R}^d} \bigl|\phi\bigl(\mathrm{e}^{-x}u\bigr)\bigr|\nu(\mathrm{d}u) \le \int_{\mathbb{R}^d} \frac{C}{\mathrm{e}^{-\zeta x}|u|^\zeta}\nu(\mathrm{d}u) \le C\mathrm{e}^{\zeta x}.$$

If we suppose also $\zeta \le \delta$, we have that

$$\bigl|\overline{\mu} * f_\phi(x)\bigr| \le \int_{\mathbb{R}} \bigl|f_\phi(x+y)\bigr|\overline{\mu}(\mathrm{d}y) \le C\mathrm{e}^{\zeta x} \int_{\mathbb{R}^+} a^{-\zeta}\mu_A(\mathrm{d}a) \le C\mathrm{e}^{\zeta x}.$$

Thus $\psi_\phi = \overline{\mu} *_{\mathbb{R}} f_\phi - f_\phi$ is well defined, continuous and $|\psi_\phi(x)| \le C\mathrm{e}^{\zeta x}$, that gives the required estimates for negative x. In order to prove (4.2) we divide the integral into two parts. For negative x we use the estimates given above:

$$\begin{aligned}
&\int_{-\infty}^0 \int_G \int_{\mathbb{R}^d} \bigl|\phi\bigl(\mathrm{e}^{-x}(au+b)\bigr) - \phi\bigl(\mathrm{e}^{-x}au\bigr)\bigr|\nu(\mathrm{d}u)\mu(\mathrm{d}b\,\mathrm{d}a)\,\mathrm{d}x \\
&\quad\le \int_{-\infty}^0 \int_{\mathbb{R}^d} \bigl|\phi\bigl(\mathrm{e}^{-x}u\bigr)\bigr|\nu(\mathrm{d}u)\,\mathrm{d}x + \int_{-\infty}^0 \int_G \int_{\mathbb{R}^d} \bigl|\phi\bigl(\mathrm{e}^{-x}au\bigr)\bigr|\nu(\mathrm{d}u)\mu(\mathrm{d}b\,\mathrm{d}a)\,\mathrm{d}x \\
&\quad\le \int_{-\infty}^0 \bigl|f_{|\phi|}(x)\bigr|\,\mathrm{d}x + \int_{-\infty}^0 \bigl|\overline{\mu} * f_{|\phi|}(x)\bigr|\,\mathrm{d}x < \infty.
\end{aligned}$$

To estimate the integral of $|\phi(\mathrm{e}^{-x}au) - \phi(\mathrm{e}^{-x}(au+b))|$ for x positive, we use the Lipschitz property of ϕ to obtain the following inequality for $0 \le \theta \le 1$

$$\bigl|\phi(s) - \phi(r)\bigr| \le C|s-r|^\theta \max_{\xi \in \{|s|,|r|\}} \frac{1}{(1+\xi)^{\beta(1-\theta)}}.$$

Again we divide the integral into two parts. First we consider the integral over the set where $|au+b| \ge \frac{1}{2}a|u|$. We choose $\theta < \min\{\delta/2, 1\}$, $\gamma < \min\{\theta/2, \beta(1-\theta)\}$. Then, in view of $\mathbf{M}(\delta)$, we have

$$\begin{aligned}
&\int\int_{|au+b|\ge(1/2)|au|} \bigl|\phi\bigl(\mathrm{e}^{-x}au\bigr) - \phi\bigl(\mathrm{e}^{-x}(au+b)\bigr)\bigr|\mu(\mathrm{d}b\,\mathrm{d}a)\nu(\mathrm{d}u) \\
&\quad\le \int_G \int_{\mathbb{R}^d} \frac{C|\mathrm{e}^{-x}b|^\theta}{(1+|\mathrm{e}^{-x}au|)^{\beta(1-\theta)}}\nu(\mathrm{d}u)\mu(\mathrm{d}b\,\mathrm{d}a) \le \int_G \int_{\mathbb{R}^d} \frac{C|\mathrm{e}^{-x}b|^\theta}{|\mathrm{e}^{-x}au|^\gamma}\nu(\mathrm{d}u)\mu(\mathrm{d}b\,\mathrm{d}a) \\
&\quad\le C\mathrm{e}^{-(\theta-\gamma)x} \int_G |b|^\theta|a|^{-\gamma}\mu(\mathrm{d}b\,\mathrm{d}a) \int_{\mathbb{R}^d} |u|^{-\gamma}\nu(\mathrm{d}u) \\
&\quad\le C\mathrm{e}^{-(\theta-\gamma)x} \int_G \bigl(|b|^{2\theta} + |a|^{-2\gamma}\bigr)\mu(\mathrm{d}b\,\mathrm{d}a) \le C\mathrm{e}^{-\gamma x}.
\end{aligned}$$

If $|au+b| < \frac{1}{2}a|u|$ then $|u| \le \frac{2|b|}{a}$. Therefore choosing θ as above and $\gamma < \frac{\delta}{2} - \theta$, in view of Proposition 2.1, for the remaining part of the integral, applying again $\mathbf{M}(\delta)$, we have

$$\begin{aligned}
&\int\int_{|au+b|\le(1/2)|au|} \bigl|\phi\bigl(\mathrm{e}^{-x}au\bigr) - \phi\bigl(\mathrm{e}^{-x}(au+b)\bigr)\bigr|\mu(\mathrm{d}b\,\mathrm{d}a)\nu(\mathrm{d}u)\\
&\quad\le \int\int_{|u|\le 2|b|/a} \bigl|\mathrm{e}^{-x}b\bigr|^{\theta}\nu(\mathrm{d}u)\mu(\mathrm{d}b\,\mathrm{d}a)\\
&\quad\le C\int_G \bigl|\mathrm{e}^{-x}b\bigr|^{\theta}\biggl(1+\frac{2|b|}{a}\biggr)^{\gamma}\mu(\mathrm{d}b\,\mathrm{d}a) \le C\mathrm{e}^{-\theta x}\int_G |b|^{\theta}\biggl(1+\frac{2|b|}{a}\biggr)^{\gamma}\mu(\mathrm{d}b\,\mathrm{d}a)\\
&\quad\le C\mathrm{e}^{-\theta x}\int_G \bigl(|b|^{\theta} + |b|^{2(\theta+\gamma)} + a^{-2\gamma}\bigr)\mu(\mathrm{d}b\,\mathrm{d}a) \le C\mathrm{e}^{-\theta x}.
\end{aligned}$$

