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Variations and Hurst index estimation for a Rosenblatt process using longer filters

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Abstract: The Rosenblatt process is a self-similar non-Gaussian process which lives in second Wiener chaos, and occurs as the limit of correlated random sequences in so-called "non-central limit theorems". It shares the same covariance as fractional Brownian motion. We study the asymptotic distribution of the quadratic variations of the Rosenblatt process based on long filters, including filters based on high-order finite-difference and wavelet-based schemes. We find exact formulas for the limiting distributions, which we then use to devise strongly consistent estimators of the self-similarity parameter H. Unlike the case of fractional Brownian motion, no matter now high the filter orders are, the estimators are never asymptotically normal, converging instead in the mean square to the observed value of the Rosenblatt process at time 1.

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Contents

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1. Introduction

Self-similar stochastic processes are of practical interest in various applications, including econometrics, internet traffic, and hydrology. These are processes $X = \{X(t) : t \geq 0\}$ whose dependence on the time parameter t is self-similar, in the sense that there exists a (self-similarity) parameter $H \in (0,1)$ such that for any constant $c \geq 0$, $\{X(ct) : t \geq 0\}$ and $\left\{c^H X(t) : t \geq 0\right\}$ have the same distribution. These processes are often endowed with other distinctive properties.

The fractional Brownian motion (fBm) is the usual candidate to model phenomena in which the selfsimilarity property can be observed from the empirical data. This fBm B^H is the continuous centered Gaussian process with covariance function

$$R^H(t,s) := \mathbf{E}\left[B^H(t)\, B^H(s)\right] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{1}$$

The parameter H characterizes all the important properties of the process. In addition to being self-similar with parameter H, which is evident from the covariance function, fBm has correlated increments: in fact, from (1) we get, as $n \to \infty$,

$$\mathbf{E}\left[\left(B^H(n) - B^H(1)\right) B^H(1)\right] = H(2H-1)\, n^{2H-2} + o\left(n^{2H-2}\right); \tag{2}$$

when $H < 1/2$, the increments are negatively correlated and the correlation decays more slowly than quadratically; when $H > 1/2$, the increments are positively correlated and the correlation decays so slowly that they are not summable, a situation which is commonly known as the long memory property. The covariance structure (1) also implies

$$\mathbf{E}\left[\left(B^H(t) - B^H(s)\right)^2\right] = |t-s|^{2H}; \tag{3}$$

this property shows that the increments of fBm are stationary and self-similar; its immediate consequence for higher moments can be used, via the so-called Kolmogorov continuity criterion, to imply that B^H has paths which are almost-surely $(H-\varepsilon)$-Hölder-continuous for any $\varepsilon > 0$.

It turns out that fBm is the *only* continuous Gaussian process which is self-similar with stationary increments. However, there are many more stochastic processes which, except for the Gaussian character, share all the other properties above for $H > 1/2$ (i.e. (1) which implies (2), the long-memory property, (3), and in many cases the Hölder-continuity). In some models the Gaussian assumption may be implausible and in this case one needs to use a different self-similar process with stationary increments to model the phenomenon. Natural candidates are the Hermite processes: these non-Gaussian stochastic processes appear as limits in the so-called Non-Central Limit Theorem (see [5, 8, 25]) and do indeed have all the properties listed above. While fBm can be expressed as a Wiener integral with respect to the standard Wiener process, i.e. the integral of a deterministic kernel w.r.t. a standard Brownian motion, the Hermite process of order $q \geq 2$ is a qth iterated integral of a deterministic function with q variables with respect to a standard Brownian motion. When $q = 2$, the Hermite process is called the Rosenblatt process. This stochastic process typically appears as a limiting model in various applications such as unit the root testing problem (see [31]), semiparametric approach to hypothesis test (see [13]), or long-range dependence estimation (see [15]). On the other hand, since it is non-Gaussian and self-similar with stationary increments, the Rosenblatt process can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian. The need of non-Gaussian self-similar processes in practice (for example in hydrology) is mentioned in the paper [26] based on the study of stochastic modeling for river-flow time series in [16]. Recent interest in the Rosenblatt and other Hermite processes, due in part to their non-Gaussian character, and in part for their independent mathematical value, is evidenced by the following references: [4, 6, 10, 18, 19, 20, 27, 28].

The results in these articles, and in the previous references on the non-central limit theorem, have one point in common: of all the Hermite processes, the most important one in terms of limit theorem, apart from fBm, is the Rosenblatt process. As such, it should be the first non-Gaussian self-similar process for which to develop a full statistical estimation theory. This is one motivation for writing this article.

Since the Hurst parameter H, thus called in reference to the hydrologist who discovered its original practical importance (see [14]), characterizes all the

important properties of a Hermite process, its proper statistical estimation is of the utmost importance. Several statistics have been introduced to this end in the case of fBm, such as variograms, maximum likelihood estimators, or spectral methods, k-variations and wavelets. Information on these various approaches, apart from wavelets, for fBm and other long-memory processes, can be found in the book of Beran [3]. More details about the wavelet-based approach can be found in [2, 11] and [30].

In this article, we will concentrate on one of the more popular methods to estimate H: the study of power variations; it is particularly well-adapted to the non-Gaussian Hermite processes, because explicit calculations can be performed via Wiener chaos analysis. In its simplest form, the kth power variation statistic of a process $\{X_t : t \in [0,1]\}$, calculated using N data points, is defined as following quantity (the absolute value of the increment may be used in the definition for non-even powers):

$$V_N := \frac{1}{N}\left[\sum_{i=0}^{N-1} \frac{\left(X_{\frac{i+1}{N}} - X_{\frac{i}{N}}\right)^k}{\mathbf{E}\left(X_{\frac{i+1}{N}} - X_{\frac{i}{N}}\right)^k} - 1\right]. \tag{4}$$

There exists a direct connection between the behavior of the variations and the convergence of an estimator for the selfsimilarity order based on these variations (see [7, 28]): if the renormalized variation satisfies a central limit theorem then so does the estimator, a desirable fact for statistical purposes.

The recent paper [28] studies the quadratic variation of the Rosenblatt process Z (the V_N above with $k = 2$), exhibiting the following facts: the normalized sequence $N^{1-H}V_N$ satisfies a non-central limit theorem, it converges in the mean square to the Rosenblatt random variable $Z(1)$ (value of the process Z at time 1); from this, we can construct an estimator for H whose behavior is still non-normal. The same result is also obtained in the case of the estimators based on the wavelet coefficients (see [2]). In the simpler case of fBm, this situation still occurs when $H > 3/4$ (see for instance [29]). For statistical applications, a situation in which asymptotic normality holds might be preferable. To achieve this for fBm, it has been known for some time that one may use "longer filters" (that means, replacing the increments $X_{\frac{i+1}{N}} - X_{\frac{i}{N}}$ by the second-order increments $X_{\frac{i+1}{N}} - 2X_{\frac{i}{N}} + X_{\frac{i-1}{N}}$, or higher order increments for instance; see [7]). To have asymptotic normality in the case of the Rosenblatt process, it was shown in [28] that one may perform a compensation of the non-normal component of the quadratic variation. In fact, this is possible only in the case of the Rosenblatt process; it is not possible for higher-order Hermite processes, and is not possible for fBm with $H > 3/4$ [recall that the case of fBm with $H \leq 3/4$ does not require any compensation]. The compensation technique for the Rosenblatt process yields asymptotic variances which are difficult to calculate and may be very high.

The question then arises to find out whether using longer filters for the Rosenblatt process might yield asymptotically normal estimators, and/or might result in low asymptotic variances. In this article, using recent results on limit theorems

for multiple stochastic integrals based on the Malliavin calculus (see [22, 23]), we will see that the answer to the first question is negative, while the answer to the second question is affirmative. We will use quadratic variations ($k = 2$) for simplicity. A summary of our results is as follows. Here Ω denotes the underlying probability space, and $L^1(\Omega)$ and $L^2(\Omega)$ are the usual spaces of integrable and square-integrable random variables.

- $V_N = T_2 + T_4$ where T_i is in the ith Wiener chaos (Proposition 2).
- $\frac{\sqrt{N}}{c_{1,H}} T_4$ converges in distribution to a standard normal (Theorem 2), where $c_{1,H}$ is given in Proposition 4.
- $\frac{N^{1-H}}{\sqrt{c_{2,H}}} V_N$ and $\frac{N^{1-H}}{\sqrt{c_{2,H}}} T_2$ both converge in $L^2(\Omega)$ to the Rosenblatt random variable $Z(1)$ (Theorem 3); the asymptotic variance $c_{2,H}$ is given explicitly in formula (16) in Proposition 3.
- There exists a strongly consistent estimator $\hat{H}_N$ for H based on V_N (Theorem 5), and $2\, c_{2,H}^{-1/2} (\log N) N^{1-\hat{H}_N} \left(\hat{H}_N - H\right)$ converges in $L^1(\Omega)$ to a Rosenblatt random variable (Theorem 7). Here $c_{2,H}$ is again given in (16). Note that while the rate of convergence of the estimator, of order $N^{-1+H} \log^{-1} N$, depends on H, the convergence result above can be used without knowledge of H since one may plug in $\hat{H}_N$ instead of H in the convergence rate.
- The asymptotic variance $c_{2,H}$ in the above convergence decreases as the length of the filter increases; this decrease is much faster for wavelets-based filters than for finite-difference-based filters: for values of $H < 0.95$, $c_{2,H}$ reaches values below 5% for wavelet filters of length less than 6, but for finite-difference filters of length no less than 16.
- When $H \in (1/2, 2/3)$, then $\frac{N}{c_{3,H}} \left[V_N - \frac{\sqrt{c_{2,H}}}{N^{1-H}} Z(1)\right]$ converges in distribution to a standard normal, where $c_{2,H}$ is given explicitly in formula (16) and $c_{3,H}$ in formula (19). Similarly, for the estimator we have that $\frac{N}{c_{3,H}} \left[-2\log(\hat{H}_N - H) - \frac{\sqrt{c_{2,H}}}{N^{1-H}} Z(1)\right]$ converges in distribution to the same standard normal. However, no mater how much we increase the order and/or the length of the filter, we cannot improve the threshold of $2/3$ for H.

What prevents the normalization of V_N from converging to a Gaussian, no matter how long the filter is, is the distinction between the two terms T_2 and T_4. In the case of fractional Brownian motion, V_N contains only one "T_2"-type term (second chaos), but this term has a behavior similar to our term T_4, and does converge to a normal when the filter is long enough; this fact has been noted before (see [7]). In our case, the normalized T_2 always converges (in $L^2(\Omega)$) to a Rosenblatt random variable; the piece that sometimes has normal asymptotics is T_4, but since T_2 always dominates it, V_N's behavior is always that of T_2. This sort of phenomenon was already noted in [6] with the order-one filter for all non-Gaussian Hermite processes, but now we know it occurs also for the simplest Hermite process that is not fBm, for filters of all orders.

The organization of our paper is as follows. Section 2 summarizes the stochastic analytic tools we will use, and gives the definitions of the Rosenblatt process and the filter variations. Therein we also establish a specific representation of the 2-power variation as the sum of two terms, one in the second Wiener chaos, which we call T_2, and another, T_4, in the fourth Wiener chaos. Section 3 establishes the correct normalizing factors for the variations, by computing second moments, showing in particular that T_2 is the dominant term. Section 4 proves that the renormalized T_4 is asymptotically normal. Section 5 proves that T_2 converges in $L^2(\Omega)$ to the value $Z(1)$ of the Rosenblatt process at time 1. In Section 6 it is shown that the variation obtained by subtracting this observed limit of T_2 leads to a correction term which is asymptotically normal. Section 7 establishes the strong consistency of the estimator $\hat{H}$ for H based on the variations, and proves that the renormalized estimator converges to a Rosenblatt random variable in $L^1(\Omega)$. Its asymptotic variance is given explicitly for any filter, thanks to the calculations in Section 3. In Section 8, we compare the numerical values of the asymptotic variances for various choices of filters, including finite-difference filters and wavelet-based filters, concluding that the latter are more efficient.

2. Preliminaries

2.1. Basic tools on multiple Wiener-Itô integrals

Let $\{W_t : t \in [0,1]\}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, P)$. If a symmetric function $f \in L^2([0,1]^n)$ is given, the multiple Wiener-Itô integral $I_n(f)$ of f with respect to W is constructed and studied in detail in [21, Chapter 1]. Here we collect the results we will need. For the most part, the results in this subsection will be used in the technical portions of our proofs, which are in the Appendix. One can construct the multiple integral starting from simple functions of the form $f := \sum_{i_1,\ldots,i_n} c_{i_1,\ldots i_n} 1_{A_{i_1}\times\cdots\times A_{i_n}}$ where the coefficient $c_{i_1,\ldots,i_n}$ is zero if two indices are equal and the sets A_{i_j} are disjoint intervals, by setting

$$I_n(f) := \sum_{i_1,\ldots,i_n} c_{i_1,\ldots i_n} W(A_{i_1}) \ldots W(A_{i_n})$$

where we put $W\left(1_{[a,b]}\right) = W([a,b]) = W_b - W_a$; then the integral is extended to all symmetric functions in $L^2([0,1]^n)$ by a density argument. It is also convenient to note that this construction coincides with the iterated Itô stochastic integral

$$I_n(f) = n! \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \ldots, t_n) dW_{t_1} \ldots dW_{t_n}.$$

The application I_n is extended to non-symmetric functions f via

$$I_n(f) = I_n\left(\tilde{f}\right) \tag{5}$$

where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1,\ldots,x_x)=\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} f(x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$

The map $(n!)^{-1/2}I_n$ can then be seen to be an isometry from $L^2([0,1]^n)$ to $L^2(\Omega)$. The nth Wiener chaos is the set of all integrals $\{I_n(f): f\in L^2([0,1]^n)\}$; the Wiener chaoses form orthogonal sets in $L^2(\Omega)$. Summarizing, we have

$$\begin{aligned}&\mathbf{E}\left(I_n(f)I_m(g)\right)=n!\langle f,g\rangle_{L^2([0,1]^n)} \quad \text{if } m=n,\\ &\mathbf{E}\left(I_n(f)I_m(g)\right)=0 \quad \text{if } m\neq n.\end{aligned} \tag{6}$$

The product for two multiple integrals can be expanded explicitly (see [21]): if $f\in L^2([0,1]^n)$ and $g\in L^2([0,1]^m)$ are symmetric, then it holds that

$$I_n(f)I_m(g)=\sum_{\ell=0}^{m\wedge n}\ell!C_m^\ell C_n^\ell I_{m+n-2\ell}(f\otimes_\ell g) \tag{7}$$

where the contraction $f\otimes_\ell g$ belongs to $L^2([0,1]^{m+n-2\ell})$ for $\ell=0,1,\ldots,m\wedge n$ and is given by

$$\begin{aligned}&(f\otimes_\ell g)(s_1,\ldots,s_{n-\ell},t_1,\ldots,t_{m-\ell})\\ =\;&\int_{[0,1]^\ell} f(s_1,\ldots,s_{n-\ell},u_1,\ldots,u_\ell)g(t_1,\ldots,t_{m-\ell},u_1,\ldots,u_\ell)du_1\ldots du_\ell.\end{aligned}$$

Note that the contraction $(f\otimes_\ell g)$ is not necessary symmetric. We will denote by $(f\tilde{\otimes}_\ell g)$ its symmetrization.

