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Spin needlets spectral estimation*

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Abstract: We consider the statistical analysis of random sections of a *spin fibre bundle* over the sphere. These may be thought of as random fields that at each point $p \in \mathbb{S}^2$ take as a value a curve (e.g. an ellipse) living in the tangent plane at that point $T_p\mathbb{S}^2$, rather than a number as in ordinary situations. The analysis of such fields is strongly motivated by applications, for instance polarization experiments in Cosmology. To investigate such fields, spin needlets were recently introduced by [21] and [20]. We consider the use of spin needlets for spin angular power spectrum estimation, in the presence of noise and missing observations, and we provide Central Limit Theorem results, in the high frequency sense; we discuss also tests for bias and asymmetries with an asymptotic justification.

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Contents

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1. Introduction

The analysis of (random or deterministic) functions defined on the sphere by means of wavelets has recently been the object of a number of theoretical and applied papers, see for instance [3, 4, 5, 60, 41, 42, 22, 23, 24, 7, 8, 27]. Many of these works have found their motivating rationale in recent developments in the applied sciences, such as Medical Imaging, Geophysics, Atmospheric Sciences, Astrophysics and Cosmology. These same fields of applications are now prompting stochastic models which are more sophisticated (and more intriguing) than ordinary, scalar valued random fields. In this paper, we shall be especially concerned with astrophysical and cosmological applications, but several similar issues can be found in other disciplines, see for instance [54] for related mathematical models in the field of brain mapping.

Concerning astrophysics, there are now many mathematical papers which have been motivated by the analysis of so-called Cosmic Microwave Background radiation (CMB); the latter can be very loosely viewed as a relic electromagnetic radiation which permeates the Universe providing a map of its status from 13.7 billion years ago, in the immediate adjacency of the Big Bang. Almost all statistics papers in this area have been concerned with the temperature component of CMB, which can be represented as a standard spherical random field (see [13] for a review). We recall that a scalar random field on the sphere may be thought of as a collection of random variables $\{T(p) : p \in \mathbb{S}^2\}$, where $\mathbb{S}^2 = \{p : \|p\|^2 = 1\}$ is the unit sphere of $\mathbb{R}^3$ and $\|.\|$ denotes Euclidean norm. $T(p)$ is isotropic if its law is invariant with respect to the group of rotations, $T(p) \stackrel{d}{=} T(gp)$ for all $g \in SO(3)$, where $\stackrel{d}{=}$ denotes equality in distribution of random fields and $SO(3)$ can be realized as the set of orthonormal 3×3 matrices with unit determinant.

However, most recent and forthcoming experiments (such as Planck, which was launched on May 14, 2009, the CLOVER, QUIET and QUAD experiments or the projected mission CMBPOL) are focussing on a much more elusive and sophisticated feature, i.e. the so-called polarization of CMB. The physical significance of the latter is explained for instance in [12, 31, 55]; we do not enter into these motivations here, but we do stress how the analysis of this feature is expected to provide extremely rewarding physical information. Just to provide a striking example, detection of a non-zero angular power spectrum for the so-called B-modes of polarization data (to be defined later) would provide the first experimental evidence of primordial gravitational waves; this would result in an impressive window into the General Relativity picture of the primordial Big

Bang dynamics and as such it is certainly one of the most interesting perspectives of current physical research. Polarization is also crucial in the understanding of the so-called reionization optical depth, for which very little information is available from temperature data, see [20] for more discussion on details.

Here, however, we shall not go deeper into these physical perspectives, as we prefer to focus instead on the new mathematical ideas which are forced in by the analysis of these datasets. A rigorous understanding requires some technicalities which are postponed to the next Section; however we hope to convey the general idea as follows. We can imagine that experiments recording CMB radiation are measuring on each direction $p \in \mathbb{S}^2$ a random ellipse living on $T_p\mathbb{S}^2$, the tangent plane at that point. The "magnitude" of this ellipse ($= c^2 = a^2 + b^2$ in standard ellipse notation), which is a standard random variable, corresponds to temperature data, on which the statistics literature has so far concentrated. The other identifying features of this ellipse (elongation and orientation) are collected in polarization data, which can be thought of as a random field taking values in a space of algebraic curves. In more formal terms (to be explained later), this can be summarized by saying that we shall be concerned with random sections of fibre bundles over the sphere; from a more group-theoretic point of view, we shall show that polarization random fields are related to so-called spin-weighted representations of the group of rotations $SO(3)$. A further mathematical interpretation, which is entirely equivalent but shall not be pursued here, is to view these data as realizations of random matrix fields (see again [54]). Quite interestingly, there are other, unrelated situations in physics where the mathematical and statistical formalism turns out to be identical. In particular gravitational lensing data, which have currently drawn much interest in Astrophysics and will certainly make up a core issue for research in the next two decades, can be shown to have the same (spin 2, see below) mathematical structure, see for instance [10]. More generally, similar issues may arise when dealing with random deformations of shapes, as dealt with for instance by [2].

The construction of a wavelet system for spin functions was first addressed in [21]; the idea in that paper is to develop the needlet approach of [41, 42] and [22, 23, 24] to this new, broader geometrical setting, and investigate the stochastic properties of the resulting spin needlet coefficients, thus generalizing results from [7, 8]. A wide range of possible applications to the analysis of polarization data is discussed in [20]. Here, we shall focus in particular on the possibility of using spin needlets for angular power spectrum estimation for spin fields, an idea that for the scalar case was suggested by [7]; in [47], needlets were used for the estimation of cross-angular power spectra of CMB and Large Scale Structure data, in [18, 19] the estimator was considered for CMB temperature data in the presence of faint noise and gaps, while in [48] the procedure was implemented on disjoint subsets of the sphere as a probe of asymmetries in CMB radiation.

The plan of this paper is as follows: in Section 2 we present the motivations for our analysis, i.e. some minimal physical background on polarization. In Section 3 and 4 we introduce the geometrical formalism on spin line bundles and spin needlets, respectively, and we define spin random fields. Sections 5, 6 and 7

are devoted to the spin needlets spectral estimator and the derivation of its asymptotic properties in the presence of missing observations and noise, including related statistical tests for bias and asymmetries. Some technical results are collected in an Appendix. Throughout this paper, given two positive sequences $\{a_j\}, \{b_j\}$ we shall write $a_j \approx b_j$ if there exist positive constants c_1, c_2 such that $c_1 a_j \leq b_j \leq c_2 a_j$ for all $j \geq 1$.

2. Motivations

The classical theory of electromagnetic radiation entails a characterization in terms of the so-called Stokes' parameters Q and U, which are defined as follows. An electromagnetic wave propagating in the z direction has components

$$E_x(z,t) = E_{0x}\cos(\tau + \delta_x)\ ,\ E_y(z,t) = E_{0y}\cos(\tau + \delta_y)\ , \tag{1}$$

where $\tau := \omega t - kz$ is the so-called propagator and $\nu = 2\pi\omega/k$ is the frequency of the wave. (1) can be viewed as the parametric equations of an ellipse which is the projection of the incoming radiation on the plane perpendicular to the direction of motion. Indeed, some elementary algebra yields

$$\frac{E_x^2(z,t)}{E_{0x}^2} + \frac{E_y^2(z,t)}{E_{0y}^2} - 2\frac{E_x(z,t)}{E_{0x}}\frac{E_y(z,t)}{E_{0y}}\cos\delta = \sin^2\delta\ ,\ \delta := \delta_y - \delta_x.$$

The magnitude of the ellipse (i.e., the sums of the squares of its semimajor and semiminor axes) is given by

$$T = E_{0x}^2 + E_{0y}^2\ ;$$

T has the nature of a scalar quantity, that is to say, it is readily seen to be invariant under rotation of the coordinate axis x and y. It can hence be viewed as an intrinsic quantity measuring the total intensity of radiation; from the physical point of view, this is exactly the nature of CMB temperature observations which have been the focus of so much research over the last decade. It should be noted that, despite the non-negativity constraint, in the physical literature on CMB experiments T is usually taken to be Gaussian around its mean, in excellent agreement with observations. This apparent paradox is explained by the fact that the variance of T is several orders of magnitude smaller than its mean, so the Gaussian approximation is justifiable.

The characterization of the polarization ellipse is completed by introducing Stokes' parameters Q and U, which are defined as

$$Q = E_{0x}^2 - E_{0y}^2\ ,\ U = 2E_{0x}E_{0y}\cos\delta\ . \tag{2}$$

To provide a flavour of their geometrical meaning, we recall from elementary geometry that the parametric equations of a circle are obtained from (1) in the special case $E_{0x} = E_{0y}$, $\delta_x = \delta_y + \pi/2$, whence the circle corresponds to $Q = U = 0$. On the other hand, it is not difficult to see that a segment aligned

on the x axis is characterized by $Q = T$, a segment aligned on the y axis by $Q = -T$, for a segment on the line $y = \pm x$ we have $\delta_x - \delta_y = 0, \pi$, and hence $Q = 0, U = \pm T$, respectively. The key feature to note, however, is the following: while T does not depend on any choice of coordinates, this is not the case for Q and U, i.e. the latter are not geometrically intrinsic quantities. However, as these parameters identify an ellipse, it is natural to expect that they will be invariant under rotations by 180° degrees and multiples thereof. This is the first step in understanding the introduction of spin random fields below.

Indeed, it is convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by focussing on $w = x + iy$; a change of coordinates corresponding to a rotation γ can then be expressed as $w' = \exp(i\gamma)w$, and some elementary algebra shows that the induced transform on (Q, U) can be written as

$$\begin{pmatrix} Q' \\ U' \end{pmatrix} = \begin{pmatrix} \cos 2\gamma & \sin 2\gamma \\ -\sin 2\gamma & \cos 2\gamma \end{pmatrix} \begin{pmatrix} Q \\ U \end{pmatrix} ,$$

or more compactly

$$Q' + iU' = \exp(i2\gamma)(Q + iU) \ . \tag{3}$$

In the physicists' terminology, (3) identifies the Stokes' parameters as spin 2 objects, that is, a rotation by an angle γ changes their value by $\exp(i2\gamma)$. As mentioned before, this can be intuitively visualized by focussing on an ellipse, which is clearly invariant by rotations of 180°. To compare with other situations, standard (scalar) random fields do not depend on the choice of coordinate axes in the local tangent plane, and as such they are spin zero fields; a vector field is spin 1, while we can envisage random fields taking values in higher order algebraic curves and thus having any integer spin $s \geq 2$.

As mentioned earlier, it is very important to notice that polarization is not the only possible motivation for the analysis of spin random fields. For instance, an identical formalism is derived when dealing with gravitational lensing, i.e. the deformation of images induced by gravity according to Einstein's laws. Gravitational lensing is now the object of very detailed experimental studies, which have led to huge challenges on the most appropriate statistical methods to be adopted (see for instance [10]). We defer to future work a discussion on the statistical procedures which are made possible by the application of spin needlets to lensing data.

3. Geometric background

In this Section, we will provide a more rigorous background on spin functions. Despite the fact that our motivating applications are limited to the case $s = 2$, we will discuss here the case of a general integer $s \in \mathbb{Z}$, which does not entail any extra difficulty.

A more rigorous point of view requires some background in Differential Geometry, for which we refer for instance to [9] and [1]. The construction of spin functions is discussed in more detail by [21], which builds upon a well-established

physical literature described for instance in [43, 55, 12]. We refer also to [34] for a very recent contribution from a stochastic point of view.

To proceed further to spin random fields, we need to recall from Geometry the notion of a fibre bundle. The latter consists of the family (E, B, π, F), where E, B, and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying a local triviality condition outlined below. The space B is called the base space of the bundle, E the total space, and F the fibre; the map π is called the projection map (or bundle projection). In our case, the base space is simply the unit sphere $B = \mathbb{S}^2$, and the fibre is homeomorphic to the complex line, see below for further details.

