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CONSTRUCTION OF NESTED SPACE-FILLING DESIGNS

BY PETER Z. G. QIAN[1], MINGYAO AI[2] AND C. F. JEFF WU[3]

University of Wisconsin–Madison, Peking University and Georgia Institute of Technology

New types of designs called *nested space-filling designs* have been proposed for conducting multiple computer experiments with different levels of accuracy. In this article, we develop several approaches to constructing such designs. The development of these methods also leads to the introduction of several new discrete mathematics concepts, including nested orthogonal arrays and nested difference matrices.

1. Introduction. Computer models are widely used in business, engineering and sciences to study complex real-world systems. The corresponding physical experimentation might otherwise be time-consuming, costly or even infeasible to conduct. Space-filling designs [Fang, Li and Sudjianto (2006) and Santner, Williams and Notz (2003)] have been widely used for conducting computer experiments. They include Latin hypercube designs [McKay, Conover and Beckman (1979)] and their improvements and variants [Butler (2001), Owen (1992, 1994b), Steinberg and Lin (2006), Tang (1993, 1998) and Ye (1998)]. Statistical properties of such designs have been studied in Loh (1996a, 1996b, 2008), Owen (1994a) and Stein (1987). Other types of space-filling designs are uniform designs [Fang et al. (2000)], quasi-Monte Carlo sequences [Niederreiter (1992)] and designs with uniform coverage [Dalal and Mallows (1998) and Lam, Welch and Young (2002)].

A large computer code, like a finite element analysis model, is often run at variable degrees of sophistication, resulting in multiple computer experiments with different levels of accuracy and varying computational times. In this article, we consider the situation in which two such experiments are available, and one source is generally more accurate than the other but also more expensive to run. As in Qian and Wu (2008), the two experiments considered are called the *high-accuracy experiment* (HE) and *low-accuracy experiment* (LE). The problem of modeling data from HE and LE has attracted a recent surge of interests. Related work includes Goldstein and Rougier (2004), Higdon et al. (2004), Kennedy and O'Hagan (2000,

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2001), Reese et al. (2004), Qian et al. (2006) and Qian and Wu (2008), among others. Most of these methods are based on flexible Gaussian process models [Fang, Li and Sudjianto (2006), Sacks et al. (1989), Santner, Williams and Notz (2003) and Welch et al. (1992)].

The sets of design points for LE and HE are denoted by D_l and D_h. Throughout the paper, LE and HE are assumed to share the same set of factors and the design region, for both D_l and D_h, are assumed to be the unit hypercube. As a suitable choice for D_l and D_h, the notion of *nested space-filling designs* (NSFDs) was introduced in Qian, Tang and Wu (2009) (referred to as QTW hereinafter). The basic idea is to construct a special orthogonal array A_1 and use it to obtain an OA-based Latin hypercube design [Tang (1993)] for D_l. Take A_2 to be a subset of A_1 that becomes an orthogonal array itself after some level-collapsing, then obtain D_h as the subarray of D_l corresponding to A_2. The constructed D_l and D_h achieve low-dimensional uniformity. The nested relationship $D_h \subset D_l$ is appealing, which is also adopted in Kennedy and O'Hagan (2000), Qian et al. (2006) and Qian and Wu (2008). It implies that the size of D_h is smaller than that of D_l which is desirable because LE is cheaper than HE, and more LE runs can be afforded. From the modeling standpoint, this structure ensures that for every point in D_h, the outputs from both HE and LE are available, thus making it easier to model the differences of outputs between the two sources, and perform model adjustment.

We call the above special orthogonal array *nested orthogonal array* (NOA). Its formal definition will be given in the next section. A family of NOAs with fixed levels was constructed in QTW based on the Rao–Hamming method which will be reviewed in Section 2.3. In this article, we propose a new approach to constructing such arrays. The principal idea is to first construct nested difference matrices and then take the Kronecker product of a nested difference matrix and a standard orthogonal array to obtain an NOA. This method is motivated by the fact that constructing a nested difference matrix is probably easier than the direct construction of its corresponding NOA. Similar considerations have been used in constructing orthogonal arrays from difference matrices [Hedayat, Sloane and Stufken (1999), referred to as HSS hereinafter]. As a modification of this approach, we provide another method that uses existing NOAs to obtain new ones. These methods can produce many new NOAs and therefore new NSFDs. Several approaches for constructing NOAs with mixed levels will also be discussed.

The remainder of the article will unfold as follows. In Section 2, some notation and definitions are introduced. In Section 3, an approach based on multiplication tables of Galois fields to constructing nested difference matrices is proposed. In Section 4, a general approach to constructing NOAs with Kronecker product is presented. In Section 5, a method is introduced for constructing new NOAs from existing ones. In Section 6, construction of NOAs with nonprime power number of levels is considered. Construction of NOAs with mixed levels is given in Section 7. In Section 8, the problem of using NOAs to obtain NSFDs is discussed. Some discussions and concluding remarks are provided in Section 9.

2. Notation and definitions.

2.1. *Preliminaries.* Let $A = (a_{ij})$ be a Latin hypercube of n runs for m factors that is an $n \times m$ matrix where each column is a permutation of $1, \dots, n$. Following McKay, Conover and Beckman (1979), a Latin hypercube design of n runs in m factors in the unit cube $[0, 1)^m$ is generated through $x_{ij} = (a_{ij} - u_{ij})/n, 1 \le i \le n, 1 \le j \le m$, where u_{ij}'s are independent $U(0, 1]$ random variables, and the n design points are given by $(x_{i1}, \dots, x_{im})$, $i = 1, \dots, n$. When such a design is projected onto each of the m factors, one and only one of the n points falls within each of the n small intervals defined by $[0, 1/n), [1/n, 2/n), \dots, [(n-1)/n, 1)$.

A symmetrical orthogonal array (OA) of size n, m constraints, s levels, and strength $t \ge 2$, is an $n \times m$ matrix with entries from a set of s levels, usually taken as $1, \dots, s$, such that for every $n \times t$ submatrix, the s^t level combinations occurs equally often. Regular fractional factorial designs, as discussed in Wu and Hamada (2000), are the most familiar examples of orthogonal arrays. In the article, we consider only OAs with strength two, denoted by $\mathrm{OA}(n, m, s)$. Asymmetrical OAs will be discussed in Section 7.

Let A be an $\mathrm{OA}(n, m, s)$ with its s levels denoted by $1, \dots, s$. Then in every column of A, each level occurs $q = n/s$ times. For each column of A, if we replace the q ones by a permutation of $1, \dots, q$, replace the q twos by a permutation of $q + 1, \dots, 2q$, and so on, we obtain an OA-based Latin hypercube [Tang (1993)]. In addition to achieving maximum stratification in one dimension, OA-based Latin hypercubes have attractive space-filling properties when projected onto 2 dimensions.

A difference matrix (DM) is a $b \times c$ array with entries from a finite abelian group $\mathcal{A}$ with g elements, such that every element of $\mathcal{A}$ appears equally often in the vector difference between any two columns of the array [Bose and Bush (1952)]. We will denote such an array by $D(b, c, g)$. If $\mathcal{A}$ is the additive group associated with a Galois field, we simply say its elements come from the associated field. For any $D(b, c, g)$, a column is defined to be *uniform* in $\mathcal{A}$ if it contains each element of $\mathcal{A}$ equally often. By subtracting the first column from all columns, any $D(b, c, g)$ can always be converted to a difference matrix of the form

$$[\,0_b \quad D^{(0)}\,], \tag{1}$$

where 0_b is the b-dimensional zero vector and every column of $D^{(0)}$ is uniform in $\mathcal{A}$.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be, respectively, $m \times n$ and $u \times v$ matrices with entries from an abelian group $\mathcal{A}$ with binary operation $*$ (usually addition or multiplication). The *Kronecker product* of A and B [Shrikhande (1964)], denoted by $A \otimes B$, is defined to be the $mu \times nv$ matrix

$$A \otimes B = \begin{bmatrix} a_{11} * B & \cdots & a_{1n} * B \\ \vdots & & \vdots \\ a_{m1} * B & \cdots & a_{mn} * B \end{bmatrix},$$

where $a_{ij} * B$ denotes the $u \times v$ matrix with entries $a_{ij} * b_{rs}$, $1 \leq r \leq u$, $1 \leq s \leq v$. Throughout this article $*$ always denotes addition.

2.2. *Galois field projections*. For every prime p and every integer $u \geq 1$, there exists a Galois field (or finite field) GF(p^u) of order p^u. The additive group GF(p^u) is cyclic, and the multiplicative group GF(p^u)/{0} is cyclic, allowing easy calculations under multiplication. Throughout, the elements of any Galois field or any subset of a Galois field are arranged in lexicographical order.

Unless stated otherwise, let $s_1 = p^{u_1}$ and $s_2 = p^{u_2}$ be powers of the same prime p with integers $u_1 > u_2 \geq 1$. Throughout, let F denote GF(s_1) with an irreducible polynomial $p_1(x)$, and G denote GF(s_2) with an irreducible polynomial $p_2(x)$. Let $f(x)$ denote the elements of F and $g(x)$ the elements of G, respectively. In condensed notation, let $\alpha_0, \ldots, \alpha_{s_1-1}$ denote the elements of F and $\beta_0, \ldots, \beta_{s_2-1}$ the elements of G with $\alpha_0 = 0$ and $\beta_0 = 0$. Next, we discuss two projections from F to G, serving as a basis for later development.

The first projection, denoted by ϕ, is taken from Bose and Bush (1952). For any $f(x) = a_0 + a_1 x + \cdots + a_{u_2-1}x^{u_2-1} + \cdots + a_{u_1-1}x^{u_1-1} \in F$, $\phi(f(x))$ is defined by

$$\phi(f(x)) = a_0 + a_1 x + \cdots + a_{u_2-1}x^{u_2-1}. \tag{2}$$

Because ϕ works by truncating all x powers of degree u_2 or higher, we call it the *truncation projection*.

The second projection, denoted by φ, is proposed in QTW. For any $f(x) \in F$, $\varphi(f(x))$ is defined by

$$\varphi(f(x)) = f(x) (\text{mod}\, p_2(x)). \tag{3}$$

Because φ works by taking modulus residues, we call it the *modulus projection*.