Our analysis will be based on the following result, due to Nualart and Peccati (see Theorem 1 in [22]).

Proposition 1. *Let n be a fixed integer. Let $I_n(f_N)$ be a sequence of symmetric square integrable random variables in the nth Wiener chaos such that $\lim_{N\to\infty}\mathbf{E}\left[I_n(f_N)^2\right]=1$. Then the following are equivalent:*

(i) *As $N\to\infty$, the sequence $\{I_n(f_N): N\geq 1\}$ converges in distribution to a standard Gaussian random variable.*

(ii) *For every $\tau=1,\ldots,n-1$*

$$\lim_{N\to\infty}||f_N\otimes_\tau f_N||^2_{L^2[[0,1]^{(2n-2\tau)}]}=0.$$

2.2. Rosenblatt process and filters: definitions, notation, and chaos representation

The Rosenblatt process is the (non-Gaussian) Hermite process of order 2 with Hurst index $H\in(\frac{1}{2},1)$. It is self-similar with stationary increments, lives in the

second Wiener chaos and can be represented as a double Wiener-Itô integral of the form

$$Z^{(H)}(t) := Z(t) = \int_0^t \int_0^t L_t(y_1, y_2) dW_{y_1} dW_{y_2}. \tag{8}$$

Here $\{W_t, t \in [0,1]\}$ is a standard Brownian motion and $L_t(y_1, y_2)$ is the kernel of the Rosenblatt process

$$L_t(y_1, y_2) = d(H) 1_{[0,t]}(y_1) 1_{[0,t]}(y_2) \int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du, \tag{9}$$

where

$$H' = \frac{H+1}{2} \text{ and } d(H) = \frac{1}{H+1} \left(\frac{H}{2(2H-1)} \right)^{-1/2}$$

and K^H is the standard kernel of fBm, defined for $s < t$ and $H \in (\frac{1}{2}, 1)$ by

$$K^H(t,s) := c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \tag{10}$$

where $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}}$ and $\beta(\cdot,\cdot)$ is the beta function. For $t > s$, we have the following expression for the derivative of K^H with respect to its first variable:

$$\frac{\partial K^H}{\partial t}(t,s) := \partial_1 K^H(t,s) = c_H \left(\frac{s}{t} \right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}. \tag{11}$$

The term *Rosenblatt random variable* denotes any random variable which has the same distribution as $Z(1)$. Note that this distribution depends on H.

Definition 1. A filter α of length $\ell \in \mathbb{N}$ and order $p \in \mathbb{N} \setminus 0$ is an $(\ell+1)$-dimensional vector $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$ such that

$$\sum_{q=0}^{\ell} \alpha_q q^r = 0, \quad \text{for} \quad 0 \le r \le p-1, \ r \in \mathbf{Z}$$
$$\sum_{q=0}^{\ell} \alpha_q q^p \neq 0$$

with the convention $0^0 = 1$.

If we associate such a filter α with the Rosenblatt process we get the filtered process V^α according to the following scheme:

$$V^\alpha \left(\frac{i}{N} \right) := \sum_{q=0}^{\ell} \alpha_q Z \left(\frac{i-q}{N} \right), \ \text{ for } i = \ell, \ldots, N-1.$$

Some examples are the following:

1. For $\alpha = \{1, -1\}$

$$V^{\alpha}\left(\frac{i}{N}\right) = Z\left(\frac{i}{N}\right) - Z\left(\frac{i-1}{N}\right).$$

This is a filter of length 1 and order 1.

2. For $\alpha = \{1, -2, 1\}$

$$V^{\alpha}\left(\frac{i}{N}\right) = Z\left(\frac{i}{N}\right) - 2Z\left(\frac{i-1}{N}\right) + Z\left(\frac{i-2}{N}\right).$$

This is a filter of length 2 and order 2.

3. More generally, longer filters produced by finite-differencing are such that the coefficients of the filter α are the binomial coefficients with alternating signs. Therefore, borrowing the notation ∇ from time series analysis, $\nabla Z\left(i/N\right) = Z\left(i/N\right) - Z\left(\left(i-1\right)/N\right)$, we define $\nabla^j = \nabla\nabla^{j-1}$ and we may write the jth-order finite-difference-filtered process as follows

$$V^{\alpha_j}\left(\frac{i}{N}\right) := \left(\nabla^j Z\right)\left(\frac{i}{N}\right).$$

From now on we assume the filter order is strictly greater than 1 ($p \geq 2$).

For such a filter α the quadratic variation statistic is defined as

$$V_N := \frac{1}{N-\ell}\sum_{i=\ell}^{N-1}\left[\frac{\left|V^{\alpha}\left(\frac{i}{N}\right)\right|^2}{\mathbf{E}\left|V^{\alpha}\left(\frac{i}{N}\right)\right|^2} - 1\right].$$

Using the definition of the filter, we can compute the covariance of the filtered process $V^{\alpha}\left(\frac{i}{N}\right)$:

$$\begin{aligned}
\pi_H^{\alpha}(j) &:= \mathbf{E}\left[V^{\alpha}\left(\frac{i}{N}\right)V^{\alpha}\left(\frac{i+j}{N}\right)\right] \\
&= \sum_{q,r=0}^{\ell}\alpha_q\alpha_r\mathbf{E}\left[Z\left(\frac{i-q}{N}\right)Z\left(\frac{i+j-r}{N}\right)\right] \\
&= \frac{N^{-2H}}{2}\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left(|i-q|^{2H} + |i+j-r|^{2H} - |j+q-r|^{2H}\right) \\
&= -\frac{N^{-2H}}{2}\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|j+q-r|^{2H} \\
&\quad + \frac{N^{-2H}}{2}\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left(|i-q|^{2H} + |i+j-r|^{2H}\right).
\end{aligned}$$

Since the term $\sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left(|i-q|^{2H} + |i+j-r|^{2H}\right)$ vanishes we get that

$$\pi_H^{\alpha}(j) = -\frac{N^{-2H}}{2} \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |j+q-r|^{2H}. \tag{12}$$

Therefore, we can rewrite the variation statistic as follows

$$\begin{aligned} V_N &= \frac{1}{N-\ell} \sum_{i=\ell}^{N-1} \left[\frac{\left|V^{\alpha}\left(\frac{i}{N}\right)\right|^2}{\pi_H^{\alpha}(0)} - 1 \right] \\ &= \frac{2N^{2H}}{N-\ell} \left(- \sum_{q,r=0}^{\ell} \alpha_r \alpha_q |q-r|^{2H} \right)^{-1} \sum_{i=\ell}^{N-1} \left[\left| V^{\alpha}\left(\frac{i}{N}\right) \right|^2 - \pi_H^{\alpha}(0) \right] \\ &= \frac{2N^{2H}}{c(H)(N-\ell)} \sum_{i=\ell}^{N-1} \left[\left| V^{\alpha}\left(\frac{i}{N}\right) \right|^2 - \pi_H^{\alpha}(0) \right], \end{aligned}$$

where

$$c(H) = - \sum_{q,r=0}^{\ell} \alpha_r \alpha_q |q-r|^{2H}. \tag{13}$$

The next lemma is informative, and will be useful in the sequel.

Lemma 1. *$c(H)$ is positive for all $H \in (0,1]$. Also, $c(0) = 0$.*

Proof. For $H < 1$, we may rewrite $c(H)$ by using the representation of the function $|q-r|^{2H}$ via fBm B^H, as its canonical metric given in (3), and its covariance function R_H given in (1). Indeed we have

$$\begin{aligned} c(x) &= - \sum_{q,r=0}^{\ell} \alpha_r \alpha_q \mathbf{E}\left[\left(B^H(q) - B^H(r) \right)^2 \right] \\ &= - \sum_{q,r=0}^{\ell} \alpha_r \alpha_q \left(R_H(q,q) + R_H(r,r) - 2R_H(q,r) \right) \\ &= -2 \left(\sum_{q=0}^{\ell} \alpha_q \right) \left(\sum_{r=0}^{\ell} \alpha_r R_H(r,r) \right) + 2 \sum_{q,r=0}^{\ell} \alpha_r \alpha_q R_H(q,r) \\ &= 0 + 2 \sum_{q,r=0}^{\ell} \alpha_r \alpha_q R_H(q,r) = \mathbf{E}\left[\left(\sum_{q=0}^{\ell} \alpha_q B^H(q) \right)^2 \right] > 0 \end{aligned}$$

where in the second-to-last line we used the filter property which implies $\sum_{q=0}^{\ell} \alpha_q = 0$, and the last inequality follows from the fact that $\sum_{q=0}^{\ell} \alpha_q B^H(q)$ is Gaussian and non-constant. When $H = 1$, the same argument as above holds because the Gaussian process X such that $X(0) = 0$ and $\mathbf{E}\left[(X(t) - X(s))^2\right] = |t-s|^2$ is evidently equal in law to $X(t) = tN$ where N is a fixed standard normal r.v. The assertion that $c(0) = 0$ comes from the filter property. □

Observe that we can write the filtered process as an integral belonging to the second Wiener chaos

$$V^{\alpha}\left(\frac{i}{N}\right)=\sum_{q=0}^{\ell}\alpha_q Z\left(\frac{i-q}{N}\right)=I_2\left(\sum_{q=0}^{\ell}\alpha_q L_{\frac{i-q}{N}}\right):=I_2\left(C_i\right),$$

where

$$C_i:=\sum_{q=0}^{\ell}\alpha_q L_{\frac{i-q}{N}}. \tag{14}$$

Using the product formula (7) for multiple stochastic integrals now results in the Wiener chaos expansion of V_N.

Proposition 2. *With C_i as in (14), the variation statistic V_N is given by*

$$\begin{aligned}V_N&=\frac{2N^{2H}}{c(H)(N-l)}\sum_{i=\ell}^{N-1}\left[\left|I_2(C_i)\right|^2-\pi_H^{\alpha}(0)\right]\\&=\frac{2N^{2H}}{c(H)(N-\ell)}\left[\sum_{i=\ell}^{N-1}I_4\left(C_i\otimes C_i\right)+4\sum_{i=\ell}^{N-1}I_2\left(C_i\otimes_1 C_i\right)\right]\\&:=T_4+T_2,\end{aligned}$$

where T_4 is a term belonging to the 4th Wiener chaos and T_2 a term living in the 2nd Wiener chaos.

In order to prove that a variation statistic has a normal limit we may use the characterization of $\mathcal{N}(0,1)$ by Nualart and Ortiz-Latorre (Proposition 1). Thus, we need to start by calculating $\mathbf{E}\left[|V_N|^2\right]$ so that we can then scale appropriately, in an attempt to apply the said proposition.

3. Scale constants for T_2 and T_4

In order to determine the convergence of V_N, using the orthogonality of the integrals belonging in different chaoses, we will study each term separately. This section begins by calculating the second moments of T_2 and T_4.

In this section we use an alternative expression for the filtered process. More specifically, denoting $b_q:=\sum_{r=0}^{q}\alpha_r$, we rewrite C_i as follows, for any $i=\ell,\ldots,N-1$:

$$\begin{aligned}C_{i,\ell}:=C_i&=\sum_{q=0}^{\ell}\alpha_q L_{\frac{i-q}{N}}\\&=\alpha_0\left(L_{\frac{i}{N}}-L_{\frac{i-1}{N}}\right)+(\alpha_0+\alpha_1)\left(L_{\frac{i-1}{N}}-L_{\frac{i-2}{N}}\right)+\cdots\\&\quad+(\alpha_0+\cdots+\alpha_{\ell-1})\left(L_{\frac{i-(\ell-1)}{N}}-L_{\frac{i-\ell}{N}}\right)\\&=\sum_{q=0}^{\ell}b_q\left(L_{\frac{i-(q-1)}{N}}-L_{\frac{i-q}{N}}\right).\end{aligned} \tag{15}$$

Recall that the filter properties imply $\sum_{q=0}^{\ell}\alpha_q=0$ and $\alpha_\ell=-\sum_{q=0}^{\ell-1}\alpha_q$.

3.1. Term T_2

By Proposition 2, we can express $\mathbf{E}(T_2^2)$ as:

$$\begin{aligned}\mathbf{E}(T_2^2) &= \frac{64\,N^{4H}}{c(H)^2(N-\ell)^2}\mathbf{E}\left[\left(\sum_{i=\ell}^{N-1} I_2\left(C_i \otimes_1 C_i\right)\right)^2\right] \\ &= \frac{2!\,64\,N^{4H}}{c(H)^2(N-\ell)^2}\sum_{i,j=\ell}^{N-1}\langle C_i \otimes_1 C_i,\ C_j \otimes_1 C_j\rangle_{L^2([0,1]^2)}\end{aligned}$$

Proposition 3. *We have*

$$\lim_{N\to\infty}\mathbf{E}\left[\left|N^{1-H}\,T_2\right|^2\right] = c_{2,H}.$$

where

$$\begin{aligned}c_{2,H} &= \frac{64}{c(H)^2}\left(\frac{2H-1}{H\,(H+1)^2}\right) \\ &\quad\times\left\{\sum_{q,r=0}^{\ell} b_q b_r\left[|1+q-r|^{2H'}+|1-q+r|^{2H'}-2|q-r|^{2H'}\right]\right\}^2. \quad (16)\end{aligned}$$

This proposition is proved in the Appendix.