The basic intuition behind fibre bundles is that they behave locally as simple Cartesian products $B \times F$. The former intuition is implemented by requiring that for all $p \in \mathbb{S}^2$ there exist a neighbourhood $U = U(p)$ such that $\pi^{-1}(U)$ is homeomorphic to $U \times F$, in such a way that π carries over to the projection onto the first factor. In particular, the following diagram should commute:

$$\begin{array}{ccc} \pi^{-1}(U) & \overset{\phi}{\longmapsto} & U \times F \\ \pi \downarrow & \swarrow_{proj} & \\ U & & \end{array} ,$$

where ϕ is a homeomorphism and $proj$ is the natural projection. The set $\pi^{-1}(x)$ is homeomorphic to F and is called the fibre over x. The fibre bundles we shall consider are smooth, that is, E, B, and F are required to be smooth manifolds and all the projections above are required to be smooth maps.

In our case, we shall be dealing with a complex line bundle which is uniquely identified by fixing transition functions to express the transformation laws under changes of coordinates. Following [21] (see also [28, 43]), we define $U_I := \mathbb{S}^2 \setminus \{N, S\}$ to be the chart covering the sphere with the exception of the North and South Poles, with the usual coordinates (ϑ, φ). We define also the rotated charts $U_R = RU_I$; in this new charts, we will use the natural coordinates (ϑ_R, φ_R). At each point p of U_R, we take as a reference direction in the tangent plane T_pS^2, the tangent vector $\partial/\partial\varphi_R$, (which points in the direction of increasing φ_R and is tangent to a circle $\theta_R = \text{constant}$). Again as in [21], we let let $\psi_{pR_2R_1}$ be the (oriented) angle from $\partial/\partial\varphi_{R_1}$ to $\partial/\partial\varphi_{R_2}$ (for a careful discussion of which is the oriented angle, see [21]); this angle is independent of any choice of coordinates. We define a complex line bundle on S^2 by letting $\exp(is\psi_{pR_2R_1})$ be the transition function from the chart U_{R_1} to U_{R_2}. A smooth spin function f is a smooth section of this line bundle. f may simply be thought of as a collection of complex-valued smooth functions $(f_R)_{R\in SO(3)}$, with f_R defined and smooth on U_R,, such that for all $R_1, R_2 \in SO(3)$, we have

$$f_{R_2}(p) = \exp(is\psi_{pR_2R_1}) f_{R_1}(p)$$

for all p in the intersection of U_{R_1} and U_{R_2}.

An alternative, group theoretic point of view can be motivated as follows. Consider the group of rotations $SO(3)$; it is a well-known that, by elementary

geometry, each element g can be expressed as

$$g = R_z(\alpha)R_x(\beta)R_z(\gamma)\ ,\ 0 \le \alpha \le \pi\ ,\ 0 \le \beta, \gamma \le 2\pi\ , \tag{4}$$

where $R_z(.)$ and $R_x(.)$ represent rotations around the z and x axis, respectively; in words, (4) is stating that each rotation can be realized by rotating first by an angle γ around the z axis, then by an angle β around the x axis, then again by an angle α around the z axis. We denote as usual by $\left\{D^l(.)\right\}_{l=0,1,2,...}$ the Wigner family of irreducible matrix representations for $SO(3)$; in terms of the Euler angles, the elements of these matrices can be expressed as

$$D^l_{m_1 m_2}(g) = \exp(-im_1\alpha)d^l_{m_1 m_2}(\beta)\exp(-im_2\gamma)\ ,$$

where $d^l_{m_1 m_2}$ is the so-called Wigner's $d(.)$ function, see [57] for analytic expressions and more details. Note that $\overline{D^l_{m_1 m_2}}(g) = (-1)^{m_1-m_2}\overline{D^l_{-m_1,-m_2}}(g)$; standard results from group representation theory ([17, 57, 58]) yield

$$\sum_{m_2} D^l_{m_1 m_2}(g)\overline{D^{l'}_{m_1' m_2}}(g) = \delta^{l'}_{l}\,\delta^{m_1'}_{m_1}\ ,$$

and

$$\int_{SO(3)} D^l_{m_1 m_2}(g)\overline{D^{l'}_{m_1' m_2'}}(g)dg = \frac{8\pi^2}{2l+1}\delta^{l'}_{l}\,\delta^{m_1'}_{m_1}\delta^{m_2'}_{m_2}\ ,$$

dg denoting the standard uniform (Haar) measure on $SO(3)$. The elements of $\left\{D^l(.)\right\}_{l=0,1,2,...}$ thus make up an orthogonal system which is also complete, i.e., it is a consequence of the Peter-Weyl theorem [17] that all square integrable functions on $SO(3)$ can be expanded, in the mean square sense, as

$$f(g) = \sum_{l}\sum_{m_1 m_2}\frac{2l+1}{8\pi^2}b^l_{m_1 m_2}D^l_{m_1 m_2}(g)\ ,$$

where the coefficients $\left\{b^l_{m_1 m_2}\right\}$ can be recovered from the inverse Fourier transform

$$b^l_{m_1 m_2} = \int_{SO(3)} f(g)\overline{D^l_{m_1 m_2}(g)}dg\ .$$

By elementary geometry, we can view the unit sphere as the quotient space $\mathbb{S}^2 = SO(3)/SO(2)$ and the functions on the sphere as those which are constants with respect to the third Euler angle γ, i.e. $f(\alpha,\beta,\gamma) = f(\alpha,\beta,\gamma')$ for all γ,γ'. It follows that

$$\begin{aligned}
&\int_{SO(3)} f(g)\overline{D^l_{m_1 m_2}(g)}\,dg \\
&= (-1)^{m_1-m_2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(g)\exp(im_1\alpha)d^l_{-m_1-m_2}(\beta)\exp(im_2\gamma)\sin\beta d\alpha d\beta d\gamma \\
&= \begin{cases} 0 & \text{for } m_2 \neq 0 \\ 2\pi b^l_{m_1 0} & \text{otherwise} \end{cases}
\end{aligned}$$

In view of the well-known identity

$$Y_{lm}(\beta,\alpha) = \sqrt{\frac{2l+1}{4\pi}} d^l_{m0}(\beta) e^{im\alpha}$$
$$= (-1)^m \sqrt{\frac{2l+1}{4\pi}} d^l_{-m0}(\beta) e^{im\alpha} = (-1)^m \sqrt{\frac{2l+1}{4\pi}} D^l_{-m,0}(\alpha,\beta,\gamma)$$

(where we have used $d^l_{mn}(\beta) = (-1)^{m-n} d^l_{-m,-n}(\beta)$, see [57], equation 4.4.1), we immediately obtain the expansion of functions on the sphere into spherical harmonics, i.e.

$$f(p) = \sum_l \sum_m \frac{2l+1}{4\pi} b^l_{m0} D^l_{m0}(p) = \sum_{lm} a_{lm} Y_{lm}(p) \ , \ a_{lm} = \sqrt{\frac{2l+1}{4\pi}} b^l_{-m0} \ .$$

We can hence loosely say that standard scalar functions on the sphere "live in the space generated by the column $s = 0$ of the Wigner's D matrices of irreducible representations", see also [38]. Now from the Peter-Weyl Theorem we know that each of the columns $s = -l, \ldots, l$ spans a space of irreducible representations, and these spaces are mutually orthogonal; it is then a natural, naive question to ask what is the physical significance of these further spaces. It turns out that these are strictly related to spin functions; indeed we can expand a smooth spin s functions as

$$f_s(\vartheta,\varphi) = \sum_l \sum_m \frac{2l+1}{4\pi} b^l_{ms} D^l_{ms}(\varphi,\vartheta,\gamma)\Big|_{\gamma=0} \ . \tag{5}$$

Spin s functions can then be related to the so-called spin weighted representations of $SO(3)$, see for instance [11]. Now by standard group representation properties we have that

$$\begin{aligned} f_s((R_z(\gamma)p) &= \sum_l \sum_m \frac{2l+1}{4\pi} b^l_{ms} D^l(R_z(\gamma)) D^l_{ms}(\varphi,\vartheta,\gamma) \\ &= \sum_l \sum_m \frac{2l+1}{4\pi} b_{lms} \exp(is\gamma) D^l_{ms}(\varphi,\vartheta,\gamma) \\ &= \exp(is\gamma) f_s(p) \ , \end{aligned}$$

as expected.

The analogy with the scalar case can actually be pursued further than that. It is well-known that the elements D^l_{m0}, $m = -l, \ldots, l$ of the Wigner's D matrices are proportional to the spherical harmonics Y_{lm}, i.e. the eigenfunctions of the spherical Laplacian operator $\Delta_{\mathbb{S}^2} Y_{lm} = -l(l+1)Y_{lm}$. It turns out that this equivalence holds in much greater generality and for all integer s and $l \geq s$ there exist a differential operator $\eth\overline{\eth}$ such that $-\eth\overline{\eth} D^l_{ms} = e_{ls} D^l_{ms}$, where $\{e_{ls}\}_{l=s,s+1,} = \{(l-s)(l+s+1)\}_{s,s+1,\ldots}$ is the associated sequence of eigenvalues (note that for $s = 0$ we are back to the usual expressions for the scalar

case, as expected). The operators $\eth, \overline{\eth}$ are defined as follows, in terms of their action on any spin s function $f_s(.)$,

$$\eth f_s(\vartheta,\varphi) = -(\sin\vartheta)^s \left[\frac{\partial}{\partial\vartheta} + \frac{i}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right](\sin\vartheta)^{-s} f_s(\vartheta,\varphi)\ , \tag{6}$$

$$\overline{\eth} f_s(\vartheta,\varphi) = -(\sin\vartheta)^{-s} \left[\frac{\partial}{\partial\vartheta} - \frac{i}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right](\sin\vartheta)^{s} f_s(\vartheta,\varphi)\ . \tag{7}$$

In (6) one should more rigorously write $(\eth f_s)_I$ on the left side and $(f_s)_I$ on the right side. In fact, if on the right side of (6) we replace (ϑ,φ) by (ϑ_R,φ_R) and f_s by $(f_s)_R$, the result is in fact $(\eth f_s)_R(\vartheta_R,\varphi_R)$ (see [21]); similarly in (7).

