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LARGE DEVIATIONS FOR INTERSECTION LOCAL TIMES IN CRITICAL DIMENSION

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Let $(X_t, t \geq 0)$ be a continuous time simple random walk on $\mathbb{Z}^d$ ($d \geq 3$), and let $l_T(x)$ be the time spent by $(X_t, t \geq 0)$ on the site x up to time T. We prove a large deviations principle for the q-fold self-intersection local time $I_T = \sum_{x \in \mathbb{Z}^d} l_T(x)^q$ in the critical case $q = \frac{d}{d-2}$. When q is integer, we obtain similar results for the intersection local times of q independent simple random walks.

1. Introduction.

Position of the problem. Let $(X_t, t \geq 0)$ be a continuous time simple random walk on $\mathbb{Z}^d$, whose generator is denoted by $\triangle$ [where $\triangle f(x) \overset{\triangle}{=} \sum_{y \sim x}(f(y) - f(x))$]. Let

$$l_T(x) = \int_0^T \delta_x(X_s)\, ds.$$

The quantity of interest in this paper is the so called q-fold self-intersection local time

$$I_T = \sum_{x \in \mathbb{Z}^d} l_T(x)^q.$$

When q is integer, then

$$I_T = q! \int_{0 \leq s_1 \leq \cdots \leq s_q \leq T} \delta_{X_{s_1} = X_{s_2} = \cdots = X_{s_q}}\, ds_1 \cdots ds_q,$$

which measures the amount of time the random walk spends on sites visited at least q-times. Quantities measuring how much a random walk does intersect itself, such as the range of the random walk, or the self-intersection local time, appear in many models in physics. Far from being exhaustive, we can cite the Polaron problem (see, for instance, [18, 30]), models of polymers (see, for instance, [8, 38–40]), or models of diffusion in random environments [3, 4, 7, 11, 12, 24, 25]. Partly motivated by the understanding of these models, many studies have been devoted to such quantities for more than twenty years. To describe the known

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TABLE 1
Typical behavior of I_T for $q=2$

d	Order of $E(I_T)$	Convergence in law	References
$d=1$	$T^{3/2}$	$\frac{I_T}{T^{3/2}} \xrightarrow{(d)} \gamma_1$	[9, 10, 13, 33]
$d=2$	$T\log(T)$	$\frac{I_T-E(I_T)}{T} \xrightarrow{(d)} \underline{\gamma}_1$	[19, 27, 28, 34, 37]
$d\geq 3$	T	$\frac{I_T-E(I_T)}{\sqrt{\operatorname{var}(I_T)}} \xrightarrow{(d)} \mathcal{N}(0,1)$,	[14, 21, 22]
		$\operatorname{var}(I_T)\sim \begin{cases}\sigma(3)T\log(T), & \text{si } d=3,\\ \sigma(d)T, & \text{si } d\geq 4,\end{cases}$	

results, we focus on I_T in the case $q=2$, where the literature is more complete, and we refer the reader to the monograph [15] in preparation for a very complete exposition of the subject, including results on the range, or intersection local times of independent random walks.

Regarding the typical behavior of I_T for large T, the results depend of course on the dimension d, and of the transience/recurrence of the random walk. They are summarized in Table 1, where γ_1 and $\underline{\gamma}_1$ are, respectively, the intersection local time and renormalized intersection local time of the Brownian motion up to time 1, and $\sigma(d)$ is a constant depending on the dimension d:

Once we know the typical behavior, on can ask for untypical ones, that is, for the large and moderate deviations for I_T. In many models, such as the Polaron problem or polymers models, this is actually the question of interest. The table below is an attempt to summarize the results for $q=2$, achieved in recent years concerning this problem.

In Table 2, $\kappa_c(2,d)$ is the best constant c in the Gagliardo–Nirenberg inequality:

$$\forall d\leq 3, \exists c\in]0,\infty[, \text{ s.t. } \forall f:\mathbb{R}^d\mapsto\mathbb{R}, \qquad \|f\|_4\leq c\|f\|_2^{1-d/4}\|\nabla f\|_2^{d/4},$$

TABLE 2
Large and moderate deviations results for I_T for $q=2$

d	$P[I_T-E(I_T)\geq b_T^2]$	Value of b_T	References
$d\leq 2$	$\exp(-\frac{2}{\kappa_c(2,d)^{8/d}}T^{(d-4)/d}b_T^{4/d})$	$T^{2-d/2}\ll b_T^2\ll T^2$	[5, 6, 13, 29, 30]
$d=3$	$\exp(-\frac{b_T^4}{2\sigma(3)T\log(T)})$	$\sqrt{T\log(T)}\ll b_T^2\ll\sqrt{T\log(T)^{3/2}}$	[15]
	$\exp(-\frac{2}{\kappa_c(2,d)^{8/d}}T^{(d-4)/d}b_T^{4/d})$	$\sqrt{T\log(T)^{3/2}}\ll b_T^2\ll T^2$	[1, 15]
$d=4$	$\exp(-\frac{b_T^4}{2\sigma(4)T})$	$\sqrt{T}\ll b_T^2\leq\sqrt{T\log\log T}$	[23]
$d\geq 5$	$\exp(-\frac{b_T^4}{2\sigma(d)T})$	$\sqrt{T}\ll b_T^2\leq\sqrt{T\log\log T}$	[23]
	$\exp(-c(d)b_T)$	$T\leq b_T^2\ll T^2$	[2, 4]

while $c(d)$ is an explicit constant related to discrete variational inequalities.

So the picture is now almost complete, except for the dimensions $d \geq 4$. Note the coexistence of two different regimes in dimensions $d = 3$ and $d \geq 5$. The first one is an extension of the central limit theorem describing the typical behavior, the second one corresponds to the same pattern than in dimension $d \leq 2$. To understand it, we give some heuristics in the general case for q, where we want to control $P[I_T - E(I_T) \geq b_T^q]$. For I_T to be atypically high, one possible strategy for the random walk is to remain during a time $\tau \leq T$, in a box of size R. If $\tau \gg R^2$, this event has a probability of order $\exp(-\tau/R^2)$. If $\tau \gg R^d$, one can expect that on the box of size R, the local time $l_\tau(x)$ is now of order τ/R^d, so that I_T has increased of an amount of order $\tau^q/R^{d(q-1)} = b_T^q$. Hence, $\tau = b_T R^{d/q'}$ where q' is the conjugate exponent of q. Therefore, this strategy has a probability of order $\exp(-b_T R^{d/q'-2})$. The best choice for R is now the choice that maximizes $\exp(-b_T R^{d/q'-2})$, under the constraint $T \geq \tau \gg R^{\max(2,d)}$.

- If $d < 2q'$ or equivalently $q < \frac{d}{(d-2)_+}$, the bigger is R, the bigger is $\exp(-b_T \times R^{d/q'-2})$, so that the best strategy for the random walk to make I_T of order b_T^q, is to remain all the time T in a ball of radius of order $(T/b_T)^{q'/d}$, leading to the result of Table 2 for $d \leq 2$ and the second regime in $d = 3$.
- If $d > 2q'$, the smaller is R, the bigger is $\exp(-b_T R^{d/q'-2})$, so that the best strategy for the random walk to make I_T of order b_T^q, is now to remain during a time τ of order b_T in a ball of radius R of order 1, leading to the second regime of Table 2 in $d \geq 5$.
- The case $d = 2q'$ is critical. In that case $\exp(-b_T R^{d/q'-2})$ does not depend on R, so that whatever the order of R, $1 \leq R \ll \sqrt{T/b_T}$, the strategy consisting to remain a time $\tau = b_T R^2$ in a ball of size R has a probability of order $\exp(-b_T)$. The critical feature of $d = 2q'$ is also reflected in the fact that the Gagliardo–Nirenberg inequality appearing in the results for $d < 2q'$, is now replaced by the Sobolev inequality. For these reasons, there is no result concerning the large and moderate deviations of I_T for $d = 2q'$.

Main results. This paper is a contribution to the large and very large deviations for I_T in the critical case $d = 2q'$. By large deviations, we mean deviations of the order of the mean $E(I_T)$, and by very large, we mean deviations of order much larger than the order of the mean. When q is an integer (i.e., when $d = 3$ and $q = 3$, or when $d = 4$ and $q = 2$), we obtain also similar results for the mutual intersection Q_T of q independent random walks $(X_t^{(i)}; t \geq 0, 1 \leq i \leq q)$, defined by:

$$Q_T = \sum_{x \in \mathbb{Z}^d} \prod_{i=1}^{q} l_T^{(i)}(x) = \int_{0 \leq s_1, \ldots, s_q \leq T} \delta_{X_{s_1}^{(1)} = X_{s_2}^{(2)} = \cdots = X_{s_q}^{(q)}} \, ds_1 \cdots ds_q,$$

where $l_T^{(i)}(x) = \int_0^T \delta_x(X_s^{(i)})\, ds$. To state our main results, we introduce some notation. For any function $f : \mathbb{Z}^d \mapsto \mathbb{R}$, $\|f\|_p$ is the l_p norm of f $[\|f\|_p^p =$

$\sum_{x\in\mathbb{Z}^d} |f|^p(x)]$, and ∇f is the discrete gradient of f [for all $j \in \{1,\dots,d\}$, for all $x \in \mathbb{Z}^d$, $\nabla_j f(x) = f(x+e_j) - f(x)$].

PROPOSITION 1. *For $d \geq 3$, let $C_S(d) \in]0;+\infty[$ be the best constant in the discrete Sobolev's inequality*

$$\forall f \in l^{2d/(d-2)}(\mathbb{Z}^d) \qquad \|f\|_{2d/(d-2)} \leq C_S(d)\|\nabla f\|_2.$$

1. *Exponential moments for I_T.*
   Let $d \geq 3$, and let $q = \frac{d}{d-2}$.

$$\text{(1)} \qquad \text{If } T^{1/q} \ll b_T, \forall \theta \in \left[0; \frac{1}{C_S^2(d)}\right[\qquad \limsup_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta I_T^{1/q})] = 0.$$

$$\text{(2)} \qquad \text{If } b_T \ll T, \forall \theta > \frac{1}{C_S^2(d)} \qquad \liminf_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta I_T^{1/q})] = +\infty.$$

2. *Exponential moments for Q_T.*
   Assume that $d = 4$ and $q = 2$, or $d = 3$ and $q = 3$.

$$\text{(3)} \qquad \text{If } T^{1/q} \ll b_T, \forall \theta \in \left[0; \frac{q}{C_S^2(d)}\right[\qquad \limsup_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta Q_T^{1/q})] = 0.$$

$$\text{(4)} \qquad \text{If } b_T \ll T, \forall \theta > \frac{q}{C_S^2(d)} \qquad \liminf_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta Q_T^{1/q})] = +\infty.$$

From Proposition 1, it is straightforward to obtain very large deviations upper bounds for I_T and Q_T. However, due to the degenerate form of the log-Laplace of $I_T^{1/q}$, the corresponding lower bounds are not a direct consequence of Proposition 1. These lower bounds are actually the main statement of the following theorem.

THEOREM 2 (Very large deviations).