That proves (4.2) and finally

$$\bigl|\psi_{\phi}(x)\bigr| \le \int_G\int_{\mathbb{R}^d} \bigl|\phi\bigl(\mathrm{e}^{-x}au\bigr) - \phi\bigl(\mathrm{e}^{-x}(au+b)\bigr)\bigr|\nu(\mathrm{d}u)\mu(\mathrm{d}b\,\mathrm{d}a) < C\mathrm{e}^{-\zeta|x|}$$

for $\zeta < \min\{\delta/4, \beta, 1\}$. □

The following proposition contains key arguments of the proof of our main results.

Proposition 4.3. *Assume that hypothesis* $(\mathbf{H})$ *and* $\mathbf{M}(\delta)$ *are satisfied. The family of measures* $\delta_{(0,\mathrm{e}^{-x})} *_G \nu$ *is relatively compact in the weak topology on* $\mathbb{R}^d \setminus \{0\}$.

Suppose that r *belongs to* $\mathcal{F}(\overline{\mu})$ *and has an exponential decay. Let* ζ *be a nonnegative Lipschitz function on* $\mathbb{R}^d \setminus \{0\}$ *such that* $\zeta(u) \le \mathrm{e}^{-\gamma|\log|u||}$ *for* $\gamma > 0$ *and set*

$$\phi(u) := \int_{\mathbb{R}} r(t)\zeta\bigl(\mathrm{e}^{t}u\bigr)\,\mathrm{d}t. \tag{4.3}$$

Then, the limit

$$\lim_{x\to+\infty}\int_{\mathbb{R}^d}\phi\bigl(u\mathrm{e}^{-x}\bigr)\nu(\mathrm{d}u)$$

exists, it is finite and equal to $\eta(\phi)$ *for any limit measure* η.

Proof. *Step 1.* First we suppose that μ satisfies (4.1). We are going to show that for functions of type (4.3) the limit

$$\lim_{x\to+\infty}\int_{\mathbb{R}^d}\phi\bigl(u\mathrm{e}^{-x}\bigr)\nu(\mathrm{d}u) = T(\phi) := -2\sigma^{-2}K(\psi_{\phi})$$

exists and it is finite. To do this we will prove that ψ_{ϕ} is an element of $\mathcal{F}(\overline{\mu})$ and $J(\psi_{\phi}) = 0$. Thus, by Corollary 3.2, the function $f_{\phi}(x) = \int_{\mathbb{R}^d}\phi(u\mathrm{e}^{-x})\nu(du)$ is the solution of the corresponding Poisson equation, it is bounded and it has a limit when x converge to $+\infty$.

First observe that by Lemma 3.3 (in view of $\mathbf{M}(\delta)$ its assumptions are satisfied), for $\beta < \min\{\delta, \gamma\}$, we have

$$\begin{aligned}
\bigl|\phi(u)\bigr| &\le C\int_{\mathbb{R}} \mathrm{e}^{-\beta|t|}\mathrm{e}^{-\gamma|t+\log|u||}\,\mathrm{d}t \le C\int_{\mathbb{R}} \mathrm{e}^{-\beta(|t-\log|u||)}\mathrm{e}^{-\gamma|t|}\,\mathrm{d}t\\
&\le C\int_{\mathbb{R}} \mathrm{e}^{-\beta(-|t|+|\log|u||)}\mathrm{e}^{-\gamma|t|}\,\mathrm{d}t = C\mathrm{e}^{-\beta|\log|u||}.
\end{aligned}$$

Thus by Lemma 4.2, f_ϕ, f_ζ, $\overline{\mu} * f_\phi$ and $\overline{\mu} * f_\zeta$ are well defined. Furthermore, since ζ is Lipschitz ψ_ζ is bounded, and $x^2\psi_\zeta(x)$ is integrable on $\mathbb{R}$. We cannot guarantee that ϕ is Lipschitz, but we can observe that

$$f_\phi(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}} r(t)\zeta\left(\mathrm{e}^{-x+t}u\right)\mathrm{d}t\,\nu(\mathrm{d}u) = \int_{\mathbb{R}} r(t+x) f_\zeta(-t)\,\mathrm{d}t = r *_{\mathbb{R}} f_\zeta(x)$$

and

$$\overline{\mu} * f_\phi(x) = r *_{\mathbb{R}} (\overline{\mu} * f_\zeta)(x).$$

Hence

$$\psi_\phi = f_\phi - \overline{\mu} * f_\phi = r * (f_\zeta - \overline{\mu} * f_\zeta) = r *_{\mathbb{R}} \psi_\zeta$$

and, by Lemma 3.3, $\psi_\phi \in \mathcal{F}(\overline{\mu})$.