The spin s spherical harmonics can then be identified as

$$\begin{aligned} Y_{lms}(\vartheta,\varphi) &= (-1)^m \sqrt{\frac{2l+1}{4\pi}} D^l_{-ms}(\varphi,\vartheta,-\psi)\exp(-is\psi) \\ &= (-1)^m \sqrt{\frac{2l+1}{4\pi}} \exp(im\varphi) d^l_{-ms}(\vartheta)\ ; \end{aligned} \tag{8}$$

again, the previous expression should be understood as $Y_{lms;I}(\vartheta,\varphi)$, i.e. spin spherical harmonics are clearly affected by coordinate transformation, but we drop the reference to the choice of chart for ease of notation whenever this can be done without the risk of confusion. The spin spherical harmonics can be shown to satisfy

$$Y_{lm,s+1} = \left[(l-s)(l+s+1)\right]^{-1/2} \eth Y_{lm,s}\ , \tag{9}$$

$$Y_{lm,s-1} = -\left[(l+s)(l-s+1)\right]^{-1/2} \overline{\eth} Y_{lm,s}\ , \tag{10}$$

which motivates the name of spin raising and spin lowering operators for $\eth, \overline{\eth}$. Iterating, it can be shown also that (see [21])

$$Y_{lms} = \left\{\frac{(l-s)!}{(l+s)!}\right\}^{1/2} (\eth)^s Y_{lm}\ , \text{ for } s>0\ ,$$

$$Y_{lms} = \left\{\frac{(l+s)!}{(l-s)!}\right\}^{1/2} (-\overline{\eth})^{-s} Y_{lm}\ , \text{ for } s<0\ .$$

Further properties of the spin spherical harmonics follow easily from their proportionality to elements of Wigner's D matrices; indeed we have (orthonormality)

$$\int_{\mathbb{S}^2} Y_{lms}(p)\overline{Y_{l'm's}}(p)dp = \int_0^{2\pi}\int_0^{\pi} Y_{lms}(\vartheta,\varphi)\overline{Y_{l'm'}(\vartheta,\varphi)}\sin\vartheta d\vartheta d\varphi = \delta_l^{l'}\delta_m^{m'}\ .$$

Viewing spin-spherical harmonics as functions on the group $SO(3)$ (i.e. identifying $p=(\vartheta,\varphi)$ as the corresponding rotation by means of Euler angles), using

(8) and the group addition properties we obtain easily, for $p, p' \in \mathbb{S}^2$, that

$$\sum_{m=-l}^{l} Y_{lms}(p)\overline{Y_{lms}(p')} = \frac{2l+1}{4\pi}\sum_{m} D^l_{-ms}(\varphi,\vartheta,0)\overline{D^l_{-ms}}(\varphi',\vartheta',0)$$
$$= \frac{2l+1}{4\pi} D^l_{-ss}(\psi(p,p'))\ ,$$

where $\psi(p,p')$ denotes the composition of the two rotations (explicit formulae can be found in [57]). In the special case $p = p'$ and $R = R'$, we have immediately

$$\sum_{m=-l}^{l} Y_{lms}(p)\overline{Y_{lms}(p)} = \frac{2l+1}{4\pi}\ , \tag{11}$$

see also [21] for an alternative proof.

By combining (5) and (8) the spectral representation of spin functions is derived:

$$f_s(\vartheta,\varphi) = \sum_l \sum_m a_{l;ms} Y_{l;ms}(\vartheta,\varphi)\ . \tag{12}$$

From (12), a further, extremely important characterization of spin functions was first introduced by [43], see also [21] for a more mathematically oriented treatment. In particular, it can be shown that there exists a scalar complex-valued function $g(\vartheta,\varphi) = \mathrm{Re}\{g\}(\vartheta,\varphi) + i\mathrm{Im}\{g\}(\vartheta,\varphi)$, such that, such that

$$f_s(\vartheta,\varphi) = f_E(\vartheta,\varphi) + if_B(\vartheta,\varphi)$$
$$= \sum_{lm} a_{lm;E} Y_{lms}(\vartheta,\varphi) + i\sum_{lm} a_{lm;B} Y_{lms}(\vartheta,\varphi)\ , \tag{13}$$

where

$$f_E(\vartheta,\varphi) = (\eth)^s \mathrm{Re}\{g\}(\vartheta,\varphi)\ ,\ f_B(\vartheta,\varphi) = (\eth)^s \mathrm{Im}\{g\}\ .$$

Note that $a_{l;ms} = a_{lm;E} + ia_{lm;B}$, where $a_{lm;E} = \overline{a}_{l-m;E}$, $a_{lm;B} = \overline{a}_{l-m;B}$. It is also readily seen that

$$a_{l;ms} + \overline{a_{l;-ms}} = a_{lm;E} + ia_{lm;B} + a_{lm;E} - ia_{lm;B} = 2a_{lm;E}\ ,$$

$$a_{l;ms} - \overline{a_{l;-ms}} = a_{lm;E} + ia_{lm;B} - a_{lm;E} + ia_{lm;B} = 2ia_{lm;B}\ .$$

In the cosmological literature, $\{a_{lm;E}\}$ and $\{a_{lm;B}\}$ are labelled the E and B modes (or the electric and magnetic components) of CMB polarization.

4. Spin needlets and spin random fields

We are now in a position to recall the construction of spin needlets, as provided by [21]. We start by reviewing a few basic facts about standard (scalar) needlets. Needlets have been defined by [41, 42] as

$$\psi_{jk}(p) = \sqrt{\lambda_{jk}}\sum_l b\left(\frac{l}{B^j}\right)\sum_{m=-l}^{l} Y_{lm}(p)\overline{Y_{lm}}(\xi_{jk})\ ,\ p \in \mathbb{S}^2, \tag{14}$$

where $\{\xi_{jk}, \lambda_{jk}\}$ are a set of cubature points and weights ensuring that

$$\sum_{jk} \lambda_{jk} Y_{lm}(\xi_{jk}) \overline{Y_{l'm'}}(\xi_{jk}) = \int_{\mathbb{S}^2} Y_{lm}(p) \overline{Y_{l'm'}}(p)\, dp = \delta_l^{l'} \delta_m^{m'} ,$$

$b(.)$ is a compactly supported, C^∞ function, and $B > 1$ is a user-chosen "bandwidth" parameter. The general cases of non-compactly supported functions $b(.)$ and more abstract manifolds than the sphere were studied by [22, 23, 24]. The stochastic properties of needlet coefficients and their use for the analysis of spherical random fields were first investigated by [7, 8], see also [32, 39, 33, 18] for further developments. Several applications have already been provided to CMB data analysis, see for instance [47, 37, 14, 19, 48, 49, 52, 50, 53].

For a fixed $B > 1$, we shall denote by $\{\mathcal{X}_j\}_{j=0}^{\infty}$ the nested sequence of cubature points corresponding to the space $\mathcal{K}_{[2B^{j+1}]}$, where $[.]$ represents as usual integer part and $\mathcal{K}_L = \oplus_{l=0}^{L} H_l$ is the space spanned by spherical harmonics up to order L. It is known that $\{\mathcal{X}_j\}_{j=0}^{\infty}$ can be taken such that the cubature points for each j are almost uniformly ϵ_j-distributed with $\epsilon_j := \kappa B^{-j}$, the coefficients $\{\lambda_{jk}\}$ are such that $cB^{-2j} \leq \lambda_{jk} \leq c' B^{-2j}$, where c, c' are finite real numbers, and $card\{\mathcal{X}_j\} \approx B^{2j}$. Exact cubature points can be defined for the spin as for the scalar case, see [6] for details; for practical CMB data analysis, these cubature points can be identified with the centre pixels provided by [29], with only a minor approximation.

Spin needlets are then defined as (see [21])

$$\psi_{jk;s}(p) = \sqrt{\lambda_{jk}} \sum_l b\left(\frac{\sqrt{e_{ls}}}{B^j}\right) \sum_{m=-l}^{l} Y_{l;ms}(p) \overline{Y_{l;ms}}(\xi_{jk}) \ . \tag{15}$$

As before, $\{\lambda_{jk}, \xi_{jk}\}$ are cubature points and weights, $b(\cdot) \in C^\infty$ is nonnegative, and has a compact support in $[1/B, B]$. The expression (15) bears an obvious resemblance with (14), but it is also important to point out some crucial differences. Firstly, we note that the square root of the eigenvalues $\sqrt{e_{ls}}$ has replaced the previous l. This formulation is instrumental for the derivation of the main properties of spin needlets by means of differential arguments in [21]; we stress, however, that this is actually a minor difference, as all our results are asymptotic and of course

$$\lim_{l\to\infty} \frac{\sqrt{e_{ls}}}{l} = \lim_{l\to\infty} \frac{\sqrt{(l-s)(l+s+1)}}{l} = 1 \text{ for all fixed } s \ .$$

A much more important feature is as follows: (15) cannot be viewed as a well-defined scalar or spin function, because $Y_{l;ms}(p), \overline{Y_{l;ms}}(\xi_{jk})$ are spin(s and $-s$) functions defined on different point of $\mathbb{S}^2$, and as such they cannot be multiplied in any meaningful way (their product depends on the local choice of coordi-

nates). Hence, (15) should be written more rigorously as

$$\psi_{jk;s}(p) = \sqrt{\lambda_{jk}} \sum_l b\left(\frac{\sqrt{e_{ls}}}{B^j}\right) \sum_{m=-l}^{l} \left\{ Y_{l;ms}(p) \otimes \overline{Y_{l;ms}}(\xi_{jk}) \right\} ,$$

$$\overline{\psi_{jk;s}}(p) = \sqrt{\lambda_{jk}} \sum_l b\left(\frac{\sqrt{e_{ls}}}{B^j}\right) \sum_{m=-l}^{l} \left\{ \overline{Y_{l;ms}}(p) \otimes Y_{l;ms}(\xi_{jk}) \right\} ,$$

where we denoted by $\otimes$ the tensor product of spin functions; spin needlets can the be viewed as spin $\{-s, s\}$ operators (written $T_{-s,s}$), which act on a space of spin s functions square integrable functions to produce a sequence of spin s square-summable coefficients, i.e. $T_{-s,s} : L^2_s \to \ell^2_s$. This action is actually an isometry, as a consequence of the tight frame property, see [6] and [25].

For any spin s function f_s, the spin needlet transform is defined by

$$\int_{\mathbb{S}^2} f_s(p)\overline{\psi_{jk;s}}(p)dp = \beta_{jk;s} ,$$

and the same inversion property holds as for standard needlets, i.e.

$$f_s(p) = \sum_{jk} \beta_{jk;s}\psi_{jk;s}(p) ,$$

the equality holding in the L^2 sense. The coefficients of spin needlets can be written explicitly as

$$\beta_{jk;s} = \int_{\mathbb{S}^2} f_s(p)\overline{\psi_{jk;2}}(p)dp = \sqrt{\lambda_{jk}} \sum_l b\left(\frac{\sqrt{e_{ls}}}{B^j}\right) \sum_{m=-l}^{l} a_{l;ms} Y_{l;ms}(\xi_{jk}) . \quad (16)$$

Remark 1. To illustrate the meaning of these projection operations, and using a notation closer to the physical literature, we could view spin s quantities as "bra" entities, i.e. write $\langle T(p) , \langle \beta_{jk;s} ,$ and spin $-s$ as "ket" quantities, i.e. write for instance $\overline{Y_{l;ms}}(p)\rangle$. Then we would obtain

$$\begin{aligned}
&\int_{\mathbb{S}^2} f_s(p)\overline{\psi_{jk;2}}(p)dp \\
&\quad = \sqrt{\lambda_{jk}} \sum_l b\left(\frac{\sqrt{e_{ls}}}{B^j}\right) \sum_{m=-l}^{l} \int_{\mathbb{S}^2} \langle f_s(p) , \overline{Y_{l;ms}}(p)\rangle \langle Y_{l;ms}(\xi_{jk}) \, dp \\
&\quad = \sqrt{\lambda_{jk}} \sum_l b\left(\frac{\sqrt{e_{ls}}}{B^j}\right) \sum_{m=-l}^{l} a_{l;ms} \langle Y_{l;ms}(\xi_{jk}) ,
\end{aligned}$$

which is a well-defined spin quantities, as the inner product $\left\langle f_s(p), \overline{Y_{l;ms}}(p)\right\rangle$ yields a well-defined, complex-valued scalar. However we shall not use this "Dirac" notation later in this paper, as we hope the meaning of our manipulations will remain clear by themselves.

The absolute value of spin needlets is indeed a well-defined scalar function, and this allows to discuss localization properties. In this framework, the main result is established in [21], where it is shown that for any $M \in \mathbb{N}$ there exists a constant $c_M > 0$ s.t., for every $\xi \in \mathbb{S}^2$:

$$|\psi_{jk;s}(\xi)| \leq \frac{c_M B^j}{(1 + B^j \arccos(\langle \xi_{jk}, \xi \rangle))^M} \text{ uniformly in } (j,k) \ , \tag{17}$$

i.e. the tails decay quasi-exponentially.

