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EXIT FROM A BASIN OF ATTRACTION FOR STOCHASTIC WEAKLY DAMPED NONLINEAR SCHRÖDINGER EQUATIONS

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We consider weakly damped nonlinear Schrödinger equations perturbed by a noise of small amplitude. The small noise is either complex and of additive type or real and of multiplicative type. It is white in time and colored in space. Zero is an asymptotically stable equilibrium point of the deterministic equations. We study the exit from a neighborhood of zero, invariant under the flow of the deterministic equation, in L^2 or in H^1. Due to noise, large fluctuations from zero occur. Thus, on a sufficiently large time scale, exit from these domains of attraction occur. A formal characterization of the small noise asymptotic of both the first exit times and the exit points is given.

1. Introduction. The study of the first exit time from a neighborhood of an asymptotically stable equilibrium point, the exit place determination or the transition between two equilibrium points in randomly perturbed dynamical systems is important in several areas of the mathematical sciences, including statistical and quantum mechanics, chemical reactions, the natural sciences, macroeconomics for modeling currency crises and escape in learning models.

For a fixed noise amplitude and for diffusions, the first exit time and the distribution of the exit points on the boundary of a domain can be characterized by the Dirichlet and Poisson equations, respectively. However, when the dimension is larger than one, we may seldom explicitly solve these equations and large deviation techniques are precious tools when the noise is assumed to be small; see, for example, [10, 13]. The techniques used in the physics literature are often called *optimal fluctuations* or *instanton formalism* and are closely related to large deviations.

In that case, an energy generally characterizes the transition between two states and the exit from a neighborhood of an asymptotically stable equilibrium point of the deterministic equation. The energy is derived from the rate function of the sample path large deviation principle (LDP). When an LDP holds, the first order of the probability of rare events is that of the Boltzmann theory and the square of the amplitude of the small noise acts as the temperature. The deterministic dynamics is sometimes interpreted as the evolution at temperature 0 and the small noise as the small temperature nonequilibrium case. The exit or transition problem is then related to a deterministic least action principle. The paths that minimize the energy,

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also called *minimum action paths*, are the most likely exiting paths or transitions. When the infimum is unique, the system has a behavior which is almost deterministic, even though there is noise. Indeed, other possible exiting paths, points or transitions are exponentially less probable. In the pioneering article [11], a nonlinear heat equation perturbed by a small noise of additive type is considered. In that case, transitions prove to be the instantons of quantum mechanics. The problem is again studied in [14], where a numerical scheme is presented to compute the optimal paths. In [20], mathematical and numerical predictions for a noisy exit problem are confirmed experimentally.

In this article, we consider the case of weakly damped nonlinear Schrödinger (NLS) equations in $\mathbb{R}^d$. These equations provide a generic model for the propagation of the envelope of a wave packet in weakly nonlinear and dispersive media. They appear, for example, in nonlinear optics, hydrodynamics, biology, field theory, crystals and Fermi–Pasta–Ulam chains of atoms. The equations are perturbed by a small noise. In optics, the noise corresponds to the spontaneous emission noise due to amplifiers placed along the fiber line in order to compensate for loss, corresponding to the weak damping, in the fiber. We shall consider here that there remains a small weak damping term. In the context of crystals or of Fermi–Pasta–Ulam chains of atoms, the noise accounts for thermal effects. The relevance of the study of the exit from a domain in nonlinear optics is discussed in [19]. The noise is of additive or multiplicative type. We define it as the time derivative in the sense of distributions of a Hilbert-space-valued Wiener process $(W_t)_{t\geq 0}$. The evolution equation could be written in Itô form,

$$i\,du^{\varepsilon,u_0} = (\Delta u^{\varepsilon,u_0} + \lambda|u^{\varepsilon,u_0}|^{2\sigma}u^{\varepsilon,u_0} - i\alpha u^{\varepsilon,u_0})\,dt + \sqrt{\varepsilon}\,dW, \tag{1.1}$$

where α and ε are positive and u_0 is an initial datum in L^2 or H^1. When the noise is of multiplicative type, the product is a Stratonovich product and the equation may be written

$$i\,du^{\varepsilon,u_0} = (\Delta u^{\varepsilon,u_0} + \lambda|u^{\varepsilon,u_0}|^{2\sigma}u^{\varepsilon,u_0} - i\alpha u^{\varepsilon,u_0})\,dt + \sqrt{\varepsilon}u^{\varepsilon,u_0} \circ dW. \tag{1.2}$$

In contrast to the heat equation, the linear part has no smoothing effects. In our case, it defines a linear group which is an isometry on the L^2-based Sobolev spaces. Thus, we cannot treat spatially rough noises and consider colored-in-space Wiener processes. This latter property is required to obtain bona fide Wiener processes in infinite dimensions. The white noise often considered in physics seems to give rise to ill-posed problems.