1. *Very large deviations for I_T.*
   Assume that $d \geq 3$, $q = \frac{d}{d-2}$, and $T \gg b_T \gg T^{1/q}$.

$$\text{(5)} \qquad \lim_{T\to\infty} \frac{1}{b_T} \log P[I_T \geq b_T^q] = -\frac{1}{C_S^2(d)}.$$

2. *Very large deviations for Q_T.*
   Assume that $d = 4$ and $q = 2$, or $d = 3$ and $q = 3$, and that $T \gg b_T \gg T^{1/q}$.

$$\text{(6)} \qquad \lim_{T\to\infty} \frac{1}{b_T} \log P[Q_T \geq b_T^q] = -\frac{q}{C_S^2(d)}.$$

Concerning the large deviations, our result is less precise since the lower and upper bounds are different. To state it, we recall that for $d \geq 3$ and $q > 1$, $\lim_{T\to\infty} \frac{1}{T} E[I_T]$ exists in $\mathbb{R}^+$ [when q is integer, this limit is equal to $q!G_d(0)^{q-1}$, where G_d is the Green kernel of the simple random walk on $\mathbb{Z}^d$].

THEOREM 3 (Large deviations for I_T). *Assume that $d \geq 3$, $q = \frac{d}{d-2}$. There exists a constant $c(d) > 0$ such that $\forall y > c(d)$*

$$
\begin{aligned}
-\frac{y^{1/q}}{C_S^2(d)} &\leq \liminf_{T\to\infty} \frac{1}{T^{1/q}} \log P[I_T \geq Ty] \\
&\leq \limsup_{T\to\infty} \frac{1}{T^{1/q}} \log P[I_T \geq Ty] = -\frac{1}{c(d)} y^{1/q}.
\end{aligned}
\tag{7}
$$

REMARK 1. Unfortunately, our proof does not allow to obtain the result for all $y > \lim_{T\to\infty} \frac{E(I_T)}{T}$.

REMARK 2. As in Theorem 2, we could obtain similar results for Q_T. However, such a result would not correspond to a large deviations result for Q_T, since $E(Q_T)$ is of order $\log(T)$ for $d \geq 3$ and $q = d/(d-2)$. Concerning Q_T, we should also mention that papers [31] and [35] give moderate deviations estimates $P[Q_T - E(Q_T) \geq \log(T) b_T]$ for scales b_T up to $\log\log\log(T)$.

Sketch of the proof. The proof of the lower bounds is easy and relies heavily on the large deviations results for $\frac{l_T}{T}$ proved by Donsker and Varadhan. Namely, let $\mathcal{F} = \{\mu : \mathbb{Z}^d \mapsto \mathbb{R}^+; \sum_{x\in\mathbb{Z}^d} \mu(x) = 1\}$. $\mathcal{F}$ is endowed with the weak topology of probability measures. By the results of Donsker and Varadhan [17], l_T/T satisfy a restricted large deviations principle in $\mathcal{F}$ (by "restricted," it is meant that the large deviations upper bound is only true for compact sets), with rate function $\mathcal{I}(\mu) = \|\nabla\sqrt{\mu}\|_2^2$. Now, for any M satisfying $Mb_T \leq T$, $\frac{I_T}{b_T^q} \geq \frac{I_{Mb_T}}{b_T^q} = M^q \|\frac{l_{Mb_T}}{Mb_T}\|_q^q$. Moreover, the function $\mu \in \mathcal{F} \mapsto \|\mu\|_q = \sup\{\sum_x \mu(x) f(x); f \text{ compactly supported}, \|f\|_{q'} = 1\}$ is lower semicontinuous in weak topology. The large deviations lower bound for $\frac{l_{Mb_T}}{Mb_T}$ [with the change of variable $\mu(x) = g^2(x)$], leads therefore to

$$
\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{b_T} \log P[I_T > b_T^q] \\
&\qquad \geq -M \inf\left\{\|\nabla g\|_2^2; g \text{ such that } \|g\|_2 = 1 \text{ and } \|g\|_{2q}^2 > \frac{1}{M}\right\}
\end{aligned}
\tag{8}
$$

for all $M < \liminf \frac{T}{b_T}$. For $b_T \ll T$, all the values of M are allowed, and taking the supremum in M in (8) leads to the lower bound in (5). Actually, this argument remains valid for any scale b_T such that $1 \ll b_T \ll T$ (see Proposition 11).

For the very large deviations upper bound for I_T, the results of Donsker and Varadhan are not sufficient, since on one hand, the large deviations upper bound for l_T/T is only true for compact sets of $\mathcal{F}$, and on the other hand, the function $\mu \in \mathcal{F} \mapsto \|\mu\|_q$ is not continuous. We present now the main ingredients of the proof of the upper bound (1). First of all, it is easy to see that $I_T \le I_T(R)$, the intersection local time of the random walk folded on the torus of radius R. Now, the main tool in the proof is the mysterious Dynkin isomorphism theorem, according to which the law of the local times of a symmetric recurrent Markov process stopped at an independent exponential time, is related to the law of the square of a Gaussian process whose covariance function is the Green kernel of the stopped Markov process. This allows us to control the exponential moments of $I_T^{1/q}$, with the exponential moments of $N_T(R) = \frac{1}{2}(\sum_{x \in \mathbb{T}_R} Z_x^{2q})^{1/q} = \frac{1}{2}\|Z\|_{2q,R}^2$ where:

- $\mathbb{T}_R$ is the torus of radius R;
- $(Z_x, x \in \mathbb{T}_R)$ is a centered Gaussian process whose covariance function is given by $G_{R,\lambda}(x, y)$, the Green kernel of the simple random walk on $\mathbb{T}_R$, stopped at an independent exponential time with parameter $\lambda \sim b_T/T$, (Lemmas 4, 5 and 6);
- $\|\cdot\|_{2q,R}$ denotes the norm in $l^{2q}(\mathbb{T}_R)$.

We can now rely on concentration inequalities for norms of Gaussian processes. Let $M_{R,T}$ denote the median of $\|Z\|_{2q,R}$. For small α,

$$\exp\left[\frac{\theta}{2}\|Z\|_{2q,R}^2\right] \le \exp\left[\frac{\theta(1+\alpha)}{2}(\|Z\|_{2q,R} - M_{R,T})^2\right]\exp\left[\frac{\theta(1+\alpha)}{2\alpha}M_{R,T}^2\right].$$

By concentration inequalities, the tail behavior of $\|Z\|_{2q,R} - M_{R,T}$ is that of a centered Gaussian variable with variance

$$\rho = \sup\{\langle f, G_{R,\lambda} f\rangle; \|f\|_{(2q)',R} = 1\}.$$

Therefore, for $\theta < \frac{1}{(1+\alpha)\rho}$,

$$\exp\left[\frac{\theta(1+\alpha)}{2}(\|Z\|_{2q,R} - MR, T)^2\right] \le \frac{1}{\sqrt{1-\theta(1+\alpha)\rho}}.$$

Besides, one can prove that $M_{R,T}$ is of order $R^{d/(2q)}$ as soon as $\lambda R^d \gg 1$, and that $\rho \sim \frac{1}{C_S^2(d)}$ if $\lambda R^2 \gg 1$. We therefore obtain the result in (1), if R is chosen so that $b_T \gg R^{d/q}$ and $\lambda R^2 \sim \frac{b_T}{T} R^2 \gg 1$. The best choice for R is now to take $R^{d/q} = T/R^2$, i.e., $R = T^{1/d}$ since $q = \frac{d}{d-2}$, leading to $b_T \gg T^{1/q}$.

An open question. The large, very large and moderate deviations for I_T and Q_T in the subcritical case (i.e., $d \le 2$, or $d = 3$ and $q < \frac{d}{d-2}$) are linked to Gagliardo–Nirenberg inequality in a continuous setting (i.e., for functions f from $\mathbb{R}^d$ to $\mathbb{R}$), while the same problem in supercritical case $d \ge 3$ and $q > \frac{d}{d-2}$, is linked to functional inequality in a discrete setting. One can therefore think that in the critical case $q=\frac{d}{d-2}$, the moderate deviations of $I_T-E[I_T]$ are at least up to some scale, related to the Sobolev inequality in a continuous setting. However, since the best constants in the discrete and continuous Sobolev inequality are the same, this would not change the statement. Therefore, we do believe that in the critical case $d=2q'$, there are only two regimes of deviations from the mean:

$$P[I_T-E(I_T)\geq b_T^q]\asymp\begin{cases}\exp\left(-\dfrac{b_T^{2q}}{2\sigma(d)T}\right), & \text{for } \sqrt{T}\ll b_T^q\ll T^{q/(2q-1)},\\ \exp\left(-\dfrac{1}{C_S^2(d)}b_T\right), & \text{for } T^{q/(2q-1)}\ll b_T^q\ll T^q.\end{cases}$$

We do not know how to prove this result. Actually, the same question is also open in the supercritical case (with $\frac{1}{C_S^2(d)}$ replaced by the constant $c(d)$ given in [2]).

The paper is organized as follows. Section 2 is devoted to the proof of exponential moments lower bounds (2) and (4). In Section 3, we prove the exponential moments upper bounds (1) and (3). In Section 4, we give the proof of the large and very large deviations lower bounds. With Proposition 1, this ends the proof of Theorem 2. Finally, Section 5 is devoted to the proof of the upper bound in (7), which ends the proof of Theorem 3.

2. Exponential moments lower bound.

This section is devoted to the proof of the lower bounds (2) and (4) in Proposition 1.

Lower bound for I_T. Fix $M>0$. Since $b_T\ll T$, for T sufficiently large $[T\geq T_0(M)]$ $Mb_T\leq T$, and $I_T\geq I_{Mb_T}$. For any f such that $\|f\|_{q'}=1$,

$$E[\exp(\theta I_T^{1/q})]\geq E[\exp(\theta I_{Mb_T}^{1/q})]\geq E\left[\exp\left(\theta\sum_x f(x)l_{Mb_T}(x)\right)\right]. \tag{9}$$

It is a standard result that the occupation measure of X satisfies a weak large deviations principle in $\mathcal{F}$, in τ-topology (i.e., the topology defined by duality with bounded measurable functions), with rate function $\mathcal{J}(\mu)=\|\nabla\sqrt{\mu}\|^2$ (see, for instance, Theorem 5.3.10, page 210 in [16]). Since f is bounded by 1 as soon as $\|f\|_{q'}=1$, the function $\mu\in\mathcal{F}\mapsto\sum_{x\in\mathbb{Z}^d}f(x)\mu(x)$ is continuous in τ-topology and the large deviations lower bound for $\frac{1}{Mb_T}\int_0^{Mb_T}\delta_{X_s}\,ds$ (written with the change of variable $g=\sqrt{\mu}$) yields: $\forall\theta\geq 0$, $\forall M>0$, $\forall f\in l_{q'}(\mathbb{Z}^d)$ such that $\|f\|_{q'}=1$,

$$\liminf_{T\to\infty}\frac{1}{b_T}\log E[\exp(\theta I_T^{1/q})]\geq M\sup_{g,\|g\|_2=1}\left\{\theta\sum_x f(x)g^2(x)-\|\nabla g\|_2^2\right\}. \tag{10}$$

Assume now that $\theta>\frac{1}{C_S^2(d)}=\inf\frac{\|\nabla f\|_2^2}{\|f\|_{2q}^2}$ for $q=\frac{d}{d-2}$. Since the infimum can be reduced to the infimum over compactly supported functions f, we can find g_0 with compact support in $\mathbb{Z}^d$, such that $\theta>\frac{\|\nabla g_0\|_2^2}{\|g_0\|_{2q}^2}$. Dividing g_0 by its l_2-norm if necessary, we can moreover assume that $\|g_0\|_2 = 1$. We now take $f = \frac{g_0^{2(q-1)}}{\|g_0\|_{2q}^{2(q-1)}}$ (note that $\|f\|_{q'} = 1$), $g = g_0$ in (10). $\forall M > 0$,

$$\begin{aligned}\liminf_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta I_T^{1/q})] &\geq M\Big(\theta \sum_x f(x)g^2(x) - \|\nabla g\|_2^2\Big)\\ &= M\bigg(\theta \frac{\sum_x g_0^{2q}(x)}{\|g_0\|_{2q}^{2(q-1)}} - \|\nabla g_0\|_2^2\bigg)\\ &= M(\theta\|g_0\|_{2q}^2 - \|\nabla g_0\|_2^2).\end{aligned}$$

But $\theta\|g_0\|_{2q}^2 - \|\nabla g_0\|_2^2 > 0$, so that (2) is proved by sending M to infinity.