We are now able to focus on the core of this paper, which is related to the analysis of spin random fields. As mentioned in the previous discussion, we have in mind circumstances where stochastic analysis must be developed on polarization random fields $\{Q \pm iU\}$, which are spin ± 2 random functions.

Hence we shall now assume we deal with random isotropic spin functions f_s, by which we mean that there exist a probability space $(\Omega, \Im, P)$, such that for all choices of charts U_R, the ordinary random function $(f_s)_R$, defined on $\Omega \times \mathbb{S}^2$, is jointly $\Im \times \mathcal{B}(U_R)$ measurable, where $\mathcal{B}(U_R)$ denotes the Borel sigma-algebra on U_R. In particular, for the spin 2 random function $(Q+iU)(p)$ as for the scalar case, the following representation holds, in the mean square sense ([21])

$$\{Q + iU\} = \sum_{lm} a_{lm;2} Y_{l;m2} \ ,$$

i.e.

$$\lim_{L \to \infty} E \int_{\mathbb{S}^2} \left| \{Q + iU\}(p) - \sum_{l=1}^{L} \sum_{m=-l}^{l} a_{lm;2} Y_{l;m2}(p) \right|^2 dp = 0 \ .$$

Note that the quantity on the left-hand side is a well-defined scalar, for all L. The sequence $\{a_{lm2} = a_{lm;E} + i a_{lm;B}\}$ is complex-valued and is such that, for all l_1, l_2, m_1, m_2 ,

$$E a_{l_1 m_1;E} a_{l_2 m_2;E} = E a_{l_1 m_1;B} a_{l_2 m_2;E} = E a_{l_1 m_1;E} a_{l_2 m_2;B} = E a_{l_1 m_1;E} \overline{a_{l_2 m_2;B}} = 0 \ ,$$

and

$$E a_{lm;E} \overline{a_{l'm';E}} = C_{lE} \delta_l^{l'} \delta_m^{m'} \ , \ E a_{lm;B} \overline{a_{l'm';B}} = C_{lB} \delta_l^{l'} \delta_m^{m'} ,$$

where

$$\sum_l \frac{2l+1}{4\pi} C_{lE} \ , \ \sum_l \frac{2l+1}{4\pi} C_{lB} < \infty \ .$$

The spin (or total) angular power spectrum is defined as

$$E|a_{lm;2}|^2 =: C_l = \{C_{lE} + C_{lB}\} \ .$$

The angular power spectrum is the key statistic for the analysis of isotropic random fields, both in the spin and the scalar case. Indeed, in the Gaussian case it provides complete information on the dependence structure of the field; moreover, it is in general a function of physical parameters of interest, which can be recovered by matching the observed and predicted angular power spectra

by various forms of minimum distance estimators. In the case of polarization, a non-zero value of the C_{lB} component would yield experimental evidence on the existence of primordial gravitational waves, one of the most elusive implications of Big Bang theories. We refer to [13] and the references therein for a more detailed discussion on techniques and motivations for angular power spectrum estimation in a CMB framework; our purpose in next section is to extend to the spin case some needlet-based procedures, which were advocated in the scalar case by [7, 47, 8] and [18, 19, 48].

5. Spin needlets spectral estimator

In this section, we shall establish an asymptotic result for the spectral estimator of spin needlets in the high resolution sense, i.e. we will investigate the asymptotic behaviour of our statistics as the frequency band goes higher and higher. We note first, however, one very important issue. As we mentioned earlier, spin needlet coefficients are not in general scalar quantities. It is possible to choose a single chart to cover all points other than the North and South Pole; these two points can be clearly neglected without any effect on asymptotic results. The resulting spin coefficients will in general depend on the chart, and should hence be written as $\{\beta_{R;jks}\}$; however the choice of the chart will only produce an arbitrary phase factor $\exp(is\gamma_k)$. The point is that, because in this paper we are only concerned with quadratic statistics, the phase factor is automatically lost and our statistics for the spin spectral estimator will be invariant with respect to the choice of coordinates. In view of this, from now on we can neglect the issues relative to the choice of charts; we will deal with needlet coefficients as scalar-valued complex quantities, i.e. we will take the chart as fixed, and for notational simplicity we write $\{\beta_{jks}\}$ rather than $\{\beta_{R;jks}\}$.

We begin by introducing some regularity conditions on the polarization angular power spectrum Γ_l, which are basically the same as in [21], see also [7, 8] and [32, 18, 33, 39] for closely related assumptions.

Condition 2. *The random field* $\{Q+iU\}(p)$ *is Gaussian and isotropic with angular power spectrum such that*

$$C_l = l^{-\alpha} g(l) > 0 \ , \ \textit{where} \ c_0^{-1} \leq g(l) \leq c_0 \ , \ \alpha > 2 \ , \ \textit{for all} \ l \in \mathbb{N} \ ,$$

and for every $r \in \mathbb{N}$ *there exist* $c_r > 0$ *such that*

$$\left| \frac{d^r}{du^r} g(u) \right| \leq c_r u^{-r} \ , \ u \in (|s|, \infty) \ .$$

Remark 3. The condition is fulfilled for instance by angular power spectra of the form

$$C_l = \frac{F_1(l)}{l^\beta F_2(l)} \ ,$$

where $F_1(l), F_2(l) > 0$ are polynomials of degree $q_1, q_2 > 0$, $\beta + q_2 - q_1 = \alpha$.

By (16), it is readily seen that

$$\begin{aligned} &E\beta_{jk;s}\beta_{j'k';s} \\ &= \sqrt{\lambda_{jk}}\sqrt{\lambda_{j'k'}}\sum_{l,l'} b\left(\frac{\sqrt{e_{ls}}}{B^j}\right) b\left(\frac{\sqrt{e_{l's}}}{B^{j'}}\right) \\ &\quad \times \sum_{m,m'} Ea_{l;ms}a_{l';m's}Y_{l;ms}\left(\xi_{jk}\right)Y_{l';m's}\left(\xi_{j'k'}\right) = 0 \end{aligned}$$

because

$$Ea_{l;ms}a_{l';m's} = Ea_{l_1m_1;E}a_{l_2m_2;E} + 2Ea_{l_1m_1;B}a_{l_2m_2;E} + Ea_{l_1m_1;E}a_{l_2m_2;B} = 0 \ .$$

On the other hand, the covariance $Cov\left(\beta_{jk;s},\overline{\beta_{jk';s}}\right) = E\beta_{jk;s}\overline{\beta_{jk';s}}$ is in general non-zero. In view of (16, it is immediate to see that

$$\left|Cov\left(\beta_{jk;s},\overline{\beta_{jk';s}}\right)\right| = \left|\sqrt{\lambda_{jk}}\sqrt{\lambda_{jk'}}\sum_{l} b^2\left(\frac{\sqrt{e_{ls}}}{B^j}\right) C_l\frac{(2l+1)}{4\pi}K^{ls}\left(\xi_{jk},\xi_{jk'}\right)\right| \ ; \tag{18}$$

where

$$K^{ls}\left(p,p'\right) = \sum_{m=-l}^{l} Y_{lms}\left(p\right)\overline{Y_{lms}\left(p'\right)} \ . \tag{19}$$

For $k = k'$ we obtain as a special case from (11) that

$$E\left|\beta_{jk;s}\right|^2 = \lambda_{jk}\sum_{l} b^2\left(\frac{\sqrt{e_{ls}}}{B^j}\right) C_l\frac{(2l+1)}{4\pi} \ . \tag{20}$$

From (18) and (20) we obtain

$$\left|Corr\left(\beta_{jk;s},\overline{\beta_{jk';s}}\right)\right| = \frac{\left|\sum_l b^2\left(\frac{\sqrt{e_{ls}}}{B^j}\right) C_l\frac{(2l+1)}{4\pi}K^{ls}\left(\xi_{jk},\xi_{jk'}\right)\right|}{\sum_l b\left(\frac{\sqrt{e_{ls}}}{B^j}\right) C_l\frac{(2l+1)}{4\pi}} \ . \tag{21}$$

The key result for the development of the high-frequency asymptotic theory in the next sections is the following uncorrelation result, which was provided by [21]; under Condition 2,

$$\left|Corr\left(\beta_{jk;s},\overline{\beta_{jk';s}}\right)\right| \leq \frac{C_M}{\left\{1+B^j d(\xi_{jk},\xi_{jk'})\right\}^M} \ , \text{ for all } M\in\mathbb{N} \ , \text{ some } C_M > 0 \ . \tag{22}$$

The analogous result for the scalar case is due to [7], see also [33, 39] for some generalizations. We recall also the following inequality ([42], *Lemma 4.8*), valid for some c_M depending only on M, which will be used in the following discussion:

$$\sum_{k'} \frac{1}{\left\{1+B^j d(\xi_{jk},\xi_{jk'})\right\}^M}\frac{1}{\left\{1+B^j d(\xi_{jk'},\xi_{jk''})\right\}^M} \leq \frac{c_M}{\left\{1+B^j d(\xi_{jk},\xi_{jk''})\right\}^M} \ . \tag{23}$$

In view of (20), let us now denote

$$\Theta_{j;s} := \sum_k E\left|\beta_{jk;s}\right|^2 = \sum_k \lambda_{jk} \sum_l b^2\left(\frac{\sqrt{e_{ls}}}{B^j}\right) C_l \frac{(2l+1)}{4\pi}$$
$$= \sum_l b^2\left(\frac{\sqrt{e_{ls}}}{B^j}\right) C_l\,(2l+1)\ .$$

Under Condition 2, it is immediate to see that

$$C_0 B^{(2-\alpha)j} \leq \Theta_{j;s} \leq C_1 B^{(2-\alpha)j}. \tag{24}$$

A question of great practical relevance is the asymptotic behaviour of $\sum_k \left|\beta_{jk;s}\right|^2$ as an estimator for $\Theta_{j;s}$; for the scalar case, this issue was dealt with by [7], where a Functional Central Limit Theorem result is established and proposed as a test for goodness of fit on the angular power spectrum. In [47], the needlets estimator was applied to the cross-spectrum of CMB and Large Scale Structure data, while [18, 19] have considered the presence of missing observations and observational noise, establishing a consistency result and providing further applications to CMB data. In the spin case, angular power spectrum estimation was considered by [21], under the unrealistic assumptions that the spin random field $P = Q + iU$ is observed on the whole sphere and without noise. Here we shall be concerned with the much more realistic case where some parts of the domain $\mathbb{S}^2$ are "masked" by the presence of foreground contamination; more precisely, we assume data are collected only on a subset $\mathbb{S}^2 \setminus G$, G denoting the masked region. In this section, we do not consider the presence of observational noise, which shall be dealt with in the following section. In the sequel, for some (arbitrary small) constant $\varepsilon > 0$, we define $G^\varepsilon = \left\{x \in \mathbb{S}^2 : d\left(x, G\right) \leq \varepsilon\right\}$. The introduction of this parameter allows to get rid of some notational difficulties in case some grid points should happen to belong to the boundary of the observed regions. For practical data analysis, the choice of any reasonably small ε has negligible impact on final results; indeed there are usually some degrees of freedom in the width of the mask to be applied to CMB data, see for instance [53]. Consider

$$\widehat{\Theta}^*_{j;sG} := \left\{\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G^\varepsilon} \lambda_k\right\}^{-1} \sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G^\varepsilon} \left|\beta^*_{jk;s}\right|^2 \tag{25}$$

where

$$\beta^*_{jk;s} = \int_{\mathbb{S}^2\setminus G} P(x)\overline{\psi_{jk;s}}(x)dx\ .$$