3.2. Term T_4

In this paragraph we estimate the second moment of T_4, the fourth chaos term appearing in the decomposition of the variation V_N. Here the function $\sum_{i=\ell}^{N-1}(C_i \otimes C_i)$ is no longer symmetric and we need to symmetrize this kernel to calculate T_4's second moment. In other words, by Proposition 2, we have that

$$\begin{aligned}\mathbf{E}\left(T_4^2\right) &= \frac{4N^{4H}}{c(H)^2(N-\ell)^2}\mathbf{E}\left[\left(\sum_{i=\ell}^{N-1} I_4(C_i\otimes C_i)\right)^2\right] \\ &= \frac{4N^{4H}}{c(H)^2(N-\ell)^2}4!\sum_{i,j=\ell}^{N-1}\langle C_i\tilde{\otimes}C_i, C_j\tilde{\otimes}C_j\rangle_{L^2([0,1]^4)}\end{aligned}$$

where $C_i\tilde{\otimes}C_i := \widetilde{C_i\otimes C_i}$. Thus, we can use the following combinatorial formula: If f and g are two symmetric functions in $L^2([0,1]^2)$, then

$$\begin{aligned}&4!\langle f\tilde{\otimes}f, g\tilde{\otimes}g\rangle_{L^2([0,1]^4)} \\ &= (2!)^2\langle f\otimes f, g\otimes g\rangle_{L^2([0,1]^4)} + (2!)^2\langle f\otimes_1 g, g\otimes_1 f\rangle_{L^2([0,1]^2)}.\end{aligned}$$

It implies

$$\begin{aligned}\mathbf{E}\left(T_4^2\right) &= \frac{4N^{4H}}{c(H)^2(N-\ell)^2}4!\sum_{i,j=\ell}^{N-1}\langle C_i\tilde{\otimes}C_i, C_j\tilde{\otimes}C_j\rangle_{L^2([0,1]^4)}\\ &= \frac{4N^{4H}}{c(H)^2(N-\ell)^2}4\sum_{i,j=\ell}^{N-1}\langle C_i\otimes C_i, C_j\otimes C_j\rangle_{L^2([0,1]^4)}\\ &+ \frac{4N^{4H}}{c(H)^2(N-\ell)^2}4\sum_{i,j=\ell}^{N-1}\langle C_i\otimes_1 C_j, C_j\otimes_1 C_i\rangle_{L^2([0,1]^2)}\\ &:= T_{4,(1)}+T_{4,(2)}.\end{aligned}$$

The proof of the next proposition, in the Appendix, shows that the two terms $T_{4,(1)}$ and $T_{4,(2)}$ have the same order of magnitude, with only the normalizing constant being different.

Proposition 4. *Recall the constant $c(H)$ defined in (13). Let*

$$\begin{aligned}\tau_{1,H} := \sum_{k=\ell}^{\infty}\sum_{q_1,q_2,r_1,r_1=0}^{\ell} b_{q_1}b_{q_2}b_{r_1}b_{r_2}\int_{[0,1]^4}dudvdu'dv'\\ \Big[\,|u-v+k-q_1+r_1|^{2H'-2}\,|u'-v'+k-q_2+r_2|^{2H'-2}\\ |u-u'+k-q_1+q_2|^{2H'-2}\,|v-v'+k-r_1+r_2|^{2H'-2}\Big].\end{aligned}$$

and

$$\rho_H^{\alpha}(k) := \frac{\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\,|k+q-r|^{2H}}{c(H)}$$

Then we have the following asymptotic variance for $\sqrt{N}T_4$:

$$\lim_{N\to\infty}\mathbf{E}\left[\left|\sqrt{N}\,T_4\right|^2\right] = c_{1,H} := 4!\left(1+\sum_{k=0}^{\infty}|\rho_H^{\alpha}(k)|^2\right)+\tau_{1,H}. \tag{17}$$

This proposition is proved in the Appendix. Observe that in the Wiener chaos decomposition of V_N the leading term is the term in the second Wiener chaos (i.e. T_2) since it is of order N^{H-1}, while T_4 is of the smaller order $N^{-1/2}$. We note that, in contrast to the case of filters of lenght 1 and power 1, the barrier $H = 3/4$ does not appear anymore in the estimation of the magnitude of T_4 Thus, the asymptotic behavior of V_N is determined by the behavior of T_2. In other words, the previous three propositions imply the following.

Theorem 1. *For all $H \in (1/2, 1)$ we have that*

$$\lim_{N\to\infty}\mathbf{E}\left[\left|N^{1-H}\,V_N\right|^2\right] = c_{2,H},$$

where $c_{2,H}$ is defined in (16).

From the practical point of view, one only needs to compute the constant $c_{2,H}$ to find the first order asymptotics of V_N. This constant is easily computed exactly from its formula (16), unlike the constant $c_{1,H}$ in Proposition (4) which can only be approximated via its unwieldy series-integral representation given therein.

4. Normality of the term T_4

We study in this section the limit of the renormalized term T_4 which lives in the fourth Wiener chaos and appears in the expression of the variation V_N. Of course, due to Theorem 1 above, this term does not affect the first order behavior of V_N but it is interesting from the mathematical point of view because its limit is similar to those of the variation based on the fractional Brownian motion ([29]). In addition, in Section 6, we will show that the asymptotics of T_4, and indeed the value of $c_{1,H}$, are not purely academic. They are needed in order to calculate the asymptotic variance of the adjusted variations, those which have a normal limit when $H \in (1/2, 2/3)$.

Define the quantity

$$\begin{aligned} G_N &:= \frac{\sqrt{N}}{c_{1,H}} T_4 = \frac{\sqrt{N}}{\sqrt{c_{1,H}}} \frac{2N^{2H}}{c(H)(N-\ell)} \sum_{i=\ell}^{N-1} I_4\left(C_i \otimes C_i\right) \\ &= I_4\left(\frac{\sqrt{N}\ 2\ N^{2H}}{\sqrt{c_{1,H}}\ c(H)\ (N-\ell)} \sum_{i=\ell}^{N-1} (C_i \otimes C_i) \right) := I_4(g_N). \end{aligned} \tag{18}$$

From the calculations above we proved that $\lim_{N\to\infty} \mathbf{E}(G_N^2) = 1$. Using the Nualart–Peccati criterion in Proposition 1, we can now prove that G_N is asymptotically standard normal.

Theorem 2. *For all $H \in (1/2, 1)$ G_N defined in (18) converges in distribution to the standard normal.*

Setup of proof of Theorem 2. To prove this theorem, by Proposition 4 and Proposition 1, it is sufficient to show that for all $\tau = 1, 2, 3$,

$$\lim_{N\to\infty} \left\| g_N \tilde{\otimes}_\tau g_N \right\|_{L^2([0,1]^{(8-2\tau)})} = 0.$$

For $\tau = 1, 2, 3$, this quantity can be written as

$$\begin{aligned} &\lim_{N\to\infty} \left(\frac{4N^{4H+1}}{c_{1,H} c(H)^2 (N-\ell)^2} \right)^2 \left\| \sum_{i,j=\ell}^{N-1} (C_i \otimes C_i) \tilde{\otimes}_\tau (C_j \otimes C_j) \right\|^2_{L^2([0,1]^{(8-2\tau)})} \\ \leq\ &\lim_{N\to\infty} \left(\frac{4N^{4H+1}}{c_{1,H} c(H)^2 (N-\ell)^2} \right)^2 \left\| \sum_{i,j=\ell}^{N-1} (C_i \otimes C_i) \otimes_\tau (C_j \otimes C_j) \right\|^2_{L^2([0,1]^{(8-2\tau)})} \end{aligned}$$

$$= \lim_{N\to\infty}\left(\frac{4N^{4H+1}}{c_{1,H}c(H)^2(N-\ell)^2}\right)^2 \sum_{i,j,m,n=\ell}^{N-1}\langle (C_i\otimes C_i)\otimes_\tau (C_j\otimes C_j),(C_m\otimes C_m)\otimes_\tau (C_n\otimes C_n)\rangle .$$

The Appendix can now be consulted for proof that for each $\tau = 1, 2, 3$ this quantity converges to 0, establishing the theorem. □

5. Anormality of the T_2 term and asymptotic distribution of the 2-variation

For the asymptotic distribution of the variation statistic we have the following proposition.

Theorem 3. *For all $H \in (1/2, 1)$, both $\frac{N^{1-H}}{\sqrt{c_{2,H}}}T_2$ and the normalized 2-variation $\frac{N^{1-H}}{\sqrt{c_{2,H}}}V_N$ converge in $L^2(\Omega)$ to the Rosenblatt random variable $Z(1)$.*

Setup of proof of Theorem 3. The strategy for proving this theorem is simple. First of all Proposition 4 implies immediately that $N^{1-H}T_4$ converges to zero in $L^2(\Omega)$. Thus if we can show the theorem's statement about T_2, the statement about V_N will following immediately from Proposition 2.

Next, to show $\frac{N^{1-H}}{\sqrt{c_{2,H}}}T_2$ converges to the random variable $Z(1)$ in $L^2(\Omega)$, recall that T_2 is a second-chaos random variable of the form $I_2(f_N)$, where $f_N(y_1, y_2)$ is a symmetric function in $L^2([0,1]^2)$, and that this double Wiener-Itô integral is with respect to the Brownian motion W used to define $Z(1)$, i.e. that $Z(1) = I_2(L_1)$ where L_1 is the kernel of the Rosenblatt process at time 1, as defined in (9). Therefore, by the isometry property of Wiener-Itô integrals (see (6)), it is necessary and sufficient to show that $\frac{N^{1-H}}{\sqrt{c_{2,H}}}f_N$ converges in $L^2([0,1]^2)$ to L_1. This is proved in the Appendix. □

6. Normality of the adjusted variations

In the previous section we proved that the distribution of the variation statistic V_N is never normal, irrespective of the order of the filter. However, in the decomposition of V_N, there is a normal part, T_4, which implies that if we subtract T_2 from V_N the remaining part will converge to a normal law. But T_2 is not observed in practice. Following the idea of the adjusted variations in [28], instead of T_2 we subtract $Z(1)$ which is observed. $Z(1)$ is the value of the Rosenblatt process at time 1. Thus, we study the convergence of the adjusted variation:

$$V_N - \frac{\sqrt{c_{2,H}}}{N^{1-H}}Z(1) = V_N - T_2 + T_2 - \frac{\sqrt{c_{2,H}}}{N^{1-H}}Z(1) := T_4 + U_2.$$

In Section 4 we showed that $\frac{\sqrt{N}}{c_{1,H}}T_4$ converges to a normal law. For the quantity U_2 we prove the following proposition

Proposition 5. *For $H \in \left(\frac{1}{2}, \frac{2}{3}\right)$, $\sqrt{N}U_2$ converges in distribution to normal with mean zero and variance given by*

$$c_{3,H} := c_{2,H} \sum_{k=1}^{\infty} (N-k-1)k^{2H} F\left(\frac{1}{k}\right), \tag{19}$$

where $c_{2,H}$ is defined as in (16) and F is defined as follows

$$\begin{aligned} F(x) = d(H)^2\alpha(H)^2 \sum_{q_1q_2r_1r_2=0}^{\ell} \int_{[0,1]^4} dudvdu'dv' \left|(u-u'+q_2-q_1)x+1\right|^{2H'-2} \\ \left[\frac{128\alpha(H)^2d(H)^2}{c_{2,H}c(H)^2} \left|u-v-q_1+r_1\right|^{2H'-2} \left|u'-v'-q_2+r_2\right|^{2H'-2}\right. \\ \left|(v-v'-r_1+r_2)x+1\right|^{2H'-2} - \frac{16d(H)\alpha(H)}{\sqrt{c_{2,H}}c(H)} \left|u-v-q_1+r_1\right|^{2H'-2} \\ \left.\left|(v-u'-q_2+r_1)x+1\right|^{2H'-2} + \left|(u-u'+q_1-q_2)x+1\right|^{2H'-2}\right]. \end{aligned}$$

Proof. The proof follows the proof of [28, Proposition 5] and is omitted here. □

Therefore, for the adjusted variation we can prove the following

Theorem 4. *Let $Z_t : t \in (0,1)$ be a Rosenblatt process with $H \in (1/2, 2/3)$. Then the adjusted variation*

$$\frac{\sqrt{N}}{c_{1,H}+c_{3,H}} \left(V_N(2,\alpha) - \frac{c_{2,H}}{N^{1-H}} Z(1)\right).$$

converges to a standard normal law. Here $c_{1,H}$, $c_{2,H}$, and $c_{3,H}$ are given in (17), (16), and (19).

Proof. The proof follows the steps of the proof of [28, Theorem 6] and is omitted. □

7. Estimators for the self-similarity index

We construct estimators for the self-similarity index of a Rosenblatt process Z based on the discrete observations at times $0, \frac{1}{N}, \frac{2}{N}, \ldots, 1$. Their strong consistency and asymptotic distribution will be consequences of the theorems above.

7.1. Setup of the estimation problem

Consider the quadratic variation statistic for a filter α of order p based on the observations of our Rosenblatt process Z:

$$S_N := \frac{1}{N}\sum_{i=\ell}^{N}\left(\sum_{q=0}^{\ell}\alpha_q Z\left(\frac{i-q}{N}\right)\right)^2. \tag{20}$$

We have already established that $\mathbf{E}[S_N] = -\frac{N^{-2H}}{2}\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|q-r|^{2H}$ (see expression (12)). By considering that $\mathbf{E}[S_N]$ can be estimated by the empirical value S_N, we can construct an estimator $\hat{H}_N$ for H by solving the following equation:

$$S_N = -\frac{N^{-2\hat{H}_N}}{2}\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|q-r|^{2\hat{H}_N}.$$

In this case, unlike the case of a filter of length 1 which was studied in [28], we cannot compute an analytical expression for the estimator. Nonetheless, the estimator $\hat{H}_N$ can be easily computed numerically by solving the following non-linear equation for fixed N, with unknown $x \in [1/2, 1]$:

$$-\frac{N^{-2x}}{2}\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|q-r|^{2x} - S_N(2,\alpha) = 0. \tag{21}$$

This equation is not entirely trivial, in the sense that one must determine whether it has a solution in $[1/2, 1]$, and whether this solution is unique. As it turns out, the answer to both questions is affirmative for large N, as seen in the next Proposition, proved further below.

Proposition 6. *Almost surely, for large N, equation (21) has exactly one solution in $[1/2, 1]$.*

Definition 2. We define the estimator $\hat{H}_N$ of H to be the unique solution of (21).

Note that Equation (21) can be rewritten as $S_N = c(x)N^{-2x}/2$ where the function c was defined in (13). The proposition is established via the following lemma.