Furthermore if ζ is radial then $J(\psi_\phi) = 0$. In fact, let ζ_r be the radial part of ζ, i.e. $\zeta_r(|u|) = \zeta(u)$, then

$$\begin{aligned}\int_{\mathbb{R}} \psi_\zeta(x)\,\mathrm{d}x &= \int_{\mathbb{R}}\int_G\int_{\mathbb{R}^d}\left[\zeta\left(au\mathrm{e}^{-x}\right) - \zeta\left(\mathrm{e}^{-x}(au+b)\right)\right]\nu(\mathrm{d}u)\mu(\mathrm{d}b\,\mathrm{d}a)\,\mathrm{d}x \\ &= \int_G\int_{\mathbb{R}^d}\int_{\mathbb{R}}\left[\zeta_r\left(\mathrm{e}^{-x+\log(|au|)}\right) - \zeta_r\left(\mathrm{e}^{-x+\log|au+b|}\right)\right]\mathrm{d}x\,\nu(\mathrm{d}u)\mu(\mathrm{d}b\,\mathrm{d}a) \\ &= \int_G\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}}\zeta_r\left(\mathrm{e}^{-x}\right)\mathrm{d}x - \int_{\mathbb{R}}\zeta_r\left(\mathrm{e}^{-x}\right)\mathrm{d}x\right)\nu(\mathrm{d}u)\mu(\mathrm{d}b\,\mathrm{d}a) = 0.\end{aligned}$$

Observe that we can apply the Fubini theorem since ζ is Lipschitz and, by Lemma 4.2, the absolute value of the integrand in the second line above is integrable. Hence

$$J(\psi_\phi) = \int_{\mathbb{R}} \psi_\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}} r * \psi_\zeta(x)\,\mathrm{d}x = \int_{\mathbb{R}} r(x)\,\mathrm{d}x \cdot \int_{\mathbb{R}} \psi_\zeta(x)\,\mathrm{d}x = 0.$$

If ζ is radial, then by Corollary 3.2, we have

$$f_\phi = A\psi_\phi + h_\phi, \tag{4.4}$$

where h_ϕ is a constant if μ_A is aperiodic and a continuous periodic function if μ_A is periodic. In any case f_ϕ is a bounded function.

In particular the same holds for f_{Φ_γ}, where

$$\Phi_\gamma(u) = \int_{\mathbb{R}} r(t)\mathrm{e}^{-\gamma|t+\log|u||}\,\mathrm{d}t.$$

For an arbitrary nonradial function ϕ of the type (4.3), there exists $\gamma > 0$ such that $\phi \le \Phi_\gamma$. Hence $f_\phi \le f_{\Phi_\gamma}$ and f_ϕ is a bounded solution of the Poisson equation associated to ψ_ϕ. Therefore, by Corollary 3.2, $J(\psi_\phi) = 0$ and $f_\phi = A\psi_\phi + h_\phi$. Since the measure ν has no mass at zero, $\lim_{x\to-\infty} f_\phi(x) = 0$ and by Theorem 3.1

$$\lim_{x\to-\infty} A\psi_\phi(x) = \sigma^{-2}K(\psi_\phi).$$

Thus when x goes to $-\infty$ the limit of h_ϕ exists which is possible only if h_ϕ is constant and equal to $-\sigma^{-2}K(\psi_\phi)$. Finally

$$\lim_{x\to+\infty} f_\phi(x) = \lim_{x\to+\infty} A\psi_\phi(x) - \sigma^{-2}K(\psi_\phi) = -2\sigma^{-2}K(\psi_\phi) =: T(\phi).$$

Step 2. The result of step 1 implies, in particular, that the family $\delta_{(0,\mathrm{e}^{-x})} *_G \nu$ of measures on $\mathbb{R}^d \setminus \{0\}$ is bounded on compact sets, hence it is relatively compact in the weak topology. Let η be a limit measure along the sequence $\{x_n\}$. We are going to show that $T(\phi) = \eta(\phi)$ for all functions ϕ of type (4.3).

If ϕ is compactly supported then

$$\delta_{(0,\mathrm{e}^{-x_n})} *_G \nu(\phi) = f_\phi(x_n) \to \eta(\phi).$$

We need now to generalize the above convergence to any function ϕ such that $\phi(u) \le \mathrm{e}^{-\gamma|\log|u||}$. Since

$$\sup_{x\in\mathbb{R}} \nu\big(\mathrm{e}^x < |u| \le \mathrm{e}^{x+1}\big) = K < \infty,$$

then, for any $M > 1$, we have

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}^d} \mathrm{e}^{-\gamma|\log|\mathrm{e}^{-x}u||} \mathbf{1}_{[1/M,M]^c}\big(\big|\mathrm{e}^{-x}u\big|\big)\nu(\mathrm{d}u) \le K\bigg(\int_{M/\mathrm{e}}^{\infty} \mathrm{e}^{-\gamma|\log(a)|}\frac{\mathrm{d}a}{a} + \int_0^{\mathrm{e}/M} \mathrm{e}^{-\gamma|\log(a)|}\frac{\mathrm{d}a}{a}\bigg).$$

Letting M go to infinity, the right-hand side of the inequality goes to zero. Thus the family of bounded measures

$$\rho_n(g) = \delta_{(0,\mathrm{e}^{-x_n})} *_G \nu(g\Phi_\gamma)$$

is tight and it converges for all continuous bounded functions g. Take $g = \phi/\Phi_\gamma$ to conclude.