Our aim will be to prove the following

Theorem 4. *Under condition (2), we have*

$$\frac{\widehat{\Theta}^*_{j;sG} - \Theta_{j;s}}{\sqrt{Var\{\widehat{\Theta}^*_{j;sG}\}}} \to_d N(0,1)\ ,\ as\ j \to \infty\ .$$

Proof. The proof will be basically in two steps; define

$$\widehat{\Theta}_{j;sG} := \left\{ \sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G^\varepsilon} \lambda_k \right\}^{-1} \sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G^\varepsilon} |\beta_{jk;s}|^2 \ , \tag{26}$$

which is clearly an unfeasible version of (25), where the $\beta^*_{jk;s}$ have been replaced by the coefficients (in the observed region) evaluated without gaps. The idea will be to show that

$$\frac{\widehat{\Theta}_{j;sG} - \Theta_{j;s}}{\sqrt{Var\{\widehat{\Theta}_{j;sG}\}}} \to_d N(0,1) \ , \ \frac{\sqrt{Var\{\widehat{\Theta}_{j;sG}\}}}{\sqrt{Var\{\widehat{\Theta}^*_{j;sG}\}}} \to 1$$
$$\text{and } \frac{\widehat{\Theta}^*_{j;sG} - \widehat{\Theta}_{j;sG}}{\sqrt{Var\{\widehat{\Theta}^*_{j;sG}\}}} \to_p 0 \ , \text{ as } j \to \infty \ .$$

The proof of these statements is provided in three separate Lemmas (14, 15, 16) in the Appendix below. □

Remark 5. In general the expressions for $Var\{\widehat{\Theta}_{j;sG}\}, Var\{\widehat{\Theta}^*_{j;sG}\}$ depend on the unknown angular power spectrum. However, the normalizing factors can be consistently estimated by subsampling techniques, following the same steps as in [8]. As revealed by a careful inspection of the proofs, it should be noted that the presence of a fixed masked region does not affect the rate of convergence of these estimators, although it does increase a variance by a factor proportional to the width of the mask. More discussion on rates of convergence can be found in the next sections.

6. Detection of asymmetries

In this Section, we shall consider one more possible application of spin needlets to problems of interest for Cosmology. In particular, a highly debated issue in modern Cosmology relates to the existence of "features", i.e. asymmetries in the distribution of CMB radiation (for instance between the Northern and the Southern hemispheres, in Galactic coordinates). These issues have been the object of dozens of physical papers, in the last few years, some of them exploiting scalar needlets, see [48].

In order to investigate this issue, we shall employ a similar technique as [8] for the scalar case. More precisely, we shall focus on the difference between the estimated angular power spectrum over two different regions of the sky. Let us consider A_1, A_2, two subsets of $\mathbb{S}^2$ such that $A_1 \cap A_2 = \emptyset$; we do not assume that $A_1 \cup A_2 = \mathbb{S}^2$, i.e. we admit the presence of missing observations. For practical applications, A_1 and A_2 can be visualized as the spherical caps centered at the north and south pole N, S (i.e. $A_1 = \left\{x \in \mathbb{S}^2 : d(x,N) \le \pi/2\right\}$, $A_2 = \left\{x \in \mathbb{S}^2 : d(x,S) \le \pi/2\right\}$, but the results would hold without any modification

for general subsets and could be easily generalized to a higher number of regions. We shall then focus on the statistic

$$\frac{\widehat{\Theta}^*_{j;sA_1} - \widehat{\Theta}^*_{j;sA_2}}{\sqrt{Var\{\widehat{\Theta}^*_{j;sA_1}\} + Var\{\widehat{\Theta}^*_{j;sA_2}\}}} ,$$

where

$$\widehat{\Theta}^*_{j;sA_1} := \left\{ \sum_{k:\xi_{jk}\in A_1^\varepsilon} \lambda_k \right\}^{-1} \sum_{k:\xi_{jk}\in A_1^\varepsilon} \left|\beta^*_{jk;s}\right|^2 ,$$

$$\widehat{\Theta}^*_{j;sA_2} := \left\{ \sum_{k:\xi_{jk}\in A_2^\varepsilon} \lambda_k \right\}^{-1} \sum_{k:\xi_{jk}\in A_2^\varepsilon} \left|\beta^*_{jk;s}\right|^2 , \text{ some } \varepsilon > 0 .$$

We are here able to establish the following Proposition, whose proof can be found in the Appendix.

Proposition 6. *As $j \to \infty$, we have*

$$\begin{pmatrix} \left[Var\left\{\widehat{\Theta}^*_{j;sA_1}\right\}\right]^{-1/2} \left(\widehat{\Theta}^*_{j;sA_1} - \Theta_{j;s}\right) \\ \left[Var\left\{\widehat{\Theta}^*_{j;sA_2}\right\}\right]^{-1/2} \left(\widehat{\Theta}^*_{j;sA_2} - \Theta_{j;s}\right) \end{pmatrix} \to_d N(0_2, I_2) ,$$

where $(0_2, I_2)$ are, respectively, the 2×1 vector of zeros and the 2×2 identity matrix.

Remark 7. An obvious consequence of Proposition 6 is

$$\tau_{j;s,A_1,A_2} = \frac{\widehat{\Theta}^*_{j;sA_1} - \widehat{\Theta}^*_{j;sA_2}}{\sqrt{Var\{\widehat{\Theta}^*_{j;sA_1}\} + Var\{\widehat{\Theta}^*_{j;sA_2}\}}} \to_d N(0,1) .$$

This result provides the asymptotic justification to implement on polarization data the same testing procedures as those considered for instance by [48] to search for features and asymmetries in CMB scalar data. More precisely, we could test for instance whether the Northern and Southern hemisphere are statistically different by taking A_1, A_2 to be the spherical caps of radius $\pi/2$ centred on the North and South pole, respectively, and then comparing $\tau_{j;s,A_1,A_2}$ with the quantiles of a Gaussian distribution. Values above the threshold would lead to a rejection of the isotropy assumption, as suggested in the temperature (scalar) case by some empirical findings in the recent cosmological literature (see again [48] and the references therein).

7. Estimation with noise

In the previous sections, we worked under a simplifying assumption, i.e. we figured that although observations on some parts of the sphere were completely

unattainable, data on the remaining part were available free of noise. In this Section, we aim at relaxing this assumption; in particular, we shall consider the more realistic circumstances where, while we still take some regions of the sky to be completely unobservable, even for those where observations are available the latter are partially contaminated by noise.

To understand our model for noise, we need to review a view basic facts on the underlying physics. A key issue about (scalar and polarized) CMB radiation experiments is that they actually measure radiation across a set of different electromagnetic frequencies, ranging from 30 GHz to nearly 900. One of the key predictions of Cosmology, whose experimental confirmation led to the Nobel Prize for J.Mather in 2006, is that CMB radiation in all its components follows a blackbody pattern of dependence over frequency. More precisely, the intensity I_A is distributed along to the various frequencies according to the Planckian curve of blackbody emission

$$I_A(v,P) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(\frac{h\nu}{k_B A}) - 1} \ . \tag{27}$$

Here, A is a scalar quantity which is the only free parameter in (27), and therefore uniquely determines the shape of the curve: we have $A = T$ for the traditional temperature data, whereas for polarization measurements one can take $A = Q, U$. Now the point is that, although there are also a number of foreground sources (such as galaxies or intergalactic dust) that emit radiation on these frequencies; all these astrophysical components (other than CMB) do not follow a blackbody curve.

We shall hence assume that D detectors are available at frequencies $\nu_1, \ldots, \nu_D$, so that the following vector random field is observed:

$$P_{v_r}(x) = P(x) + N_{v_r}(x) \ ;$$

here, both $P(x), N_v(x)$ are taken to be Gaussian zero-mean, mean square continuous random fields, independent among them and such that, while the signal $P(x)$ is identical across all frequencies, the noise $N_v(x)$ is not. More precisely, we shall assume for noise the same regularity conditions as for the signal P, again under the justification that they seem mild and general:

Condition 8. *The (spin) random field $N_v(x)$ is Gaussian and isotropic, independent from $P(x)$ and with total angular power spectrum $\{C_{lN}\}$ such that*

$$C_{lN} = l^{-\gamma} g_N(l) > 0 \ , \ \textit{where} \ c_{0N}^{-1} \leq g_N(l) \leq c_{0N} \ , \ \gamma > 2 \ , \ l \in \mathbb{N} \ ,$$

and for every $r \in \mathbb{N}$ there exist $c_r > 0$ such that

$$|\frac{d^r}{du^r} g_N(u)| \leq c_{rN} u^{-r} \ , \ u \in (|s|, \infty) \ .$$

It follows from our previous assumptions that for each frequency ν_r we shall be able to evaluate

$$\int_{\mathbb{S}^2} P_{v_r}(x)\overline{\psi_{jk;s}}(x)dx =: \beta_{jk;sr} = \beta_{jk;sP} + \beta_{jk;sN_r}$$

where clearly

$$\beta_{jk;sP} = \int_{\mathbb{S}^2} P(x)\overline{\psi_{jk;s}}(x)dx \ , \ \beta_{jk;sN_r} = \int_{\mathbb{S}^2} N_{v_r}(x)\overline{\psi_{jk;s}}(x)dx \ .$$

Now it is immediate to note that

$$\begin{aligned} E|\beta_{jk;sr}|^2 &= E|\beta_{jk;sP} + \beta_{jk;sN_r}|^2 \\ &= E\beta_{jk;sP}\overline{\beta_{jk;sP}} + E\beta_{jk;sN_r}\overline{\beta_{jk;sN_r}} + E\beta_{jk;sN_r}\overline{\beta_{jk;sP}} \\ &\quad + E\beta_{jk;sP}\overline{\beta_{jk;sN_r}} \\ &= E|\beta_{jk;sP}|^2 + E|\beta_{jk;sN_r}|^2 \ , \end{aligned}$$

so that the estimator $\sum_k |\beta_{jk;sr}|^2$ will now be upward biased. In the next subsections we shall discuss two possible solutions for dealing with this bias terms, along the lines of [51], and we will provide statistical procedures to test for estimation bias. We note first that correlation of needlet coefficients across different channels are provided by

$$E\beta_{jk;sr}\overline{\beta_{jk';sr}} = E\beta_{jk;sP}\overline{\beta_{jk';sP}} + E\beta_{jk;sN_r}\beta_{jk';sN_r} \ .$$

Denote

$$\Theta^N_{j;s} = \sum_k E|\beta_{jk;sN_r}|^2 = \sum_l b^2(\frac{\sqrt{e_{ls}}}{B^j})\frac{2l+1}{4\pi}C_{lN} \ ;$$

as before, it is easy to obtain that $C_1 B^{(2-\gamma)j} \leq \Theta^N_{j;s} \leq C_2 B^{(2-\gamma)j}$. With the same discussion as for (22) provided by [21], we have that, under Condition 2 and 8,

$$\left|Corr\left(\beta_{jk;sr}, \overline{\beta_{jk';sr}}\right)\right| \leq \frac{C_M}{\{1 + B^j d(\xi_{jk}, \xi_{jk'})\}^M} \ , \text{ for all } M \in \mathbb{N} \ . \tag{28}$$

7.1. The needlet auto-power spectrum estimator

In many circumstances, it can be reasonable to assume that the angular power spectrum of the noise component, C_{lN}, is known in advance to the experimenter. For instance, if noise is primarily dominated by instrumental components, then its behaviour may possibly be calibrated before the experimental devices are actually sent in orbit, or otherwise by observing a peculiar region where the signal has been very tightly measured by previous experiments. Assuming the angular power spectrum of noise to be known, the expected value for the bias term is immediately derived:

$$E|\beta_{jk;sN_r}|^2 = \sum_l b^2(\frac{\sqrt{e_{ls}}}{B^j})\frac{2l+1}{4\pi}C_{lN_r} \ ,$$

whence it is natural to propose the bias-corrected estimator

$$\begin{aligned}\widetilde{\Theta}_j^{AP} &:= \frac{1}{D}\sum_k\sum_r\left\{|\beta_{jk;sr}|^2 - E|\beta_{jk;sN_r}|^2\right\}\\ &= \frac{1}{D}\sum_k\sum_r\left\{\left(\beta_{jk;sP}+\beta_{jk;sN_r}\right)\left(\overline{\beta_{jk;sP}}+\overline{\beta_{jk;sN_r}}\right) - E|\beta_{jk;sN_r}|^2\right\}\\ &= \sum_k|\beta_{jk;sP}|^2 + \frac{1}{D}\left\{\sum_k\sum_r\left(\beta_{jk;sP}\overline{\beta_{jk;sN_r}} + \beta_{jk;sN_r}\overline{\beta_{jk;sP}}\right.\right.\\ &\quad \left.\left. + \left[|\beta_{jk;sN_r}|^2 - E|\beta_{jk;sN_r}|^2\right]\right)\right\}\ .\end{aligned}$$

We call the previous statistic the needlet auto-power spectrum estimator (AP, compare [51]). The derivation of the following Proposition is rather standard, and hence omitted for brevity's sake.