Lemma 2. *For any $H \in (1/2, 1)$, almost surely, $\lim_{N\to\infty} N^{2H}S_N = c(H)/2$.*

Proof. Firstly, we show that V_N converges to zero almost surely as $N \to \infty$. We already know that this is true in $L^2(\Omega)$. Consider the following

$$P\left(|V_N| > N^{-\beta}\right) \le N^{q\beta}\mathbf{E}\left(|V_N|^q\right) \le c_{q,4}\left[\mathbf{E}\left(V_N^2\right)\right]^{q/2} \le c\, N^{q\beta}N^{(H-1)q}.$$

If we choose $\beta < 1 - H$ and q large enough so that $(1 - H - \beta)q > 1$. This implies that

$$\sum_{N=0}^{\infty} P\left(|V_N| > N^{-\beta}\right) \le c\sum_{N=0}^{\infty} N^{(\beta+H-1)q} < +\infty$$

Therefore, the Borel-Cantelli lemma implies $|V_N| \to 0$ a.s., with speed of convergence equal to $N^{-\beta}$, for all $\beta < 1 - H$. Since $V_N = \frac{S_N}{\mathbf{E}(S_N)} - 1$ we have

$$1 + V_N = -\frac{2N^{2H}}{\sum_{q,r=0}^{\ell} \alpha_q \alpha_r |q-r|^{2H}} S_N = 2N^{2H} S_N / c\,(H)\,. \tag{22}$$

The almost-sure convergence of V_N to 0 yields the statement of the lemma. □

Proof of Proposition 6. For $x \in [\frac{1}{2}, 1]$ and for any fixed N, define the function

$$F_N\,(x) = \frac{c(x)}{2} N^{-2x} - S_N = -\frac{N^{-2x}}{2} \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |q-r|^{2x} - S_N.$$

Equation (21) is $F_N\,(x) = 0$. Observe that $F_N(x)$ is strictly decreasing. Indeed, we have that

$$F_N'\,(x) = \log\left(N^{-2x}\right) \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |q-r|^{2x} - N^{-2x} \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \log|q-r|\ |q-r|^{2x}.$$

Then, $F_N'\,(x) < 0$ is equivalent to

$$N > \exp\left\{ \frac{\sum_{q,r=0}^{\ell} \alpha_q \alpha_r \log|q-r|\ |q-r|^{2x}}{\sum_{q,r=0}^{\ell} \alpha_q \alpha_r |q-r|^{2x}} \right\},$$

since we know, using Lemma 1, that $c\,(x) = \sum_{q,r=0}^{\ell} \alpha_q \alpha_r |q-r|^{2x}$, which is evidently continuous on $[\frac{1}{2}, 1]$, is strictly negative on that interval. Thus, if we choose N to be large enough, i.e.

$$N > \max_{x \in [\frac{1}{2} 1]} \exp\left\{ \frac{\sum_{q,r=0}^{\ell} \alpha_q \alpha_r \log|q-r|\ |q-r|^{2x}}{\sum_{q,r=0}^{\ell} \alpha_q \alpha_r |q-r|^{2x}} \right\}$$

the function $F_N\,(x)$ is invertible on $[\frac{1}{2}, 1]$, and equation (21) has no more than one solution there.

To guarantee existence of a solution, we use Lemma 2. This lemma implies the existence of a sequence ε_N such that $2N^{2H} S_N = c(H) + \varepsilon_N$ and $\lim_{N\to\infty} \varepsilon_N = 0$ almost surely. Since in addition c is continuous, then almost surely, we can choose N large enough, so that $2N^{2H} S_N$ is in the image of $[\frac{1}{2}, 1]$ by the function c. Thus the equation $c\,(x) = 2N^{2H} S_N$ has at least one solution in $[\frac{1}{2}, 1]$. Since this equation is equivalent to (21), the proof of the proposition is complete. □

7.2. Properties of the estimator

Now, it remains to prove that any such $\hat{H}_N$ is consistent and to determine its asymptotic distribution.

Theorem 5. *For $H \in (1/2, 1)$ assume that the observed process used in the previous definition is a Rosenblatt process with Hurst parameter H. Then strong consistency holds for $\hat{H}_N$, i.e.*

$$\lim_{N\to\infty} \hat{H}_N = H, \quad a.s.$$

In fact, we have more precisely that $\lim_{N\to\infty}\big(H - \hat{H}_N\big)\log N = 0$ *a.s.*

Proof. From line (22) in the proof of Lemma 2, and using the fact that $\hat{H}_N$ solves equation (21), i.e. $c\big(\hat{H}_N\big)N^{-\hat{H}_N} = 2S_N$, we can write

$$1 + V_N = -\frac{2N^{2H}}{\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|q-r|^{2H}}S_N = \frac{c(\hat{H}_N)}{c\,(H)}N^{2(H-\hat{H}_N)}.$$

Now note that $c(\hat{H}_N)/c\,(H)$ is the ratio of two values of the continuous function c at two points in $[1/2, 1]$. However, Lemma 1 proves that on this interval, the function c is strictly positive; since it is continuous, it is bounded above and away from 0. Let $a = \min_{x\in[1/2,1]} c\,(x) > 0$ and $A = \max_{x\in[1/2,1]} c\,(x) < \infty$. These constants a and A are of course non random. Therefore $c(\hat{H}_N)/c\,(H)$ is always in the interval $[a/A, A/a]$. Thus, almost surely,

$$\left|\log\left(c(\hat{H}_N)/c\,(H)\right)\right| \le \log\frac{A}{a}.$$

We may now write

$$\log\,(1 + V_N) = 2\big(H - \hat{H}_N\big)\log N + \log\left(\frac{c(\hat{H}_N)}{c\,(H)}\right). \tag{23}$$

Since in addition $\lim_{N\to\infty}\log\,(1 + V_N) = 0$ a.s., we get that almost surely,

$$\left|H - \hat{H}_N\right| = O\left(\frac{1}{\log N}\right).$$

This implies the first statement of the proposition.

The second statement, which is more precise, is now obtained as follows. Since $\hat{H}_N \to H$ almost surely, and c is continuous, $\log\big(c(\hat{H}_N)/c\,(H)\big)$ converges to 0. The second statement now follows immediately. □

The asymptotic distribution of the estimator $\hat{H}_N$ is stated in the next result. Its proof uses Theorem 3 and Theorem 1, plus the expression (23). While novel and interesting, this proof is more technical than the proofs of the proposition and theorem above, and is therefore relegated to the Appendix.

Theorem 6. *For any $H \in (\frac{1}{2}, 1)$, the convergence*

$$\lim_{N\to\infty} 2c_{2,H}^{-1/2}N^{1-H}\big(\hat{H}_N - H\big)\log N = Z(1)$$

holds in $L^2\,(\Omega)$, where $Z(1)$ is a Rosenblatt random variable.

As can be seen from Theorem 3 and Theorem 6, the renormalization of the statistic V_N, as well as the renormalization of the difference $\hat{H}_N - H$, depend on H: it is of order of N^{1-H}. The quantities $N^{1-H}V_N$ and $N^{1-H}\hat{H}_N$ cannot be computed numerically from the empirical data, thereby compromising the use of the asymptotic distributions for statistical purposes such as model validation. Therefore one would like to have other quantities with known asymptotic distribution which can be calculated using only the data. The next theorem addresses this issue by showing that one can replace H by $\hat{H}_N$ in the term N^{1-H}, and still obtain a convergence as in Theorem 6, this time in $L^1(\Omega)$. Its proof is in the Appendix.

Theorem 7. *For any $H \in (\frac{1}{2}, 1)$, with the Rosenblatt random variable $Z(1)$,*

$$\lim_{N\to\infty} \mathbf{E}\left[\left|2\, c_{2,H}^{-1/2} N^{1-\hat{H}_N} \log N\big(\hat{H}_N - H\big) - Z(1)\right|\right] = 0.$$

8. Numerical computation of the asymptotic variance

In practice certain issues may occur when we compute the asymptotic variance. The most crucial question is what order of filter we should choose. Indeed, from (16) with $\hat{H}_N$ instead of H, it follows that the constants of the variance not only depend on the filter length/order (ℓ, p), but also on the number of observations (N). We measure the "accuracy" of the estimator $\hat{H}_N$ by its standard error which is the following quantity:

$$\frac{\sqrt{C_{2,\hat{H}_N}}}{2N^{1-\hat{H}_N}\log N}.$$

There are several types of filters that we can use. In this paper, we choose to work with finite-difference and wavelet-type filters.

- The finite-difference filters are produced by finite-differencing the process. In this case the filter length is the same as the order of the filter. The coefficients of the order-ℓ finite difference filter are given by

$$\alpha_k = (-1)^{k+1}\binom{\ell}{k}, \quad k = 0, \ldots, \ell.$$

- The wavelet filters we are using are the Daubechies filters with k-vanishing moments. (By vanishing moments we mean that all moments of the wavelet filter are zero up to a power). The Daubechies wavelets form a family of orthonormal wavelets with compact support and the maximum number of vanishing moments. In this scenario, the number of vanishing moments determines the order of the filter and the filter length is twice the order. For more details, the reader can refer to [17].

We computed the standard error for $N = 10,000$ observations, filters of order varying from 2 to 20 and Hurst parameters varying from 0.55 to 0.95. This means that the corresponding lengths of the finite-difference filters were 2 to 20 and

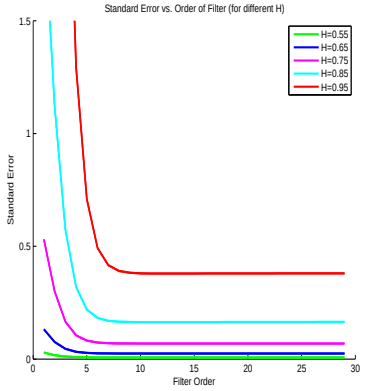

FIG 1. *Finite Difference Filters.*

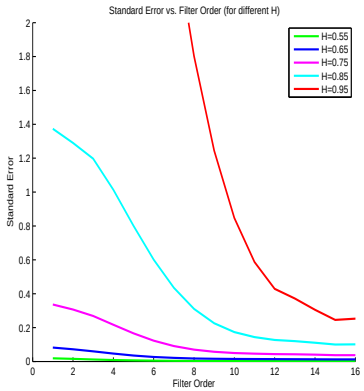

FIG 2. *Wavelet Filters.*

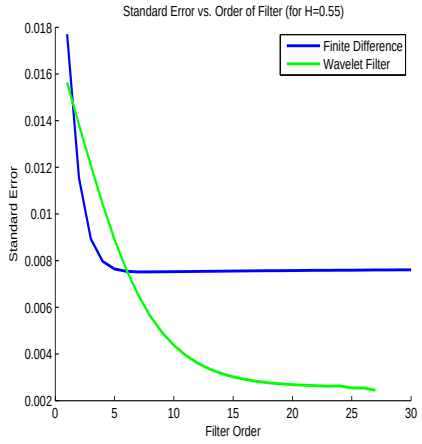

FIG 3. *Comparison between the two types of filter.*

for the wavelets 4 to 40. The code we use to simulate the Rosenblatt process is based on a Donsker-type limit theorem and was provided to us by J.M. Bardet [1]. The results are illustrated in the figures 1, 2, and 3, on the next page; these are graphs of the asymptotic standard error $\sqrt{c_{2,H}}/(2N^{1-\hat{H}_N}\log N)$ for various fixed values of H as the order of the filters increase.

We observe that the standard error decreases with the order of the filter. Furthermore, we observe that the wavelet filters are more effective than the finite-difference ones, since they have a higher impact on the decrease of the standard error for the same order, as the filter increases. Specifically, the graph in Fig. 1, with the finite difference filters, shows that for fixed H, there is no advantage to using a filter beyond a certain order p, since the standard error

tends to a constant as $p \to \infty$. This does not occur for the wavelet filters, where the standard error continues to decrease as $p \to \infty$ in all cases as seen in the graph in Fig. 2. On the other hand, the finite-difference filters have lower errors than the wavelet filters for low filter lengths; only after a certain order p^* do the latter become more effective; this comparison is seen in the graph in Fig. 3, where p^* is roughly 9.

In addition, since the order of convergence depends on the true value of the Hurst parameter H, we investigated the behavior of the error with respect to H. It seems that the higher H is, the more we lose in terms of accuracy; this is visible in all three graphs.

In general, the choice of a longer filter might lead to a smaller error, but at the same time it increases the computational time needed in order to compute $\hat{H}$ and its standard error. In a future work, we will study extensively this trade-off and other consequences of using longer filters.

9. Appendix: proofs

9.1. Proof of Proposition 3

We start by computing the contraction term $C_i \otimes_1 C_i$:

$$
\begin{aligned}
(C_i \otimes_1 C_i)(y_1, y_2) &= \int_0^1 C_i(x, y_1) C_i(x, y_2) dx \\
&= \sum_{q,r=0}^{\ell} b_q b_r \int_0^1 \left(L_{\frac{i-(q-1)}{N}}(x, y_1) - L_{\frac{i-q}{N}}(x, y_1) \right) \\
&\times \left(L_{\frac{i-(r-1)}{N}}(x, y_2) - L_{\frac{i-r}{N}}(x, y_2) \right) dx \\
&= d(H)^2 \sum_{q,r=0}^{\ell} b_q b_r 1_{[0, \frac{i-q+1}{N}]}(y_1) 1_{[0, \frac{i-r+1}{N}]}(y_2) \int_0^{\frac{i-q+1}{N} \wedge \frac{i-r+1}{N}} dx \\
&\times \left(\int_{\frac{i-q}{N}}^{\frac{i-q+1}{N}} \frac{\partial K^{H'}}{\partial u}(u, x) \frac{\partial K^{H'}}{\partial u}(u, y_1) du \right) \\
&\times \left(\int_{\frac{i-r}{N}}^{\frac{i-r+1}{N}} \frac{\partial K^{H'}}{\partial v}(v, x) \frac{\partial K^{H'}}{\partial v}(v, y_2) dv \right) \\
&= d(H)^2 \sum_{q,r=0}^{\ell} b_q b_r 1_{[0, \frac{i-q+1}{N}]}(y_1) 1_{[0, \frac{i-r+1}{N}]}(y_2) \\
&\times \int_{I_{i_q}} \int_{I_{i_r}} du\, dv \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(v, y_2) du dv \\
&\times \left(\int_0^{u \wedge v} dx \frac{\partial K^{H'}}{\partial u}(u, x) \frac{\partial K^{H'}}{\partial v}(v, x) \right)
\end{aligned}
$$

$$= \alpha(H)d(H)^2 \sum_{q,r=0}^{\ell} b_q b_r 1_{[0,\frac{i-q+1}{N}]}(y_1) 1_{[0,\frac{i-r+1}{N}]}(y_2)$$
$$\int_{I_{i_q}} \int_{I_{i_r}} du\, dv |u-v|^{2H'-2} \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial v}(v,y_2) du dv,$$

where $I_{i_q} = \left(\frac{i-q}{N}, \frac{i-q+1}{N}\right]$.