Lower bound for Q_T. Fix $M > 0$. Since $b_T \ll T$, for T sufficiently large $[T \geq T_0(M)]$ $Mb_T \leq T$, and $Q_T \geq Q_{Mb_T}$. $\forall\theta \geq 0$, and $\forall m \in \mathbb{N}$,

$$\begin{aligned}E[\exp(\theta Q_T^{1/q})] &\geq E[\exp(\theta Q_{Mb_T}^{1/q})]\\ &\geq \frac{\theta^{qm}}{(qm)!} E[Q_{Mb_T}^m]\\ &= \frac{\theta^{qm}}{(qm)!} \sum_{x_1,\dots,x_m} E\Bigg[\prod_{j=1}^q \prod_{i=1}^m l_{Mb_T}^{(j)}(x_i)\Bigg]\\ &= \frac{\theta^{qm}}{(qm)!} \sum_{x_1,\dots,x_m} E\Bigg[\prod_{i=1}^m l_{Mb_T}(x_i)\Bigg]^q\\ &\geq \frac{\theta^{qm}}{(qm)!} \Bigg[\sum_{x_1,\dots,x_m} f(x_1)\cdots f(x_m) E\Bigg[\prod_{i=1}^m l_{Mb_T}(x_i)\Bigg]\Bigg]^q\end{aligned}$$

for any $f \in l_{q'}(\mathbb{Z}^d)$, such that $\|f\|_{q'} = 1$. Therefore, $\forall\theta \geq 0$, and $\forall m \in \mathbb{N}$,

$$E[\exp(\theta Q_T^{1/q})]^{1/q} \geq \frac{\theta^m}{((qm)!)^{1/q}} E\bigg[\Big(\sum_x f(x) l_{Mb_T}(x)\Big)^m\bigg]. \tag{11}$$

It follows from Stirling's formula that there exists $C > 0$ such that $\forall m \in \mathbb{N}$, $\frac{1}{((qm)!)^{1/q}} \geq C\frac{1}{q^m m!}$. Hence, $\forall\theta \geq 0$, and $\forall m \in \mathbb{N}$,

$$E[\exp(\theta Q_T^{1/q})]^{1/q} \geq C\frac{1}{m!} E\bigg[\bigg(\frac{\theta}{q}\int_0^{Mb_T} f(X_s)\,ds\bigg)^m\bigg]. \tag{12}$$

Summing over m, we have thus proved that for $T \geq T_0(M)$, $\forall\theta \geq 0$, $\forall f \in l_{q'}(\mathbb{Z}^d)$ such that $\|f\|_{q'} = 1$,

$$E[\exp(\theta Q_T^{1/q})]^{1/q} \geq CE\bigg[\exp\bigg(\frac{\theta}{q}\int_0^{Mb_T} f(X_s)\,ds\bigg)\bigg].$$

At this point, the proof is the same as the proof of the lower bound for I_T.

3. Exponential moments upper bounds. In this section, we obtain an upper bound for the exponential moments of $I_T^{1/q}$ and $Q_T^{1/q}$.

Step 1. Comparison with the SILT of the random walk on the torus, stopped at an exponential time.

LEMMA 4. *Let $\alpha > 0$, and let τ be an exponential random variable with parameter $\lambda = \alpha \frac{b_T}{T}$, independent of the random walk $(X_s, s \geq 0)$. Let $R \in \mathbb{N}^*$, and let us denote by $X_s^{(R)} = X_s \operatorname{mod}(R)$ the simple random walk on $\mathbb{T}_R$, the d-dimensional discrete torus of radius R. Finally, let $l_\tau^{(R)}(x) = \int_0^\tau \delta_x(X_s^{(R)})\,ds$, and $I_{R,\tau} = \sum_{x\in\mathbb{T}_R}(l_\tau^{(R)}(x))^q$. Then, $\forall\theta > 0$, $\forall\alpha > 0$, $\forall R > 0$, $\forall T > 0$,*

$$E[\exp(\theta I_T^{1/q})] \leq e^{\alpha b_T} E[\exp(\theta I_{R,\tau}^{1/q})]. \tag{13}$$

PROOF.

$$\begin{aligned} I_T &= \sum_{x\in\mathbb{Z}^d} l_T^q(x) = \sum_{x\in\mathbb{T}_R}\sum_{k\in\mathbb{Z}^d} l_T^q(x+kR) \\ &\leq \sum_{x\in\mathbb{T}_R}\left(\sum_{k\in\mathbb{Z}^d} l_T(x+kR)\right)^q = \sum_{x\in\mathbb{T}_R} l_{R,T}^q(x) = I_{R,T}. \end{aligned}$$

Therefore,

$$\begin{aligned} E[\exp(\theta I_T^{1/q})]\exp(-\alpha b_T) &\leq E[\exp(\theta I_{R,T}^{1/q})]P[\tau \geq T] \\ &\leq E[\exp(\theta I_{R,T}^{1/q})\mathbb{1}_{\tau\geq T}] \\ &\leq E[\exp(\theta I_{R,\tau}^{1/q})], \end{aligned}$$

where the first inequality comes from the choice of $\lambda = \alpha\frac{b_T}{T}$, and the second one from independence of τ and X. $\square$

Step 2. *The Eisenbaum isomorphism theorem*. There are various versions of isomorphism theorems in the spirit of the Dynkin isomorphism theorem. We use here the following version due to Eisenbaum [20] (see also Corollary 8.1.2, page 364 in [32]).

THEOREM 5 (Eisenbaum). *Let α and τ be as in Lemma 4. Let us define for all $x, y \in \mathbb{T}_R$, $G_{R,\lambda}(x, y) = E_x[\int_0^\tau \delta_y(X_s^{(R)})\,ds]$. Let $(Z_x, x \in \mathbb{T}_R)$ be a centered Gaussian process with covariance matrix $G_{R,\lambda}$, independent of τ and of the random walk $(X_s, s \geq 0)$. For $s \neq 0$, consider the process $S_x := l_\tau^{(R)}(x) + \frac{1}{2}(Z_x + s)^2$. Then, for all measurable and bounded function $F : \mathbb{R}^{\mathbb{T}_R} \mapsto \mathbb{R}$,*

$$E\big[F\big((S_x; x \in \mathbb{T}_R)\big)\big] = E\left[F\left(\left(\frac{1}{2}(Z_x+s)^2; x\in\mathbb{T}_R\right)\right)\left(1+\frac{Z_0}{s}\right)\right]. \tag{14}$$

Step 3. Comparison between exponential moments of I_T and exponential moments for $\sum_x Z_x^{2q}$.

Theorem 5 allows one to control exponential moments of $I_{R,\tau}^{1/q}$ by exponential moments of $(\sum_{x\in\mathbb{T}_R} Z_x^{2q})^{1/q}$.

LEMMA 6. *For any $\alpha > 0$ and $R > 0$, let τ and $(Z_x, x \in \mathbb{T}_R)$ be defined as in Lemma 5. $\forall\alpha > 0$, $\forall\theta > 0$, $\forall\gamma > \theta$, $\forall\varepsilon \in]0; \min(1, \sqrt{\frac{\gamma}{\theta} - 1})[$, $\forall R > 0$, $\forall T > 0$, there exists a constant $C(\varepsilon) \in]0; \infty[$ depending only on ε, such that*

$$\begin{aligned}
&E[\exp(\theta I_{R,\tau}^{1/q})] \\
&\quad\le 1 + C(\varepsilon)\frac{\theta}{\gamma - \theta(1+\varepsilon)^2}\left(1 + \frac{\sqrt{T}R^{d/2q}}{\sqrt{\alpha}b_T}\right) \\
&\qquad\times \frac{E[\exp(\gamma/2\|Z\|_{2q,R}^2)]^{1/(1+\varepsilon)}}{P[\|Z\|_{2q,R} \ge 2\sqrt{2b_T\varepsilon}]}\exp(\gamma\varepsilon^2 b_T),
\end{aligned}\tag{15}$$

where $\|\cdot\|_{p,R}$ is the l_p norm of functions on $\mathbb{T}_R$.

PROOF. By independence of $(Z_x, x \in \mathbb{T}_R)$ and $(X_s, s \ge 0)$, $\forall s \neq 0$, $\forall y > 0$, $\forall\varepsilon > 0$,

$$\begin{aligned}
&P\left[\sum_{x\in\mathbb{T}_R}\frac{(Z_x+s)^{2q}}{2^q} \ge b_T^q\varepsilon^q\right]P[I_{R,\tau} \ge b_T^q y^q] \\
&\quad= P\left[\sum_{x\in\mathbb{T}_R}\frac{(Z_x+s)^{2q}}{2^q} \ge b_T^q\varepsilon^q; \sum_{x\in\mathbb{T}_R}\left(l_\tau^{(R)}(x)\right)^q \ge b_T^q y^q\right] \\
&\quad\le P\left[\sum_{x\in\mathbb{T}_R} S_x^q \ge b_T^q(y^q+\varepsilon^q)\right] \\
&\quad= E\left[\left(1+\frac{Z_0}{s}\right)\mathbb{1}_{\sum_{x\in\mathbb{T}_R}(Z_x+s)^{2q}/2^q \ge b_T^q(y^q+\varepsilon^q)}\right] \qquad \text{by Theorem 5.}
\end{aligned}\tag{16}$$

Hence, using Markov inequality,

$$\begin{aligned}
E[\exp(\theta I_{R,\tau}^{1/q})] &= 1 + \int_0^\infty \theta b_T e^{\theta b_T y}P[I_{R,\tau} \ge b_T^q y^q]\,dy \\
&\le 1 + \frac{E[(1+Z_0/s)\exp(\gamma/2\|Z+s\mathbb{1}\|_{2q,R}^2)]}{P[\|Z+s\mathbb{1}\|_{2q,R} \ge \sqrt{2b_T\varepsilon}]} \\
&\qquad\times\int_0^\infty \theta b_T e^{\theta b_T y}e^{-b_T\gamma(y^q+\varepsilon^q)^{1/q}}\,dy.
\end{aligned}\tag{17}$$