Step 3. Now we return to the general case when the condition (4.1) does not necessarily hold. Then by Lemma 4.1 there exists $\nu_0 = \delta_{x_0} *_{\mathbb{R}^d} \nu$ for which (4.1) holds and $\delta_{(0,\mathrm{e}^{-x})} *_G \nu$ and $\delta_{(0,\mathrm{e}^{-x})} *_G \nu_0$ have the same behavior on compactly supported functions when x go to $+\infty$. Since

$$\sup_{x\in\mathbb{R}} \delta_{(0,\mathrm{e}^{-x})} * \nu\big(1 \le |u| \le \mathrm{e}\big) = K < \infty, \qquad \sup_{x\in\mathbb{R}} \delta_{(0,\mathrm{e}^{-x})} * \nu_0\big(1 \le |u| \le \mathrm{e}\big) = K_0 < \infty,$$

reasoning as in the previous step, we prove that the families of measures $\rho_x(\cdot) = \delta_{(0,\mathrm{e}^{-x})} * \nu(\cdot\Phi_\gamma)$ and $\rho_x^0(\cdot) = \delta_{(0,\mathrm{e}^{-x})} * \nu_0(\cdot\Phi_\gamma)$ are tight, thus have the same limit on bounded functions. Then, for all functions ϕ of type (4.3),

$$\lim_{x\to\infty} \delta_{(0,\mathrm{e}^{-x})} *_G \nu(\phi) = \lim_{x\to\infty} \delta_{(0,\mathrm{e}^{-x})} *_G \nu_0(\phi)$$

and the proof is finished. ☐

Proof of Theorem 1.1 – Existence of the limit. We assume that μ_A is aperiodic. In view of Proposition 4.3 the family of measures $\delta_{(0,\mathrm{e}^{-x})} *_G \nu$ is relatively compact in the weak topology and if η is an accumulation point, then it is $\mathbb{R}^+$ invariant. Therefore, there exists a probability measure Σ_η on S^{d-1} and a constant C_η such that $\eta = C_\eta \frac{\mathrm{d}a}{a} \otimes \Sigma_\eta$ (see [12], Proposition 1.15). It remains to prove that C_η and Σ_η do not depend on η. We have proved in Proposition 4.3 that for any function ϕ of type (4.3), the limit exists (that is it does not depend on the subsequence along which one tends to η)

$$\lim_{x\to+\infty} \int_{\mathbb{R}^d} \phi\big(\mathrm{e}^{-x}u\big)\nu(\mathrm{d}u) = \eta(\phi) = T(\phi).$$

Consider the radial function $\Phi_\gamma(u) = \int_{\mathbb{R}} r(t)\mathrm{e}^{-\gamma|t+\log|u||}\,\mathrm{d}t$, since $\eta(\Phi_\gamma) = C_\eta \int_{\mathbb{R}^+} \Phi_\gamma(a)\frac{\mathrm{d}a}{a}$. Then:

$$C_\eta = \frac{T(\Phi_\gamma)}{\int_{\mathbb{R}^+} \Phi_\gamma(a)(\mathrm{d}a/a)}$$

does not depend on η. Set $C_+ = C_\eta$.

For any Lipschitz function ζ_0 of S^{d-1} consider the function $\zeta(u) = \mathrm{e}^{-\gamma|\log|u||}\zeta_0(u/|u|)$ and

$$\phi(u) = \int_{\mathbb{R}} r(t)\zeta\big(\mathrm{e}^t u\big)\,\mathrm{d}t = \Phi_\gamma(u) = \int_{\mathbb{R}} r(t)\mathrm{e}^{-\gamma|t+\log|u||}\zeta_0\big(\mathrm{e}^{-t}u/\big|\mathrm{e}^{-t}u\big|\big)\,\mathrm{d}t = \Phi_\gamma(u)\zeta_0\big(u/|u|\big).$$

Then

$$\eta(\phi) = C_+\Sigma_\eta(\zeta_0)\cdot\int_{\mathbb{R}^+} \Phi_\gamma(a)\frac{\mathrm{d}a}{a} = T(\phi)$$

thus $\Sigma_\eta(\zeta_0)$ does not depend on η. □

Proof of Theorem 1.2 – Existence of the limit. Assume that μ_A is periodic and $G(\mu_A) = \langle \mathrm{e}^p \rangle$. Let $D = \{w \in \mathbb{R}^d \setminus \{0\}\colon 1 \leq |w| < \mathrm{e}^p\}$ be the fundamental domain for the action of $G(\mu_A)$ on $\mathbb{R}^d \setminus \{0\}$. Then every $z \in \mathbb{R}^d \setminus \{0\}$ can be uniquely written as $z = aw$, where $a \in G(\mu_A)$ and $w \in D$. Denote by l the counting measure on $G(\mu_A)$, that is $l(\phi) = \sum_{k\in\mathbb{Z}} \phi(\mathrm{e}^{kp})$. Let η be an accumulation point of the family of measures $\delta_{(0,\mathrm{e}^{-x})} *_G \nu$. Then, in view of Proposition 4.3, η is $G(\mu_A)$ invariant. Therefore there exists a probability measure Σ_η on D and a constant C_η such that $\eta = C_\eta l \otimes \Sigma_\eta$.

Observe that any radial function ϕ on $\mathbb{R}^d$ can also be seen as a function on $\mathbb{R}^+$, thus in abuse of notation, we will use below the same symbol to denote both a radial function on $\mathbb{R}^d$ and its projection on $\mathbb{R}^+$. Let ϕ be a nonnegative element of $\mathcal{T}$, then

$$\eta(\phi) = C_\eta \sum_{k\in\mathbb{Z}} \int_D \phi\bigl(\mathrm{e}^{kp} w\bigr) \Sigma_\eta(\mathrm{d}w) = C_\eta \sum_{k\in\mathbb{Z}} \int_D \phi\bigl(\mathrm{e}^{kp} |w|\bigr) \Sigma_\eta(\mathrm{d}w) = C_\eta l(\phi).$$