Now, the inner product computes as

$$\langle C_i \otimes_1 C_i, C_j \otimes_1 C_j \rangle_{L^2[0,1]^2}$$
$$= \alpha(H)^2 d(H)^4 \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_0^1 \int_0^1 dy_1 dy_2$$
$$\int_{I_{i_{q_1}}} \int_{I_{i_{r_1}}} \int_{I_{j_{q_2}}} \int_{I_{j_{r_2}}} du dv du' dv' |u-v|^{2H'-2} |u'-v'|^{2H'-2}$$
$$\frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial v}(v,y_2) \frac{\partial K^{H'}}{\partial u'}(u',y_1) \frac{\partial K^{H'}}{\partial v'}(v',y_2) du dv du' dv'$$
$$= \alpha(H)^2 d(H)^4 \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2}$$
$$\int_{I_{i_{q_1}}} \int_{I_{i_{r_1}}} \int_{I_{j_{q_2}}} \int_{I_{j_{r_2}}} du dv du' dv' |u-v|^{2H'-2} |u'-v'|^{2H'-2}$$
$$\left(\int_0^{u \wedge u'} \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial u'}(u',y_1) dy_1 \right)$$
$$\left(\int_0^{v \wedge v'} \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial v'}(v',y_2) dy_2 \right)$$
$$= \alpha(H)^4 d(H)^4 \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_{I_{i_{q_1}}} \int_{I_{i_{r_1}}} \int_{I_{j_{q_2}}} \int_{I_{j_{r_2}}} du dv du' dv'$$
$$\times |u-v|^{2H'-2} |u'-v'|^{2H'-2} |u-u'|^{2H'-2} |v-v'|^{2H'-2}.$$

We make the following change of variables

$$\bar{u} = \left(u - \frac{i-q_1}{N} \right) N$$

and the second moment of T_2 becomes

$$\mathbf{E}\left(T_2^2\right)$$
$$= \frac{128\, \alpha(H)^4 d(H)^4}{c(H)^2} \frac{N^{4H}}{(N-\ell)^2} \sum_{i,j=\ell}^{N-1} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_{I_{i_{q_1}}} \int_{I_{i_{r_1}}}$$
$$\times \int_{I_{j_{q_2}}} \int_{I_{j_{r_2}}} du dv du' dv' |u-v|^{2H'-2} |u'-v'|^{2H'-2} |u-u'|^{2H'-2} |v-v'|^{2H'-2}$$

$$= \frac{128\,\alpha(H)^4 d(H)^4}{c(H)^2} \frac{N^{4H}}{(N-\ell)^2} \frac{1}{N^4 N^{8H'-8}} \sum_{i,j=\ell}^{N-1} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2}$$
$$\times \int_{[0,1]^4} dudvdu'dv' |u-v-q_1+r_1|^{2H'-2} |u'-v'-q_2+r_2|^{2H'-2}$$
$$\times |u-u'+i-j-q_1+q_2|^{2H'-2} |v-v'+i-j-r_1+r_2|^{2H'-2}$$
$$= \frac{128\,\alpha(H)^4 d(H)^4}{c(H)^2} \frac{1}{(N-\ell)^2} \sum_{i,j=\ell}^{N-1} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_{[0,1]^4} dudvdu'dv'$$
$$\times |u-v-q_1+r_1|^{2H'-2} |u'-v'-q_2+r_2|^{2H'-2}$$
$$\times \left(|u-u'+i-j-q_1+q_2|^{2H'-2} |v-v'+i-j-r_1+r_2|^{2H'-2} \right).$$

Let $cst. = \frac{128\,\alpha(H)^4 d(H)^4}{c(H)^2}$. We study first the diagonal terms of the above double sum

$$\begin{aligned}
&\mathbf{E}\left(T_{2-diag}^2\right) \\
&= cst. \frac{N-\ell-1}{(N-\ell)^2} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_{[0,1]^4} dudvdu'dv' \\
&\times |u-v-q_1+r_1|^{2H'-2} |u'-v'-q_2+r_2|^{2H'-2} \\
&\times |u-u'-q_1+q_2|^{2H'-2} |v-v'-r_1+r_2|^{2H'-2}.
\end{aligned}$$

We conclude that

$$\mathbf{E}\left(T_{2-diag}^2\right) = \mathcal{O}\left(N^{-1}\right).$$

Let's consider now the non-diagonal terms

$$\mathbf{E}\left(T_{2-off}^2\right) = 2cst. \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2}$$
$$\times \int_{[0,1]^4} dudvdu'dv' \times |u-v-q_1+r_1|^{2H'-2} |u'-v'-q_2+r_2|^{2H'-2}$$
$$\frac{1}{(N-\ell)^2} \sum_{i,j=\ell,\ i\neq j}^{N-1} |u-u'+i-j-q_1+q_2|^{2H'-2} |v-v'+i-j-r_1+r_2|^{2H'-2} \tag{24}$$

Observe that the term (24) can be calculated as follows:

$$\frac{1}{(N-\ell)^2} \sum_{i,j=\ell\ i\neq j}^{N-1} |u-u'+i-j-q_1+r_1|^{2H'-2} |v-v'+i-j-r_1+r_2|^{2H'-2}$$
$$= \frac{1}{(N-\ell)^2} \sum_{i=\ell}^{N-1} \sum_{k=1}^{N-i} |u-u'+k-q_1+q_2|^{2H'-2} |v-v'+k-r_1+r_2|^{2H'-2}$$

$$= \frac{1}{(N-\ell)^2} \sum_{k=\ell}^{N-1} (N-k-1)$$
$$\times |u-u'+k-q_1+q_2|^{2H'-2} |v-v'+k-r_1+r_2|^{2H'-2}$$
$$= N^{4H'-4} \frac{N}{(N-\ell)^2} \sum_{k=\ell}^{N-1} \left(1-\frac{k+1}{N}\right)$$
$$\times \left|\frac{u-u'}{N}+\frac{k}{N}-\frac{q_1-q_2}{N}\right|^{2H'-2} \left|\frac{v-v'}{N}+\frac{k}{N}-\frac{r_1-r_2}{N}\right|^{2H'-2}.$$

We may now use a Riemann sum approximation and the fact that $4H'-4 = 2H-2 > -1$. Since ℓ is fixed and q_1 and q_2 are less than ℓ, we get that the term in (24) is asymptotically equivalent to

$$\sum_{k=\ell}^{N-1} \left(1-\frac{k}{N}\right) \left|\frac{k}{N}\right|^{2H'-2} \left|\frac{k}{N}\right|^{2H'-2}$$
$$= \int_0^1 (1-x)\, x^{2H-2} dx + o(1) = \frac{1}{2H(2H-1)} + o(1).$$

We conclude that

$$\mathbf{E}\left(T_2^2\right) + o\left(N^{2H-2}\right) = \frac{cst.N^{2H-2}}{H(2H-1)} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2}$$
$$\times \int_{[0,1]^4} dudvdu'dv' |u-v-q_1+r_1|^{2H'-2} |u'-v'-q_2+r_2|^{2H'-2}.$$

Using the fact that

$$\int_{[0,1]^2} |u-v-q+r|^{2H'-2} dudv$$
$$= \frac{1}{2H'(2H'-1)} \left[|1+q-r|^{2H'} + |1-q+r|^{2H'} - 2|q-r|^{2H'}\right]$$

the proposition follows.

9.2. *Proof of Proposition 4*

9.2.1. *The term* $\mathbf{E}\left(T_{4,(1)}^2\right)$

We have

$$\mathbf{E}\left(T_{4,(1)}^2\right) = \frac{4N^{4H}}{c(H)^2(N-\ell)^2} 4! \sum_{i,j=\ell}^{N-1} \langle C_i \otimes C_i, C_j \otimes C_j \rangle_{L^2([0,1]^4)}$$

$$= \frac{4N^{4H}}{c(H)^2(N-\ell)^2} 4! \sum_{i,j=\ell}^{N-1} \left| \langle C_i, C_j \rangle_{L^2([0,1]^2)} \right|^2$$

The scalar product computes as

$$\begin{aligned}
&\langle C_i, C_j \rangle_{L^2([0,1]^2)} \\
&= \left\langle \sum_{q=0}^{\ell} \alpha_q L_{\frac{i-q}{N}}, \sum_{r=0}^{\ell} \alpha_r L_{\frac{j-r}{N}} \right\rangle_{L^2([0,1]^2)} \\
&= \int_0^1 \int_0^1 \left(\sum_{q=0}^{\ell} \alpha_q L_{\frac{i-q}{N}}(y_1, y_2) \right) \left(\sum_{r=0}^{\ell} \alpha_r L_{\frac{j-r}{N}}(y_1, y_2) \right) dy_1 dy_2 \\
&= d(H)^2 \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \int_0^1 \int_0^1 \left[\int_{y_1 \vee y_2}^{\frac{i-q}{N}} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] \\
&\quad \times \left[\int_{y_1 \vee y_2}^{\frac{j-r}{N}} \frac{\partial K^{H'}}{\partial v}(v, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) dv \right] dy_1 dy_2 \\
&= d(H)^2 \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \int_0^{\frac{i-q}{N}} \int_0^{\frac{j-r}{N}} \left(\int_0^{u \wedge v} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_1) dy_1 \right)^2 du dv \\
&= \alpha(H)^2 \, d(H)^2 \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \int_0^{\frac{i-q}{N}} \int_0^{\frac{j-r}{N}} |u-v|^{2H-2} du dv
\end{aligned}$$

where $\alpha(H) = \frac{H(H+1)}{2} = H'(2H'-1)$ and

$$\begin{aligned}
&\int_0^{\frac{i-q}{N}} \int_0^{\frac{j-r}{N}} |u-v|^{2H-2} du dv = \\
&\frac{1}{H(2H-1)} \left[\left| \frac{i-q}{N} \right|^{2H} + \left| \frac{j-r}{N} \right|^{2H} - \left| \frac{j-i+q-r}{N} \right|^{2H} \right]
\end{aligned} \tag{25}$$

Using the fact that $\frac{\alpha(H)^2 \, d(H)^2}{H(2H-1)} = \frac{1}{2}$ and (25) the scalar product becomes

$$\begin{aligned}
&\langle C_i, C_j \rangle_{L^2([0,1]^2)} \\
&= \frac{\alpha(H)^2 \, d(H)^2}{H(2H-1)} \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left[\left| \frac{i-q}{N} \right|^{2H} + \left| \frac{j-r}{N} \right|^{2H} - \left| \frac{j-i+q-r}{N} \right|^{2H} \right] \\
&= \frac{1}{2} \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left[\left| \frac{i-q}{N} \right|^{2H} + \left| \frac{j-r}{N} \right|^{2H} - \left| \frac{j-i+q-r}{N} \right|^{2H} \right] \\
&= \frac{1}{2} \left[\left(\sum_{q=0}^{\ell} \alpha_q \left| \frac{i-q}{N} \right|^{2H} \right) \left(\sum_{r=0}^{\ell} \alpha_r \right) + \right.
\end{aligned}$$

$$\left(\sum_{r=0}^{\ell}\alpha_r\left|\frac{j-r}{N}\right|^{2H}\right)\left(\sum_{q=0}^{\ell}\alpha_q\right)-\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{i-j+q-r}{N}\right|^{2H}\Bigg]$$
$$=-\frac{1}{2}\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{i-j+q-r}{N}\right|^{2H}=\pi_H^{\alpha}(i-j).$$

The last equality is true since $\sum_{q=0}^{\ell}\alpha_q=0$ by the filter definition. Therefore, we have

$$\sum_{i,j=\ell}^{N-1}\left|\langle C_i,C_j\rangle_{L^2([0,1]^2)}\right|^2=\frac{1}{4}\sum_{i,j=\ell}^{N-1}\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{i-j+q-r}{N}\right|^{2H}\right|^2$$
$$=\frac{1}{4}\sum_{i=\ell}^{N-1}\sum_{k=0}^{N-2}\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{k+q-r}{N}\right|^{2H}\right|^2$$
$$=\frac{N^{-4H}(N-\ell-1)}{4}\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|q-r|^{2H}\right|^2$$
$$+\frac{1}{4}\sum_{i=\ell}^{N-1}\sum_{k=1}^{N-2}\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{k+q-r}{N}\right|^{2H}\right|^2$$
$$=c(H)^2\frac{N^{-4H}(N-\ell-1)}{4}+\frac{1}{4}\sum_{k=0}^{N-2}(N-k-2)\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\left|\frac{k+q-r}{N}\right|^{2H}\right|^2$$
$$=c(H)^2\frac{(N-l-1)N^{-4H}}{4}+\frac{N^{-4H+1}}{4}\sum_{k=0}^{N-2}\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\,|k+q-r|^{2H}\right|^2$$
$$-2\frac{N^{-4H}}{4}\sum_{k=0}^{N-2}\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\,|k+q-r|^{2H}\right|^2$$
$$+\frac{N^{-4H}}{4}\sum_{k=0}^{N-2}k\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r\,|k+q-r|^{2H}\right|^2.$$

At this point we need the next lemma to estimate the behavior of the above quantity. This lemma is the key point which implies the fact that the longer variation statistics has, in the case when the observed process is the fractional Brownian motion, a Gaussian limit without any restriction on H (see [12]).

Lemma 3. *For all $H\in(0,1)$, we have that*

(i) $\sum_{k=1}^{\infty}\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|k+q-r|^{2H}\right|^2<+\infty$

(ii) $\sum_{k=1}^{\infty}k\left|\sum_{q,r=0}^{\ell}\alpha_q\alpha_r|k+q-r|^{2H}\right|^2<+\infty.$

Proof. *Proof of (i).* Let $f(x) = \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left(1 + (q-r)x\right)^{2H}$, so the summand can be written as

$$\sum_{q,r=0}^{\ell} \alpha_q \alpha_r |k+q-r|^{2H} = k^{2H} f\left(\frac{1}{k}\right).$$

Using a Taylor expansion at $x_0 = 0$ for the function $f(x)$ we get that

$$\begin{aligned}&\left(1 + (q-r)x\right)^{2H} \approx \\ &1 + 2H(q-r)x + \cdots + \frac{2H(2H-1)\ldots(2H-n+1)}{n!}(q-r)^n x^n.\end{aligned}$$

For small x we observe that the function $f(x)$ is asymptotically equivalent to

$$2H(2H-1)\ldots(2H-(p-1))x^{2p},$$

where p is the order of the filter. Therefore, the general term of the series is equivalent to

$$(2H)^2(2H-1)^2\ldots(2H-(p-1))^2 k^{4H-4p}$$

Therefore for all $H < p - \frac{1}{4}$ the series converges to a constant depending only on H. Due to our choice for the order of the filter $p \geq 2$, we obtain the desired result.