Now, $\forall \varepsilon > 0$, $\forall \theta > 0$, $\forall \gamma > \theta$, $\forall T > 0$,

$$\int_0^\infty \theta b_T e^{\theta b_T y} e^{-b_T \gamma (y^q + \varepsilon^q)^{1/q}} dy \le \int_0^\infty \theta b_T e^{\theta b_T y} e^{-b_T \gamma y} dy = \frac{\theta}{\gamma - \theta}. \tag{18}$$

Regarding the denominator in (17),

$$P\big[\|Z + s\mathbb{1}\|_{2q,R} \ge \sqrt{2b_T\varepsilon}\big] \ge P\big[\|Z\|_{2q,R} \ge \sqrt{2b_T\varepsilon} + \|s\mathbb{1}\|_{2q,R}\big] \tag{19}$$

$$= P\big[\|Z\|_{2q,R} \ge \sqrt{2b_T\varepsilon} + |s| R^{d/2q}\big]. \tag{20}$$

On the other hand, $\forall \varepsilon > 0$,

$$\|Z + s\mathbb{1}\|_{2q,R}^2 \le (\|Z\|_{2q,R} + \|s\mathbb{1}\|_{2q,R})^2 \le \|Z\|_{2q,R}^2 (1 + \varepsilon) + \left(1 + \frac{1}{\varepsilon}\right) \|s\mathbb{1}\|_{2q,R}^2,$$

so that

$$\begin{aligned} E\bigg[\bigg(1 + \frac{Z_0}{s}\bigg) \exp\bigg(\frac{\gamma}{2} \|Z + s\mathbb{1}\|_{2q,R}^2\bigg)\bigg] \\ &\le E\bigg[\bigg(1 + \frac{Z_0}{s}\bigg) \exp\bigg(\frac{\gamma}{2}(1+\varepsilon)\|Z\|_{2q,R}^2\bigg)\bigg] \exp\bigg(\frac{\gamma}{2} \frac{1+\varepsilon}{\varepsilon} s^2 R^{d/q}\bigg) \\ &\le E\bigg[\bigg|1 + \frac{Z_0}{s}\bigg|^{(1+\varepsilon)/\varepsilon}\bigg]^{\varepsilon/(1+\varepsilon)} E\bigg[\exp\bigg(\frac{\gamma}{2}(1+\varepsilon)^2 \|Z\|_{2q,R}^2\bigg)\bigg]^{1/(1+\varepsilon)} \\ &\quad \times \exp\bigg(\frac{\gamma}{2} \frac{1+\varepsilon}{\varepsilon} s^2 R^{d/q}\bigg), \end{aligned} \tag{21}$$

Z_0 being a centered Gaussian variable with variance $G_{R,\lambda}(0,0) \le E(\tau) = 1/\lambda$, for all $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ depending only on ε such that

$$E\bigg[\bigg|1 + \frac{Z_0}{s}\bigg|^{(1+\varepsilon)/\varepsilon}\bigg]^{\varepsilon/(1+\varepsilon)} \le C(\varepsilon)\bigg(1 + \sqrt{\frac{T}{\alpha b_T}} \frac{1}{s}\bigg). \tag{22}$$

Putting (17), (18), (20), (21) and (22) together, we have thus proved that $\forall \theta > 0$, $\forall \gamma > \theta$, $\forall \varepsilon > 0$, $\forall R > 0$, $\forall T > 0$, $\forall s \ne 0$,

$$\begin{aligned} E[\exp(\theta I_{R,\tau}^{1/q})] \\ &\le 1 + C(\varepsilon) \frac{\theta}{\gamma - \theta} \bigg(1 + \sqrt{\frac{T}{\alpha b_T}} \frac{1}{s}\bigg) \\ &\qquad \times \frac{E[\exp(\gamma(1+\varepsilon)^2/2 \|Z\|_{2q,R}^2)]^{1/(1+\varepsilon)}}{P[\|Z\|_{2q,R} \ge \sqrt{2b_T\varepsilon} + |s| R^{d/2q}]} \\ &\qquad \times \exp\bigg(\frac{\gamma}{2} \frac{1+\varepsilon}{\varepsilon} s^2 R^{d/q}\bigg). \end{aligned} \tag{23}$$

Choose $s=\sqrt{2b_T}\varepsilon^{3/2}R^{-d/2q}$ in (23). $\forall\theta>0$, $\forall\gamma>\theta$, $\forall\varepsilon>0$, $\forall R>0$, $\forall T>0$,

$$\begin{aligned} &E[\exp(\theta I_{R,\tau}^{1/q})] \\ (24)\qquad &\le 1+C(\varepsilon)\frac{\theta}{\gamma-\theta}\left(1+\frac{\sqrt{T}R^{d/2q}}{\sqrt{\alpha}b_T\varepsilon^{3/2}}\right) \\ &\quad\times\frac{E[\exp(\gamma(1+\varepsilon)^2/2\|Z\|_{2q,R}^2)]^{1/(1+\varepsilon)}}{P[\|Z\|_{2q,R}\ge\sqrt{2b_T\varepsilon}(1+\varepsilon)]}\exp(\gamma\varepsilon^2(1+\varepsilon)b_T). \end{aligned}$$

(15) is now obtained by the change of variable $\gamma\rightsquigarrow\gamma/(1+\varepsilon)^2$. $\square$

Step 4. Large deviations for $\|Z\|_{2q,R}$.

LEMMA 7. *For any $\alpha>0$ and $R>0$, let τ and $(Z_x, x\in\mathbb{T}_R)$ be defined as in Lemma 5. Let* $\rho_1(\alpha,R,T):=\inf\{\sum_{x,y\in\mathbb{T}_R}f_xG_{R,\lambda}^{-1}(x,y)f_y;\ f$ *such that* $\sum_{x\in\mathbb{T}_R}f_x^{2q}=1\}$.

1. $\forall\alpha>0$, $\forall R>0$, $\forall T>0$, $\alpha\frac{b_T}{T}\le\rho_1(\alpha,R,T)\le 2d+\alpha\frac{b_T}{T}$.
2. $\forall\alpha>0$, $\forall\varepsilon>0$, $\forall R>0$, $\forall T>0$,

$$(25)\qquad P\big[\|Z\|_{2q,R}\ge\sqrt{b_T\varepsilon}\big]\ge\frac{1-1/(b_T\varepsilon\rho_1(\alpha,R,T))}{\sqrt{2\pi b_T\varepsilon\rho_1(\alpha,R,T)}}\exp\left(-\frac{b_T\varepsilon\rho_1(\alpha,R,T)}{2}\right).$$

3. $\exists C(q)$ *such that* $\forall\alpha>0$, $\forall R>0$, $\forall T>0$, $\forall\gamma<\rho_1(\alpha,R,T)$, $\forall\varepsilon>0$ *such that* $\gamma(1+\varepsilon)<\rho_1(\alpha,R,T)$,

$$\begin{aligned} (26)\qquad E\left[\exp\left(\frac{\gamma}{2}\|Z\|_{2q,R}^2\right)\right]&\le\frac{2}{\sqrt{1-\gamma(1+\varepsilon)/(\rho_1(\alpha,R,T))}} \\ &\quad\times\exp\left(C(q)\gamma\frac{1+\varepsilon}{\varepsilon}R^{d/q}G_{R,\lambda}(0,0)\right). \end{aligned}$$

PROOF. 1. Since $G_{R,\lambda}=(\lambda\,\mathrm{Id}-\triangle)^{-1}$,

$$\rho_1(\alpha,R,T)=\inf\{\lambda\|f\|_{2,R}^2-(f,\triangle f);\ f\text{ such that }\|f\|_{2q,R}=1\}.$$

Taking $f=\delta_0$, we obtain that $\rho_1(\alpha,R,T)\le\lambda+2d=\alpha\frac{b_T}{T}+2d$. For the lower bound, note that if $\|f\|_{2q,R}=1$, for all $x\in\mathbb{T}_R$, $|f_x|\le 1$, so that $\|f\|_{2,R}^2\ge\sum_{x\in\mathbb{T}_R}f_x^{2q}=1$. Therefore, $\rho_1(\alpha,R,T)\ge\lambda$.

2. For all $(f_x, x\in\mathbb{T}_R)$, such that $\sum_x|f_x|^{2q/(2q-1)}=1$,

$$P\big[\|Z\|_{2q,R}\ge\sqrt{b_T\varepsilon}\big]\ge P\left[\sum_{x\in\mathbb{T}_R}f_xZ_x\ge\sqrt{b_T\varepsilon}\right].$$

$\sum_{x\in\mathbb{T}_R} f_x Z_x$ is a real centered Gaussian variable, with variance

$$\sigma^2_{\alpha,R,T}(f) = \sum_{x,y\in\mathbb{T}_R} G_{R,\lambda}(x,y) f_x f_y.$$

Therefore, for all $(f_x, x \in \mathbb{T}_R)$, such that $\sum_x |f_x|^{2q/(2q-1)} = 1$,

$$\begin{aligned} P\big[\|Z\|_{2q,R} \geq \sqrt{b_T\varepsilon}\big] &\geq \frac{\sigma_{\alpha,R,T}(f)}{\sqrt{2\pi}\sqrt{b_T\varepsilon}}\biggl(1 - \frac{\sigma^2_{\alpha,R,T}(f)}{b_T\varepsilon}\biggr)\exp\biggl(-\frac{b_T\varepsilon}{2\sigma^2_{\alpha,R,T}(f)}\biggr), \\ &\geq \frac{\sigma_{\alpha,R,T}(f)}{\sqrt{2\pi}\sqrt{b_T\varepsilon}}\biggl(1 - \frac{\rho_2(\alpha,R,T)}{b_T\varepsilon}\biggr)\exp\biggl(-\frac{b_T\varepsilon}{2\sigma^2_{\alpha,R,T}(f)}\biggr), \end{aligned}$$

where $\rho_2(\alpha,R,T) := \sup\{\sigma^2_{\alpha,R,T}(f);\ f$ such that $\sum_{x\in\mathbb{T}_R}|f_x|^{2q/(2q-1)} = 1\}$. Take the supremum over f, to obtain $\forall\alpha > 0$, $\forall R > 0$, $\forall T > 0$,

$$\begin{aligned} P\big[\|Z\|_{2q,R} \geq \sqrt{b_T\varepsilon}\big] &\geq \frac{\sqrt{\rho_2(\alpha,R,T)}}{\sqrt{2\pi b_T\varepsilon}}\biggl(1 - \frac{\rho_2(\alpha,R,T)}{b_T\varepsilon}\biggr) \\ &\qquad \times \exp\biggl(-\frac{b_T\varepsilon}{2\rho_2(\alpha,R,T)}\biggr). \end{aligned} \tag{27}$$