If ϕ is Lipschitz and belongs to $\mathcal{T}$ (for instance $\phi(u) = \tau(u) = (1 - |\log|u||/p)^+$, the triangular function of "base" $2p$), we can apply Proposition 4.3 to the function $\Phi(u) = \int_{\mathbb{R}} r(t)\phi(\mathrm{e}^t u)\,\mathrm{d}t$. Since Φ also belongs to $\mathcal{T}$, we conclude that the value $C_\eta = T(\Phi)/l(\Phi) =: C_+$ does not depend on η. Thus for any $\phi \in \mathcal{T}$ we have

$$\lim_{z\to+\infty} \int_{\mathbb{R}^d} \phi\bigl(z^{-1}u\bigr)\nu(\mathrm{d}u) = C_+ l(\phi).$$

We calculate now the limit of $\nu(|u| \leq z)$. Observe that for the triangular function τ one has $l(\tau) = 1$ and, for any $\varepsilon > 0$, there exits N such that $|\int_{\mathbb{R}^d} \tau(\mathrm{e}^{-kp}u)\nu(\mathrm{d}u) - C_+| \leq \varepsilon$ for all $k \geq N$. Then since

$$\sum_{k=N+1}^{[\log z/p]-1} \tau\bigl(\mathrm{e}^{-kp}u\bigr) \leq \mathbf{1}_{]\mathrm{e}^{Np},z]}(u) \leq \sum_{k=N}^{[\log z/p]+1} \tau\bigl(\mathrm{e}^{-kp}u\bigr)$$

and $\nu(|u| \leq \mathrm{e}^{Np}) < \infty$, we have

$$\limsup_{z\to+\infty} \left| \frac{p\nu(|u| \leq z)}{\log(z)} - C_+ \right| \leq \varepsilon.$$

□

We would like to remark that, in the periodic case, we could not prove that the measures of the dilated annulus $\mathcal{C}(z\alpha, z\beta) = \{u\colon z\alpha \leq |u| < z\beta\}$ converge as z goes to infinity. What is proved by the arguments above is that, if $\{z_n\}$ is a sequence along which the dilated measure converge to a limit measure η, then

$$\lim_{n\to\infty} \nu\bigl(\mathcal{C}\bigl(z_n a, z_n a\mathrm{e}^p\bigr)\bigr) = C_+$$

for all but countably many a, namely for all a such that $\eta(|u| = a) = 0$. Thus there is still open question, wheather the dilated measure $\delta_{(0,z^{-1})} * \nu$ converges weakly for $z \to \infty$.

5. Positivity of the limiting constant

In this section we are going to discuss nondegeneracy of the limit measure (1.3) and to finish the proofs of Theorems 1.1 and 1.2. A partial result was obtained in [4] in the one-dimensional case and $B \geq \varepsilon$ a.s. In this particular case positivity of the constant follows immediately from the formula defining C_+.

Now we are going to prove

Theorem 5.1. *If hypothesis* (**H**) *is satisfied, then for all* $\alpha, \beta > 0$

$$\limsup_{z\to\infty} \nu\bigl\{u \in \mathbb{R}^d\colon z\alpha < |u| \leq z\beta\bigr\} > 0. \tag{5.1}$$

To prove this theorem, we will need the following explicit construction of the measure ν obtained in [1]. Define a random walk on $\mathbb{R}$ by $S_0 = 0$ and $S_n = \log(A_1 \cdots A_n)$ for $n \geq 1$ and consider the downward ladder times of S_n: $L_0 = 0$ and $L_n = \inf\{k > L_{n-1};\ S_k < S_{L_{n-1}}\}$. Let $L = L_1$. The Markov process $\{X^x_{L_n}\}$ satisfies the recursion $X^x_{L_n} = M_n X^x_{L_{n-1}} + Q_n$, where (Q_n, M_n) is a sequence of G-valued i.i.d. random variables and $(Q_n, M_n) =_d (X_L, \mathrm{e}^{S_L})$. It is known that $-\infty < \mathbb{E}[S_L] < 0$ and $\mathbb{E}[\log^+ |X_L|] < \infty$ (see [10,14]). Therefore there exists a unique invariant probability measure ν_L of the process $\{X_{L_n}\}$ and the measure ν can be written (up to a constant) as

$$\nu(f) = \int_{\mathbb{R}^d} \mathbb{E}\left[\sum_{n=0}^{L-1} f\big(X^x_n\big)\right] \nu_L(\mathrm{d}x), \tag{5.2}$$

where X^x_n is the process defined in (1.1).

In the proof of Theorem 5.1, we will use a generalized version of the duality lemma. Let $W_i = (Y_i, Z_i)$ be a sequence of i.i.d. random variables on $\mathbb{R} \times \mathbb{R}$ and let $S_n = \sum_{i=1}^n Y_i$ if $n \geq 1$ and $S_0 = 0$ (later we will take $W_i = (\log A_i, B_i))$. We define a sequence of stopping times: $T_0 = 0$, $T_i = \inf\{n > T_{i-1}\colon\ S_n \geq S_{T_{i-1}}\}$ and we put $L = \inf\{n\colon\ S_n < 0\}$. If the events are void then the stopping times are equal to ∞.

Lemma 5.2 (Duality lemma). *Consider a sequence of nonnegative functions*

$$\alpha_n : (\mathbb{R} \times \mathbb{R})^n \to \mathbb{R}$$

for $n \geq 1$, α_0 *equal to some constant and* $\alpha_\infty = 0$. *Then*

$$\mathbb{E}\left[\sum_{i=0}^{L-1} \alpha_i(W_1, \ldots, W_i)\right] = \mathbb{E}\left[\sum_{i=0}^{\infty} \alpha_{T_i}(W_{T_i}, \ldots, W_1)\right].$$

Proof. Although the technic of proof is classical (see for instance [11]), we present here a complete argument for reader's convenience.