Results on local and global well-posedness and on the effect of a noise on the blow-up phenomenon are proved in [4–7] in the case $\alpha = 0$. The mixing property and convergence to equilibrium are studied for weakly damped cubic one-dimensional equations on a bounded domain in [9]. We consider these equations in the whole space $\mathbb{R}^d$ and assume that the power of the nonlinearity σ satisfies $\sigma < 2/d$. We may check that the above result still holds with the damping term and that for such powers of the nonlinearity the solutions do not exhibit blow-up.

In [15] and [16], we have proven sample paths LDPs for the two types of noises, but without damping, and deduced the asymptotic of the tails of the blow-up times. In [15], we also deduced the tails of the mass (defined later) of the pulse at the end of a fiber optical line. We have thus evaluated the error probabilities in optical soliton transmission when the receiver records the signal on an infinite time interval. In [8], we have applied the LDPs to the problem of the diffusion in position of the soliton and studied the tails of the random arrival time of a pulse in optical soliton transmission for noises of additive and multiplicative types. Note that, in [17], we study local well-posedness and large deviations and characterize the support when the noise is a fractional in time additive noise.

The flow defined by the above equations can be decomposed into a Hamiltonian, a gradient and a random component. The mass

$$\mathbf{N}(u) = \int_{\mathbb{R}^d} |u|^2 \, dx$$

characterizes the gradient component. The Hamiltonian denoted by $\mathbf{H}(u)$, defined for functions in H^1, has a kinetic and a potential term and may be written

$$\mathbf{H}(u) = (1/2) \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \big(\lambda/(2\sigma+2)\big) \int_{\mathbb{R}^d} |u|^{2\sigma+2} \, dx.$$

Note that the vector fields associated with the mass and Hamiltonian are orthogonal. We could rewrite, for example, equation (1.1) as

$$du^{\varepsilon,u_0} = \left(\frac{\delta \mathbf{H}(u^{\varepsilon,u_0})}{\delta \overline{u^{\varepsilon,u_0}}} - (\alpha/2) \frac{\delta \mathbf{N}(u^{\varepsilon,u_0})}{\delta u^{\varepsilon,u_0}} \right) dt - i\sqrt{\varepsilon}\, dW.$$

Also, the mass and Hamiltonian are invariant quantities of the equation without noise and damping. Other quantities, such as the linear or angular momentum, are also invariant for nonlinear Schrödinger equations.

Without noise, solutions are uniformly attracted to zero in L^2 and in H^1. In this article, we study the classical problem of exit from a bounded domain containing zero in its interior and invariant under the deterministic evolution. We prove that the behavior of the random evolution is completely different from the deterministic evolution. Although, for finite times, the probabilities of large excursions from neighborhoods of zero go to zero exponentially fast with ε if we wait long enough—the time scale is exponential—such large fluctuations occur and exit from a domain takes place. We give two types of results depending on the topology we consider, L^2 or H^1. The L^2-setting is less involved than the H^1-setting. This is due to the structure of the NLS equation and the fact that the L^2-norm is conserved for deterministic non damped equations. We have also chosen to work in H^1 because it is the mathematical framework in which to study perturbations of solitons, a problem we hope to address in future research.