EXAMPLE 1. Let $p = 2$, $u_1 = 3$ and $u_2 = 2$, giving $s_1 = 8$ and $s_2 = 4$. Use $p_1(x) = x^3 + x + 1$ for GF(8) and $p_2(x) = x^2 + x + 1$ for GF(4). Then the projection φ is given as $\{0, x^2 + x + 1\} \to 0$, $\{1, x^2 + x\} \to 1$, $\{x, x^2 + 1\} \to x$, $\{x + 1, x^2\} \to x + 1$.

Let δ be either of the projections described above. For an array D with entries from F, $\delta(D)$ denotes the array obtained from D after the levels of its entries are collapsed according to δ. Clearly, the entries of $\delta(D)$ take values in G.

Notice that for any $\alpha_i, \alpha_j \in F$,

$$\delta(\alpha_i + \alpha_j) = \delta(\alpha_i) + \delta(\alpha_j). \tag{4}$$

This means that the two operations δ and $+$ are interchangeable, which is critical to the constructions in Sections 4, 5 and 7.

2.3. *Nested space-filling designs and nested orthogonal arrays.* Now we give a formal definition of NOAs, which underly the construction of NSFDs in QTW. Let A_1 be an OA(n_1, k, s_1). Suppose there is a subarray of A_1 with size n_2, denoted by A_2, and there is a projection δ that collapses the s_1 levels of A_1 into s_2 levels. Further suppose A_2 becomes an OA(n_2, k, s_2) after the levels of its entries are collapsed according to δ. Then A_1, or more precisely (A_1, A_2), is an NOA, denoted by NOA(A_1, A_2) or NOA(A_1, A_2, δ). To be emphatic about a small OA being nested within a larger OA, we say A_1 "contains" $\delta(A_2)$.

Let (A_1, A_2) be an NOA defined above. Construction of an NSFD is done as follows. The array A_1 is used to generate an OA-based Latin hypercube design D_l. Let D_h denote the subset of D_l corresponding to A_2. Then D_l, or more precisely (D_l, D_h), is an NSFD, where both D_l and D_h achieve uniformity in low dimensions.

The family of NOA(A_1, A_2), constructed in QTW by using the Rao–Hamming method, has the following set of parameters:

(i) A_1 is an OA(n_1, m_2, s_1), where $n_1 = s_1^k$, $m_2 = (s_2^k - 1)/(s_2 - 1)$ and $k \geq 2$ is an integer;

(ii) A_2 is a subarray of A_1 and $\varphi(A_2)$ is an OA(n_2, m_2, s_2) with $n_2 = s_2^k$.

This construction works for $2u_2 \leq u_1 + 1$.

2.4. *Nested difference matrices.* Let D_1 be a $D(b_1, c, s_1)$ with entries from F. Suppose there is a subarray of D_1 with b_2 rows denoted by D_2, and a projection δ that collapses the s_1 levels of D_1 into the s_2 levels of G. Further suppose D_2 is a $D(b_2, c, s_2)$ if the levels of its entries are collapsed according to δ. Then D_1, or more precisely (D_1, D_2), is called a *nested difference matrix* (NDM), denoted by NDM(D_1, D_2) or NDM(D_1, D_2, δ). To be emphatic about a smaller DM being nested within a larger DM, we say D_1 "contains" $\delta(D_2)$.

3. Construction of nested difference matrices. In this section, we propose an approach based on multiplication tables of Galois fields to constructing NDMs. It works for any $u_1 > u_2 \geq 1$. Here the projection ϕ in (2) is used. For a scalar a and a vector $c = (c_1, \dots, c_m)'$, $a + c$ denotes $(a + c_1, \dots, a + c_m)'$, where $'$ stands for vector transpose. Similarly, $a + A$ denotes the element-wise sum of a scalar a and a matrix A. We focus on the case of $p = 2$ and briefly discuss the case of $p = 3$ in the end of the section. Two sets or vectors are defined to be *disjoint* if they have no element in common. Because the constructions in Section 4 can use a small NDM and a standard OA to generate a larger NOA, here we construct NDMs with up to 16 columns. Throughout, we use the irreducible polynomial $p(x) = x^u + x + 1$ for any GF(2^u), $u \geq 1$. Unless stated otherwise, let $r_{-1} = (0)$, $r_0 = (0, 1)'$, $r_m = (0, 1, x, x + 1, \dots, x^m + \cdots + x + 1)'$, $m \geq 1$. Note that r_m has 2^{m+1} elements.

A $D(s_1, s_1, s_1)$ can be obtained by constructing the $s_1 \times s_1$ multiplication table of GF(s_1), where the rows and columns are labeled by all distinct elements of

$GF(s_1)$. Hereinafter, in describing such a table, we call a row (or column) labeled with an element $f(x) \in GF(s_1)$ as "row (or column) $f(x)$."

3.1. *A* $D(2^{m+1}, 2^2, 2^{m+1})$ *containing a* $D(2^m, 2^2, 2^m)$ *with* $m \geq 2$. Let $F = GF(2^{u_1})$ and $G = GF(2^{u_2})$ with $u_1 = m+1$, $u_2 = m$ and $m \geq 2$. Let D_0 be the multiplication table of F. By taking columns r_1 of D_0, obtain a matrix D_1.

Collect the elements of F into two vectors:

$$g_1 = (r'_{m-2}, x^{m-1} + r'_{m-2})' \quad \text{and} \quad g_2 = x^m + g_1, \tag{5}$$

where the ith element in g_2 equals its counterpart in g_1 plus x^m. Now place the rows of D_1 in two clusters: the top one comprising those labeled with r_{m-2} and $x^m + r_{m-2}$, and the bottom one with $x^{m-1} + r_{m-2}$ and $x^m + x^{m-1} + r_{m-2}$. This arrangement may look abstract at this moment but will become clear after Theorem 1. Table 1 gives $\phi(D_1)$, where, for $m = 2$, the entries need to be taken modulo $p_1(x) = x^{u_1} + x + 1$ and then collapsed according to ϕ.

Take D_2 to be the submatrix of D_1 consisting of the rows labeled with r_{m-2} and $x^m + x^{m-1} + r_{m-2}$. Because r_{m-2} is the set of polynomials of order at most $m-2$, r_{m-2} and $x^{m-1} + r_{m-2}$ are disjoint and their union is $GF(2^m)$. The following is a simple result regarding columns x and $x+1$ of $\phi(D_1)$.

LEMMA 1. (i) *The vectors* $(x+1)r_{m-2}$ *and* $x^{m-1} + (x+1)r_{m-2}$ *are disjoint and their union is* $GF(2^m)$;

(ii) *the vectors* $(x+1)r_{m-2}$ *and* $(x^{m-1} + x + 1) + (x+1)r_{m-2}$ *are disjoint and their union is* $GF(2^m)$.

(iii) *the vectors* xr_{m-2} *and* $(x+1) + xr_{m-2}$ *are disjoint and their union is* $GF(2^m)$.

PROOF. (i) It suffices to show that $(x+1)r_{m-2}$ and $x^{m-1} + (x+1)r_{m-2}$ are disjoint. Assuming the contrary, then there are two elements α_1 and α_2 from r_{m-2} such that $(x+1)\alpha_1 = x^{m-1} + (x+1)\alpha_2$, implying $(x+1)(\alpha_1 - \alpha_2) = x^{m-1}$. This is impossible because $x+1$ does not divide x^{m-1}.

(ii) It follows from (i) by noting that $(x+1) + (x+1)r_{m-2}$ has the same set of elements as $(x+1)r_{m-2}$.

TABLE 1
The matrix $\phi(D_1)$ *obtained from* D_1 *in Theorem* 1

	0	**1**	$\boldsymbol{x}$	$\boldsymbol{x+1}$
r_{m-2}	0	r_{m-2}	xr_{m-2}	$(x+1)r_{m-2}$
$x^m + r_{m-2}$	0	r_{m-2}	$(x+1) + xr_{m-2}$	$(x+1) + (x+1)r_{m-2}$
$x^{m-1} + r_{m-2}$	0	$x^{m-1} + r_{m-2}$	xr_{m-2}	$x^{m-1} + (x+1)r_{m-2}$
$x^m + x^{m-1} + r_{m-2}$	0	$x^{m-1} + r_{m-2}$	$(x+1) + xr_{m-2}$	$(x^{m-1} + x + 1) + (x+1)r_{m-2}$

(iii) Assuming the contrary, then there are two elements α_1 and α_2 from r_{m-2} such that $\alpha_1 - \alpha_2 - 1 = x^{-1}$, a contradiction. □

THEOREM 1. *Consider D_1 and D_2 constructed above. For $m \geq 2$, we have*:

(i) *the matrix D_1 is a $D(2^{m+1}, 2^2, 2^{m+1})$*;
(ii) *the matrix $\phi(D_2)$ is a $D(2^m, 2^2, 2^m)$.*

PROOF. Only (ii) needs a proof. Because the elements $\{0, 1, x, x+1\}$, used to label the columns of D_1, form an additive group, it suffices to show that columns $1, x, x+1$ of $\phi(D_2)$ are uniform in GF(2^m). Note that, due to the grouping scheme in (5), columns 1 and x of $\phi(D_2)$ in Table 1 are exactly an half fraction of those of $\phi(D_1)$. Then it remains to show that column $x+1$ of $\phi(D_2)$ is uniform in GF(2^m). This follows from Lemma 1 as $(x+1) + (x+1)r_{m-2}$ and $(x+1)r_{m-2}$ have the same set of elements. □

EXAMPLE 2 [A $D(2^2, 2, 2^2)$ containing a $D(2, 2, 2)$]. Although this example has only two columns, we include it here because its construction is similar to those in Theorem 1. Let $F = \text{GF}(2^2)$ and $G = \text{GF}(2)$. Let D_0 be the multiplication table of F given by

	0	**1**	***x***	***x* + 1**
0	0	0	0	0
1	0	1	x	$x+1$
x	0	x	$x+1$	1
$x+1$	0	$x+1$	1	x