Proposition 9. *As $j\to\infty$, we have*

$$\frac{\widetilde{\Theta}_j^{AP}-\Theta_j}{\sqrt{Var\{\widetilde{\Theta}_j^{AP}\}}} \to_d N(0,1)\ ,$$

where

$$Var\left\{\widetilde{\Theta}_j^{AP}\right\} = O(B^{2(1-\min(\alpha,\gamma))j})\ .$$

As before, the normalizing variance in the denominator can be consistently estimated by subsampling techniques, along the lines of [8]. It should be noticed that the rate of convergence for $\left\{\widetilde{\Theta}_j^{AP}-\Theta_j\right\} = O(B^{(1-\min(\alpha,\gamma))j})$ is the same as in the noiseless case for $\gamma\geq\alpha$, whereas it slower otherwise, when the noise is asymptotically dominating. The "signal-to-noise" ratio $\Theta_j/\sqrt{Var\{\widetilde{\Theta}_j^{AP}\}}$ is easily seen to be in the order of $B^{2j-\alpha j}/B^{(1-\min(\alpha,\gamma))j} = B^{j(1+\min(\alpha,\gamma)-\alpha)}$, whence it decays to zero unless $\alpha\leq\gamma+1$.

7.2. The needlet cross-power spectrum estimator

To handle the bias term, we shall pursue here a different strategy than the previous subsection, dispensing with any prior knowledge of the spectrum of the noise component. The idea is to exploit the fact that, while the signal is perfectly correlated among the different frequency components, noise is by assumption independent. We shall hence focus on the needlets cross-angular power spectrum

estimator (CP), defined as

$$\begin{aligned}\widetilde{\Theta}_j^{CP} &:= \frac{1}{D(D-1)}\sum_k \sum_{r_1\neq r_2} \beta_{jk;sr_1}\overline{\beta_{jk;sr_2}} \\ &= \frac{1}{D(D-1)}\sum_k \sum_{r_1\neq r_2} \left(\beta_{jk;sP}+\beta_{jk;sN_{r_1}}\right)\left(\overline{\beta_{jk;sP}}+\overline{\beta_{jk;sN_{r_2}}}\right) \\ &= \sum_k |\beta_{jk;sP}|^2 + \frac{1}{D(D-1)} \\ &\quad \times \left\{\sum_k \sum_{r_1\neq r_2} \left(\beta_{jk;sP}\overline{\beta_{jk;sN_{r_2}}}+\beta_{jk;sN_{r_1}}\overline{\beta_{jk;sP}}+\beta_{jk;sN_{r_1}}\overline{\beta_{jk;sN_{r_2}}}\right)\right\} .\end{aligned}$$

In view of the previous independence assumptions, it is then immediately seen that the above estimator is unbiased for Θ_j, i.e.

$$E\widetilde{\Theta}_j^{CP} = \sum_k E\,|\beta_{jk;sP}|^2 = \sum_l b^2\left(\frac{\sqrt{e_{ls}}}{B^j}\right)\frac{2l+1}{4\pi}C_l \ .$$

We are actually able to establish a stronger result, namely

Proposition 10. *As $j\to\infty$, we have*

$$\frac{\widetilde{\Theta}_j^{CP}-\Theta_j}{\sqrt{Var\{\widetilde{\Theta}_j^{CP}\}}} \to_d N(0,1)\ ,\ Var\left\{\widetilde{\Theta}_j^{CP}\right\} = O(B^{2(1-\min(\alpha,\gamma))j}) \ .$$

We omit also this (standard) proof for brevity's sake. We can repeat here the same comments as in the previous subsection, concerning the possibility of estimating the normalizing variance by subsampling techniques, along the lines of [8], and the roles of α,γ for the rate of convergence $\left\{\widetilde{\Theta}_j^{CP}-\Theta_j\right\} = O(B^{(1-\min(\alpha,\gamma))j})$.

7.3. Hausman test for noise misspecification

In the previous two subsections, we have considered two alternate estimators for the angular power spectrum, in the presence of observational noise. It is a standard result (compare [51]) that the auto-power spectrum estimator enjoys a smaller variance, provided of course that the model for noise is correct. Loosely speaking, we can hence conclude that the auto-power spectrum estimator is more efficient when noise is correctly specified, while the cross-power spectrum estimator is more robust, as it does not depend on any previous knowledge on the noise angular power spectrum. An obvious question at this stage is whether the previous results can be exploited to implement a procedure to search consistently for noise misspecification. The answer is indeed positive, as we shall show in the Appendix along the lines of the procedure suggested by [51].

Proposition 11. *Under Assumptions 2 and 8 , we have*

$$\frac{\widetilde{\Theta}^{CP}_{j;s} - \widetilde{\Theta}^{AP}_{j;s}}{\sqrt{Var\{\widetilde{\Theta}^{CP}_{j;s} - \widetilde{\Theta}^{AP}_{j;s}\}}} \rightarrow_d N(0,1),$$

where

$$Var\left\{\widetilde{\Theta}^{CP}_{j;s} - \widetilde{\Theta}^{AP}_{j;s}\right\} = O(B^{2(1-\gamma)j})$$

Remark 12. Note that $Var\{\widetilde{\Theta}^{CP}_{j;s}\}, Var\{\widetilde{\Theta}^{AP}_{j;s}\}, 2Cov\{\widetilde{\Theta}^{CP}_{j;s}, \widetilde{\Theta}^{AP}_{j;s}\}$ are robust to misspecification of the noise, because Variance and Covariance are translation invariant. It follows that the denominator can (once again) be consistently estimated by subsampling techniques, as in [8].

Under the alternative of noise misspecification, we have easily

$$\frac{\widetilde{\Theta}^{CP}_{j;s} - \widetilde{\Theta}^{AP}_{j;s}}{\sqrt{Var\{\widetilde{\Theta}^{CP}_{j;s} - \widetilde{\Theta}^{AP}_{j;s}\}}} \rightarrow_d N(\delta_j, 1)$$

where

$$\delta_j := \frac{E|\beta_{jk;sN_r}|^2 - \Theta_{j;sN_r}}{\sqrt{Var\{\widetilde{\Theta}^{CP}_{j;s} - \widetilde{\Theta}^{AP}_{j;s}\}}}$$

where $\Theta_{j;sN_r}$ is the bias-correction term which is wrongly adopted. The derivation of the power properties of this testing procedure is then immediate.

As a final comment, we notice that throughout this paper we have only been considering estimation and testing for the total angular power spectrum $C_l = C_{lE} + C_{lB}$. The separate estimation of the two components (E and B modes) is of great interest for physical applications, and will be addressed in future work.

8. Appendix

In this paper, we deal with quadratic transforms of random needlet coefficients; as in the earlier works in this area, we use the diagram formulae (see for instance [46, 56]) extensively, and we provide here a brief overview to fix notation. Denote by H_q the $q-$th order Hermite polynomials, defined as

$$H_q(u) = (-1)^q e^{u^2/2} \frac{d^q}{du} e^{-u^2/2}.$$

Diagrams are basically mnemonic devices for computing the moments and cumulants of polynomial forms in Gaussian random variables. Our notation is the same as for instance in [35, 36], where again these techniques are applied in a CMB related framework. Let p and $l_{ij} = 1, \ldots, p$ be given integers. A diagram

γ of order $(l_1,\dots,l_p)$ is a set of points $\{(j,l) : 1 \le j \le p, 1 \le l \le l_j\}$ called vertices, viewed as a table $W = \overrightarrow{l_1} \otimes \cdots \otimes \overrightarrow{l_p}$ and a partition of these points into pairs

$$\{((j,l),(k,s)) : 1 \le j \le k \le p; 1 \le l \le l_j, 1 \le s \le l_k\},$$

called edges. We denote by $I(W)$ the set of diagrams of order $(l_1,\dots,l_p)$. If the order is $l_1 = \cdots = l_p = q$, for simplicity, we also write $I(p,q)$ instead of $I(W)$. We say that:

a) A diagram has a flat edge if there is at least one pair $\{(i,j)(i',j')\}$ such that $i = i'$; we write I_F for the set of diagrams that have at least one flat edge, and $I_{\overline{F}}$ otherwise.

b) A diagram is *connected* if it is not possible to partition the rows $\overrightarrow{l_1} \cdots \overrightarrow{l_p}$ of the table W into two parts, i.e. one cannot find a partition $K_1 \cup K_2 = \{1,\dots,p\}$ that, for each member V_k of the set of edges $(V_1,\dots,V_r)$ in a diagram γ, either $V_k \in \cup_{j\in K_1} \overrightarrow{l_j}$, or $V_k \in \cup_{j\in K_2} \overrightarrow{l_j}$ holds; we write I_C for connected diagrams, and $I_{\overline{C}}$ otherwise.

c) A diagram is paired if, considering any two sets of edges $\{(i_1,j_1)(i_2,j_2)\}$ $\{(i_3,j_3)(i_4,j_4)\}$, then $i_1 = i_3$ implies $i_2 = i_4$; in words, the rows are completely coupled two by two.

The following, well-known Diagram Formula plays a key role in some of our computations (see [46] and [56]).

Proposition 13. (Diagram Formula) *Let $(X_1,\dots,X_p)$ be a centered Gaussian vector, and let $\gamma_{ij} = E[X_iX_j], i,j = 1,\dots,p$ be their covariances, $H_{l_1},\dots,H_{l_p}$ be Hermite polynomials of degree $l_1,\dots,l_p$ respectively. Let L be a table consisting of p rows $l_1,\dots,l_p$, where l_j is the order of Hermite polynomial in the variable X_j. Then*

$$E\left[\prod_{j=1}^{p} H_{l_j}(X_j)\right] = \sum_{G\in I(l_1,\dots,l_p)} \prod_{1\le i\le j\le p} \gamma_{ij}^{\eta_{ij}(G)}$$

$$Cum(H_{l_1}(X_1),\dots,H_{l_p}(X_p)) = \sum_{G\in I_c(l_1,\dots,l_p)} \prod_{1\le i\le j\le p} \gamma_{ij}^{\eta_{ij}(G)}$$

where, for each diagram G, $\eta_{ij}(G)$ is the number of edges between rows l_i, l_j and $Cum(H_{l_1}(X_1),\dots,H_{l_p}(X_p))$ denotes the p-th order cumulant.