We have

$$\mathbb{E}\left[\sum_{i=0}^{L-1} \alpha_i(W_1, \ldots, W_i)\right] = \alpha_0 + \sum_{i=1}^{\infty} \mathbb{E}\big[\mathbf{1}_{[S_j \geq 0\, \forall j=1,\ldots,i]} \alpha_i(W_1, \ldots, W_i)\big].$$

For fixed i, consider the reversed time sequence $\overline{W}_k = W_{i-k+1}$ and observe that the vector $(\overline{W}_1, \ldots, \overline{W}_i) = (W_i, \ldots, W_1)$ has the same law as $(W_1, \ldots, W_i)$. Thus

$$\begin{aligned}
\mathbb{E}\big[\mathbf{1}_{[S_j \geq 0\, \forall j=1,\ldots,i]} \alpha_i(W_1, \ldots, W_i)\big] &= \mathbb{E}\big[\mathbf{1}_{[\sum_{k=1}^j Y_k \geq 0\, \forall j=1,\ldots,i]} \alpha_i(W_1, \ldots, W_i)\big] \\
&= \mathbb{E}\big[\mathbf{1}_{[\sum_{k=1}^j \overline{Y}_k \geq 0\, \forall j=1,\ldots,i]} \alpha_i(\overline{W}_1, \ldots, \overline{W}_i)\big] \\
&= \mathbb{E}\big[\mathbf{1}_{[\sum_{k=1}^j Y_{i-k+1} \geq 0\, \forall j=1,\ldots,i]} \alpha_i(W_i, \ldots, W_1)\big] \\
&= \mathbb{E}\big[\mathbf{1}_{[S_i \geq S_l\, \forall l=0,\ldots,i-1]} \alpha_i(W_i, \ldots, W_1)\big] = \mathbb{E}\big[\mathbf{1}_{[\exists k \geq 1: i = T_k]} \alpha_i(W_i, \ldots, W_1)\big].
\end{aligned}$$

Then

$$\mathbb{E}\left[\sum_{i=0}^{L-1} \alpha_i(W_1, \ldots, W_i)\right] = \alpha_0 + \sum_{i=1}^{\infty} \mathbb{E}\big[\mathbf{1}_{[\exists k \geq 1: i = T_k]} \alpha_i(W_i, \ldots, W_1)\big] = \mathbb{E}\left[\sum_{k=0}^{\infty} \alpha_{T_k}(W_{T_k}, \ldots, W_1)\right].$$

□

Proof of Theorem 5.1. *Step 1.* First we claim that there exist two positive constants C and M such that for every positive nonincreasing f on $\mathbb{R}^+$

$$\int_{\mathbb{R}^d} f\big(|u|\big) \nu(\mathrm{d}u) \geq C \int_M^\infty f(a) \frac{\mathrm{d}a}{a}. \tag{5.3}$$

Take $Y_i = \log A_i$ and $S_n = \sum_{i=1}^n Y_i$. Choose a ball B of $\mathbb{R}^d$ of radius R such that $\nu_L(B) = C_R > 0$. We have

$$\begin{aligned}\int_{\mathbb{R}^d} f\big(|u|\big)\nu(\mathrm{d}u) &\geq \int_B \mathbb{E}\Bigg[\sum_{n=0}^{L-1} f\big(|A_1A_2\cdots A_nx + A_2A_3\cdots A_nB_1 + \cdots + B_n|\big)\Bigg]\nu_L(\mathrm{d}x)\\ &\geq C_R\mathbb{E}\Bigg[\sum_{n=0}^{L-1} f\big(A_1A_2\cdots A_n\big(R + |B_1| + \cdots + |B_n|\big)\big)\Bigg]\\ &= C_R\mathbb{E}\Bigg[\sum_{n=0}^{L-1} f\big(\mathrm{e}^{S_n + \log(R + \sum_{i=1}^n |B_i|)}\big)\Bigg] = C_R\mathbb{E}\Bigg[\sum_{n=0}^{\infty} f\big(\mathrm{e}^{S_{T_n} + \log(R + \sum_{i=1}^{T_n} |B_i|)}\big)\Bigg].\end{aligned}$$

In the last line we have applied the duality lemma to the functions:

$$\alpha_n\big((Y_1, B_1), \ldots, (Y_n, B_n)\big) = f\big(\mathrm{e}^{\sum_{i=1}^n Y_i + \log(R + \sum_{i=1}^n |B_i|)}\big).$$

Consider two sequences of i.i.d. variables

$$U_j = \max\big\{\log\big(1 + R + |B_i|\big) \colon i = T_{j-1} + 1, \ldots, T_j\big\}$$

and

$$V_j = S_{T_j} - S_{T_{j-1}} + \log(T_j - T_{j-1}) + U_j.$$

Observe that for $n \geq 1$

$$\begin{aligned}S_{T_n} + \log\Bigg(R + \sum_{i=1}^{T_n} |B_i|\Bigg) &\leq S_{T_n} + \log\Bigg(\sum_{j=1}^n \Bigg(R + \sum_{i=T_{j-1}+1}^{T_j} |B_i|\Bigg)\Bigg)\\ &\leq \sum_{j=1}^n \Bigg((S_{T_j} - S_{T_{j-1}}) + \log\Bigg(1 + R + \sum_{T_{j-1}+1 \leq i \leq T_j} |B_i|\Bigg)\Bigg) \leq \sum_{j=1}^n V_j.\end{aligned}$$