Take D_1 be the first two columns of D_0. Obtain D_2 as the submatrix of D_1 consisting of rows 0 and 1. The matrix D_1 is a $D(2^2, 2, 2^2)$, and $\phi(D_2)$ is a $D(2, 2, 2)$ given by

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

EXAMPLE 3 [A $D(2^3, 2^2, 2^3)$ containing a $D(2^2, 2^2, 2^2)$]. Let $F = \text{GF}(2^3)$ and $G = \text{GF}(2^2)$. Take D_1 to be the columns of the multiplication table of F labeled with r_1 given by

	0	**1**	***x***	***x* + 1**
0	0	0	0	0
1	0	1	x	$x+1$
x^2	0	x^2	$x+1$	x^2+x+1
x^2+1	0	x^2+1	1	x^2
x	0	x	x^2	x^2+x
$x+1$	0	$x+1$	x^2+x	x^2+1
x^2+x	0	x^2+x	x^2+x+1	1
x^2+x+1	0	x^2+x+1	x^2+1	x

From Theorem 1, D_1 is a $D(2^3, 2^2, 2^3)$, D_2 is the submatrix of D_1 consisting of rows $0, 1, x^2+x, x^2+x+1$, and $\phi(D_2)$ is a $D(2^2, 2^2, 2^2)$ given by

	0	**1**	***x***	***x* + 1**
0	0	0	0	0
1	0	1	x	$x+1$
x^2+x	0	x	$x+1$	1
x^2+x+1	0	$x+1$	1	x

3.2. *A* $D(2^{m+2}, 2^2, 2^{m+2})$ *containing a* $D(2^m, 2^2, 2^m)$ *with* $m \geq 2$. Let $F = \mathrm{GF}(2^{u_1})$ and $G = \mathrm{GF}(2^{u_2})$ with $u_1 = m+2$, $u_2 = m$ and $m \geq 2$. Let D_0 be the multiplication table of F. By taking columns r_1 of D_0, obtain a matrix D_1.

Collect the elements of F into two vectors:

$$g_1 = (r'_{m-2}, x^{m-1} + r'_{m-2}, x^{m+1} + r'_{m-2}, x^{m+1} + x^{m-1} + r'_{m-2})' \quad \text{and} \tag{6}$$
$$g_2 = x^m + g_1.$$

Now place the rows of D_1 in four clusters. From top to bottom, their row labels are: cluster 1 with r_{m-2} and $x^m + r_{m-2}$; cluster 2 with $x^{m-1} + r_{m-2}$ and $x^m + x^{m-1} + r_{m-2}$; cluster 3 with $x^{m+1} + r_{m-2}$ and $x^{m+1} + x^m + r_{m-2}$; and cluster 4 with $x^{m+1} + x^{m-1} + r_{m-2}$ and $x^{m+1} + x^m + x^{m-1} + r_{m-2}$. Table 2 gives $\phi(D_1)$, where for $m = 2$ or 3, the entries need to be taken modulo $p_1(x) = x^{u_1} + x + 1$ and then collapsed according to ϕ.

Take D_2 to be the submatrix of D_1 consisting of the rows labeled with r_{m-2} and $x^{m+1} + x^m + x^{m-1} + r_{m-2}$.

THEOREM 2. *Consider* D_1 *and* D_2 *constructed above*. *For* $m \geq 2$, *we have*:

TABLE 2
The matrix $\phi(D_1)$ *obtained from* D_1 *in Theorem* 2

	0	**1**	***x***	***x* + 1**
r_{m-2}	0	r_{m-2}	xr_{m-2}	$(x+1)r_{m-2}$
$x^m + r_{m-2}$	0	r_{m-2}	xr_{m-2}	$(x+1)r_{m-2}$
$x^{m-1} + r_{m-2}$	0	$x^{m-1} + r_{m-2}$	xr_{m-2}	$x^{m-1} + (x+1)r_{m-2}$
$x^m + x^{m-1} + r_{m-2}$	0	$x^{m-1} + r_{m-2}$	xr_{m-2}	$x^{m-1} + (x+1)r_{m-2}$
$x^{m+1} + r_{m-2}$	0	r_{m-2}	$(x+1) + xr_{m-2}$	$(x+1) + (x+1)r_{m-2}$
$x^{m+1} + x^m + r_{m-2}$	0	r_{m-2}	$(x+1) + xr_{m-2}$	$(x+1) + (x+1)r_{m-2}$
$x^{m+1} + x^{m-1} + r_{m-2}$	0	$x^{m-1} + r_{m-2}$	$(x+1) + xr_{m-2}$	$(x^{m-1} + x + 1) + (x+1)r_{m-2}$
$x^{m+1} + x^m + x^{m-1} + r_{m-2}$	0	$x^{m-1} + r_{m-2}$	$(x+1) + xr_{m-2}$	$(x^{m-1} + x + 1) + (x+1)r_{m-2}$

(i) *the matrix* D_1 *is a* $D(2^{m+2}, 2^2, 2^{m+2})$;
(ii) *the matrix* $\phi(D_2)$ *is a* $D(2^m, 2^2, 2^m)$.

The proof of this theorem is similar to that of Theorem 1 and therefore omitted.

EXAMPLE 4 [A $D(2^4, 2^2, 2^4)$ containing a $D(2^2, 2^2, 2^2)$]. Let $F = \mathrm{GF}(2^4)$ and $G = \mathrm{GF}(2^2)$. From Theorem 2, D_1 is a $D(2^4, 2^2, 2^4)$, D_2 is the submatrix of D_1 consisting of the rows labeled with $(0, 1, x^3+x^2+x, x^3+x^2+x+1)'$, and $\phi(D_2)$ is a $D(2^2, 2^2, 2^2)$ given by

	0	**1**	$\boldsymbol{x}$	$\boldsymbol{x+1}$
0	0	0	0	0
1	0	1	x	$x+1$
x^3+x^2+x	0	x	$x+1$	1
x^3+x^2+x+1	0	$x+1$	1	x

3.3. *A* $D(2^{m+2}, 2^3, 2^{m+2})$ *containing a* $D(2^{m+1}, 2^3, 2^m)$ *with* $m \geq 2$. Let $F = \mathrm{GF}(2^{u_1})$ and $G = \mathrm{GF}(2^{u_2})$ with $u_1 = m+2$, $u_2 = m$ and $m \geq 2$. Let D_0 denote the multiplication table of F. By taking columns r_2 of D_0, obtain a matrix D_1.

Collect the elements of F into two vectors:

$$g_1 = (r'_{m-2}, x^{m-1} + r'_{m-2}, x^{m+1} + r'_{m-2}, x^{m+1} + x^{m-1} + r'_{m-2})' \quad \text{and}$$
$$g_2 = x^m + g_1. \tag{7}$$

Now place the rows of D_1 in four clusters. From top to bottom, their row labels are: cluster 1 with r_{m-2} and $x^m + r_{m-2}$; cluster 2 with $x^{m-1} + r_{m-2}$ and $x^m + x^{m-1} + r_{m-2}$; cluster 3 with $x^{m+1} + r_{m-2}$ and $x^{m+1} + x^m + r_{m-2}$; and cluster 4 with $x^{m+1} + x^{m-1} + r_{m-2}$ and $x^{m+1} + x^m + x^{m-1} + r_{m-2}$. Table 3 gives columns $x^2 + r_1$ of $\phi(D_1)$, where, for $m = 2$ or 3, the entries need to be taken modulus $p_1(x)$ and then collapsed according to ϕ, and

$$\alpha_1 = x^2(r'_{m-3}, r'_{m-3})',$$
$$\alpha_2 = \big((x^2+1)r'_{m-3}, x^{m-2} + (x^2+1)r'_{m-3}\big)',$$
$$\alpha_3 = \big((x^2+x)r'_{m-3}, x^{m-1} + (x^2+x)r'_{m-3}\big)',$$
$$\alpha_4 = \big((x^2+x+1)r'_{m-3}, x^{m-1} + x^{m-2} + (x^2+x+1)r'_{m-3}\big)'.$$

Take D_2 to be the submatrix of D_1 consisting of rows $r_{m-2}, x^m + x^{m-1} + r_{m-2}$, $x^{m+1} + r_{m-2}, x^{m+1} + x^m + x^{m-1} + r_{m-2}$.

TABLE 3
Columns $x^2 + r_1$ of $\phi(D_1)$ obtained from D_1 in Theorem 3

	$\mathbf{x^2}$	$\mathbf{x^2+1}$	$\mathbf{x^2+x}$	$\mathbf{x^2+x+1}$
r_{m-2}	α_1	α_2	α_3	α_4
$x^m + r_{m-2}$	$(x+1)+\alpha_1$	$(x+1)+\alpha_2$	$(x+1)+\alpha_3$	$(x+1)+\alpha_4$
$x^{m-1} + r_{m-2}$	α_1	$x^{m-1}+\alpha_2$	α_3	$x^{m-1}+\alpha_4$
$x^m + x^{m-1} + r_{m-2}$	$(x+1)+\alpha_1$	$(x^{m-1}+x+1)+\alpha_2$	$(x+1)+\alpha_3$	$(x^{m-1}+x+1)+\alpha_4$
$x^{m+1} + r_{m-2}$	$(x^2+x)+\alpha_1$	$(x^2+x)+\alpha_2$	$(x^2+1)+\alpha_3$	$(x^2+1)+\alpha_4$
$x^{m+1} + x^m + r_{m-2}$	$(x^2+1)+\alpha_1$	$(x^2+1)+\alpha_2$	$(x^2+x)+\alpha_3$	$(x^2+x)+\alpha_4$
$x^{m+1} + x^{m-1} + r_{m-2}$	$(x^2+x)+\alpha_1$	$(x^{m-1}+x^2+x)+\alpha_2$	$(x^2+1)+\alpha_3$	$(x^{m-1}+x^2+1)+\alpha_4$
$x^{m+1} + x^m + x^{m-1} + r_{m-2}$	$(x^2+1)+\alpha_1$	$(x^{m-1}+x^2+x)+\alpha_2$	$(x^2+x)+\alpha_3$	$(x^{m-1}+x^2+1)+\alpha_4$

THEOREM 3. *Consider D_1 and D_2 constructed above. For $m \geq 2$, we have*:

(i) *the matrix D_1 is a $D(2^{m+2}, 2^3, 2^{m+2})$*;
(ii) *the matrix $\phi(D_2)$ is a $D(2^{m+1}, 2^3, 2^m)$*.