Lemma 14. *As $j \to \infty$, under Condition 2 we have*

$$\frac{\widehat{\Theta}_{j;sG} - \Theta_{j;s}}{\sqrt{Var\{\widehat{\Theta}_{j;sG}\}}} \to_d N(0,1)\ .$$

Proof. Notice that

$$\left(\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G^\varepsilon} \lambda_k\right)^2 Var\left(\widehat{\Theta}_{j;sG}\right)$$

$$= Var\left[\sum_k |\beta_{jk;s}|^2\right] = \sum_{k,k'} \left|E\beta_{jk;s}\overline{\beta_{jk';s}}\right|^2$$
$$= \sum_{k,k'} \lambda_{jk}\lambda_{jk'} \left|\sum_l b^2\left(\frac{\sqrt{e_{ls}}}{B^j}\right) C_l \frac{(2l+1)}{4\pi} K^{ls}\left(\xi_{jk}, \xi_{jk'}\right)\right|^2 .$$

By standard manipulations we obtain the upper bound

$$Var\left[\sum_k |\beta_{jk;s}|^2\right] \leq C_M B^{2(2-\alpha)j} \sum_{k,k'} \lambda_{jk}\lambda_{jk'} \frac{1}{\left[1 + B^j d(\xi_{jk}, \xi_{jk'})\right]^{2M}}$$
$$\leq C_M B^{2(2-\alpha)j} \left[\sup_{k'} \lambda_{jk'}\right] \sum_k \lambda_{jk} \sum_{k'} \frac{1}{\left[1 + d(\xi_{jk}, \xi_{jk'})\right]^{2M}}$$
$$= \sum_k \lambda_{jk} O(B^{2(1-\alpha)j}) ,$$

in view of (22) (24) and $\lambda_{jk} \approx B^{-2j}$. On the other hand, we also have the trivial lower bound

$$\sum_{k,k'} \left|E\beta_{jk;s}\overline{\beta_{jk';s}}\right|^2 \geq \sum_k \left|E\beta_{jk;s}\overline{\beta_{jk;s}}\right|^2 = \Theta_{j;s}^2 \sum_k \lambda_{jk}^2 \geq c \sum_k \lambda_{jk} B^{2(1-\alpha)j},$$

whence we have

$$Var\left\{\sum_k |\beta_{jk;s}|^2\right\} \approx \left(\sum_k \lambda_{jk}\right)\left(B^{2(1-\alpha)j}\right) . \tag{29}$$

By recent results in [45, 44] it suffices to focus on fourth-order cumulant; the proof that

$$Cum_4\left\{\frac{\sum_k |\beta_{jk;s}|^2 - \left(\sum_k \lambda_{jk}\right)\Theta_{j;s}}{\sqrt{Var\left\{\sum_k |\beta_{jk;s}|^2\right\}}}\right\} \to 0 \text{ as } j \to \infty ,$$

is a standard application of the Diagram Formula, indeed we have

$$Cum_4\left\{\sum_k |\beta_{jk;s}|^2 - \left(\sum_k \lambda_{jk}\right)\Theta_{j;s}\right\}$$
$$= 6 \sum_{k_1,k_2,k_3,k_4} E\beta_{jk_1;s}\overline{\beta_{jk_2;s}} E\beta_{jk_2;s}\overline{\beta_{jk_3;s}} E\beta_{jk_3;s}\overline{\beta_{jk_4;s}} E\beta_{jk_4;s}\overline{\beta_{jk_1;s}}$$
$$\leq C\left(\Theta_{j;s}\right)^4 \left(\sum_k \lambda_{jk}\right)\left[\sup_{k'} \lambda_{jk'}\right]^3 = \left(\sum_k \lambda_{jk}\right) O\left(B^{(2-4\alpha)j}\right) ,$$

in view of (22) and (23). Thus the Proposition is established. □

Next we turn to the following

Lemma 15. *As $j \to \infty$, under Condition 2 we have*

$$\frac{\sqrt{Var\{\widehat{\Theta}_{j;sG}\}}}{\sqrt{Var\{\widehat{\Theta}^*_{j;sG}\}}} \to 1$$

Proof. Again in view of the Diagram Formula, it is enough to focus on

$$Var\left(\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G^\varepsilon} |\beta_{jk;s}|^2\right) - Var\left(\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G^\varepsilon} \left|\beta^*_{jk;s}\right|^2\right)$$
$$= O\left(\sum_{k,k'} \left|E\beta_{jk;s}\overline{\beta_{jk';s}}\right|^2 - \sum_{k,k'} \left|E\beta^*_{jk;s}\overline{\beta^*_{jk';s}}\right|^2\right).$$

Now notice that

$$\left|E\beta_{jk;s}\overline{\beta_{jk';s}}\right|^2 - \left|E\beta^*_{jk;s}\overline{\beta^*_{jk';s}}\right|^2$$
$$= E\beta_{jk;s}\overline{\beta_{jk';s}}\left(E\overline{\beta_{jk;s}}\beta_{jk';s} - E\overline{\beta^*_{jk;s}}\beta^*_{jk';s}\right)$$
$$+ E\overline{\beta^*_{jk;s}}\beta^*_{jk';s}\left(E\beta_{jk;s}\overline{\beta_{jk';s}} - E\beta^*_{jk;s}\overline{\beta^*_{jk';s}}\right), \tag{30}$$

and

$$E\overline{\beta_{jk;s}}\beta_{jk';s} - E\overline{\beta^*_{jk;s}}\beta^*_{jk';s} = E\overline{\beta_{jk;s}}\left(\beta_{jk';s} - \beta^*_{jk';s}\right) + E\beta^*_{jk';s}\left(\overline{\beta_{jk;s}} - \overline{\beta^*_{jk;s}}\right)$$
$$\le \left\{E|\overline{\beta_{jk;s}}|^2\right\}^{1/2}\left\{E|\beta_{jk';s} - \beta^*_{jk';s}|^2\right\}^{1/2}$$
$$+ \left\{E|\beta^*_{jk';s}|^2\right\}^{1/2}\left\{E|\overline{\beta_{jk;s}} - \overline{\beta^*_{jk;s}}|^2\right\}^{1/2}. \tag{31}$$

Hence

$$E|\beta_{jk;s} - \beta^*_{jk;s}|^2 \le E\left\{\int_G P(x)\overline{\psi_{jk;s}}(x)dx\right\}^2$$
$$\le E\left\{\sup_{x\in G}\{\overline{\psi_{jk;s}}(x)\}\int_{G^\varepsilon}|P(x)|dx\right\}^2$$
$$\le \left[\sup_{x\in G}\{\overline{\psi_{jk;s}}(x)\}\right]^2 E\left\{\int_G |P(x)|dx\right\}^2$$
$$\le \left[\sup_{x\in G}\{\overline{\psi_{jk;s}}(x)\}\right]^2 E\left\{\left[\int_G 1dx\right]\left[\int_G |P(x)|^2dx\right]\right\}$$
$$\le 4\pi\left[\sup_{x\in G}\{\overline{\psi_{jk;s}}(x)\}\right]^2 E\left\{\left[\int_G |P(x)|^2dx\right]\right\}$$
$$= O\left(\frac{B^{2j}}{[1+B^j\varepsilon]^{2M}}\right).$$

Now recall that

$$E|\beta_{jk;s}|^2 = O\left(B^{-\alpha j}\right) \ ,$$

whence $E|\beta^*_{jk;s}|^2 = O\left(B^{-\alpha j}\right)$, if $M > \alpha/2 + 1$. Hence, in view of (31)

$$\left|E\overline{\beta^*_{jk;s}}\beta^*_{jk';s} - E\overline{\beta_{jk;s}}\beta_{jk';s}\right| \leq \frac{CB^{(1-\alpha/2)j}}{[1+B^j\varepsilon]^M} \ , \tag{32}$$

for some constant $C > 0$. Also, from (30) and (32) we obtain that

$$\begin{aligned}
&\sum_{k,k'}\left(\left|E\beta_{jk;su}\overline{\beta_{jk';su}}\right|^2 - \left|E\beta^*_{jk;su}\overline{\beta^*_{jk';su}}\right|^2\right) \\
&\quad\leq \sum_{k,k'}\left(\left|E\beta_{jk;s}\overline{\beta_{jk';s}}\right| + \left|E\beta^*_{jk;s}\overline{\beta^*_{jk';s}}\right|\right) O\left(\frac{B^{-j\alpha/2}}{[1+B^j\varepsilon]^M}\right) \\
&\quad\leq O\left(\frac{B^{(1-\alpha/2)j}}{[1+B^j\varepsilon]^M}\right)\Theta_{j;s}\sum_{k,k'}\frac{C_M\sqrt{\lambda_{jk}\lambda_{jk'}}}{\{1+B^j d(\xi_{jk},\xi_{jk'})\}^M} \\
&\quad\leq O\left(\frac{B^{3(1-\alpha/2)j}}{[1+B^j\varepsilon]^M}\right)\sum_k \lambda_{jk} \ .
\end{aligned}$$

Recall from (29) that $Var\left(\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G}|\beta_{jk;s}|^2\right) = \left(\sum_k \lambda_{jk}\right) O\left(B^{2(1-\alpha)j}\right)$. Hence for M large enough, that is $M > 1+\alpha/2$, the statement of the Proposition is established. □

Lemma 16. *As $j \to \infty$, under Condition 2 we have*

$$\frac{\widehat{\Theta}^*_{j;sG} - \widehat{\Theta}_{j;sG}}{\sqrt{Var\left\{\widehat{\Theta}^*_{j;sG}\right\}}} \to_p 0 \ .$$

Proof. We have

$$E\left\{\left[\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G^\varepsilon}\lambda_k\right]\left(\widehat{\Theta}^*_{j;sG} - \widehat{\Theta}_{j;sG}\right)\right\}^2 = E\left\{\sum_k |\beta_{jk;s}|^2 - \left|\beta^*_{jk;s}\right|^2\right\}^2 \ ,$$

which we can expand as follows

$$\begin{aligned}
&E\left\{\sum_k \overline{\beta_{jk;s}}\left(\beta_{jk;s} - \beta^*_{jk;s}\right) + \sum_k \beta^*_{jk;s}\left(\overline{\beta_{jk;s}} - \overline{\beta^*_{jk;s}}\right)\right\}^2 \\
&\quad= E\left\{\sum_k \overline{\beta_{jk;s}}\left(\beta_{jk;s} - \beta^*_{jk;s}\right)\right\}^2 + E\left\{\sum_k \beta^*_{jk;s}\left(\overline{\beta_{jk;s}} - \overline{\beta^*_{jk;s}}\right)\right\}^2
\end{aligned}$$

$$
\begin{aligned}
&+2E\left\{\sum_k \overline{\beta_{jk;s}}\left(\beta_{jk;s}-\beta^*_{jk;s}\right)\right\}\left\{\sum_k \beta^*_{jk;s}\left(\overline{\beta_{jk;s}}-\overline{\beta^*_{jk;s}}\right)\right\}\\
&=\sum_{k,k'}\left[E\overline{\beta_{jk;s}}\left(\beta_{jk';s}-\beta^*_{jk';s}\right)E\overline{\beta_{jk';s}}\left(\beta_{jk;s}-\beta^*_{jk;s}\right)\right.\\
&\left.+E\beta^*_{jk;s}\left(\overline{\beta_{jk';s}}-\overline{\beta^*_{jk';s}}\right)E\beta^*_{jk';s}\left(\overline{\beta_{jk;s}}-\overline{\beta^*_{jk;s}}\right)\right]\\
&+\left\{\sum_k E\overline{\beta_{jk;s}}\left(\beta_{jk;s}-\beta^*_{jk;s}\right)\right\}^2+\left\{\sum_k E\beta^*_{jk;s}\left(\overline{\beta_{jk;s}}-\overline{\beta^*_{jk;s}}\right)\right\}^2\\
&+2\left\{\sum_k E\overline{\beta_{jk;s}}\left(\beta_{jk;s}-\beta^*_{jk;s}\right)\right\}\left\{\sum_k E\beta^*_{jk;s}\left(\overline{\beta_{jk;s}}-\overline{\beta^*_{jk;s}}\right)\right\}\\
&+2\left\{\sum_{k,k'} E\overline{\beta_{jk;s}}\left(\beta_{jk';s}-\beta^*_{jk';s}\right)E\beta^*_{jk';s}\left(\overline{\beta_{jk;s}}-\overline{\beta^*_{jk;s}}\right)\right\}\\
&+2\left\{\sum_{k,k'} E\overline{\beta_{jk;s}}\beta^*_{jk';s}E\left(\overline{\beta_{jk;s}}-\overline{\beta^*_{jk;s}}\right)\left(\beta_{jk';s}-\beta^*_{jk';s}\right)\right\}.
\end{aligned}
$$