We are now going to prove that $\forall\alpha > 0$, $\forall R > 0$, $\forall T > 0$,

$$\rho_2(\alpha,R,T) = \frac{1}{\rho_1(\alpha,R,T)}. \tag{28}$$

Indeed,

$$\begin{aligned} (G_{R,\lambda}h, h) = (G_{R,\lambda}h, G^{-1}_{R,\lambda}G_{R,\lambda}h) &\geq \rho_1(\alpha,R,T)\|G_{R,\lambda}h\|^2_{2q,R} \\ &\geq \rho_1(\alpha,R,T)\frac{(G_{R,\lambda}h,h)^2}{\|h\|^2_{2q/(2q-1),R}}, \end{aligned}$$

where the first inequality follows from the definition of $\rho_1(\alpha,R,T)$, and the second one from Hölder's inequality. Therefore, for all h, $(G_{R,\lambda}h,h) \leq \frac{1}{\rho_1(\alpha,R,T)}\|h\|^2_{2q/(2q-1),R}$. Taking the supremum over h yields $\rho_2(\alpha,R,T) \leq \frac{1}{\rho_1(\alpha,R,T)}$. For the opposite inequality, take f_0 achieving the infimum in the definition of $\rho_1(\alpha,R,T)$. Applying the Lagrange multipliers method, it is easy to see that f_0 satisfies the equation $G^{-1}_{R,\lambda}f_0 = \rho_1(\alpha,R,T)f_0^{2q-1}$. Hence, $\|G^{-1}_{R,\lambda}f_0\|_{2q/(2q-1),R} = \rho_1(\alpha,R,T)\|f_0^{2q-1}\|_{2q/(2q-1),R} = \rho_1(\alpha,R,T)\|f_0\|^{2q-1}_{2q,R} = \rho_1(\alpha,R,T)$. Moreover, $(G^{-1}_{R,\lambda}f_0, f_0) = \rho_1(\alpha,R,T)$ and

$$\rho_2(\alpha,R,T) \geq \frac{(G^{-1}_{R,\lambda}f_0, G_{R,\lambda}G^{-1}_{R,\lambda}f_0)}{\|G^{-1}_{R,\lambda}f_0\|^2_{2q/(2q-1),R}} \geq \frac{\rho_1(\alpha,R,T)}{\rho_1(\alpha,R,T)^2} = \frac{1}{\rho_1(\alpha,R,T)},$$

which ends the proof of (28) and of (25).

3. Let $M_{R,T}$ denote the median of $\|Z\|_{2q,R}$. For $\gamma < \rho_1(\alpha, R, T)$, and $\varepsilon > 0$ such that $\gamma(1+\varepsilon) < \rho_1(\alpha, R, T)$,

$$E\left[\exp\left(\frac{\gamma}{2}\|Z\|_{2q,R}^2\right)\right] \le E\left[\exp\left(\frac{\gamma(1+\varepsilon)}{2}(\|Z\|_{2q,R} - M_{R,T})^2\right)\right] \times \exp\left(\frac{\gamma}{2}\frac{1+\varepsilon}{\varepsilon}M_{R,T}^2\right).$$

But $M_{R,T} = \operatorname{median}((\sum_x Z_x^{2q})^{1/2q}) = (\operatorname{median}(\sum_x Z_x^{2q}))^{1/2q}$. Moreover, it is easy to see that for any positive r.v. X, $\operatorname{median}(X) \le 2E(X)$. Hence, using the fact that Z_x is a centered Gaussian variable with variance $G_{R,\lambda}(0,0)$,

$$M_{R,T}^2 \le 2^{1/q} E\left[\sum_{x\in\mathbb{T}_R} Z_x^{2q}\right]^{1/q} = 2^{1/q} R^{d/q} G_{R,\lambda}(0,0) E(V^{2q})^{1/q},$$

where $V \sim \mathcal{N}(0,1)$.

On the other hand,

$$E\left[\exp\left(\frac{\gamma(1+\varepsilon)}{2}(\|Z\|_{2q,R} - M_{R,T})^2\right)\right] = 1 + \int_0^\infty \frac{\gamma(1+\varepsilon)}{2} e^{\gamma(1+\varepsilon)u/2} P\big[\big|\|Z\|_{2q,R} - M_{R,T}\big| \ge \sqrt{u}\big]\,du.$$

We now use the concentration inequalities for norms of Gaussian processes (see, for instance, Lemma 3.1 in [26]): $\forall u > 0$,

$$P\big[\big|\|Z\|_{2q,R} - M_{R,T}\big| \ge \sqrt{u}\big] \le 2P\big(V \ge \sqrt{\rho_1(\alpha, R, T)u}\big).$$

Therefore, since $\gamma(1+\varepsilon) < \rho_1(\alpha, R, T)$,

$$\begin{aligned} E\left[\exp\left(\frac{\gamma(1+\varepsilon)}{2}(\|Z\|_{2q,R} - M_{R,T})^2\right)\right] &\le -1 + 2E\left[\exp\left(\frac{\gamma(1+\varepsilon)}{2\rho_1(\alpha, R, T)}V^2\right)\right] \\ &= -1 + \frac{2}{\sqrt{1 - \gamma(1+\varepsilon)/(\rho_1(\alpha, R, T))}}. \end{aligned}$$ □

Step 5. An upper bound for exponential moments of I_T and Q_T.

LEMMA 8. *Assume that* $\log(T) \ll b_T \le T$, *and that* R *depends on* T *in such a way that* $\forall\alpha > 0$, $b_T \gg R^{d/q} G_{R,\lambda}(0,0)$. *For all* $\alpha > 0$, *set*

$$\begin{aligned} \rho_1(\alpha) &= \liminf_{T\to\infty} \rho_1(\alpha, R, T) \\ &= \liminf_{T\to\infty} \inf\left\{\alpha\frac{b_T}{T}\|f\|_{2,R}^2 + \|\nabla f\|_{2,R}^2;\ f \text{ such that } \|f\|_{2q,R} = 1\right\} \end{aligned}$$

$$\rho_1 = \limsup_{\alpha\to 0} \rho_1(\alpha).$$

1. *For any* $\theta \in [0, \rho_1[$, $\limsup_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta I_T^{1/q})] = 0$.
2. *For any* $\theta \in [0, q\rho_1[$,

$$\limsup_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta Q_T^{1/q})] = 0.$$

PROOF. Point 2 is a straightforward consequence of 1, since

$$Q_T^{1/q} = \left(\sum_x \prod_{i=1}^q l_T^{(i)}(x)\right)^{1/q} \le \left(\prod_{i=1}^q \|l_T^{(i)}\|_q\right)^{1/q} \le \frac{1}{q}\sum_{i=1}^q \|l_T^{(i)}\|_q,$$

where the last inequality comes from the concavity of the log function. Hence,

$$E[\exp(\theta Q_T^{1/q})] \le E\left[\exp\left(\frac{\theta}{q}\|l_T\|_q\right)\right]^q = E\left[\exp\left(\frac{\theta}{q} I_T^{1/q}\right)\right]^q.$$

We thus focus on step 1 of Lemma 8. Let $\alpha > 0$, and $\theta < \rho_1(\alpha)$ be fixed. Take γ such that $\theta < \gamma < \rho_1(\alpha)$. Take then $\varepsilon \in]0; \min(\sqrt{\frac{\gamma}{\theta} - 1}, 1)[$ such that

$$\theta < \gamma < \gamma(1+2\varepsilon) < \rho_1(\alpha).$$

For T sufficiently large ($T \ge T_0$), $\rho_1(\alpha, R, T) \ge \gamma(1+2\varepsilon)$. Lemmas 4 and 6 lead to

$$\begin{aligned} e^{-\alpha b_T} E[e^{\theta I_T^{1/q}}] &\le 1 + C(\varepsilon)\frac{\theta}{\gamma - \theta(1+\varepsilon)^2}\left(1 + \frac{\sqrt{T} R^{d/2q}}{\sqrt{\alpha} b_T}\right) \\ &\quad \times \frac{E[\exp(\gamma/2\|Z\|_{2q,R}^2)]^{1/(1+\varepsilon)}}{P[\|Z\|_{2q,R} \ge \sqrt{8 b_T \varepsilon}]} \exp(\gamma\varepsilon^2 b_T). \end{aligned} \tag{29}$$

By Lemma 7, for $b_T \le T$, and $T \ge T_0$, $\rho_1(\alpha, R, T) \ge \gamma(1+2\varepsilon)$, and

$$\begin{aligned} &P\big[\|Z\|_{2q,R} \ge \sqrt{8 b_T \varepsilon}\big] \\ &\quad \ge \frac{1}{\sqrt{16\pi b_T \varepsilon(2d+\alpha)}}\left(1 - \frac{1}{8 b_T \varepsilon \rho_1(\alpha, R, T)}\right)\exp\big(-4 b_T \varepsilon(2d+\alpha)\big), \\ &\quad \ge \frac{1}{\sqrt{16\pi b_T \varepsilon(2d+\alpha)}}\left(1 - \frac{1}{8 b_T \varepsilon \gamma(1+2\varepsilon)}\right)\exp\big(-4 b_T \varepsilon(2d+\alpha)\big). \end{aligned}$$

Moreover, for $T \ge T_0$, (26) of Lemma 7 yields

$$\begin{aligned} &E\left[\exp\left(\frac{\gamma}{2}\|Z\|_{2q,R}^2\right)\right]^{1/(1+\varepsilon)} \\ &\quad \le \left(2\sqrt{\frac{1+2\varepsilon}{\varepsilon}}\right)^{1/(1+\varepsilon)} \exp\left(C(q)\frac{\gamma}{\varepsilon} R^{d/q} G_{R,\lambda}(0,0)\right). \end{aligned}$$

Therefore, for $R^{d/q}G_{R,\lambda}(0,0) \ll b_T$, and $b_T \gg \log(T)$,

$$\limsup_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta I_T^{1/q})] \le \alpha + 4\varepsilon(2d+\alpha) + \gamma\varepsilon^2.$$

Sending ε to 0, we thus obtain that $\forall\alpha > 0$, $\forall\theta < \rho_1(\alpha)$,

$$\limsup_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta I_T^{1/q})] \le \alpha. \tag{30}$$

Take now $\theta < \rho_1 = \limsup_{\alpha\to 0} \rho_1(\alpha)$. Let (α_n) be a sequence converging to 0, such that $\lim_{n\to\infty} \rho_1(\alpha_n) = \rho_1$. For sufficiently large n, $\rho_1(\alpha_n) > \theta$, and by (30),

$$\limsup_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta I_T^{1/q})] \le \alpha_n.$$

Point 1 is now proved by letting n go to infinity. □

Step 6. Study of ρ_1 and $G_{R,\lambda}(0,0)$.

By Lemma 8 and (2), we know that if R is such that $b_T \gg R^{d/q}G_{R,\lambda}(0,0)$, then $\rho_1 \le \frac{1}{C_S^2(d)}$. It could however happen that $\rho_1 = 0$. It remains thus to determine the values of R for which $\rho_1 > 0$, and to study the behavior of $G_{R,\lambda}(0,0)$.