Its proof is similar to that of Theorem 1 and therefore omitted.

EXAMPLE 5 [A $D(2^5, 2^3, 2^5)$ containing a $D(2^4, 2^3, 2^3)$]. Let $F = \mathrm{GF}(2^5)$ and $G = \mathrm{GF}(2^3)$. We have $g_1 = (0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1, x^4, x^4+1, x^4+x, x^4+x+1, x^4+x^2, x^4+x^2+1, x^4+x^2+x, x^4+x^2+x+1)'$ and $\alpha_1 = (0, x^2, 0, x^2)'$, $\alpha_2 = (0, x^2+1, x, x^2+x+1)'$, $\alpha_3 = (0, x^2+x, x^2, x)'$, $\alpha_4 = (0, x^2+x+1, x^2+x, 1)'$. From Theorem 3, D_1 is a $D(2^5, 2^3, 2^5)$, D_2 is the submatrix of D_1 consisting of rows $0, 1, x, x+1, x^3+x^2, x^3+x^2+1, x^3+x^2+x, x^3+x^2+x+1, x^4, x^4+1, x^4+x, x^4+x+1, x^4+x^3+x^2, x^4+x^3+x^2+1, x^4+x^3+x^2+x, x^4+x^3+x^2+x+1$, and $\phi(D_2)$ is a $D(2^4, 2^3, 2^3)$, with columns $x^2 + r_1$ of $\phi(D_2)$ given by

	x^2	x^2+1	x^2+x	x^2+x+1
0	0	0	0	0
1	x^2	x^2+1	x^2+x	x^2+x+1
x	0	x	x^2	x^2+x
$x+1$	x^2	x^2+x+1	x	1
x^3+x^2	$x+1$	x^2+x+1	$x+1$	x^2+x+1
x^3+x^2+1	x^2+x+1	x	x^2+1	0
x^3+x^2+x	$x+1$	x^2+1	x^2+x+1	1
x^3+x^2+x+1	x^2+x+1	0	1	x^2+x
x^4	x^2+x	x^2+x	x^2+1	x^2+1
x^4+1	x	$x+1$	$x+1$	x
x^4+x	x^2+x	x^2	1	$x+1$
x^4+x+1	x	1	x^2+x+1	x^2
$x^4+x^3+x^2$	x^2+1	1	x^2+x	x
$x^4+x^3+x^2+1$	1	x^2	0	x^2+1
$x^4+x^3++x^2+x$	x^2+1	$x+1$	x	x^2
$x^4+x^3+x^2+x+1$	1	x^2+x	x^2	$x+1$

3.4. *Some extensions*. Some extensions of the proposed method are considered here. Similar to Sections 3.1–3.3, we can construct the following two families of NDMs: (a) a $D(2^{m+3}, 2^3, 2^{m+3})$ containing a $D(2^{m+1}, 2^3, 2^m)$ and (b) a $D(2^{m+3}, 2^4, 2^{m+3})$ containing a $D(2^{m+2}, 2^4, 2^m)$ with $m \geq 2$. For brevity we present the case with $m = 2$, where $F = \mathrm{GF}(2^5)$ and $G = \mathrm{GF}(2^2)$. By taking columns r_3 of the multiplication table of F, we obtain a matrix D_1. Clearly, D_1 is a $D(2^5, 2^4, 2^5)$.

Collect the elements of F into

$$g_1 = (r_0', x + r_0', x^3 + r_0', x^3 + x + r_0', x^4 + r_0', x^4 + x + r_0', x^4 + x^3 + r_0', x^4 + x^3 + x + r_0')'$$

and $g_2 = x^2 + g_1$. Note that, for the columns labeled with r_2, the ith row in g_1 is the same as its counterpart in g_2. Let D_2 be the submatrix of D_1 consisting of rows $(0, 1, x^3 + x, x^3 + x + 1, x^4, x^4 + 1, x^4 + x^3 + x, x^4 + x^3 + x + 1)$. It is easy to see that $\phi(D_2)$ is a $D(2^3, 2^3, 2^2)$ given by

	0	**1**	$\boldsymbol{x}$	$\boldsymbol{x+1}$	$\boldsymbol{x^2}$	$\boldsymbol{x^2+1}$	$\boldsymbol{x^2+x}$	$\boldsymbol{x^2+x+1}$
0	0	0	0	0	0	0	0	0
1	0	1	x	$x+1$	0	1	x	$x+1$
x^3+x	0	x	0	x	$x+1$	1	$x+1$	1
x^3+x+1	0	$x+1$	x	1	$x+1$	0	1	x
x^4	0	0	$x+1$	$x+1$	x	x	1	1
x^4+1	0	1	1	0	x	$x+1$	$x+1$	x
x^4+x^3+x	0	x	$x+1$	1	1	$x+1$	x	0
x^4+x^3+x+1	0	$x+1$	1	x	1	x	0	$x+1$

Take D_3 to be the submatrix of D_1 consisting of rows $(0, 1), x^2 + (x, x+1), x^2 + (x^3, x^3+1), (x^3+x, x^3+x+1), x^2 + (x^4, x^4+1), (x^4+x, x^4+x+1), (x^4 + x^3, x^4+x^3+1), x^2 + (x^4+x^3+x, x^4+x^3+x+1)$ of D_1. It is easy to verify that $\phi(D_3)$ is a $D(2^4, 2^4, 2^2)$ with columns $x^3 + r_2$ of $\phi(D_3)$ given by

	$\boldsymbol{x^3}$	$\boldsymbol{x^3+1}$	$\boldsymbol{x^3+x}$	$\boldsymbol{x^3+x+1}$	$\boldsymbol{x^3+x^2}$	$\boldsymbol{x^3+x^2+1}$	$\boldsymbol{x^3+x^2+x}$	$\boldsymbol{x^3+x^2+x+1}$
0	0	0	0	0	0	0	0	0
1	0	1	x	$x+1$	0	1	x	$x+1$
x^2+x	$x+1$	1	$x+1$	1	$x+1$	1	$x+1$	1
x^2+x+1	$x+1$	0	1	x	$x+1$	0	1	x
x^2+x^3	1	1	1	1	x	x	x	x
x^2+x^3+1	1	0	$x+1$	x	x	$x+1$	0	1
x^3+x	x	0	x	0	1	$x+1$	1	$x+1$
x^3+x+1	x	1	0	$x+1$	1	x	$x+1$	0
x^2+x^4	$x+1$	$x+1$	0	0	1	1	x	x
x^2+x^4+1	$x+1$	x	x	$x+1$	1	0	0	1
x^4+x	0	x	$x+1$	1	x	0	1	$x+1$
x^4+x+1	0	$x+1$	1	x	x	1	$x+1$	0
x^4+x^3	x	x	1	1	$x+1$	$x+1$	0	0
x^4+x^3+1	x	$x+1$	$x+1$	x	$x+1$	x	x	$x+1$
$x^2+x^4+x^3+x$	1	$x+1$	x	0	0	x	$x+1$	1
$x^2+x^4+x^3+x+1$	1	x	0	$x+1$	0	$x+1$	1	x

The proposed method can be extended to construct NDMs with $p = 3$. Note that the presentation of this extension is more involved because the irreducible polynomials for GF(3^u), $u \geq 1$, do not have a unified form. [In contrast, we can use $p(x) = x^u + x + 1$ for any GF(2^u), $u \geq 1$.] For brevity we provide examples from a useful family: a $D(3^{m+1}, 3^2, 3^{m+1})$ containing a $D(3^m, 3^2, 3^m)$ with $m \geq 1$.

Here let $r_{-1} = (0)$, $r_0 = (0, 1, 2)'$, $r_m = (0, 1, 2, x, x+1, x+2, \ldots, 2x^m, 2x^m + 1, \ldots, 2x^m + 2x^{m-1} + \cdots + 2x + 2)'$ with $m \geq 1$. Note that here r_m has 3^{m+1} elements. Let $F = \mathrm{GF}(3^{u_1})$ with an irreducible polynomial $p_1(x)$ and $G = \mathrm{GF}(3^{u_2})$ with an irreducible polynomial $p_2(x)$, where $u_1 = m + 1$, $u_2 = m$ and $m \geq 1$. Let D_0 be the multiplication table of F. By taking columns r_1 of D_0, obtain a matrix D_1. Clearly, D_1 is a $D(3^{m+1}, 3^2, 3^{m+1})$.

Collect the elements of F into three vectors:

$$g_1 = (r'_{m-2}, x^{m-1} + r'_{m-2}, 2x^{m-1} + r'_{m-2})',$$
$$g_2 = x^m + g_1 \quad \text{and} \quad g_3 = 2x^m + g_1. \tag{8}$$

As a consequence of this grouping scheme, for any column labeled with r_0 in $\phi(D_1)$, the rows labeled with g_1 are the same as those labeled with g_2 or those labeled with g_3. This convenient structure implies that we only need to focus on columns $x + r_0$ and $2x + r_0$ in the construction. The key is to find a subset of D_1 in which these columns are uniform in G. Some examples are given.