Now recall again

$$
E\left|\beta_{jk;s}\right|^2, E\left|\beta^*_{jk;s}\right|^2 \le CB^{-\alpha j}, \text{ and } E\left|\beta_{jk;s}-\beta^*_{jk;s}\right|^2 \le \frac{C'B^{2j}}{[1+B^j\varepsilon]^M},
$$

whence from the same steps as in the previous Proposition, we have

$$
E\overline{\beta_{jk;s}}\left(\beta_{jk';s}-\beta^*_{jk';s}\right), E\overline{\beta^*_{jk;s}}\left(\beta_{jk';s}-\beta^*_{jk';s}\right) \le \frac{CB^{(1-\alpha/2)j}}{[1+B^j\varepsilon]^M}.
$$

It follows that

$$
E\left\{\sum_k \left|\beta_{jk;s}\right|^2-\left|\beta^*_{jk;s}\right|^2\right\}^2 \le \frac{CB^{(6-\alpha)j}}{[1+B^j\varepsilon]^{2M}}.
$$

By arguments in the previous Propositions, we know that

$$
Var\left\{\left[\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G^\varepsilon}\lambda_k\right]\widehat{\Theta}^*_{j;sG}\right\} \approx \left(\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus G^\varepsilon}\lambda_{jk}\right)B^{2(1-\alpha)j};
$$

thus the statement is established, provided we take $M > 2+\alpha/2$. □

Proof. (Proposition 6). By the Cramer-Wold device, the proof can follow very much the same steps as for the univariate case. We first establish the asymptotic uncorrelation of the two components, i.e. we show that

$$
\begin{aligned}
&\lim_{j\to\infty}\left[Var\left\{\widehat{\Theta}^*_{j;sA_1}\right\}Var\left\{\widehat{\Theta}^*_{j;sA_2}\right\}\right]^{-1/2}E\left\{\left(\widehat{\Theta}^*_{j;sA_1}-\Theta_{j;s}\right)\left(\widehat{\Theta}^*_{j;sA_2}-\Theta_{j;s}\right)\right\}\\
&=0. \qquad (33)
\end{aligned}
$$

Now

$$E\left(\widehat{\Theta}^*_{j;sA_1}-\Theta_{j;s}\right)\left(\widehat{\Theta}^*_{j;sA_2}-\Theta_{j;s}\right)=E\left(\widehat{\Theta}^*_{j;sA_1}-\Theta_{j;s}\right)E\left(\widehat{\Theta}^*_{j;sA_2}-\Theta_{j;s}\right)$$
$$+\left\{\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus A_1^\varepsilon}\lambda_k\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus A_2^\varepsilon}\lambda_k\right\}^{-1}\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus A_1^\varepsilon}\sum_{k':\xi_{jk'}\in\mathbb{S}^2\setminus A_2^\varepsilon}\left|E\beta^*_{jk;s}\overline{\beta^*_{jk';s}}\right|^2. \tag{34}$$

In view of (22) and Proposition 16, we have

$$|(34)|\leq\left(\Theta_{j;s}\right)^2\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus A_1^\varepsilon}\sum_{k':\xi_{jk'}\in\mathbb{S}^2\setminus A_2^\varepsilon}\frac{C\lambda_{jk}\lambda_{jk'}}{\left[1+B^jd(\xi_{jk},\xi_{jk'})\right]^{2M}}$$
$$\leq\frac{C\left(\Theta_{j;s}\right)^2\left[\sup_k\lambda_{jk}\right]^2}{\left[1+2B^j\varepsilon\right]^{2(M-1)}}=O\left(B^{2(1-\alpha-M)j}\right).$$

Thus (33) is established, in view of (29) and Propositions (15), (16). For the fourth order cumulant, given any generic constants u,v, we shall write

$$X=u\left[Var\left\{\widehat{\Theta}^*_{j;sA_1}\right\}\right]^{-1/2}\left(\widehat{\Theta}^*_{j;sA_1}-\Theta_{j;s}\right), \tag{35}$$

and

$$Y=v\left[Var\left\{\widehat{\Theta}^*_{j;sA_2}\right\}\right]^{-1/2}\left(\widehat{\Theta}^*_{j;sA_2}-\Theta_{j;s}\right). \tag{36}$$

Recall that

$$Cum_4\left(X+Y\right)=Cum_4\left(X\right)+Cum_4\left(Y\right)+4Cum(X,Y,Y,Y)$$
$$+6Cum(X,X,Y,Y)+4Cum(X,X,X,Y);$$

by results in the previous Section, we have immediately $Cum_4\left(X\right),Cum_4\left(Y\right)\to 0$, as $j\to\infty$. On the other hand, in view of Proposition 16 and the equivalence between convergence in probability and in L^p for Gaussian subordinated processes (see [30]), we can replace $\widehat{\Theta}^*_{j;sA_i}$ by $\widehat{\Theta}_{j;sA_i}$ in (35) and (36), and we have easily

$$Cum(X,Y,Y,Y)$$
$$\leq CB^{4(\alpha-1)j}\left(\Theta_{j;s}\right)^2\sum_{k:\xi_{jk}\in\mathbb{S}^2\setminus A_1^\varepsilon}\sum_{\xi_{jk_1},..,\xi_{jk_3}\in\mathbb{S}^2\setminus A_2^\varepsilon}\frac{\lambda_{jk}\lambda_{jk_1}\lambda_{jk_3}\lambda_{jk_3}}{\left[1+B^jd(\xi_{jk},\xi_{jk_1})\right]^M}$$
$$\times\frac{1}{\left[1+B^jd(\xi_{jk_2},\xi_{jk_1})\right]^M\left[1+B^jd(\xi_{jk_3},\xi_{jk_2})\right]^M\left[1+B^jd(\xi_{jk},\xi_{jk_3})\right]^M}$$
$$\leq\frac{CB^{4(\alpha-1)j}\left(\Theta_{j;s}\right)^2\left[\sup_k\lambda_{jk}\right]^4}{\left[1+2B^j\varepsilon\right]^{2(M-1)}}=O\left(B^{-2(M+1)j}\right).$$

Similarly, we have

$$Cum(X,X,X,Y),Cum(X,X,Y,Y)\leq CB^{-2(M+1)j}.$$

Thus the Proposition is established, provided we choose $M>2+\alpha$. □

Proof. (Proposition 11). The proof is again quite standard, and we only need to provide the main details. Notice first that

$$\begin{aligned}
&\widetilde{\Theta}_{j;s}^{CP} - \widetilde{\Theta}_{j;s}^{AP} \\
&= \frac{1}{D(D-1)} \sum_k \sum_{r_1 \neq r_2} \beta_{jk;sr_1} \overline{\beta_{jk;sr_2}} - \frac{1}{D} \sum_k \sum_r \left\{ |\beta_{jk;sr}|^2 - E|\beta_{jk;sN_r}|^2 \right\} \\
&= \frac{1}{D(D-1)} \sum_k \left\{ (D-1) \sum_r E|\beta_{jk;sN_r}|^2 - \sum_{r_1 \neq r_2} |\beta_{jk;sr_1} - \beta_{jk;sr_2}|^2 \right\},
\end{aligned}$$

and applying again the Diagram Formula, we have that

$$\begin{aligned}
&Var\left(\widetilde{\Theta}_{j;s}^{CP} - \widetilde{\Theta}_{j;s}^{AP}\right) \\
&= \frac{1}{D^2(D-1)^2} \sum_{k_1,k_2} \sum_{r_1 \neq r_2,\, r_3 \neq r_4,} \left| E\left(\beta_{jk_1;sr_1} - \beta_{jk_1;sr_2}\right) \left(\overline{\beta_{jk_2;sr_3}} - \overline{\beta_{jk_2;sr_4}}\right) \right|^2 .
\end{aligned}$$

Similarly to the discussion for (29), we can show that

$$Var\left(\widetilde{\Theta}_{j;s}^{CP} - \widetilde{\Theta}_{j;s}^{AP}\right) = O\left(D^2 B^{2(1-\gamma))j}\right).$$

Once again, the next step is to consider the fourth order cumulants,

$$\begin{aligned}
&Cum_4 \left\{ \sum_k \left(\sum_{r_1 \neq r_2} |\beta_{jk;sr_1} - \beta_{jk;sr_2}|^2 - (D-1) \sum_r E|\beta_{jk;sN_r}|^2 \right) \right\} \\
&= 6 \sum_{k_1,..,k_4} \sum_{r_{2n} \neq r_{2n-1},\, n=1,..,4} E\left(\beta_{jk_1;sr_1} - \beta_{jk_1;sr_2}\right) \left(\overline{\beta_{jk_2;sr_3}} - \overline{\beta_{jk_2;sr_4}}\right) \\
&\quad \times E\left(\beta_{jk_2;sr_3} - \beta_{jk_2;sr_4}\right) \left(\overline{\beta_{jk_3;sr_5}} - \overline{\beta_{jk_3;sr_6}}\right) \\
&\quad \times E\left(\beta_{jk_3;sr_5} - \beta_{jk_3;sr_6}\right) \left(\overline{\beta_{jk_4;sr_7}} - \overline{\beta_{jk_4;sr_8}}\right) \\
&\quad \times E\left(\beta_{jk_4;sr_7} - \beta_{jk_4;sr_8}\right) \left(\overline{\beta_{jk_1;sr_1}} - \overline{\beta_{jk_1;sr_2}}\right) \\
&\leq C_M D^4 \left(\Theta_{j;s}^N\right)^4 \sum_{k_1,..,k_4} \frac{\lambda_{jk_1}\lambda_{jk_2}\lambda_{jk_3}\lambda_{jk_4}}{\left[1 + d(\xi_{jk_1}, \xi_{jk_2})\right]^M \left[1 + d(\xi_{jk_2}, \xi_{jk_3})\right]^M} \\
&\quad \times \frac{1}{\left[1 + d(\xi_{jk_3}, \xi_{jk_4})\right]^M \left[1 + d(\xi_{jk_4}, \xi_{jk_1})\right]^M} \\
&\leq C D^4 B^{(2-4\gamma)j}.
\end{aligned}$$

in view of (28), choosing $M \geq 3$. Now it is easy to see that

$$Cum_4 \left\{ \frac{\widetilde{\Theta}_{j;s}^{CP} - \widetilde{\Theta}_{j;s}^{AP}}{\sqrt{Var\left\{\widetilde{\Theta}_{j;s}^{CP} - \widetilde{\Theta}_{j;s}^{AP}\right\}}} \right\} \to 0 \ ,$$

whence the Proposition is established, again resorting to results in [45]. □

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