Proof of (ii). Similarly as before, we can write the general term of the series as

$$\begin{aligned}k\left|\sum_{q,r=0}^{\ell} \alpha_q \alpha_r |k+q-r|^{2H}\right|^2 &= k\left|k^{2H} f\left(\frac{1}{k}\right)\right|^2 \\ &\approx (2H)^2(2H-1)^2\ldots(2H-(p-1))^2 k^{4H-4p-1}\end{aligned}$$

Therefore for all $H < p$ the series converges to a constant depending only on H. □

Combining all the above we have

$$\begin{aligned}\mathbf{E}\left(T^2_{4,(1)}\right) &= \frac{4\,N^{4H}}{c(H)^2(N-\ell)^2} 4! \sum_{i,j=1}^{N} \left|\langle C_i, C_j\rangle_{L^2([0,1]^2)}\right|^2 \\ &= \frac{4\,N^{4H}}{c(H)^2(N-\ell)^2} 4! \Bigg[\frac{1}{4}\, c(H)^2 (N-\ell-1) N^{-4H} \\ &+ \frac{N^{-4H+1}}{4} \sum_{k=0}^{N-2} \left|\sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H}\right|^2 \\ &- 2\,\frac{N^{-4H}}{4} \sum_{k=0}^{N-2} \left|\sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H}\right|^2\end{aligned}$$

$$
\begin{aligned}
&+ \frac{N^{-4H}}{4} \sum_{k=0}^{N-2} k \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H} \right|^2 \Bigg] \\
&= \frac{4!}{c(H)^2} \Bigg[c(H)^2 \, \frac{N-\ell-1}{(N-\ell)^2} + \left(\frac{N^1}{(N-\ell)^2} - 2\frac{1}{(N-\ell)^2} \right) \\
&\sum_{k=0}^{N-2} \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H} \right|^2 \\
&+ \frac{1}{(N-\ell)^2} \sum_{k=0}^{N-2} k \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H} \right|^2 \Bigg] \\
&= \frac{4!}{c(H)^2} \Bigg[c(H)^2 \, \left(\frac{N}{(N-\ell)^2} - \frac{l+1}{(N-\ell)^2} \right) + \frac{N-2}{(N-\ell)^2} \\
&\sum_{k=0}^{N-2} \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H} \right|^2 \\
&+ \frac{1}{(N-\ell)^2} \sum_{k=0}^{N-2} k \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H} \right|^2 \Bigg] \\
&\approx \frac{4!}{c(H)^2} \Bigg[c(H)^2 \, \left(N^{-1} - (\ell+1)N^{-2} \right) + \left(N^{-1} - 2N^{-2} \right) \\
&\sum_{k=0}^{N-2} \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H} \right|^2 \\
&+ N^{-2} \sum_{k=0}^{N-2} k \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H} \right|^2 \Bigg].
\end{aligned}
$$

Since the leading term is of order N^{-1} we have that

$$
\mathbf{E}\left(T^2_{4,(1)}\right) \simeq 4!\, c(H)^{-2} N^{-1} \left[c(H)^2 + \sum_{k=0}^{N-2} \left| \sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H} \right|^2 \right].
$$

If we define the correlation function of the filtered process as

$$
\rho^{\alpha}_H(k) = \frac{\pi^{\alpha}_H(k)}{\pi^{\alpha}_H(0)} = \frac{\sum_{q,r=0}^{\ell} \alpha_q \alpha_r \left|k+q-r\right|^{2H}}{c(H)}
$$

we can express the asymptotic variance $\lim_{N\to\infty} N\, \mathbf{E}\big(T^2_{4,(1)}\big)$ in terms of a series involving $\rho^{\alpha}_H(k)$.

9.2.2. The term $\mathbf{E}\left(T_{4,(2)}^2\right)$

In order to handle this term we use the alternate expression (15) of C_i. Therefore, following similar calculations as in the T_2 case we get that

$$\begin{aligned}
&\mathbf{E}\left(T_{4,(2)}^2\right)\\
&= \frac{c_{4,H}^{(1)}}{(N-\ell)^2}\sum_{q_1,q_2,r_1,r_1=0}^{\ell} b_{q_1}b_{q_2}b_{r_1}b_{r_2}\int_{[0,1]^4} dudvdu'dv'\\
&\times\sum_{i,j=\ell}^{N-1}\Big[\left|u-v+i-j-q_1+r_1\right|^{2H'-2}\left|u'-v'+i-j-q_2+r_2\right|^{2H'-2}\\
&\left|u-u'+i-j-q_1+q_2\right|^{2H'-2}\left|v-v'+i-j-r_1+r_2\right|^{2H'-2}\Big]\\
&= \frac{c_{4,H}^{(2)}}{(N-\ell)^2}\sum_{q_1,q_2,r_1,r_1=0}^{\ell} b_{q_1}b_{q_2}b_{r_1}b_{r_2}\int_{[0,1]^4} dudvdu'dv'\\
&\times\sum_{i=\ell}^{N-1}\sum_{k=0}^{N-\ell-i}\Big[\left|u-v+k-q_1+r_1\right|^{2H'-2}\left|u'-v'+k-q_2+r_2\right|^{2H'-2}\\
&\left|u-u'+k-q_1+q_2\right|^{2H'-2}\left|v-v'+k-r_1+r_2\right|^{2H'-2}\Big]\\
&= \frac{c_{4,H}^{(3)}}{(N-\ell)^2}\sum_{q_1,q_2,r_1,r_1=0}^{\ell} b_{q_1}b_{q_2}b_{r_1}b_{r_2}\int_{[0,1]^4} dudvdu'dv'\\
&\times\sum_{k=0}^{N-\ell}(N-k-1)\Big[\left|u-v+k-q_1+r_1\right|^{2H'-2}\left|u'-v'+k-q_2+r_2\right|^{2H'-2}\\
&\left|u-u'+k-q_1+q_2\right|^{2H'-2}\left|v-v'+k-r_1+r_2\right|^{2H'-2}\Big].
\end{aligned}$$

We study the convergence of the above series as $N\to\infty$

$$\begin{aligned}
&\sum_{k=0}^{N-1}(N-k-1)\Big[\left|u-v+k-q_1+r_1\right|^{2H'-2}\left|u'-v'+k-q_2+r_2\right|^{2H'-2}\\
&\left|u-u'+k-q_1+q_2\right|^{2H'-2}\left|v-v'+k-r_1+r_2\right|^{2H'-2}\Big]\\
&= (N-1)\sum_{k=0}^{N-1}\Big[\left|u-v+k-q_1+r_1\right|^{2H'-2}\left|u'-v'+k-q_2+r_2\right|^{2H'-2}\\
&\left|u-u'+k-q_1+q_2\right|^{2H'-2}\left|v-v'+k-r_1+r_2\right|^{2H'-2}\Big]\\
&-\sum_{k=0}^{N-1}k\Big[\left|u-v+k-q_1+r_1\right|^{2H'-2}\left|u'-v'+k-q_2+r_2\right|^{2H'-2}
\end{aligned}$$

$$\left|u-u'+k-q_1+q_2\right|^{2H'-2}\left|v-v'+k-r_1+r_2\right|^{2H'-2}\Bigg]$$
$$:=(I)\ +\ (II).$$

Therefore the general term of the series is asymptotically equivalent to

$$\left(\frac{(2H'-2)\dots(2H'-2p-1)}{(2p)!}\right)^4(u-v-q_1+r_1)^{2p}\,(u'-v'-q_2+r_2)^{2p}$$
$$\cdot\,(u-u'-q_1+q_2)^{2p}\,(v-v'-r_1+r_2)^{2p}\ \ k^{4H-4-8p},$$

which converges for all $H\in(\frac{1}{2},1)$. We treat the second series (II) in the same way and we get that it is asymptotically equivalent to $cst.\ k^{4H-4-8p}$. Combining all the above we have

$$\mathbf{E}\left(T^2_{4,(2)}\right)$$
$$=\frac{c'_{4,H}}{(N-\ell)^2}\sum_{q_1,q_2,r_1,r_1=0}^{\ell}b_{q_1}b_{q_2}b_{r_1}b_{r_2}\int_{[0,1]^4}dudvdu'dv'$$
$$\Bigg\{(N-\ell)\sum_{k=\ell}^{N-1}\Bigg[\left|u-v+k-q_1+r_1\right|^{2H'-2}\left|u'-v'+k-q_2+r_2\right|^{2H'-2}$$
$$\left|u-u'+k-q_1+q_2\right|^{2H'-2}\left|v-v'+k-r_1+r_2\right|^{2H'-2}\Bigg]$$
$$-\sum_{k=\ell}^{N-1}k\Bigg[\left|u-v+k-q_1+r_1\right|^{2H'-2}\left|u'-v'+k-q_2+r_2\right|^{2H'-2}$$
$$\left|u-u'+k-q_1+q_2\right|^{2H'-2}\left|v-v'+k-r_1+r_2\right|^{2H'-2}\Bigg]\Bigg\}.$$

The leading term in $\mathbf{E}\big(T^2_{4,(2)}\big)$ is of order N^{-1} and the constant computes as

$$\tau_{1,H}=\sum_{k=\ell}^{\infty}\sum_{q_1,q_2,r_1,r_1=0}^{\ell}b_{q_1}b_{q_2}b_{r_1}b_{r_2}\int_{[0,1]^4}dudvdu'dv'$$
$$\Bigg[\left|u-v+k-q_1+r_1\right|^{2H'-2}\left|u'-v'+k-q_2+r_2\right|^{2H'-2}$$
$$\left|u-u'+k-q_1+q_2\right|^{2H'-2}\left|v-v'+k-r_1+r_2\right|^{2H'-2}\Bigg].$$

Therefore, combining the two terms we get the statement of the proposition.

9.3. End of proof of Theorem 2

Recall that we only need to show that for $\tau=1,2,3$ the terms $||g_N\otimes_\tau g_N||^2_{L^2([0,1]^{8-2\tau})}$ converge to 0 as N tends to infinity.

- *Term* for $\tau = 1$.

$$\begin{aligned} J_1 &= \left(\frac{4N^{4H+1}}{c_{1,H}c(H)^2(N-\ell)^2}\right)^2 \\ &\quad \times \sum_{i,j,m,n=\ell}^{N-1} \langle (C_i \otimes C_i) \otimes_1 (C_j \otimes C_j), (C_m \otimes C_m) \otimes_1 (C_n \otimes C_n)\rangle \\ &= \left(\frac{4N^{4H+1}}{c_{1,H}c(H)^2(N-\ell)^2}\right)^2 \sum_{i,j,m,n=\ell}^{N-1} \langle C_i, C_m\rangle_{L^2([0,1]^2)} \\ &\quad \times \langle C_j, C_n\rangle_{L^2([0,1]^2)} \langle C_i \otimes_1 C_j, C_m \otimes_1 C_n\rangle_{L^2([0,1]^2)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &J_1 = \\ &\leq cst.\frac{N^{8H+2}}{(N-\ell)^4}\frac{1}{N^4} \sum_{i,j,m,n=\ell}^{N-1} \sum_{q_1,r_1,q_2,r_2,q_3,r_3,q_4,r_4=0}^{\ell} b_{q_1}b_{r_1}b_{q_2}b_{r_2}b_{q_3}b_{r_3}b_{q_4}b_{r_4} \\ &\times \left|\frac{i-m+q_1-r_1}{N}\right|^{2H} \left|\frac{j-n+q_2-r_2}{N}\right|^{2H} \Bigg[\int_{[0,1]^4} dudvdu'dv' \\ &\times \left|\frac{u-v+i-j-q_3+r_3}{N}\right|^{2H'-2} \left|\frac{u'-v'+m-n-q_4+r_4}{N}\right|^{2H'-2} \\ &\times \left|\frac{u-u'+i-m-q_3+q_4}{N}\right|^{2H'-2} \left|\frac{v-v'+j-n+r_3+r_4}{N}\right|^{2H'-2}\Bigg] \\ &\leq cst.\frac{N^2}{(N-\ell)^4} \sum_{i,j,m,n=\ell}^{N-1} \sum_{q_1,r_1,q_2,r_2,q_3,r_3,q_4,r_4=0}^{\ell} b_{q_1}b_{r_1}b_{q_2}b_{r_2}b_{q_3}b_{r_3}b_{q_4}b_{r_4} \\ &\times |i-m+q_1-r_1|^{2H} \ |j-n+q_2-r_2|^{2H} \Bigg[\int_{[0,1]^4} dudvdu'dv' \\ &\times |u-v+i-j-q_3+r_3|^{2H'-2} |u'-v'+m-n-q_4+r_4|^{2H'-2} \\ &\times |u-u'+i-m-q_3+q_4|^{2H'-2} |v-v'+j-n+r_3+r_4|^{2H'-2}\Bigg]. \end{aligned}$$

As in the computations for $T_{4,(2)}$ we can show that the above series converges and thus $J_1 = \mathcal{O}(N^{-2})$, which implies that for all $H \in (\frac{1}{2}, 1)$

$$\lim_{N\to\infty} J_1 = 0.$$

- *Term* for $\tau = 2$

$$J_2 = \left(\frac{4N^{4H+1}}{c_{1,H}c(H)^2(N-\ell)^2}\right)^2$$

$$\sum_{i,j,m,n=\ell}^{N-1} \langle (C_i \otimes C_i) \otimes_2 (C_j \otimes C_j), (C_m \otimes C_m) \otimes_2 (C_n \otimes C_n) \rangle$$

$$= \left(\frac{4N^{4H+1}}{c_{1,H} c(H)^2 (N-\ell)^2} \right)^2 \sum_{i,j,m,n=\ell}^{N-1} \langle C_i, C_j \rangle_{L^2([0,1]^2)}$$

$$\times \langle C_m, C_n \rangle_{L^2([0,1]^2)} \langle C_i, C_m \rangle_{L^2([0,1]^2)} \langle C_j, C_n \rangle_{L^2([0,1]^2)}.$$

$$J_2 \leq cst. \frac{N^{8H+2}}{(N-\ell)^4}$$

$$\sum_{i,j,m,n=\ell}^{N-1} \langle C_i, C_j \rangle_{L^2[0,1]^2} \langle C_i, C_m \rangle_{L^2[0,1]^2} \langle C_m, C_n \rangle_{L^2[0,1]^2} \langle C_j, C_n \rangle_{L^2[0,1]^2}$$

$$= cst. \frac{N^{8H+2}}{(N-\ell)^4} \sum_{i,j,m,n=\ell}^{N-1} \sum_{q_1 q_2 q_3 q_4 = 0}^{\ell} \alpha_{q_1} \alpha_{q_2} \alpha_{q_3} \alpha_{q_4} \left| \frac{i-j+q_1-q_2}{N} \right|^{2H}$$

$$\times \left| \frac{i-m+q_1-q_3}{N} \right|^{2H} \left| \frac{m-n+q_3-q_4}{N} \right|^{2H} \left| \frac{j-n+q_2-q_4}{N} \right|^{2H}$$

$$= cst. \frac{N^2}{(N-\ell)^4} \sum_{i,j,m,n=\ell}^{N-1} \sum_{q_1 q_2 q_3 q_4 = 0}^{\ell} \alpha_{q_1} \alpha_{q_2} \alpha_{q_3} \alpha_{q_4} |i-j+q_1-q_2|^{2H}$$

$$\times |i-m+q_1-q_3|^{2H} |m-n+q_3-q_4|^{2H} |j-n+q_2-q_4|^{2H}.$$

The series converges for all $H \in (1/2, 1)$, so the whole term is of order $\mathcal{O}(N^{-2})$ which means that goes to zero as $N \to \infty$.