EXAMPLE 6 [A $D(3^3, 3^2, 3^3)$ containing a $D(3^2, 3^2, 3^2)$]. Let $F = \mathrm{GF}(3^3)$ with $p_1(x) = x^3 + 2x + 1$ and $G = \mathrm{GF}(3^2)$ with $p_2(x) = x^2 + x + 2$. Let D_0 be the multiplication table of F. By taking columns r_1 of D_0, obtain a matrix D_1 which is a $D(3^3, 3^2, 3^3)$. Let D_2 be the submatrix of D_1 consisting of rows $r_0, 2x^2 + x + r_0$ and $x^2 + 2x + r_0$. It is easy to see that $\phi(D_2)$ is a $D(2^3, 2^2, 2^3)$ with columns $x, x+1, x+2, 2x, 2x+1, 2x+2$ given by

	$\boldsymbol{x}$	$\boldsymbol{x+1}$	$\boldsymbol{x+2}$	$\boldsymbol{2x}$	$\boldsymbol{2x+1}$	$\boldsymbol{2x+2}$
0	0	0	0	0	0	0
1	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$
2	$2x$	$2x+2$	$2x+1$	x	$x+2$	$x+1$
$2x^2+x$	$2x+1$	1	$x+1$	$x+2$	$2x+2$	2
$2x^2+x+1$	1	$x+2$	$2x$	2	x	$2x+1$
$2x^2+x+2$	$x+1$	$2x$	2	$2x+2$	1	x
x^2+2x	$x+2$	2	$2x+2$	$2x+1$	$x+1$	1
x^2+2x+1	$2x+2$	x	1	$x+1$	2	$2x$
x^2+2x+2	2	$2x+1$	x	1	$2x$	$x+2$

EXAMPLE 7 [A $D(3^4, 3^2, 3^4)$ containing a $D(3^3, 3^2, 3^3)$]. Let $F = \mathrm{GF}(3^4)$ with $p_1(x) = x^4 + x + 2$ and $G = \mathrm{GF}(3^3)$ with $p_2(x) = x^3 + 2x + 1$. Let D_0 be the multiplication table of F. By taking columns r_1 of D_0, obtain a matrix D_1, which is a $D(3^4, 3^2, 3^4)$. Let D_2 be the submatrix of D_1 consisting of rows r_1, $2x^3+x^2+r_1$ and $x^3+2x^2+r_1$. Columns $x, x+1, x+2, 2x, 2x+1, 2x+2$ of $\phi(D_2)$ are given by

	x	$x+1$	$x+2$
r_1	xr_1	$(x+1)r_1$	$(x+2)r_1$
$2x^3+x^2+r_1$	$(x+2)+xr_1$	$(x^2+x+2)+(x+1)r_1$	$(2x^2+x+2)+(x+2)r_1$
$x^3+2x^2+r_1$	$(2x+1)+xr_1$	$(2x^2+2x+1)+(x+1)r_1$	$(x^2+2x+1)+(x+2)r_1$
	$2x$	$2x+1$	$2x+2$
r_1	$2xr_1$	$(2x+1)r_1$	$(2x+2)r_1$
$2x^3+x^2+r_1$	$(2x+1)+2xr_1$	$(x^2+2x+1)+(2x+1)r_1$	$(2x^2+2x+1)+(2x+2)r_1$
$x^3+2x^2+r_1$	$(x+2)+2xr_1$	$(2x^2+x+2)+(2x+1)r_1$	$(x^2+x+2)+(2x+2)r_1$

Similar to Lemma 1 it is easy to show that any two of $(x+1)r_1$, $(x+2)r_1$ and $(2x+1)r_1$ are disjoint and the union of the three is r_2. Hence the columns of $\phi(D_2)$ are uniform in r_2 and $\phi(D_2)$ is a $D(3^3, 3^2, 3^3)$.

4. Constructing nested orthogonal arrays with Kronecker products. In this section we present a general approach to constructing NOAs. It generates an NOA by taking the Kronecker product of an NDM and a standard OA. Let δ be either ϕ in (2) or φ in (3) unless stated otherwise.

The following lemma [Bose and Bush (1952)] says that taking the Kronecker product of an OA and a DM gives a larger OA.

LEMMA 2. *If D is a $D(b, c, s)$ and A is an* OA(n, k, s), *and both are based on the same abelian group $\mathcal{A}$, then the array $H = A \otimes D$ is an* OA(nb, kc, s).

For $\delta(A \otimes D)$, we have:

LEMMA 3. *If D is a $D(b, c, s)$ and A is an* OA(n, k, s), *and both are based on* GF(s), *then*

$$\delta(A \otimes D) = \delta(A) \otimes \delta(D). \tag{9}$$

This lemma can be readily verified by using the definition of δ and (4). It basically says the two operations δ and $\otimes$ in (9) are interchangeable, which is key to the constructions to be proposed later.

Now recall a classical result from Addleman and Kempthorne (1961).

LEMMA 4. *If a factor in an OA has s_1 levels and $s_2|s_1$, then it can be replaced by a new factor with s_2 levels by partitioning the s_1 symbols into s_2 groups of size*

s_1/s_2 and by replacing the symbols in the same group with a common symbol. The resulting array is still an OA.

Note that if the s_1 and s_2 levels in this lemma come from $\mathrm{GF}(s_1)$ and $\mathrm{GF}(s_2)$, respectively, the condition $s_2|s_1$ clearly holds and the required level collapsing can be done through using δ.

Here is a similar result for difference matrices.

LEMMA 5. *If D is a $D(b, c, s_1)$ based on $\mathrm{GF}(s_1)$, then $\delta(D)$ is a $D(b, c, s_2)$.*

This lemma can be readily proved by following the definitions of difference matrices and δ.

Now we are ready to present the details of the proposed construction. Let A be an $\mathrm{OA}(n, m, s_1)$ based on $\mathrm{GF}(s_1)$. Let (D_1, D_2, δ) be an NDM constructed in Section 3, where D_1 is a $D(b_1, c, s_1)$ based on $\mathrm{GF}(s_1)$ and D_2 is a submatrix of D_1, and $\delta(D_2)$ is a $D(b_2, c, s_2)$ based on $\mathrm{GF}(s_2)$. Put

$$H_1 = A \otimes D_1 \quad \text{and} \quad H_2 = A \otimes D_2. \tag{10}$$

THEOREM 4. *For H_1 and H_2 in (10), the array (H_1, H_2, δ) is an NOA, where H_1 is an $\mathrm{OA}(nr_1, mc, s_1)$, H_2 is a submatrix of H_1 and $\delta(H_2)$ is an $\mathrm{OA}(nr_2, mc, s_2)$.*

This theorem can be readily verified by following Lemmas 2–5.

EXAMPLE 8. Let $p = 2$, $u_1 = 3$, $u_2 = 2$, giving $s_1 = 8$ and $s_2 = 4$. Take an $\mathrm{NDM}(D_1, D_2, \phi)$ from Example 3, where D_1 is a $D(8, 4, 8)$ and $\phi(D_2)$ is a $D(4, 4, 4)$. The projection ϕ is as follows: $\{0, x^2\} \to 0$, $\{1, x^2 + 1\} \to 1$, $\{x, x^2 + x\} \to x$, $\{x + 1, x^2 + x + 1\} \to x + 1$. Let A be a trivial orthogonal array $\mathrm{OA}(8, 1, 8)$, the column vector listing all elements of $\mathrm{GF}(8)$. From Theorem 4, H_1 is an $\mathrm{OA}(64, 4, 8)$, H_2 is a submatrix of H_1, and $\phi(H_2)$ is an $\mathrm{OA}(32, 4, 4)$.

Note that the construction (10) is not restricted to use NDMs from Section 3. Here is an example.

EXAMPLE 9. Let D_0 be the $D(12, 12, 4)$ [Seberry (1979)] given in the Appendix. Take D_1 to be the submatrix of D_0 consisting of columns $1, 3, 4$ and 5. It can be verified that D_1 is a $D(12, 4, 4)$. Take D_2 to be the submatrix of D_1 consisting of rows 1,2,4,5. Let δ be a projection by deleting the first digits of the entries in D_2. Clearly, $\delta(D_2)$ is a $D(4, 4, 2)$. Let A be the $\mathrm{OA}(64, 21, 4)$ constructed by using the Rao–Hamming method (HSS). Put $H_1 = A \otimes D_1$ and $H_2 = A \otimes D_2$.

Then the array (H_1, H_2, δ) is an NOA, where H_1 is an OA(768, 84, 4), H_2 is a submatrix of H_1 and $\delta(H_2)$ is an OA(256, 84, 2).

5. Obtaining new nested orthogonal arrays from existing ones. As a modification of the method in the previous section, we discuss here a procedure for obtaining new NOAs from existing ones. Let (A_1, A_2, φ) be an arbitrary NOA constructed in QTW, where A_1 is an OA(n_1, m, s_1), A_2 is a submatrix of A_1 and $\varphi(A_2)$ is an OA(n_2, m, s_2). Let D be a $D(b, c, s_1)$ based on GF(s_1). Put

$$H_1 = A_1 \otimes D \quad \text{and} \quad H_2 = A_2 \otimes D. \tag{11}$$

THEOREM 5. *For H_1 and H_2 in* (11), *we have*:

(i) *the matrix H_1 is an* OA$(n_1 b, mc, s_1)$;
(ii) *the matrix H_2 is a submatrix of H_1 and $\varphi(H_2)$ is an* OA$(n_2 b, mc, s_2)$.

This theorem can be readily verified by following Lemmas 2, 3 and 5.