We claim that the variables V_j are integrable. In fact since $Y_i = \log A_i$ has a moment of order $2 + \varepsilon$, then classical results guarantee that $S_{T_j} - S_{T_{j-1}}$ is integrable and $T_j - T_{j-1}$ has a moment of order $1/(2 + \varepsilon)$. So we need only to prove that the variable U_j has the first moment (see [7], p. 1279). By the Borel–Cantelli Lemma it sufficient to show that

$$\limsup_{n\to\infty} \frac{1}{n} U_n < M \quad \text{a.s.}$$

for some constant M. We have

$$\frac{1}{n} U_n = \frac{\sum_{j=1}^n (T_j - T_{j-1})^{1/(2+\varepsilon)}}{n} \cdot \frac{U_n}{\sum_{j=1}^n (T_j - T_{j-1})^{1/(2+\varepsilon)}}.$$

By the strong law of large numbers the first term converges. For the second term we have

$$\Bigg(\frac{U_n}{\sum_{j=1}^n (T_j - T_{j-1})^{1/(2+\varepsilon)}}\Bigg)^{2+\varepsilon} \leq \frac{U_n^{2+\varepsilon}}{T_n} \leq \frac{\sum_{k=1}^{T_n} \log(1 + R + |B_k|)^{2+\varepsilon}}{T_n}$$

which converges since $(\log^+ |B_1|)^{2+\varepsilon}$ is integrable.

Let $U(y,x)=\sum_{n=1}^{\infty}\mathbb{E}[\mathbf{1}_{(y,x]}(\sum_{i=1}^{n}V_i)]$. Since $0<\mathbb{E}V_1<\infty$, by the renewal theorem

$$\lim_{x\to\infty}\frac{U(0,x)}{x}=\frac{1}{\mathbb{E}V_1}>0.$$

Hence for any $m>1$ there exist large N such that $\inf_{k\ge N}\frac{U(m^k,m^{k+1})}{m^k}=C_1>0$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^d} f\big(|x|\big)\nu(\mathrm{d}x) &\ge C_R\mathbb{E}\Bigg[\sum_{n=0}^{\infty} f\big(\mathrm{e}^{\sum_{i=1}^{n}V_i}\big)\Bigg] \ge C_R\sum_{k>N}U\big(m^k,m^{k+1}\big)f\big(\mathrm{e}^{m^{k+1}}\big)\\ &\ge C_RC_1\sum_{k>N}m^k f\big(\mathrm{e}^{m^{k+1}}\big)\ge\frac{C_RC_1}{m^2}\sum_{k>N}\int_{m^{k+1}}^{m^{k+2}}f\big(\mathrm{e}^x\big)\,\mathrm{d}x\ge C\int_{m^{N+1}}^{\infty}f\big(\mathrm{e}^x\big)\,\mathrm{d}x,\end{aligned}$$

that proves (5.3).

Step 2. Consider now the functions $f_n=\mathbf{1}_{[0,\beta^{n+1}/\alpha^n]}$ on $\mathbb{R}^+$. Observe that

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\int_{\mathbb{R}^d}f_n\big(|u|\big)\nu(\mathrm{d}u)&=\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\int_{\mathbb{R}^d}\mathbf{1}_{((\beta^k/\alpha^k)\alpha,(\beta^k/\alpha^k)\beta]}\big(|u|\big)\nu(\mathrm{d}u)\\ &\le\limsup_{z\to\infty}\nu\big\{u\colon z\alpha<|u|\le z\beta\big\}.\end{aligned}$$

Thus by (5.3)

$$\begin{aligned}\limsup_{z\to\infty}\nu\big\{u\colon z\alpha<|u|\le z\beta\big\}&\ge C\limsup_{n\to\infty}\frac{1}{n}\int_{M}^{\infty}f_n(a)\frac{\mathrm{d}a}{a}\\ &=C\limsup_{n\to\infty}\frac{1}{n}\bigg(\log\bigg(\frac{\beta^{n+1}}{\alpha^n}\bigg)-\log M\bigg)=C\log(\beta/\alpha)>0.\end{aligned}$$

□

Appendix: Sketch of the proof of Theorem 3.1

For $0<\lambda<1$ let

$$G^\lambda*\psi=\sum_{n=0}^{\infty}\lambda^n\overline{\mu}^{*n}*\psi\quad\text{and}\quad A^\lambda\psi=J(\psi)G^\lambda*g(0)-G^\lambda*\psi$$

for some fixed positive function g in $\mathcal{F}(\overline{\mu})$ such that $J(g)=1$. Observe that the Fourier transform of the measure G^λ is $\widehat{G}^\lambda(\theta)=\frac{1}{1-\lambda\widehat{\overline{\mu}}(\theta)}$. Thus $G^\lambda*\psi(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\theta x}\frac{\widehat{\psi}(-\theta)}{1-\lambda\widehat{\overline{\mu}}(\theta)}\,\mathrm{d}\theta$. The class of functions $\mathcal{F}(\overline{\mu})$ is chosen in such a way that $\frac{\widehat{\psi}(-\theta)}{1-\lambda\widehat{\overline{\mu}}(\theta)}$ is integrable outside a neighborhood of zero uniformly for all $\lambda\le 1$. Thus the only obstruction to integrability is at zero and can be dealt using the methods introduced in [18,19].