- *Term* for $\tau = 3$.

$$J_3 = \left(\frac{4N^{4H+1}}{c_{1,H} c(H)^2 (N-\ell)^2} \right)^2$$

$$\sum_{i,j,m,n=\ell}^{N-1} \langle (C_i \otimes C_i) \otimes_3 (C_j \otimes C_j), (C_m \otimes C_m) \otimes_3 (C_n \otimes C_n) \rangle$$

$$= \left(\frac{4N^{4H+1}}{c_{1,H} c(H)^2 (N-\ell)^2} \right)^2 \sum_{i,j,m,n=\ell}^{N-1} \langle C_i, C_j \rangle_{L^2([0,1]^2)}$$

$$\times \langle C_m, C_n \rangle_{L^2([0,1]^2)} \langle C_i \otimes_1 C_j, C_m \otimes_1 C_n \rangle.$$

With similar computations as in the case of T_4 we conclude that $J_3 = \mathcal{O}(N^{-2})$.

9.4. Proof of Theorem 3

According to our previous computations we can write

$$f_N(y_1,y_2) = \frac{8N^{2H}}{c(H)(N-\ell)} \sum_{i=\ell}^{N-1} (C_i \otimes_1 C_i)(y_1,y_2)$$
$$= \frac{8d(H)^2\alpha(H)}{c(H)} \frac{N^{2H}}{(N-\ell)} \sum_{i=\ell}^{N-1} \sum_{q,r=0}^{\ell} b_q b_r 1_{[0,\frac{i-q+1}{N}]}(y_1) 1_{[0,\frac{i-r+1}{N}]}(y_2)$$
$$\times \int_{I_{i_q}} \int_{I_{i_r}} dudv|u-v|^{2H'-2}\, \partial_1 K^{H'}(u,y_1)\partial_1 K^{H'}(v,y_2)$$

Let us show first that we can reduce this function to the interval $y_1 \in [0, \frac{i-q}{N}]$ and $y_2 \in [0, \frac{i-r}{N}]$. We will show that if $y_1 \in I_{i_q}, y_2 \in [0, \frac{i-r}{N}]$ (and similarly for the situations $y_1 \in [0, \frac{i-q}{N}], y_2 \in I_{i_r}$ and $y_1 \in I_{i_q}, y_2 \in I_{i_r}$) the corresponding terms goes to zero as $N \to \infty$. We have, due to the fact that the intervals I_{i_q} are disjoint,

$$\| \frac{N^{1-H}N^{2H}}{(N-\ell)} \sum_{i=\ell}^{N-1} \sum_{q,r=0}^{\ell} b_q b_r 1_{I_{i_q}}(y_1) 1_{[0,\frac{i-r}{N}]}(y_2)$$
$$\int_{I_{i_q}} \int_{I_{i_r}} dudv|u-v|^{2H'-2}\, \partial_1 K^{H'}(u,y_1)\partial_1 K^{H'}(v,y_2) \|^2_{L^2([0,1]^2)}$$
$$= \frac{N^{2+2H}}{(N-\ell)^2} \sum_{i=\ell}^{N} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_{I_{i_{q_1}}} \int_{I_{i_{r_1}}} \int_{I_{i_{q_2}}} \int_{I_{i_{r_2}}} dv'du'dvdu$$
$$\times \left(|u-v| \cdot |u'-v'| \cdot |u-u'| \cdot |v-v'|\right)^{2H'-2}$$
$$= \frac{N^{2+2H}}{(N-\ell)^2} \frac{1}{N^4} \frac{1}{N^{4(2H'-2)}} \sum_{i=\ell}^{N} \sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1} b_{r_1} b_{q_2} b_{r_2} \int_{[0,1]^4} dudvdu'dv'$$
$$\times |u-v-q_1+r_1|^{2H'-2} |u'-v'-q_2+r_2|^{2H'-2}$$
$$|u-u'-q_1+q_2|^{2H'-2} |v-v'-r_1+r_2|^{2H'-2} \asymp N^{1-2H}$$

which tends to zero because $2H > 1$.

This proves the following asymptotic equivalence in $L^2([0,1]^2)$:

$$f_N(y_1,y_2) \simeq \frac{8d(H)^2\alpha(H)}{c(H)} \frac{N^{2H}}{(N-\ell)} \sum_{i=\ell}^{N-1} \sum_{q,r=0}^{\ell} b_q b_r 1_{[0,\frac{i-q}{N}]}(y_1) 1_{[0,\frac{i-r}{N}]}(y_2)$$
$$\times \int_{I_{i_q}} \int_{I_{i_r}} dudv|u-v|^{2H'-2}\, \partial_1 K^{H'}(u,y_1)\partial_1 K^{H'}(v,y_2).$$

We will show that the above term, normalize by $\frac{N^{1-H}}{\sqrt{c_{2,H}}}$, converges pointwise for $y_1,\ y_2 \in [0,1]$ to the kernel of the Rosenblatt random variable.

On the interval $I_{i_q} \times I_{i_r}$ we may attemp to replace the evaluation of $\partial_1 K^{H'}$ at u and v by setting $u = (i-q)/N$ and $v = (i-r)/N$. More precisely, we can write

$$\partial_1 K^{H'}(u,y_1)\partial_1 K^{H'}(v,y_2) = \left(\partial_1 K^{H'}(u,y_1) - \partial_1 K^{H'}\left(\frac{i-q}{N},y_1\right)\right)\partial_1 K^{H'}(v,y_2)$$
$$+ \partial_1 K^{H'}\left(\frac{i-q}{N},y_1\right)\left(\partial_1 K^{H'}(v,y_2) - \partial_1 K^{H'} - \partial_1 K^{H'}\left(\frac{i-r}{N},y_2\right)\right)$$

and all the above summand above can be treated in the same manner. For the first one, using the definition of the derivative of $K^{H'}$ with respect to the first variable, we get for any $y_1 \in [0, \frac{i-q}{N}]$,

$$\begin{aligned}
&\partial_1 K^{H'}(u,y_1) - \partial_1 K^{H'}\left(\frac{i-q}{N},y_1\right)\\
&= c_H y_1^{\frac{1}{2}-H}\left((u-y_1)^{H-\frac{3}{2}}u^{H-\frac{1}{2}} - \left(\frac{i-q}{N}-y_1\right)^{H-\frac{3}{2}}\left(\frac{i-q}{N}\right)^{H-\frac{1}{2}}\right)\\
&\leq c_H y_1^{\frac{1}{2}-H}\left(\frac{i-q}{N}-y_1\right)^{H-\frac{3}{2}}\left(u^{H-\frac{1}{2}} - \left(\frac{i-q}{N}\right)^{H-\frac{1}{2}}\right)\\
&\leq c_H y_1^{\frac{1}{2}-H}\left(\frac{i-q}{N}-y_1\right)^{H-\frac{3}{2}}\left(u - \left(\frac{i-q}{N}\right)\right)^{H-\frac{1}{2}}\\
&\leq c_H N^{\frac{1}{2}-H} y_1^{\frac{1}{2}-H}\left(\frac{i-q}{N}-y_1\right)^{H-\frac{3}{2}}
\end{aligned}$$

and for any $y_2 \in [0, \frac{i-r}{N}]$

$$\begin{aligned}
\partial_1 K^{H'}(v,y_2) &= c_H y_2^{\frac{1}{2}-H}(v-y_2)^{H-\frac{3}{2}}v^{H-\frac{1}{2}}\\
&\leq c_H y_2^{\frac{1}{2}-H}\left(\frac{i-r}{N}-y_1\right)^{H-\frac{3}{2}}(\frac{i-r+1}{N})^{H-\frac{1}{2}}.
\end{aligned}$$

As a consequence of the above estimates,

$$\begin{aligned}
&N^{1-H}\frac{N^{2H}}{N-\ell}\sum_{i=\ell}^{N-1}\sum_{q,r=0}^{\ell} b_q b_r 1_{[0,\frac{i-q}{N}]}(y_1)1_{[0,\frac{i-r}{N}]}(y_2)\\
&\times \int_{I_{i_q}}\int_{I_{i_q}} dvdu|u-v|^{2H'-2}\left(\partial_1 K^{H'}(u,y_1) - \partial_1 K^{H'}\left(\frac{i-q}{N},y_1\right)\right)\partial_1 K^{H'}(v,y_2)\\
&\leq cN^{\frac{1}{2}-H}\frac{N^{1+H}}{N-\ell}\sum_{i=\ell}^{N-1}\sum_{q,r=0}^{\ell} b_q b_r 1_{[0,\frac{i-q}{N}]}(y_1)1_{[0,\frac{i-r}{N}]}(y_2)\left(\frac{i-q}{N}-y_1\right)^{H-\frac{3}{2}}\\
&\times \left(\frac{i-r}{N}-y_2\right)^{H-\frac{3}{2}}\left(\frac{i-r+1}{N}\right)^{H-\frac{1}{2}}\int_{I_{i_q}}\int_{I_{i_q}} dvdu|u-v|^{2H'-2}
\end{aligned}$$

$$\leq cN^{\frac{1}{2}-H}\frac{1}{N-\ell}\sum_{i=\ell}^{N-1}\sum_{q,r=0}^{\ell} b_q b_r 1_{[0,\frac{i-q}{N}]}(y_1)1_{[0,\frac{i-r}{N}]}(y_2)$$
$$\times\left(\frac{i-q}{N}-y_1\right)^{H-\frac{3}{2}}\left(\frac{i-r}{N}-y_2\right)^{H-\frac{3}{2}}\left(\frac{i-r+1}{N}\right)^{H-\frac{1}{2}}. \tag{26}$$

The quantity $\frac{1}{N-\ell}\sum_{i=\ell}^{N-1} 1_{[0,\frac{i-q}{N}]}(y_1)1_{[0,\frac{i-r}{N}]}(y_2)\left(\frac{i-q}{N}-y_1\right)^{H-\frac{3}{2}}\left(\frac{i-r}{N}-y_2\right)^{H-\frac{3}{2}}\times(\frac{i-r+1}{N})^{H-\frac{1}{2}}$ is comparable, for large N, to the integral $\int_{y_1\vee y_2}^1(u-y_1)^{H-\frac{3}{2}}(u-y_2)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}$ and the term $N^{\frac{1}{2}-H}$ in front gives the convergence to zero of (26) for any fixed y_1, y_2.

This means we have proved the following pointwise asymptotically equivalent for $f_N(y_1,y_2)$:

$$\frac{N^{1-H}}{\sqrt{c_{2,H}}}f_N(y_1,y_2)\simeq\frac{8d(H)^2\alpha(H)}{\sqrt{c_{2,H}}\,c(H)}\frac{N^{1+H}}{(N-\ell)}\sum_{i=\ell}^{N-1}\sum_{q,r=0}^{\ell}1_{[0,\frac{i-q}{N}]}(y_1)1_{[0,\frac{i-r}{N}]}(y_2)b_q b_r$$
$$\partial_1 K^{H'}\left(\frac{i-q}{N},y_1\right)\partial_1 K^{H'}\left(\frac{i-r}{N},y_2\right)\int_{I_{i_q}}\int_{I_{i_r}}dudv|u-v|^{2H'-2}.$$

Recall that

$$\int_{I_{i_q}}\int_{I_{i_r}}dvdu|u-v|^{2H'-2}$$
$$=\frac{N^{-(1+H)}}{2H'(2H'-1)}\left\{|1-q+r|^{2H'}+|1+q-r|^{2H'}-2|q-r|^{2H'}\right\}.$$

Thus we get

$$\frac{N^{1-H}}{\sqrt{c_{2,H}}}f_N(y_1,y_2)$$
$$\simeq\frac{8d(H)^2\alpha(H)}{c_{2,H}\,c(H)}\sum_{q,r=0}^{\ell}b_q b_r\left\{|1-q+r|^{2H'}+|1+q-r|^{2H'}-2|q-r|^{2H'}\right\}$$
$$\times\frac{1}{(N-\ell)}\sum_{i=\ell}^{N-1}\partial_1 K^{H'}\left(\frac{i-q}{N},y_1\right)\partial_1 K^{H'}\left(\frac{i-r}{N},y_2\right).$$

Further, we can ignore the terms q/N and r/N in comparison with i/N in the last line above, and thus invoke a Riemann sum approximation, which proves that, for every $y_1, y_2\in(0,1)^2$

$$\lim_{N\to\infty}\frac{N^{1-H}}{c_{2,H}}f_N(y_1,y_2)$$
$$=\frac{8d(H)^2\alpha(H)}{c_{2,H}\,c(H)}\sum_{q,r=0}^{\ell}b_q b_r\left\{|1-q+r|^{2H'}+|1+q-r|^{2H'}-2|q-r|^{2H'}\right\}$$

$$\frac{1}{(N-\ell)}\lim_{N\to\infty}\sum_{i=\ell}^{N-1}\partial_1 K^{H'}(\frac{i-q}{N},y_1)\partial_1 K^{H'}(\frac{i-r}{N},y_2)$$
$$= d(H)\int_{y_1\vee y_2}\partial_1 K^{H'}(\frac{u}{N},y_1)\partial_1 K^{H'}(\frac{u}{N},y_2)du$$
$$= L_1(y_1,y_2).$$