EXAMPLE 10. Let $p = 2$, $u_1 = 3$, $u_2 = 2$, giving $s_1 = 8$ and $s_2 = 4$. We use $p_1(x) = x^3 + x + 1$ for GF(8) and $p_2(x) = x^2 + x + 1$ for GF(4). The condition $2u_2 \le u_1 + 1$ is satisfied and the projection φ is as follows. $\{0, x^2 + x + 1\} \to 0$, $\{1, x^2 + x\} \to 1$, $\{x, x^2 + 1\} \to x$, $\{x + 1, x^2\} \to x + 1$. Take an NOA(A_1, A_2, φ) from Section 2.3, where A_1 is an OA(64, 5, 8) and A_2 is the following submatrix of A_1

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & x & x+1 \\
x & 0 & x & x^2 & x^2+x \\
x+1 & 0 & x+1 & x^2+x & x^2+1 \\
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & x+1 & x \\
x & 1 & x+1 & x^2+1 & x^2+x+1 \\
x+1 & 1 & x & x^2+x+1 & x^2 \\
0 & x & x & x & x \\
1 & x & x+1 & 0 & 1 \\
x & x & 0 & x^2+x & x^2 \\
x+1 & x & 1 & x^2 & x^2+x+1 \\
0 & x+1 & x+1 & x+1 & x+1 \\
1 & x+1 & x & 1 & 0 \\
x & x+1 & 1 & x^2+x+1 & x^2+1 \\
x+1 & x+1 & 0 & x^2+1 & x^2+x
\end{bmatrix}.$$

We have $\varphi(A_2)$ is an OA(16, 5, 4). Let D be the multiplication table of GF(8). Then D is a $D(2^3, 2^3, 2^3)$ and $\varphi(D)$ is a $D(2^3, 2^3, 2^2)$ given by

	0	**1**	$\boldsymbol{x}$	$\boldsymbol{x+1}$	$\boldsymbol{x^2}$	$\boldsymbol{x^2+1}$	$\boldsymbol{x^2+x}$	$\boldsymbol{x^2+x+1}$
0	0	0	0	0	0	0	0	0
1	0	1	x	$x+1$	0	1	x	$x+1$
x	0	x	0	x	$x+1$	1	$x+1$	1
$x+1$	0	$x+1$	x	1	$x+1$	0	1	x
x^2	0	0	$x+1$	$x+1$	x	x	1	1
x^2+1	0	1	1	0	x	$x+1$	$x+1$	x
x^2+x	0	x	$x+1$	1	1	$x+1$	x	0
x^2+x+1	0	$x+1$	1	x	1	x	0	$x+1$

From Theorem 5, H_1 is an OA(512, 40, 8), H_2 is a submatrix of H_1, and $\varphi(H_2)$ is an OA(128, 40, 8).

6. Construction of nested orthogonal arrays with nonprime power number of levels. In this section, we construct NOAs with nonprime power number of levels. This construction complements the methods in the previous two sections, where NOAs with prime power number of levels are constructed. First we introduce a simple projection, denoted by ρ_a, for any integer $a \geq 1$, to be

$$\rho_a(u) = u(\operatorname{mod} a). \tag{12}$$

The following lemma gives some properties of ρ_a:

LEMMA 6. (i) *If $a, b \geq 1$ are integers with $b|a$, then $\rho_b(\rho_a(u)) = \rho_b(u)$;* (ii) *for any integer $a \geq 1$, $\rho_a(u_1 + u_2) = \rho_a(\rho_a(u_1) + \rho_a(u_2))$.*

We now use this projection to construct a family of NOAs based on the *zero-sum array* (HSS). For an integer s, let $\mathbb{Z}$ denote the residue classes modulo s. Let $s_1, s_2 \geq 1$ be integers with $s_2|s_1$. Let F denote $\mathbb{Z}_{s_1}$ and G denote $\mathbb{Z}_{s_2}$. Obtain an $s_1^2 \times 3$ matrix A_1, where the first two columns have each of the s_1^2 possible 2-tuples from $F \times F$ as a row, and for row (i, j) in the first two columns, its corresponding entry in the third column is taken as $-(i + j)(\operatorname{mod} s_1)$. Take A_2 to be the submatrix of A_1 consisting of rows (i, j), $0 \leq i, j \leq s_2 - 1$, in the first two columns.

THEOREM 6. *For A_1 and A_2 constructed above, we have*:

(i) *the matrix A_1 is an* OA$(s_1^2, 3, s_1)$;
(ii) *the matrix A_2 is a submatrix of A_1 and $\rho_{s_2}(A_2)$ is an* OA$(s_2^2, 3, s_2)$.

This theorem can be readily verified by following Lemma 6.

As a straightforward extension of Theorem 5, we can take the Kronecker product of an NOA from Theorem 6 and a standard DM to obtain a new NOA. By extending Theorem 4, we can take the Kronecker product of an NDM with nonprime power number of levels and a standard OA to obtain an NOA. Here is an example.

EXAMPLE 11. Obtain a matrix D_1 by suppressing the first digits of all entries of the $D(12,6,12)$ in the Appendix. It can be verified that D_1 is a $D(12,6,6)$. Let D_2 be the submatrix of D_1 consisting of rows 1, 4, 5, 6, 8 and 12. Clearly, $\rho_3(D_2)$ is a $D(6,6,3)$. Let A be the OA(36, 3, 6) obtained by taking the first three columns of Table 7C.8 in Wu and Hamada (2000). Put $H_1 = A \otimes D_1$ and $H_2 = A \otimes D_2$. The array (H_1, H_2, ρ_3) is an NOA, where H_1 is an OA(432, 18, 6), H_2 is a submatrix of H_1 and $\rho_3(H_2)$ is an OA(216, 18, 3).

7. Construction of nested orthogonal arrays with mixed levels. In this section, we discuss the issue of constructing NOAs with mixed levels. The key here is to embed nested structures in the constructions of asymmetrical (mixed) OAs, like those in Wang and Wu (1991) (referred to as WW hereinafter) and Wang (1996). Such an embedding can be done in various ways as described in the remainder of the section. We use $\mathrm{OA}(n, s_1^{\gamma_1} \cdots s_k^{\gamma_k})$ to denote an asymmetrical OA.

7.1. *Using nested orthogonal arrays and Wang–Wu method.* This construction makes use of the Kronecker products in (11) and the Wang–Wu method in WW. For $1 \le j \le v$, let s_{j1} and s_{j2} be powers of the same prime p_j with integers $u_{j1} > u_{j2} \ge 1$. The primes p_j's are assumed to be all distinct. Suppose A_1 is an $\mathrm{OA}(n_1, s_{11}^{k_1} \cdots s_{v1}^{k_v})$ and can be partitioned as

$$A_1 = [\, A_{11} \quad \cdots \quad A_{v1} \,],$$

where each A_{j1} comes from an $\mathrm{NOA}(A_{j1}, A_{j2}, \delta_j)$, A_{j1} is an $\mathrm{OA}(n_1, k_j, s_{j1})$ based on $\mathrm{GF}(s_{j1})$, A_{j2} is a submatrix of A_{j1} and $\delta_j(A_{j2})$ is an $\mathrm{OA}(n_2, k_j, s_{j2})$ based on $\mathrm{GF}(s_{j2})$.

For $1 \le j \le v$, let $D(j)$ denote a $D(b, c_j, s_{j1})$ with entries from $\mathrm{GF}(s_{j1})$. Put

$$H_1 = [A_{11} \otimes D(1) \cdots A_{v1} \otimes D(v) B_1]$$

and

$$H_2 = [A_{12} \otimes D(1) \cdots A_{v2} \otimes D(v) B_2],$$

where $C = (0, \ldots, b-1)'$, $B_1 = (C', \ldots, C')'$ represents a b-level factor with C appearing n_1 times and $B_2 = (C', \ldots, C')'$ represents a b-level factor with C appearing n_2 times.

THEOREM 7. *For H_1 and H_2 constructed above, we have*:

(i) *the matrix H_1 is an* $\mathrm{OA}(bn_1, b^1 s_{11}^{k_1 c_1} \cdots s_{v1}^{k_v c_v})$;

(ii) *the matrix H_2 is a submatrix of H_1 and H_2 is an* $\mathrm{OA}(bn_2, b^1 s_{12}^{k_1 c_1} \cdots s_{v2}^{k_v c_v})$ *after the levels of the s_{j1}-level factors are collapsed according to δ_j for $j = 1, \ldots, v$.*

This theorem can be readily verified by following the result on the generalized Kronecker product in WW and the definition of δ_j.

EXAMPLE 12 [An OA$(288, 6^6 4^{12})$ containing an OA$(72, 3^6 2^{12})$]. Let A_1 be an OA$(24, 6^1 4^1)$ formed by taking all level combinations of a factor at six levels, $0, 1, 2, 3, 4, 5$, and a factor at four levels, $00, 01, 10, 11$. Take A_2 to be the subarray of A_1 consisting of all level combinations of $0, 1, 2$ and $00, 01$. Let D_1 be the $D(12, 6, 6)$ and D_2 the $D(12, 12, 4)$ from the Appendix. Put $H_1 = [A_{11} \otimes D_1, A_{21} \otimes D_2]$ and $H_2 = [A_{12} \otimes D_1, A_{22} \otimes D_2]$. From Theorem 7, H_1 is an OA$(288, 6^6 4^{12})$ and H_2 becomes an OA$(72, 3^6 2^{12})$ after the following level collapsing: for the 6-level factors, using $\{0, 3\} \to 0$, $\{1, 4\} \to 1$, $\{2, 5\} \to 2$; for the 4-level factors, deleting the first digit and retaining the second, for example, both 01 and 11 are projected to 1.

7.2. *Using nested difference matrices and Wang–Wu method*. This construction makes use of the Kronecker products in (10) and the Wang–Wu method in WW.

For $1 \le j \le v$, let s_{j1} and s_{j2} be powers of the same prime p_j with integers $u_{j1} > u_{j2} \ge 1$. The primes p_j's are assumed to be all different. Suppose A is an OA$(n, s_{11}^{k_1} \cdots s_{v1}^{k_v})$ and can be partitioned as

$$A = [\, A_1 \quad \cdots \quad A_v \,],$$

where A_j is an OA(n, k_j, s_{j1}) based on GF(s_{j1}). Let D be a partitioned matrix

$$[\, D_1(1) \quad \cdots \quad D_1(v) \,],$$

where $D_1(j)$ comes from an NDM$(D_1(j), D_2(j), \delta_j)$, $D_1(j)$ is a $D(b_1, c_j, s_{j1})$ based on GF(s_{j1}), $D_2(j)$ is a submatrix of $D_1(j)$, and $\delta_j(D_2(j))$ is a $D(b_2, c_j, s_{j2})$ based on GF(s_{j2}).

Put

$$H_1 = [A_1 \otimes D_1(1) \cdots A_v \otimes D_1(v) B_1]$$

and

$$H_2 = [A_1 \otimes D_2(1) \cdots A_v \otimes D_2(v) B_2],$$

where $C_1 = (0, \ldots, b_1 - 1)'$, $B_1 = (C_1', \ldots, C_1')'$ represents a b_1-level factor with C_1 appearing n times, $C_2 = (0, \ldots, b_2 - 1)'$ is a subvector of C_1 with b_2 elements and $B_2 = (C_2', \ldots, C_2')'$ represents a b_2-level factor with C_2 appearing n times.