LEMMA 9 [Behavior of $\rho_1(\alpha, R, T)$]. *Let $d \ge 3$, and $q = \frac{d}{d-2}$. Let ρ_1 be defined as in Lemma 8.*

1. *Assume that R depends on T in such a way that $\forall\alpha > 0$, $\lambda R^2 \gg 1$. Then $\rho_1 \ge \frac{1}{C_S^2(d)}$.*
2. *Assume that R depends on T in such a way that $\lim_{T\to\infty} \lambda R^2 = l(\alpha) \in]0; +\infty[$. Then there exists a constant C such that $\forall\alpha > 0$, $\rho_1(\alpha) > C\min(1, l(\alpha))$.*

PROOF. Let $f_0 \in l_{2q}(\mathbb{T}_R)$ achieve the minimum in the definition of $\rho_1(\alpha, R, T)$. f_0 is viewed as a periodic function on $\mathbb{Z}^d$, and by definition

$$\rho_1(\alpha, R, T) = \lambda\|f_0\|_{2,R}^2 + \|\nabla f_0\|_{2,R}^2; \qquad \|f_0\|_{2q,R} = 1.$$

Let $0 < r < R$, and define

$$\mathcal{C}_{r,R} = \bigcup_{i=1}^{d} \{x \in \mathbb{Z}^d; 0 \le x_i \le r \text{ or } R - r \le x_i \le R\}.$$

Then one can find $a \in \mathbb{Z}^d$ such that $\sum_{x\in\mathcal{C}_{r,R}} f_0^{2q}(x-a) \le \frac{2dr}{R}$. Indeed, on one hand,

$$\begin{aligned}\sum_{a\in[0,R]^d} \sum_{x\in\mathcal{C}_{r,R}} f_0^{2q}(x-a) &= \sum_{x\in\mathcal{C}_{r,R}} \sum_{a\in[0,R]^d} f_0^{2q}(x-a)\\ &= \sum_{x\in\mathcal{C}_{r,R}} \sum_{x\in\mathbb{T}_R} f_0^{2q}(x) = \operatorname{card}(\mathcal{C}_{r,R}) \le 2drR^{d-1}.\end{aligned}$$

On the other hand,

$$\sum_{a\in[0,R]^d}\sum_{x\in\mathcal{C}_{r,R}} f_0^{2q}(x-a) \geq R^d \inf_{a\in[0;R]^d}\sum_{x\in\mathcal{C}_{r,R}} f_0^{2q}(x-a).$$

Set $f_{0,a}(x) \triangleq f_0(x-a)$. $f_{0,a}$ is a periodic function of period R. Note that $\|\nabla f_{0,a}\|_{2,R} = \|\nabla f_0\|_{2,R}$, $\|f_{0,a}\|_{2q,R} = \|f_0\|_{2q,R}$, and that $\|f_{0,a}\|_{2,R} = \|f_0\|_{2,R}$. We can therefore assume without loss of generality, that f_0 achieving the minimum in the definition of $\rho_1(\alpha, R, T)$, satisfies also

$$\sum_{x\in\mathcal{C}_{r,R}} f_0^{2q}(x) \leq \frac{2dr}{R}.$$

Let $\psi : \mathbb{Z}^d \mapsto [0,1]$ a truncature function satisfying

$$\begin{cases} \psi(x) = 0, & \text{if } x \notin [0;R]^d; \\ \psi(x) = 1, & \text{if } x \in [0;R]^d \setminus \mathcal{C}_{r,R}; \\ |\nabla_i \psi(x)| \leq \dfrac{1}{r}, & \forall x \in \mathbb{Z}^d, \forall i \in \{1,\dots,d\}. \end{cases}$$

Fix $\varepsilon > 0$, and take $r = \frac{\varepsilon R}{2d}$. By definition, for $q = \frac{d}{d-2}$,

$$\frac{1}{C_s^2(d)} \leq \frac{\|\nabla(\psi f_0)\|_2^2}{\|\psi f_0\|_{2q}^2}.$$

Regarding the denominator,

$$\|\psi f_0\|_{2q}^{2q} \geq \sum_{x\in[0;R]^d} f_0^{2q}(x) - \sum_{x\in\mathcal{C}_{r,R}} f_0^{2q}(x) \geq 1 - \frac{2dr}{R} = 1-\varepsilon. \tag{31}$$

It remains to control $\|\nabla(\psi f_0)\|_2$,

$$\begin{aligned} \|\nabla(\psi f_0)\|_2^2 &= \sum_{x\in[0;R]^d}\sum_{i=1}^d \bigl(\nabla_i\psi(x) f_0(x+e_i) + \psi(x)\nabla_i f_0(x)\bigr)^2 \\ &= \sum_{x\in[0;R]^d}\sum_{i=1}^d (\nabla_i\psi(x))^2 f_0^2(x+e_i) + \psi^2(x)(\nabla_i f_0(x))^2 \\ &\quad + 2\sum_{x\in[0;R]^d}\sum_{i=1}^d \nabla_i\psi(x)\psi(x) f_0(x+e_i)\nabla_i f_0(x) \\ &\leq \frac{d}{r^2}\|f_0\|_{2,R}^2 + \|\nabla f_0\|_{2,R}^2 + \frac{2\sqrt{d}}{r}\|f_0\|_{2,R}\|\nabla f_0\|_{2,R} \\ &\leq \|\nabla f_0\|_{2,R}^2(1+\varepsilon) + \frac{d}{r^2}\|f_0\|_{2,R}^2(1+1/\varepsilon). \\ &\leq (1+\varepsilon)\max\left(1, \frac{d}{\lambda r^2\varepsilon}\right)\rho_1(\alpha, R, T). \end{aligned} \tag{32}$$

It follows from (31) and (32) that $\forall\varepsilon \in]0;1[$, $\forall\alpha > 0$, $\forall T > 0$,

$$\frac{1}{C_S^2(d)} \le \frac{1+\varepsilon}{(1-\varepsilon)^{1/q}} \max\left(1, \frac{4d^3}{\varepsilon^3}\frac{1}{\lambda R^2}\right)\rho_1(\alpha, R, T). \tag{33}$$

Case 1. Since R is such that $b_T \gg \frac{T}{R^2}$, $\forall\varepsilon > 0$, $\forall\alpha > 0$, $\rho_1(\alpha) \ge \frac{1}{C_S^2(d)}\frac{(1-\varepsilon)^{1/q}}{1+\varepsilon}$. Hence, letting ε go to 0, $\forall\alpha > 0$, $\rho_1(\alpha) \ge \frac{1}{C_S^2(d)}$, so that $\rho_1 \ge \frac{1}{C_S^2(d)}$.

Case 2. Take $\varepsilon = 1/2$ in (33), and let $l(\alpha) = \lim_{T\to\infty} \lambda R^2$. Then $\forall\alpha > 0$,

$$\rho_1(\alpha) \ge \frac{2^{1-1/q}}{3}\frac{1}{C_s^2(d)} \min\left(1, \frac{l(\alpha)}{32d^3}\right) \ge C \min(l(\alpha), 1). \qquad \square$$

LEMMA 10 [Behavior of $G_{R,\lambda}(0,0)$]. *Assume that $d \ge 3$, that $\lambda \ll 1$, and that R depends on T in such a way that $\lambda R^d \gg 1$. Then $\lim_{T\to\infty} G_{R,\lambda}(0,0) = G_d(0,0)$, where $G_d(0,0)$ is the expected amount of time the simple random walk on $\mathbb{Z}^d$ spends on site* 0.

PROOF. Let $p_t^R(x,y)$ be the transition probability of $X_t^{(R)}$. Then

$$G_{R,\lambda}(0,0) = \int_0^\infty \exp(-\lambda t) p_t^R(0,0)\,dt.$$

It follows from Nash inequality (see, for instance, Theorems 2.3.1 and 3.3.15 in [36]) that there exists a constant $C(d)$ such that $\forall R > 0$, $\forall t > 0$,

$$\left| p_t^R(0,0) - \frac{1}{R^d} \right| \le \frac{C(d)}{t^{d/2}}.$$

Therefore, $\forall S > 0$,

$$\begin{aligned}
&\int_S^{+\infty} \exp(-\lambda t) p_t^R(0,0)\,dt \\
&\qquad \le \frac{1}{R^d}\int_0^\infty \exp(-\lambda t)\,dt + \int_S^{+\infty} \frac{C(d)}{t^{d/2}}\,dt \\
&\qquad \le \frac{1}{\lambda R^d} + \frac{C(d)}{S^{d/2-1}}.
\end{aligned}$$

Thus, when $\lambda R^d \gg 1$, and $S \gg 1$,

$$\lim_{T\to\infty} \int_S^{+\infty} \exp(-\lambda t) p_t^R(0,0)\,dt = 0. \tag{34}$$

For the values of t less than S,

$$\begin{aligned}
p_t^R(0,0) &= P_0\big(X_t^{(R)} = 0\big) \\
&\le P_0\left[X_t^{(R)} = 0; \sup_{s\le S} \|X_s\| \le \frac{R}{2}\right] + P_0\left[\sup_{s\le S} \|X_s\| \ge \frac{R}{2}\right]
\end{aligned}$$

$$= P_0\Big[X_t = 0; \sup_{s \le S} \|X_s\| \le \frac{R}{2}\Big] + P_0\Big[\sup_{s \le S} \|X_s\| \ge \frac{R}{2}\Big]$$

$$\le P_0[X_t = 0] + C(d) \exp\Big(-\frac{R^2}{C(d)S}\Big).$$

The third equality comes from the fact that as long as X does not exit a ball of radius $R/2$, then X and $X^{(R)}$ are the same. The fourth one follows from standard results on simple random walks. Thus,

$$\int_0^S \exp(-\lambda t) p_t^R(0,0)\, dt \le \int_0^\infty p_t(0,0)\, dt + C(d) S \exp\Big(-\frac{R^2}{C(d)S}\Big).$$

On the other hand, $p_t^R(0,0) = P_0(X_t^{(R)} = 0) \ge p_t(0,0)$, so that

$$\begin{aligned}\int_0^S \exp(-\lambda t) p_t^R(0,0)\, dt &\ge \int_0^S p_t(0,0)\, dt - \int_0^S \big(1 - \exp(-\lambda t)\big)\, dt \\ &= \int_0^S p_t(0,0)\, dt + \frac{\exp(-\lambda S) - 1 + \lambda S}{\lambda}.\end{aligned}$$

Hence, if S is chosen so that $S \gg 1$, $S \ll R^2/(\log(R))^{1+\varepsilon}$, and $\lambda S^2 \ll 1$,

$$\lim_{T\to\infty} \int_0^S \exp(-\lambda t) p_t^R(0,0)\, dt = \int_0^\infty p_t(0,0)\, dt = G_d(0,0). \tag{35}$$

Now, for $\lambda \ll 1$, and $\lambda R^d \gg 1$ (which implies $R \gg 1$), one can always choose S such that $1 \ll S \ll \min(R^2/(\log(R))^{1+\varepsilon}, 1/\sqrt{\lambda})$. For such a choice of S, it follows from (34) and (35) that

$$\lim_{T\to\infty} G_{R,\lambda}(0,0) = G_d(0,0) < \infty \qquad \text{for } d \ge 3. \qquad \square$$

Step 7. End of proof of Proposition 1.