To finish the proof it suffices to check that $N^{1-H}f_N$ is a Cauchy sequence in $L^2([0,1]^2)$. Up to a constant depending on H we have that for all M, N,

$$||N^{1-H}f_N - M^{1-H}f_M||^2_{L^2[0,1]^2}$$
$$= N^{2-2H}||f_N||^2_{L^2[0,1]^2} + M^{2-2H}||f_M||^2_{L^2([0,1]^2)} - 2N^{1-H}M^{1-H}\langle f_N, f_M\rangle_{L^2[0,1]^2}$$
$$= cst.\frac{N^{2H+2}}{(N-\ell)^2}\sum_{i,j=\ell}^{N-1}\sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1}b_{r_1}b_{q_2}b_{r_2}\int_{I^N_{i_{q_1}}}\int_{I^N_{i_{r_1}}}\int_{I^N_{j_{q_2}}}\int_{I^N_{j_{r_2}}} dudvdu'dv'$$
$$\times|u-v|^{2H'-2}|u'-v'|^{2H'-2}|u-u'|^{2H'-2}|v-v'|^{2H'-2}$$
$$+ cst.\frac{M^{2H+2}}{(M-\ell)^2}\sum_{i,j=\ell}^{M-1}\sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1}b_{r_1}b_{q_2}b_{r_2}\int_{I^M_{i_{q_1}}}\int_{I^M_{i_{r_1}}}\int_{I^M_{j_{q_2}}}\int_{I^M_{j_{r_2}}} dudvdu'dv'$$
$$\times|u-v|^{2H'-2}|u'-v'|^{2H'-2}|u-u'|^{2H'-2}|v-v'|^{2H'-2}$$
$$- cst.\frac{M^{1+H}N^{1+H}}{(M-\ell)(N-\ell)}\sum_{i=\ell}^{N-1}\sum_{j=\ell}^{M-1}\sum_{q_1,r_1,q_2,r_2=0}^{\ell} b_{q_1}b_{r_1}b_{q_2}b_{r_2}\int_{I^N_{i_{q_1}}}\int_{I^N_{i_{r_1}}}$$
$$\int_{I^M_{j_{q_2}}}\int_{I^M_{j_{r_2}}} dudvdu'dv'|u-v|^{2H'-2}|u'-v'|^{2H'-2}|u-u'|^{2H'-2}|v-v'|^{2H'-2}.$$

The first two terms have already been studied and will converge to the same constant as $M,N\to\infty$. Concerning the inner product, by making the usual change of variable we have

$$\frac{(MN)^{H+1}}{(M-\ell)(N-\ell)}\frac{(NM)^{2H'-2}}{N^2M^2}\sum_{i=\ell}^{N-1}\sum_{j=\ell}^{M-1}\sum_{q_1,r_1,q_2,r_2=0}^{\ell}\int_{[0,1]^4} dudvdu'dv'$$
$$\times|u-v-q_1+r_1|^{2H'-2}|u'-v'-q_3+r_3|^{2H'-2}$$
$$\times\left|\frac{u}{N}-\frac{u'}{M}+\frac{i}{N}-\frac{j}{N}-\frac{q_1}{N}+\frac{q_2}{N}\right|^{2H'-2}\left|\frac{v}{N}-\frac{v'}{M}+\frac{i}{N}-\frac{j}{N}-\frac{r_1}{N}+\frac{r_2}{N}\right|^{2H'-2}.$$

For large i,j we can ignore the terms $\frac{u}{N}$, $\frac{u'}{N}$, $\frac{q_1}{N}$, etc., compared to $\frac{i}{N}$ and $\frac{j}{N}$. Therefore, the above quantity is a Riemann sum that converges to the same constant as the squared terms, as $M,N\to\infty$. This finishes the proof of the theorem.

9.5. Proof of Theorem 6

We wish to show that, as $N \to \infty$,

$$E := \mathbf{E}\left[\left(Z(1) - 2c_{2,H}^{-1/2} N^{1-H}\left(\hat{H}_N - H\right)\log N\right)^2\right] \to 0.$$

A minor technical difficulty occurs when V_N is not small. We deal with this as follows. We decompose the above expectation E according to whether or not $|V_N| \leq 1/2$: we have $E = E_1 + E_2$ where

$$E_1 = \mathbf{E}\left[\mathbf{1}_{|V_N|>1/2}\left(Z(1) - 2c_{2,H}^{-1/2} N^{1-H}\left(\hat{H}_N - H\right)\log N\right)^2\right].$$

Dealing with this term first, Schwarz's and Minkowski's inequalities yields

$$E_1 \leq 2\mathbf{P}^{1/2}\left[|V_N| > 1/2\right] \times \left(\mathbf{E}^{1/2}\left[Z(1)^4\right] + 4c_{2,H}^{-1} N^{2-2H}\log^2 N\ \mathbf{E}^{1/2}\left[\left(\hat{H}_N - H\right)^4\right]\right).$$

Since $\hat{H}_N$ is bounded, the sum of the two rooted expectation terms above is bounded above by a constant multiple of N^{2-2H}. Therefore to deal with E_1, one only needs to show that $\mathbf{P}\left[|V_N| > 1/2\right] \ll N^{-4+4H}$. It is well known that any random variable X which can be written as a finite sum of Wiener chaos terms up to order q satisfies, for any integer n, $\mathbf{E}\left[X^{2n}\right] \leq K_{n,q}\left(\mathbf{E}\left[X^2\right]\right)^n$ where $K_{n,q}$ depends only on n and q. This can be proved iteratively by using formula (7), for instance. Therefore, since V_N is a sum of terms in the second and 4th chaos ($q = 4$), by Chebyshev's inequality, and using Theorem 1, with N large enough,

$$\begin{aligned}\mathbf{P}\left[|V_N| > 1/2\right] &\leq 4^n \mathbf{E}\left[|V_N|^{2n}\right] \leq 4^n c_{n,4}\left(\mathbf{E}\left[|V_N|^2\right]\right)^n \\ &\leq 8^n K_{n,4} c_{2,H}^n N^{2Hn-2n}.\end{aligned}$$

It is thus sufficient to choose $n = 3$ to guarantee that $E_1 \to 0$.

We now only need to study E_2. We invoke the mean value theorem to express $\left(\hat{H}_N - H\right)\log N$ more explicitly. For any $x, y \in [1/2, 1]$, there exists $\xi \in (x, y)$ such that

$$\log\frac{c(x)}{c(y)} = (x - y)\left(\log c\right)'(\xi).$$

Here the function $\left(\log c\right)'$ is bounded on $[1/2, 1]$, because c' is bounded and c is bounded below. Therefore, denoting by $\xi_N \in [1/2, 1]$ the value corresponding to $x = H$ and $y = \hat{H}_N$, and using line (23) in the proof of Theorem 5, we can write

$$\log(1 + V_N) = \left(\hat{H}_N - H\right)\left(2\log N + \left(\log c\right)'(\xi_N)\right)$$

and thus

$$\left(\hat{H}_N - H\right)(2\log N) = \log(1 + V_N) - \frac{\log(1 + V_N)}{2\log N + \left(\log c\right)'(\xi_N)}.$$

Since $\left|(\log c)'(\xi_N)\right|$ is bounded (by a non-random value), by choosing N large enough, an upper bound for the last fraction above, in absolute value, is $2\frac{V_N}{\log N}$. Therefore (using Minkowski's inequality),

$$\begin{aligned}\sqrt{E_2} &= \mathbf{E}^{1/2}\left[\mathbf{1}_{|V_N|\le 1/2}\left(Z(1) - 2c_{2,H}^{-1/2}N^{1-H}\left(\hat{H}_N - H\right)\log N\right)^2\right] \\ &\le \mathbf{E}^{1/2}\left[\mathbf{1}_{|V_N|\le 1/2}\left(Z(1) - c_{2,H}^{-1/2}N^{1-H}\left(\log\left(1+V_N\right)\right)\right)^2\right] \qquad (27) \\ &+ \mathbf{E}^{1/2}\left[\mathbf{1}_{|V_N|\le 1/2}\left(2c_{2,H}^{-1/2}N^{1-H}V_N/\log N\right)^2\right]. \qquad (28)\end{aligned}$$

By Theorem 1, the term in line (28) is bounded above by $1/\log^2 N$, and thus converges to 0. For the term in line (27), because of the indicator $\mathbf{1}_{|V_N|\le 1/2}$, we use the fact that when $|x| \le 1/2$, we have $|x - \log(1+x)| \le x^2$. Thus this line is bounded above by

$$\begin{aligned}&\mathbf{E}^{1/2}\left[\mathbf{1}_{|V_N|\le 1/2}\left(Z(1) - c_{2,H}^{-1/2}N^{1-H}V_N\right)^2\right] \qquad (29) \\ &+ \mathbf{E}^{1/2}\left[\mathbf{1}_{|V_N|\le 1/2}\left(c_{2,H}^{-1/2}N^{1-H}\left|V_N\right|^2\right)^2\right]. \qquad (30)\end{aligned}$$

The term in line (29) converges to 0 by Theorem 3. Finally, by Theorem 1 again, and the earlier statement about higher powers of random variables with finite chaos expansions, the term in line (30) is of order N^{2H-2}, and therefore converges to 0 as well. This proves that E_2 converges to 0, finishing the proof of the theorem.

9.6. *Proof of Theorem 7*

It is sufficient to prove that

$$\lim_{N\to\infty}\mathbf{E}\left[\left|\left(N^{1-\hat{H}_N} - N^{1-H}\right)\left(\hat{H}_N - H\right)\log N\right|\right] = 0.$$

We decompose the probability space depending on whether $\hat{H}_N$ is far or not from its mean. For a fixed value $\varepsilon > 0$ it is convenient to define the event

$$D = \left\{\hat{H}_N > \varepsilon + 2H - 1\right\}.$$

We have

$$\begin{aligned}&\mathbf{E}\left[\left|\left(N^{1-\hat{H}_N} - N^{1-H}\right)\left(\hat{H}_N - H\right)\log N\right|\right] = \\ &= \mathbf{E}\left[\mathbf{1}_D\left|\left(N^{1-\hat{H}_N} - N^{1-H}\right)\left(\hat{H}_N - H\right)\log N\right|\right] + \\ &+ \mathbf{E}\left[\mathbf{1}_{D^c}\left|\left(N^{1-\hat{H}_N} - N^{1-H}\right)\left(\hat{H}_N - H\right)\log N\right|\right] \\ &:= A + B.\end{aligned}$$

Proof.

Term A:

Introduce the notation
$x = \max\big(1-H, 1-\hat{H}_N\big)$ and $y = \min\big(1-H, 1-\hat{H}_N\big)$.

$$\begin{aligned}\left|N^{1-\hat{H}_N} - N^{1-H}\right| &= e^{x\log N} - e^{y\log N} = e^{y\log N}\left(e^{(x-y)\log N} - 1\right)\\ &\le N^y(\log N)(x-y)N^{x-y} = 2\log N N^x\left|H-\hat{H}_N\right|\\ &= \log N N^x\left|H-\hat{H}_N\right|.\end{aligned}$$

Thus,

$$\begin{aligned}A &\le \mathbf{E}\left[\mathbf{1}_D N^x\left|H-\hat{H}_N\right|^2\log^2 N\right]\\ &= \mathbf{E}\left[N^{x-(2-2H)}\mathbf{1}_D N^{2-2H}\left|H-\hat{H}_N\right|^2\log^2 N\right]\end{aligned}$$

Now, choose $\varepsilon \in (0, 1-H)$. In this case, if $\omega \in D$ and $x = 1-H$, we get $x-(2-2H) = -x < -\varepsilon$. On the other hand, for $\omega \in D$ and $x = 1-\hat{H}_N$ we get $x-(2-2H) = 2-2\hat{H}_N-(2-2H) < -\varepsilon$. In conclusion, on D, $x-(2-2H) < -\varepsilon$ which implies immediately

$$A \le N^{-\varepsilon}\mathbf{E}\left[N^{2-2H}\left|\hat{H}_N - H\right|^2\log^2 N\right].$$

and since the last expectation is bounded

$$\lim_{N\to\infty} A = 0.$$

Term B

Now, let $\omega \in D^c$ then $H-\hat{H}_N > 1-H-\varepsilon$. Since $\varepsilon < 1-H$ it implies $H > \hat{H}_N$. Consequently, it is not sufficient to bound $\left|N^{1-\hat{H}_N} - N^{1-H}\right|$ above by $N^{1-\hat{H}_N}$. In the same fashion we bound $\left|\hat{H}-H\right|$ above by H. Using Hölder's inequality with powers $\frac{1}{4}$ and $\frac{3}{4}$

$$\begin{aligned}B &\le H\log N\mathbf{E}\left[\mathbf{1}_{D^c}N^{1-\hat{H}_N}\right]\\ &\le H\log N\left[P(D^c)\right]^{3/4}\left(\mathbf{E}\left[N^{(1-\hat{H}_N)4}\right]\right)^{1/4}.\end{aligned}$$

By Chebyshev's inequality, we have

$$\mathbf{P}^{3/4}\left[D^c\right] \le \frac{\mathbf{E}^{3/4}\left[\left|H-\hat{H}\right|^2\right]}{(1-H-\varepsilon)^{3/2}} \le cN^{-3(2-2H)/4} \tag{31}$$

for some constant c depending only H. Dealing with the other term in the upper bound for B is a little less obvious. We must return to the definition of $\hat{H}$. We have

$$1 + V_N = N^{2\left(H-\hat{H}\right)} = N^{4\left(H-\hat{H}\right)} = N^{4\left(1-\hat{H}\right)} N^{-4(1-H)}.$$

Therefore,

$$\mathbf{E}^{1/4}\left[N^{\left(1-\hat{H}\right)4}\right] \leq N^{1-H}\mathbf{E}^{1/(4)}\left[1+V_N\right] \leq 2N^{1-H}.$$

Finally, we get

$$B \leq 2Hc\left(\log N\right) N^{-(1-H)}.$$

Finally, B goes to 0 as $N \to \infty$. This finishes the proof of the theorem.

□

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