THEOREM 8. *For H_1 and H_2 constructed above, we have*:

(i) *the matrix H_1 is an* OA$(b_1 n, b_1^1 s_{11}^{k_1 c_1} \cdots s_{v1}^{k_v c_v})$;

(ii) *the matrix* H_2 *is a submatrix of* H_1 *and* H_2 *becomes an* $\mathrm{OA}(b_2 n, b_2^1 s_{12}^{k_1 c_1} \cdots s_{v2}^{k_v c_v})$ *after the levels of the* s_{j1}*-level factors are collapsed according to* δ_j *for* $j = 1, \ldots, v$.

This theorem can be readily verified by following the result on the generalized Kronecker product in WW and the definition of δ_j.

7.3. *Using a nested nonorthogonal mixed matrix and a special mixed difference matrix.* Wang (1996) constructs an asymmetrical OA using a mixed DM and a nonorthogonal matrix with mixed levels. Unlike the Wang–Wu method, this construction does not use OAs and therefore can give asymmetrical OAs with more flexible run sizes. Here we modify it to construct NOAs with mixed levels.

For $j = 1, 2$, let s_{j1} and s_{j2} be powers of the same prime p_j with integers $u_{j1} > u_{j2} \geq 1$. For $j = 1, 2$, choose an $\mathrm{NDM}(D_{j1}, D_{j2}, \delta_j)$, where D_{j1} is a $D(n_1, k_j, s_{j1})$ based on $\mathrm{GF}(s_{j1})$, D_{j2} is a submatrix of D_{j1} and $\delta_j(D_{j2})$ is a $D(n_2, k_j, s_{j2})$ with entries from $\mathrm{GF}(s_{j2})$. Construct an $\mathrm{NDM}(D_{01}, D_{02}, \delta_0)$, where $\delta_0 = \delta_1 \times \delta_2$, D_{01} is a $D(n_1, k_0, s_{11}s_{21})$ based on $\mathrm{GF}(s_{11}) \times \mathrm{GF}(s_{21})$, D_{02} is a submatrix of D_{01} and $\delta_0(D_{02})$ is a $D(n_2, k_0, s_{12}s_{22})$ based on $\mathrm{GF}(s_{12}) \times \mathrm{GF}(s_{22})$. For $j = 1, 2$, let $\sigma_j(\cdot)$ denote the operation of taking the jth component of every entry in a matrix whose entries are represented by two digits. For $j = 1, 2$, further assume the augmented matrix $[\sigma_j(D_{01}), D_{j1}]$ is a $D(n_1, k_0 + k_j, s_{j1})$ and $\delta_j[\sigma_j(D_{02}), D_{j2}]$ is a $D(n_2, k_0 + k_j, s_{j2})$. For $j = 1, 2$, let C_j be the column vector comprising all level combinations of $\mathrm{GF}(s_{1j})$ and $\mathrm{GF}(s_{2j})$.

Put

$$H_1 = [C_1 \otimes D_{01}, \sigma_1(C_1) \otimes D_{11}, \sigma_2(C_1) \otimes D_{21}] \quad \text{and}$$
$$H_2 = [C_2 \otimes D_{02}, \sigma_1(C_2) \otimes D_{12}, \sigma_2(C_2) \otimes D_{22}]. \tag{13}$$

THEOREM 9. *For* H_1 *and* H_2 *in* (13), *we have*:

(i) *the matrix* H_1 *is an* $\mathrm{OA}(n_1 s_{11} s_{21}, (s_{11}s_{21})^{k_0} s_{11}^{k_1} s_{21}^{k_2})$;

(ii) *the matrix* H_2 *becomes an* $\mathrm{OA}(n_2 s_{12} s_{22}, (s_{12}s_{22})^{k_0} s_{12}^{k_1} s_{22}^{k_2})$ *after the levels of the* $s_{11}s_{21}$*-level factors are collapsed according to* δ_0 *and the levels of* s_{j1}*-level factors are collapsed according to* δ_j *for* $j = 1, 2$.

This theorem can be readily verified by following the theorem in Wang (1996) and the definition of δ_j.

We now give a simple method for constructing a type of matrix $D = [D_0, D_1, D_2]$ required by the preceding theorem. (As a side note, this construction is related to Problem 6.17 in HSS, page 144.) For $j = 1, 2$, take D_j to be a $D(b_j, c_j, s_j)$ based on $\mathrm{GF}(s_j)$. Let c_0 be any integer between 1 and $\min(c_1, c_2)$. For $j = 1, 2$, partition D_j as $D_j = [D_{j0}, D_{j1}]$, where both D_{10} and D_{20} have c_0 columns. Let $\alpha_{i,k}$ denote the (i, k)th entry of D_1 and $\beta_{j,k}$ the (j, k)th entry of D_2. Construct a

$b_1b_2 \times c_0$ matrix D_0 whose entry in the $((i-1)b_2+j)$th row and kth column is $(\alpha_{i,k}, \beta_{j,k})$, $1 \le i \le b_1$, $1 \le j \le b_2$, $1 \le k \le c_0$. Define $D = [D_0, D_{11}^*, D_{21}^*]$, where the $((i-1)b_2+j)$th row of D_{11}^* is the ith row of D_{11} and the $((i-1)b_2+j)$th row of D_{21}^* is the jth row of D_{21}.

LEMMA 7. *For D constructed above, we have*:

(i) *the matrix D_0 is a difference matrix $D(b_1b_2, c_0, s_1s_2)$*;
(ii) *for $j = 1, 2$, the matrix D_{j1}^* is a difference matrix $D(b_1b_2, c_j - c_0, s_j)$*;
(iii) *for $j = 1, 2$, $[\sigma_j(D_0), D_{j1}^*]$ is a difference matrix $D(b_1b_2, c_j, s_j)$.*

EXAMPLE 13 [An OA$(144, 12^2 4^2 3^1)$ containing an OA$(72, 6^2 2^2 3^1)$]. Let C_{10} be the vector listing all level combinations of GF(4) and GF(3). Denote by $0, 1, x, x+1$ the elements of GF(4) are $0, 1, 2$ the elements of GF(3). For $j = 1, 2$, let C_{1j} be the column vector listing all jth digits of C_{10}. The transpose of the matrix $C_1 = (C_{10}, C_{11}, C_{12})$ is given by

$$\begin{bmatrix} 00 & 01 & 02 & 10 & 11 & 12 & x0 & x1 & x2 & (x+1)0 & (x+1)1 & (x+1)2 \\ 0 & 0 & 0 & 1 & 1 & 1 & x & x & x & x+1 & x+1 & x+1 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix}.$$

Let $C_2 = (C_{20}, C_{21}, C_{22})$ be the submatrix of C_1 consisting of the first six rows. Take D_1 to be the following $D(4, 4, 4)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & x & x+1 \\ 0 & x & x+1 & 1 \\ 0 & x+1 & 1 & x \end{bmatrix}$$

and D_2 to be the following $D(3, 3, 3)$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

From Lemma 7, $D = [D_0, D_1, D_2]$ is

$$\begin{bmatrix} 00 & 00 & 0 & 0 & 0 \\ 00 & 01 & 0 & 0 & 2 \\ 00 & 02 & 0 & 0 & 1 \\ 00 & 10 & x & x+1 & 0 \\ 00 & 11 & x & x+1 & 2 \\ 00 & 12 & x & x+1 & 1 \\ 00 & x0 & x+1 & 1 & 0 \\ 00 & x1 & x+1 & 1 & 2 \\ 00 & x2 & x+1 & 1 & 1 \\ 00 & (x+1)0 & 1 & x & 0 \\ 00 & (x+1)1 & 1 & x & 2 \\ 00 & (x+1)2 & 1 & x & 1 \end{bmatrix}.$$

Put $H_1 = [C_{10} \otimes D_0, C_{11} \otimes D_1, C_{12} \otimes D_2]$ and $H_2 = [C_{20} \otimes D_0, C_{21} \otimes D_1, C_{22} \otimes D_2]$. From Theorem 9, H_1 is an OA(144, $12^2 4^2 3^1$) and H_2 becomes an OA(72, $6^2 2^2 3^1$) after the following level collapsing: for the 12-level factors, using $\{00, x0\} \to 00$, $\{10, (x+1)0\} \to 10$, $\{01, x1\} \to 01$, $\{11, (x+1)1\} \to 11$, $\{02, x2\} \to 02$, $\{12, (x+1)2\} \to 12$; for the 4-level factors, using $\{0, 2\} \to 0$ and $\{1, 3\} \to 1$.

8. Generation of nested space-filling designs. In this section, we discuss the problem of using NOAs to generate NSFDs. Throughout, we assume the factors are quantitative and each of them takes values in the interval [0, 1]. When we say that a design is space-filling or achieves uniformity in low dimensions, we mean that, when projected onto low dimensions, the design points are evenly scattered in the design region. For this problem, we present an approach following the procedure in QTW used for the same problem. Unlike QTW, the present approach covers both NOAs with equal levels and with mixed levels. Consider an NOA(H_1, H_2), where H_1 is an OA($n_1, s_{11}^{\gamma_1} \cdots s_{k1}^{\gamma_k}$) with $m = \sum_{i=1}^{k} \gamma_i$, H_2 is a submatrix of H_1 and H_2 becomes an OA($n_2, s_{12}^{\gamma_1} \cdots s_{k2}^{\gamma_k}$) after the levels of the s_{j1}-level factors are collapsed into s_{j2} levels according to a projection δ_j. If $k = 1$, this array reduces to an NOA with equal levels.

The first step in constructing an OA-based Latin hypercube design D_l using H_1 is to relabel the s_{j1} levels of H_1, currently represented by the elements of a Galois field (or other mathematical structures), as $1, \ldots, s_{j1}$. Note that the projection δ_j divides the s_{j1} levels into s_{j2} groups, each of size $e_j = s_{j1}/s_{j2}$, and two levels belong to the same group if their projected values match. To ensure that the subset of D_l corresponding to H_2 has good space-filling properties, we label the s_{j1} levels of any s_{j1}-level factor in H_1 in such a way that the group of levels that are mapped to the same level should form a consecutive subset of $\{1, \ldots, s_{j1}\}$. The s_{j2} groups are arbitrarily labeled as groups $1, \ldots, s_{j2}$, and the e_j levels within the ith group are arbitrarily labeled as $(i-1)e_j + 1, \ldots, (i-1)e_j + e_j$ for $i = 1, \ldots, s_{j2}$.