Choose R such that

$$\frac{T}{R^2} \ll b_T, \qquad b_T \gg R^{d/q}.$$

Then, on one hand, $\forall \alpha > 0$, $\lambda b_T \ll R^2$, and $\rho_1 \ge \frac{1}{C_S^2(d)}$ by 1. of Lemma 9. On the other hand, $\lambda R^d = \alpha \frac{b_T}{T} R^d \gg \alpha \frac{b_T}{T} R^2 \gg 1$. Hence, by Lemma 10, $G_{R,\lambda}(0,0) \simeq G_d(0,0)$ and it follows from Lemma 8 that $\rho_1 \le \frac{1}{C_S^2(d)}$. Therefore, for such a choice of R, $\rho_1 = \frac{1}{C_s^2(d)}$ and

$$\forall \theta \in \Big[0; \frac{1}{C_s^2(d)}\Big[\qquad \liminf_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta I_T^{1/q})] = 0,$$

$$\forall \theta \in \Big[0; \frac{q}{C_s^2(d)}\Big[\qquad \liminf_{T\to\infty} \frac{1}{b_T} \log E[\exp(\theta Q_T^{1/q})] = 0.$$

The best choice for R corresponds to $T/R^2 = R^{d/q} = R^{d-2}$, i.e., $R^d = T$, leading to $b_T \gg T^{1-2/d} = T^{1/q}$.

4. Large and very large deviations lower bounds.

The aim of this section is to prove the lower bounds in Theorems 2 and 3. We have actually the following result.

PROPOSITION 11. 1. *Lower bound for I_T.*
Assume that $d \geq 3$, $q = \frac{d}{d-2}$, and $T \gg b_T \gg 1$.

$$\liminf_{T\to\infty} \frac{1}{b_T} \log P[I_T \geq b_T^q] \geq -\frac{1}{C_S^2(d)}. \tag{36}$$

2. *Lower bound for Q_T.*
Assume that $d = 4$ and $q = 2$, or $d = 3$ and $q = 3$, and that $1 \ll b_T \ll T$.

$$\liminf_{T\to\infty} \frac{1}{b_T} \log P[Q_T \geq b_T^q] \geq -\frac{q}{C_S^2(d)}. \tag{37}$$

PROOF OF (36). Fix $M > 0$. Let T_0 be such that for all $T \geq T_0$, $\frac{T}{b_T} > M$. For $T \geq T_0$,

$$P[I_T \geq b_T^q] \geq P[I_{Mb_T} \geq b_T^q] \geq P\left[\left\|\frac{l_{Mb_T}}{Mb_T}\right\|_q \geq \frac{1}{M}\right].$$

The function $\mu \in \mathcal{F} \mapsto \|\mu\|_q = \sup_{f;\|f\|_{q'}=1} \sum_x \mu(x) f(x)$ is lower semicontinuous in τ-topology, so that $\forall t > 0$, $\{\mu \in \mathcal{F}, \|\mu\|_q > t\}$ is an open subset of $\mathcal{F}$. Therefore, $\forall \varepsilon > 0$,

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{Mb_T} \log P\left[\left\|\frac{l_{Mb_T}}{Mb_T}\right\|_q \geq \frac{1}{M}\right] \\
&\quad \geq \liminf_{T\to\infty} \frac{1}{Mb_T} \log P\left[\left\|\frac{l_{Mb_T}}{Mb_T}\right\|_q > \frac{1-\varepsilon}{M}\right] \\
&\quad \geq -\inf\left\{\|\nabla f\|_2^2; \|f\|_2 = 1, \|f\|_{2q}^2 > \frac{1-\varepsilon}{M}\right\}.
\end{aligned}$$

We have thus proved that $\forall M > 0$, $\forall \varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{b_T} \log P[I_T \geq b_T^q] \geq -M\rho_3\left(\frac{1-\varepsilon}{M}\right),$$

where $\rho_3(y) := \inf\{\|\nabla f\|_2^2; \|f\|_{2q}^2 > y, \|f\|_2 = 1\}$. To end the proof of (36), it remains to show that when $q = \frac{d}{d-2}$, $\forall y > 0$,

$$\inf_{M>0} M\rho_3(y/M) = \frac{y}{C_S^2(d)}. \tag{38}$$

But, if $q = \frac{d}{d-2}$, $\forall y > 0$,

$$\inf_{M>0} M\rho_3(y/M) = y \inf_{M>0} M\rho_3(1/M) \tag{39}$$

$$= y \inf_{M>0} \inf_{f} \left\{ M\|\nabla f\|_2^2; \|f\|_2 = 1, \|f\|_{2q}^2 > \frac{1}{M} \right\} \tag{40}$$

$$= y \inf_{f;\|f\|_2=1} \inf_{M>0} \left\{ M\|\nabla f\|_2^2; M > \frac{1}{\|f\|_{2q}^2} \right\} \tag{41}$$

$$= y \inf_{f;\|f\|_2=1} \left\{ \frac{\|\nabla f\|_2^2}{\|f\|_{2q}^2} \right\}; \tag{42}$$

$$= \frac{y}{C_S^2(d)}. \tag{43}$$

□

PROOF OF (37). The proof of (37) cannot be done as the proof of (36), since the function $(\mu_1, \dots, \mu_q) \mapsto \sum_{x\in\mathbb{Z}^d} \mu_1(x)\cdots\mu_q(x)$ is not lower semicontinuous in the product of τ-topology.

Let $\varepsilon > 0$ be fixed. Let h be a function approaching the infimum in the definition of $C_S(d)$, i.e., h is such that

$$\|\nabla h\|_2^2 \le \frac{\|h\|_{2q}^2}{C_S(d)^2}(1+\varepsilon), \qquad q = \frac{d}{d-2}.$$

Dividing h by its l_2-norm if necessary, we may and we do assume that $\|h\|_2 = 1$.

Set $\eta = 2^{(q+1)/q}\varepsilon^{1/q}$, and $M = \frac{1}{(2-(1+\eta)^q)^{1/q}\|h\|_{2q}^2}$ [ε is chosen small enough in order that M is strictly positive; actually, one has to choose $\varepsilon < \varepsilon_0(q) = (2^{1/q} - 1)^q 2^{-(q+1)}$]. For T large enough, $T \ge Mb_T$, and

$$P[Q_T \ge b_T^q] \ge P[Q_{Mb_T} \ge b_T^q].$$

Assume that $\forall i \in \{1, \dots, q\}$, $\|\frac{l_{Mb_T}^{(i)}}{Mb_T} - h^2\|_q < \eta\|h\|_{2q}^2$. Then

$$\begin{aligned}
&\left| \frac{Q_{Mb_T}}{(Mb_T)^q} - \|h\|_{2q}^{2q} \right| \\
&\quad = \left| \sum_{x\in\mathbb{Z}^d} \prod_1^q \frac{l_{Mb_T}^{(i)}(x)}{Mb_T} - h^{2q}(x) \right| \\
&\quad \le \sum_{x\in\mathbb{Z}^d} \sum_{j=1}^q \left(\prod_{i=1}^{j-1} h^2(x) \right) \left| \frac{l_{Mb_T}^{(j)}(x)}{Mb_T} - h^2(x) \right| \left(\prod_{l=j+1}^q \frac{l_{Mb_T}^{(l)}(x)}{Mb_T} \right) \\
&\quad \le \sum_{j=1}^q \left\| \frac{l_{Mb_T}^{(j)}}{Mb_T} - h^2 \right\|_q \|h\|_{2q}^{2(j-1)} \prod_{l=j+1}^q \left\| \frac{l_{Mb_T}^{(l)}}{Mb_T} \right\|_q
\end{aligned}$$

$$\leq \eta \|h\|_{2q}^{2q} \sum_{j=1}^{q} (1+\eta)^{q-j} = \eta \|h\|_{2q}^{2q} \frac{(1+\eta)^q - 1}{\eta}$$
$$= [(1+\eta)^q - 1] \|h\|_{2q}^{2q}.$$

Therefore, $Q_{Mb_T} \geq b_T^q M^q \|h\|_{2q}^{2q} (2 - (1+\eta)^q) = b_T^q$, by the choice of M. Hence, for T large enough,

$$\begin{aligned} P[Q_T \geq b_T^q] &\geq P\left[\forall i \in \{1, \dots, q\}, \left\| \frac{l_{Mb_T}^{(i)}}{Mb_T} - h^2 \right\|_q < \eta \|h\|_{2q}^2 \right] \\ &= P\left[\left\| \frac{l_{Mb_T}}{Mb_T} - h^2 \right\|_q < \eta \|h\|_{2q}^2 \right]^q. \end{aligned} \tag{44}$$

But,

$$\begin{aligned} \left\| \frac{l_{Mb_T}}{Mb_T} - h^2 \right\|_q^q &= \sum_{x \in \mathbb{Z}^d} \left(\frac{l_{Mb_T}(x)}{Mb_T} - h^2(x) \right)^q \\ &= \sum_{x \in \mathbb{Z}^d} \sum_{j=0}^{q} (-1)^{q-j} C_j^q \frac{l_{Mb_T}^j(x)}{(Mb_T)^j} h^{2(q-j)}(x) \\ &= \left\| \frac{l_{Mb_T}}{Mb_T} \right\|_q^q + (-1)^q \|h\|_{2q}^{2q} - F_q\left(\frac{l_{Mb_T}}{Mb_T} \right), \end{aligned}$$

where the function F_q is defined by $F_q(\mu) = \sum_{j=1}^{q-1} (-1)^{q+1-j} C_j^q \sum_x \mu^j(x) \times h^{2(q-j)}(x)$. Hence, for T large enough,

$$\begin{aligned} &P[Q_T \geq b_T^q]^{1/q} \\ &\quad \geq P\left[F_q\left(\frac{l_{Mb_T}}{Mb_T} \right) > \left\| \frac{l_{Mb_T}}{Mb_T} \right\|_q^q + \left((-1)^q - \eta^q\right) \|h\|_{2q}^{2q} \right] \end{aligned} \tag{45}$$

$$\begin{aligned} &\quad \geq P\left[F_q\left(\frac{l_{Mb_T}}{Mb_T} \right) > \left\| \frac{l_{Mb_T}}{Mb_T} \right\|_q^q + \left((-1)^q - \eta^q\right) \|h\|_{2q}^{2q}; \right. \\ &\qquad\qquad \left. \left\| \frac{l_{Mb_T}}{Mb_T} \right\|_q^q < \left(1 + \frac{\eta^q}{2}\right) \|h\|_{2q}^{2q} \right] \end{aligned} \tag{46}$$

$$\begin{aligned} &\quad \geq P\left[F_q\left(\frac{l_{Mb_T}}{Mb_T} \right) > \left(1 + (-1)^q - \frac{\eta^q}{2}\right) \|h\|_{2q}^{2q} \right] \\ &\qquad - P\left[\left\| \frac{l_{Mb_T}}{Mb_T} \right\|_q^q \geq \left(1 + \frac{\eta^q}{2}\right) \|h\|_{2q}^{2q} \right]. \end{aligned} \tag{47}$$

The second term is controlled by the large deviations upper bound for I_T, and we have

$$\limsup_{T\to\infty}\frac{1}{b_T}\log P\left[\left\|\frac{l_{Mb_T}}{Mb_T}\right\|_q^q \geq \left(1+\frac{\eta^q}{2}\right)\|h\|_{2q}^{2q}\right] \tag{48}$$
$$\leq -M\frac{(1+\eta^q/2)^{1/q}}{C_S^2(d)}\|h\|_{2q}^2 = -\frac{(1+\eta^q/2)^{1/q}}{(2-(1+\eta)^q)^{1/q}C_S^2(d)},$$

by the choice of M.