If $\psi\in\mathcal{F}(\overline{\mu})$, then using the fact that

$$\big|1-\lambda\widehat{\overline{\mu}}(\theta)\big|\ge\lambda\big|1-\widehat{\overline{\mu}}(\theta)\big|\ge\lambda c|\theta|^2, \tag{A.1}$$

one can write

$$\frac{\widehat{\psi}(-\theta)}{1-\lambda\widehat{\overline{\mu}}(\theta)}=\frac{J(\psi)-\mathrm{i}K(\psi)\theta}{1-\lambda\widehat{\overline{\mu}}(\theta)}\mathbf{1}_{[-a,a]}(\theta)+\psi_0^\lambda(\theta),$$

where $\psi_0^\lambda(\theta)$ is a family of functions in $L^1(\mathrm{d}\theta)$ bounded uniformly for $1/2 \le \lambda \le 1$. For all $\psi, \phi \in \mathcal{F}(\overline{\mu})$ and $x, y \in \mathbb{R}$, a standard calculation gives the decomposition

$$G^\lambda * \phi(-y) - G^\lambda * \psi(x-y) = -\bigl(K(\phi) - K(\psi) + xJ(\psi)\bigr)C^\lambda_{\overline{\mu}}(y) + \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}y\theta} h^\lambda_{\psi,\phi,x}(\theta)\,\mathrm{d}\theta, \tag{A.2}$$

where the functions $h^\lambda_{\psi,\phi,x}(\theta)$ are bounded uniformly for all $\lambda \in [1/2, 1]$ by $(1+x^2)H_{\phi,\psi}(\theta)$ for some integrable function $H_{\phi,\psi}$ and

$$C^\lambda_{\overline{\mu}}(\theta)(y) = \frac{\mathrm{i}}{2\pi} \int_{|\theta|<a} \frac{\mathrm{e}^{-\mathrm{i}y\theta}\theta}{1 - \lambda\widehat{\overline{\mu}}(\theta)}\,\mathrm{d}\theta.$$

By Theorem 3.1" in [18] the limit $\lim_{\lambda \nearrow 1} C^\lambda_{\overline{\mu}}(y) = C^1_{\overline{\mu}}(y)$ exists and $\lim_{y\to\pm\infty} C^1_{\overline{\mu}}(y) = \pm\sigma^{-2}$.

By Lebesgue's dominated convergence theorem, the following limit exists

$$\lim_{\lambda\nearrow 1} G^\lambda * \phi(-y) - G^\lambda * \psi(x-y) = -\bigl(K(\phi) - K(\psi) + xJ(\psi)\bigr)C^1_{\overline{\mu}}(y) + \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}y\theta} h^1_{\psi,\phi,x}(\theta)\,\mathrm{d}\theta. \tag{A.3}$$

For $\phi = J(\psi)g$ and $y = 0$, we have $A^\lambda\psi(x) = G^\lambda * \phi(0) - G^\lambda * \psi(x)$ thus

$$A\psi(x) = \lim_{\lambda\nearrow 1} A^\lambda\psi(x) = -\bigl(J(\phi)K(g) - K(\psi) + xJ(\psi)\bigr)C^1_{\overline{\mu}}(0) + \int_{\mathbb{R}} h^1_{\psi,\phi,x}(\theta)\,\mathrm{d}\theta.$$

Hence we have proved the existence of the recurrent potential kernel. The continuity of $A\psi$ follows from uniform integrability. Furthermore since $C^1_{\overline{\mu}}(0)$ is finite, we also have

$$\bigl|A^\lambda\psi(x)\bigr| \le C'\bigl(1 + x^2\bigr). \tag{A.4}$$

Take now $\phi = \psi$ then by (A.2), (A.3) and the Riemann–Lebesgue lemma

$$A\psi(x-y) - A\psi(-y) = -xJ(\psi)C^1_{\overline{\mu}}(y) + \widehat{h}^1_{\psi,\psi,x}(-y) \to \mp xJ(\psi)\sigma^{-2} \tag{A.5}$$

when $y \to \pm\infty$. If $J(\psi) > 0$ and x goes to $+\infty$ then

$$\frac{A\psi(x)}{x} = \frac{A\psi(\{x\}) + \sum_{k=1}^{[x]}(A\psi(k+\{x\}) - A\psi(k+\{x\}-1))}{x} \to J(\psi)\sigma^{-2},$$

where $[x]$ is the integer part of x and $\{x\} = x - [x]$.

If $J(\psi) = 0$ then $A^\lambda\psi = -G^\lambda * \psi$, taking $\phi = 0$, $x = 0$ we have

$$A\psi(-y) = K(\psi)C_{\overline{\mu}}(y) + \widehat{h}^1_{\psi,0,0}(-y)$$

and passing with y to $\pm\infty$ we obtain the expected limit.

To prove that $A\psi$ is a solution of the Poisson equation observe that

$$\overline{\mu} * A^\lambda\psi = c_\lambda J(\psi) - \sum_{n=0}^{\infty} \lambda^n \overline{\mu}^{*n+1} * \psi = A^\lambda\psi + G^\lambda * (\psi - \overline{\mu} * \psi).$$

Notice that

$$G^\lambda * (\psi - \overline{\mu} * \psi)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\theta} \widehat{\psi}(-\theta) \frac{1 - \widehat{\overline{\mu}}(\theta)}{1 - \lambda\widehat{\overline{\mu}}(\theta)}\,\mathrm{d}\theta,$$

and, by (A.1), the integrand is dominated by $2|\widehat{\psi}| \in L^1(\mathrm{d}\theta)$ for all $1/2 < \lambda \le 1$. Therefore, by Lebesgue's dominated convergence theorem $\lim_{\lambda\nearrow 1} \overline{\mu} * A^\lambda\psi = A\psi + \psi$. By (A.4) and dominate convergence, we conclude

$$\overline{\mu} * A\psi(x) = \int_{\mathbb{R}} \lim_{\lambda\nearrow 1} A^\lambda\psi(x+y)\overline{\mu}(\mathrm{d}y) = \lim_{\lambda\nearrow 1} \int_{\mathbb{R}} A^\lambda\psi(x+y)\overline{\mu}(\mathrm{d}y) = A\psi(x) + \psi(x).$$

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