After labeling the levels of the s_j-level factors of H_1 as $1, \ldots, s_{j1}$, $j = 1, \ldots, k$, as discussed above, we now use this array to obtain an OA-based Latin hypercube design as described in Section 2.1. Let D_l denote the set of points and D_h be the subset of D_l corresponding to H_2. Then (i) D_l achieves maximum uniformity in one dimension and, when D_l is projected onto the dimensions of an s_{j1}-level factor and an s_{k1}-level factor, the points achieve uniformity on $s_{j1} \times s_{k1}$ grids; and (ii) D_h is a subset of D_l and, when D_l is projected onto the dimensions of an s_{j1}-level factor and an s_{k1}-level factor, the points achieve uniformity on $s_{j2} \times s_{k2}$ grids.

An example is given to illustrate the above procedure.

EXAMPLE 14. Consider the NOA in Example 8, where H_1 is an OA(64, 4, 8), H_2 is a submatrix of H_1 and $\phi(H_2)$ is an OA(32, 4, 4). The four groups of levels of H_1 are $\{0, x^2\}$, $\{1, x^2+1\}$, $\{x, x^2+x\}$ and $\{x+1, x^2+x+1\}$. We label $\{0, x^2\}$ as levels 1 and 2, $\{1, x^2+1\}$ as levels 3 and 4, $\{x, x^2+x\}$ as levels

TABLE 4
The H_1 matrix in Example 14

Run #	x_1	x_2	x_3	x_4	Run #	x_1	x_2	x_3	x_4
1	1	1	1	1	33	2	2	2	2
2	1	3	5	7	34	2	4	6	8
3	1	2	7	8	35	2	1	8	7
4	1	4	3	2	36	2	3	4	1
5	1	5	2	6	37	2	6	1	5
6	1	7	6	4	38	2	8	5	3
7	1	6	8	3	39	2	5	7	4
8	1	8	4	5	40	2	7	3	6
9	3	3	3	3	41	4	4	4	4
10	3	1	7	5	42	4	2	8	6
11	3	4	5	6	43	4	3	6	5
12	3	2	1	4	44	4	1	2	3
13	3	7	4	8	45	4	8	3	7
14	3	5	8	2	46	4	6	7	1
15	3	8	6	1	47	4	7	5	2
16	3	6	2	7	48	4	5	1	8
17	5	5	5	5	49	6	6	6	6
18	5	7	1	3	50	6	8	2	4
19	5	6	3	4	51	6	5	4	3
20	5	8	7	6	52	6	7	8	5
21	5	1	6	2	53	6	2	5	1
22	5	3	2	8	54	6	4	1	7
23	5	2	4	7	55	6	1	3	8
24	5	4	8	1	56	6	3	7	2
25	7	7	7	7	57	8	8	8	8
26	7	5	3	1	58	8	6	4	2
27	7	8	1	2	59	8	7	2	1
28	7	6	5	8	60	8	5	6	7
29	7	3	8	4	61	8	4	7	3
30	7	1	4	6	62	8	2	3	5
31	7	4	2	5	63	8	3	1	6
32	7	2	6	3	64	8	1	5	4

5 and 6 and $\{x + 1, x^2 + x + 1\}$ as levels 7 and 8. Table 4 presents the array H_1 after using such labeling, where H_2 correspond to runs 1, 2, 7–10, 15–18, 23–26, 31–34, 39–42, 47–50, 55–58, 63–64. We then use H_1 to construct an OA-based Latin hypercube design D_l for x_1 to x_4. Now choose D_h to be the subset of D_l corresponding to H_2. The points in any bivariate projection of D_h achieve uniformity on the 4×4 grids in two dimensions. The points in the bivariate projections of D_l also achieve similar uniformity.

9. Discussions and concluding remarks. Multiple computer experiments with different levels of accuracy have become prevalent in business, engineering and science for studying complex real world systems. NSFDs are attractive for such experiments. Several methods are proposed for constructing various families of NOAs, which can be used to generate many new NSFDs. In the development of these methods, two new discrete mathematics concepts, called nested orthogonal arrays and nested difference matrices, are introduced. These concepts should be further studied in their own right.

NSFDs can also be used in validation of computer models, that is, testing the accuracy of a computer model against some field data [Bayarri et al. (2007), Kennedy and O'Hagan (2001) and Oberkampf and Trucano (2007)]. Let D_c denote the set of design points for the computer model and D_f denote the set of design points for the corresponding physical experiment used as a benchmark in the validation. Unlike the situation of D_l and D_h, D_c should have more columns than D_f because of the need of accommodating calibration (tuning) parameters that appear in the computer model only. Precisely, construction of D_c and D_f is guided by the following requirements:

(i) D_c contains all factors of D_f and has additional columns to accommodate the calibration parameters.
(ii) When restricted to the shared factors, $D_f \subset D_c$.
(iii) Both D_c and D_f have good space-filling properties.

With slight modifications, our construction methods for D_h and D_l can give D_f and D_c that satisfy the above requirements. For illustration, we modify the construction in Section 3.1, where $F = \mathrm{GF}(2^{m+1})$, $G = \mathrm{GF}(2^m)$, $m \geq 2$, D_0 is a $D(2^{m+1}, 2^{m+1}, 2^{m+1})$ and D_1 is a $D(2^{m+1}, 2^4, 2^{m+1})$. Take D_1^* to be D_0. Then D_1^* has more columns than D_1. Next replace D_1 by D_1^* and follow through the steps in Section 3.1 and the construction in (10). Let A be the $\mathrm{OA}(n, c, s_1)$ used in Theorem 4. Then we have the following results: for $m \geq 2$,

(i) the matrix D_1^* is a $D(2^{m+1}, 2^{m+1}, 2^{m+1})$;
(ii) the matrix $\phi(D_2)$ is a $D(2^m, 2^2, 2^m)$;
(iii) the matrix $H_1^* = A \otimes D_1^*$ is an $\mathrm{OA}(n2^{m+1}, 2^{m+1}c, 2^{m+1})$;
(iv) for the shared $4m_2$ factors, $H_2 = A \otimes D_2$ is a submatrix of H_1^* and $\delta(H_2)$ is an $\mathrm{OA}(n2^m, 4c, 2^m)$.

As in Section 8, we use (H_1^*, H_2) to generate a pair of nested designs for D_c and D_f, where D_c has $2^{m+1}c$ columns and D_f has $4c$ columns and both have good space-filling properties.

Extensions of the present work can be made in several directions. First, similar to the construction of OA-based Latin hypercubes designs [Tang (1993, 1994) and Leary, Bhaskar and Keane (2003)], it is possible to produce multiple NSFDs based on a given NOA. In a separate article, we plan to use both distance and correlation criteria to construct optimal NSFDs. Second, the constructed NOAs in this article have strength 2 that can guarantee uniformity in two dimensions only.

The proposed methods can be extended to produce NOAs with higher-dimensional stratification by exploring nesting in difference matrices with strength 3 or higher [Hedayat, Stufken and Su (1996)]. Another possibility is to use quasi-Monte Carlo sequences, like nets [Niederreiter (1992)]. A paper in preparation will address the issue of constructing nested nets. Third, it is worth studying the sampling properties of NSFDs. Fourth, a natural extension of the present work is to construct NSFDs for experiments with more than two levels of accuracy. One way to achieve this is to extend the method in QTW to directly construct NOAs with more sophisticated nesting, that is, a 32-run OA contains a 16-run OA that contains an 8-run OA. Another possibility is to modify the method in Section 3 to obtain NDMs with nesting at more than two levels and then use them to produce the desired NOAs. Finally, given the close connections between OAs and coding theory, it should be possible to use coding-theoretical techniques to construct new NOAs. We are currently exploring this issue.

APPENDIX

$D(12, 12, 4)$ from Seberry (1979)

00	00	00	00	00	00	00	00	00	00	00	00
00	00	00	01	01	01	11	11	11	10	10	10
00	00	00	11	11	11	10	10	10	01	01	01
00	11	01	10	01	11	01	10	00	11	00	00
00	11	01	11	10	01	00	01	10	10	11	00
00	11	01	01	11	10	10	00	01	00	10	11
00	01	10	11	00	10	01	00	11	01	11	10
00	01	10	10	11	00	11	01	00	10	01	11
00	01	10	00	10	11	00	11	01	11	10	01
00	10	11	01	10	00	01	11	10	01	00	11
00	10	11	00	01	10	10	01	11	11	01	00
00	10	11	10	00	01	11	10	01	00	11	01

$D(12, 6, 12)$ based on $(\mathbb{Z}_2 \oplus \mathbb{Z}_6, +)$ [Dulmage, Johnson and Mendelsohn (1961)]

00	00	00	00	00	00
00	01	03	12	04	10
00	02	10	01	15	12
00	03	01	15	14	02
00	04	13	05	02	11
00	05	15	13	11	01
00	10	02	03	12	13
00	11	12	14	10	15
00	12	05	02	13	04
00	13	04	11	01	14
00	14	11	10	03	05
00	15	14	04	05	03

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P. Z. G. QIAN
DEPARTMENT OF STATISTICS
UNIVERSITY OF WISCONSIN–MADISON
MADISON, WISCONSIN 53706
USA
E-MAIL: peterq@stat.wisc.edu

M. AI
LMAM, SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING 100871
CHINA
E-MAIL: myai@math.pku.edu.cn

C. F. J. WU
H. MILTON STEWART SCHOOL OF INDUSTRIAL
AND SYSTEMS ENGINEERING
GEORGIA INSTITUTE OF TECHNOLOGY
755 FERST DRIVE NW, ATLANTA
GEORGIA 30332-0205
USA
E-MAIL: jeffwu@isye.gatech.edu