On the other hand, the function $\mu\in\mathcal{F}\mapsto F_q(\mu)$ is lower semicontinuous in τ-topology. Indeed:

- For $d=4$ and $q=2$, $F_2(\mu)=2\sum_x\mu(x)h^2(x)$ is continuous.
- For $d=3$ and $q=3$, $F_3(\mu)=3\sum_x\mu^2(x)h^2(x)-3\sum_x\mu(x)h^4(x)=3\times\sup_{g;\|g\|_2=1}\{\sum_x\mu(x)h(x)g(x)\}^2-3\sum_x\mu(x)h^4(x)$ is lower semicontinuous.

Using the large deviations lower bound in $\mathcal{F}$ for $\frac{l_{Mb_T}}{Mb_T}$, we get that

$$\liminf_{T\to\infty}\frac{1}{b_T}\log P\left[F_q\left(\frac{l_{Mb_T}}{Mb_T}\right)>\left(1+(-1)^q-\frac{\eta^q}{2}\right)\|h\|_{2q}^{2q}\right] \tag{49}$$
$$\geq -M\inf\left\{\|\nabla g\|_2^2;\ \|g\|_2=1,\ F_q(g^2)>\left(1+(-1)^q-\frac{\eta^q}{2}\right)\|h\|_{2q}^{2q}\right\}.$$

Note that:

- For $d=4$ and $q=2$, $F_2(h^2)=2\|h\|_4^4>(1+(-1)^2-\frac{\eta^2}{2})\|h\|_4^4$.
- For $d=3$ and $q=3$, $F_3(h^2)=0>(1+(-1)^3-\frac{\eta^3}{2})\|h\|_6^6$.

Therefore, in any case,

$$\liminf_{T\to\infty}\frac{1}{b_T}\log P\left[F_q\left(\frac{l_{Mb_T}}{Mb_T}\right)>\left(1+(-1)^q-\frac{\eta^q}{2}\right)\|h\|_{2q}^{2q}\right]$$
$$\geq -M\|\nabla h\|_2^2 = -\frac{\|\nabla h\|_2^2}{(2-(1+\eta)^q)^{1/q}\|h\|_{2q}^2} \tag{50}$$
$$\geq -\frac{1+\varepsilon}{C_S^2(d)(2-(1+\eta)^q)^{1/q}},$$

by the choice of M and h. Putting (47), (48) and (50) together, we get that

$$\frac{1}{q}\liminf_{T\to+\infty}\frac{1}{b_T}\log P[Q_T\geq b_T^q]\geq -\frac{\min(1+\varepsilon;(1+\eta^q/2)^{1/q})}{(2-(1+\eta)^q)^{1/q}C_S^2(d)}. \tag{51}$$

But for $\varepsilon\in]0;1]$, $(1+\varepsilon)^q=\sum_{k=0}^q C_q^k\varepsilon^k\leq 1+\varepsilon\sum_{k=1}^q C_q^k=1+\varepsilon(2^q-1)<1+\varepsilon 2^q=1+\frac{\eta^q}{2}$. We have thus proved that $\forall\varepsilon\in]0;1\wedge\varepsilon_0(q)[$,

$$\liminf_{T\to+\infty}\frac{1}{b_T}\log P[Q_T\geq b_T^q]\geq -\frac{q(1+\varepsilon)}{C_S^2(d)(2-(1+\eta)^q)^{1/q}}. \tag{52}$$

(37) is then obtained by letting ε go to zero. □

5. Large deviations upper bound. The only thing that remains to prove now is the upper bound in Theorem 3.

Let $\alpha > 0$ and $A > 0$ to be chosen later. We take here

$$\lambda = \alpha \frac{T^{1/q}}{T}; \qquad R^d = AT.$$

Let τ be an exponential time with parameter λ, independent on the random walk. Exactly as in (16), $\forall s > 0$, $\forall \varepsilon > 0$,

$$\begin{aligned}
&\exp(-\alpha T^{1/q}) P[I_T \geq Ty] \\
&\quad \leq P[I_{R,\tau} \geq Ty] \\
&\quad \leq \frac{E[(1 + Z_0/s); \|Z + s\mathbb{1}\|_{2q,R} \geq \sqrt{2} T^{1/2q} (y+\varepsilon)^{1/2q}]}{P[\|Z + s\mathbb{1}\|_{2q,R} \geq \sqrt{2} T^{1/2q} \varepsilon^{1/2q}]} \\
&\quad \leq E\biggl[\biggl(1 + \frac{Z_0}{s}\biggr)^{(1+\varepsilon)/\varepsilon}\biggr]^{\varepsilon/(1+\varepsilon)} \\
&\qquad \times \frac{P[\|Z\|_{2q,R} \geq \sqrt{2} T^{1/2q} (y+\varepsilon)^{1/2q} - sR^{d/2q}]^{1/(1+\varepsilon)}}{P[\|Z\|_{2q,R} \geq \sqrt{2} T^{1/2q} \varepsilon^{1/2q} + sR^{d/2q}]}.
\end{aligned} \tag{53}$$

We now choose $sR^{d/2q} = \sqrt{2} T^{1/2q} \varepsilon^{1/2q}$, i.e., $s = \sqrt{2} A^{-1/2q} \varepsilon^{1/2q}$.

$$\begin{aligned}
&P[I_T \geq Ty] \\
&\quad \leq \exp(\alpha T^{1/q}) E\biggl[\biggl(1 + \frac{Z_0}{s}\biggr)^{(1+\varepsilon)/\varepsilon}\biggr]^{\varepsilon/(1+\varepsilon)} \\
&\qquad \times \frac{P[\|Z\|_{2q,R} \geq \sqrt{2} T^{1/2q} ((y+\varepsilon)^{1/2q} - \varepsilon^{1/2q})]^{1/(1+\varepsilon)}}{P[\|Z\|_{2q,R} \geq 2\sqrt{2} T^{1/2q} \varepsilon^{1/2q}]}.
\end{aligned} \tag{54}$$

Using the fact that Z_0 is a centered Gaussian variable with variance $G_{R,\lambda}(0,0)$, we obtain that $\forall \varepsilon > 0$,

$$\begin{aligned}
E\biggl[\biggl(1 + \frac{Z_0}{s}\biggr)^{(1+\varepsilon)/\varepsilon}\biggr]^{\varepsilon/(1+\varepsilon)} &\leq C(\varepsilon)\biggl(1 + \frac{\sqrt{G_{R,\lambda}(0,0)}}{s}\biggr) \\
&\leq C(\varepsilon)\bigl(1 + \sqrt{G_{R,\lambda}(0,0)} A^{1/2q}\bigr).
\end{aligned}$$

But, $\lambda R^d = \alpha A T^{1/q} \gg 1$, so that $\limsup_{T \to \infty} G_{R,\lambda}(0,0) < \infty$ by Lemma 10. Therefore, $\forall \varepsilon > 0$, $\forall \alpha > 0$, $\forall A > 0$,

$$\limsup_{T \to \infty} \frac{1}{T^{1/q}} \log E\biggl[\biggl(1 + \frac{Z_0}{s}\biggr)^{(1+\varepsilon)/\varepsilon}\biggr]^{\varepsilon/(1+\varepsilon)} = 0.$$

Let us treat the numerator of the ratio appearing in the left-hand side of (54). Using again that

$$M_{R,T} = \text{median}(\|Z\|_{2q,R}) \le 2^{1/2q} E\Big[\sum_x Z_x^{2q}\Big]^{1/2q}$$
$$\le C(q) R^{d/2q} G_{R,\lambda}(0,0)^{1/2}$$
$$\sim C(q) A^{1/2q} T^{1/2q} G_d(0,0)^{1/2},$$

we conclude that there exists a constant $C(q)$ such that $\forall \alpha > 0$, $\forall A > 0$, for T large enough, $\forall \varepsilon > 0$,

$$\begin{aligned} &P\big[\|Z\|_{2q,R} \ge \sqrt{2} T^{1/2q}\big((y+\varepsilon)^{1/2q} - \varepsilon^{1/2q}\big)\big] \\ &\quad \le P\big[\|Z\|_{2q,R} - M_{R,T} \ge \sqrt{2} T^{1/2q}\big((y+\varepsilon)^{1/2q} - \varepsilon^{1/2q} - C(q)A^{1/2q}\big)\big] \\ &\quad \le 2\exp\big(-T^{1/q}\rho_1(\alpha, R, T)\big((y+\varepsilon)^{1/2q} - \varepsilon^{1/2q} - C(q)A^{1/2q}\big)_+^2\big). \end{aligned} \tag{55}$$

But $\lambda R^2 = \alpha A^{2/d}$, and it follows from Lemma 9 that $\forall \alpha > 0$, $\forall A > 0$, for $\forall \varepsilon > 0$,

$$\begin{aligned} &\limsup_{T\to\infty} \frac{1}{T^{1/q}} \log P\big[\|Z\|_{2q,R} \ge \sqrt{2} T^{1/2q}\big((m+y+\varepsilon)^{1/2q} - \varepsilon^{1/2q}\big)\big] \\ &\quad \le -c(q)\min(1, \alpha A^{2/d})\big((y+\varepsilon)^{1/2q} - \varepsilon^{1/2q} - C(q)A^{1/2q}\big)_+^2. \end{aligned} \tag{56}$$

For the denominator in (54), using (27), (28) and part 1 of Lemma 7, we get that

$$\liminf_{T\to\infty} \frac{1}{T^{1/q}} \log P\big[\|Z\|_{2q,R} \ge 2\sqrt{2} T^{1/2q}\varepsilon^{1/2q}\big] \ge -C(q)\varepsilon^{1/q}. \tag{57}$$

We have thus proved that $\forall \alpha > 0$, $\forall A > 0$, for $\forall \varepsilon > 0$,

$$\begin{aligned} &\limsup_{T\to\infty} \frac{1}{T^{1/q}} P[I_T \ge Ty] \\ &\quad \le \alpha + C(q)\varepsilon^{1/q} - c(q)\min(1, \alpha A^{2/d}) \\ &\qquad \times \big((y+\varepsilon)^{1/2q} - \varepsilon^{1/2q} - C(q)A^{1/2q}\big)_+^2. \end{aligned} \tag{58}$$

We send ε to zero and take $\alpha = A^{-2/d}$, to obtain that $\forall A > 0$,

$$\limsup_{T\to\infty} \frac{1}{T^{1/q}} P[I_T \ge Ty] \le A^{-2/d} - c(q)\big(y^{1/2q} - C(q)A^{1/2q}\big)_+^2. \tag{59}$$

We now choose A such that $C(q)A^{1/2q} = \frac{1}{2} y^{1/2q}$. $\forall y > 0$,

$$\limsup_{T\to\infty} \frac{1}{T^{1/q}} P[I_T \ge Ty] \le -c(q)(y^{1/q} - y^{-2/d}) \le -c(q) y^{1/q} \tag{60}$$

for $y^{-2/d} \le y^{1/q}/2$, that is, $y > 2$.

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