We give a formal characterization of the small noise asymptotic of the first exit time and exit points. The main tool is a uniform large deviation principle at the level of the paths of the solutions. The behavior of the process is proved to be exponentially equivalent to that of the process starting from a little ball around zero. Thus, if a multiplicative noise and the L^2-topology is considered, such balls are also invariant by the stochastic evolution and the exit problem is not interesting. In infinite dimensions, we are faced with two major difficulties. Primarily, the domains under consideration are not relatively compact. In bounded domains of $\mathbb{R}^d$, it is sometimes possible to use compact embedding and the regularizing properties of the semigroup. In [12], where the case of the heat semigroup and a space variable in a unidimensional torus is treated, these properties are available. Also, in [2], the neighborhood is defined for a strong topology of β-Hölder functions and is relatively compact for a weaker topology, and the space variable is again in a bounded subset of $\mathbb{R}^d$. We are not able to use the above properties here since the Schrödinger linear group is an isometry on every Sobolev space based on L^2 and we work on the whole space $\mathbb{R}^d$. Another difficulty in infinite dimensions and with unbounded linear operators is that, unlike ODEs, continuity of the linear flow with respect to the initial data holds in a weak sense. The semigroup is strongly continuous and not in general uniformly continuous. We see that we may use arguments other than those used in the finite-dimensional setting, some of which are taken from [3], and that the expected results still hold. We are also faced with particular difficulties arising from the nonlinear Schrödinger equation, including the fact that the nonlinearity is locally Lipschitz only in H^1 for $d = 1$. To this end, we use the hypercontractivity governed by the Strichartz inequalities which is related to the dispersive properties of the equation.

In this article, we do not address the control problems for the controlled deterministic PDE. We could expect that the upper and lower bound on the expected first exit time are equal and could be written in terms of the usual quasi-potential. The exit points could be related to solitary waves. These issues will be studied in future works.

The article is organized as follows. In the first section, we introduce the main notation and tools, the proof of the uniform large deviation principle is given in the Appendix. In the next section, we consider the exit from a domain in L^2 for equations with additive noise, while in the last section, we consider the exit from domains in H^1 for equations with an additive or multiplicative noise.

2. Preliminaries. Throughout the paper, the following notation will be used.

The set of positive integers and positive real numbers are denoted by $\mathbb{N}^*$ and $\mathbb{R}_+^*$, respectively. For $p \in \mathbb{N}^*$, L^p is the Lebesgue space of complex-valued functions. For k in $\mathbb{N}^*$, $W^{k,p}$ is the Sobolev space of L^p-functions with partial derivatives up to level k, in the sense of distributions, in L^p. For $p = 2$ and s in $\mathbb{R}_+^*$, H^s is the Sobolev space of tempered distributions v with Fourier transform $\hat{v}$ such that $(1 + |\xi|^2)^{s/2}\hat{v}$ belongs to L^2. We denote the spaces by $L^p_{\mathbb{R}}$, $W^{k,p}_{\mathbb{R}}$ and $H^s_{\mathbb{R}}$ when the functions are real-valued. The space L^2 is endowed with the inner product

$(u,v)_{\mathrm{L}^2} = \mathfrak{Re}\int_{\mathbb{R}^d} u(x)\overline{v}(x)\,dx$. If I is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ is a Banach space and r belongs to $[1,\infty]$, then $\mathrm{L}^r(I;E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$.

The space of linear continuous operators from B into $\tilde{B}$, where B and $\tilde{B}$ are Banach spaces, is $\mathcal{L}_c(B,\tilde{B})$. When $B = H$ and $\tilde{B} = \tilde{H}$ are Hilbert spaces, such an operator is Hilbert–Schmidt when $\sum_{j\in\mathbb{N}} \|\Phi e_j^H\|_{\tilde{H}}^2 < \infty$ for every complete orthonormal system $(e_j)_{j\in\mathbb{N}}$ of H. The set of such operators is denoted by $\mathcal{L}_2(H,\tilde{H})$, or $\mathcal{L}_2^{s,r}$ when $H = \mathrm{H}^s$ and $\tilde{H} = \mathrm{H}^r$. When $H = \mathrm{H}^s_{\mathbb{R}}$ and $\tilde{H} = \mathrm{H}^r_{\mathbb{R}}$, we denote it by $\mathcal{L}^{s,r}_{2,\mathbb{R}}$. When $s = 0$ or $r = 0$, the Hilbert space is L^2 or $\mathrm{L}^2_{\mathbb{R}}$.

We also denote by B^0_ρ and S^0_ρ the open ball and the sphere centered at 0 of radius ρ in L^2, respectively. We denote these by B^1_ρ and S^1_ρ in H^1. We write $\mathcal{N}^0(A,\rho)$ for the ρ-neighborhood of a set A in L^2 and $\mathcal{N}^1(A,\rho)$ for the neighborhood in H^1. In the following, we require that compact sets satisfy the Hausdorff property.

In Lemma 3.6 below, we use the integrability of the Schrödinger linear group which is related to the dispersive property. Recall that $(r(p),p)$ is an admissible pair if p is such that $2 \le p < 2d/(d-2)$ when $d > 2$ ($2 \le p < \infty$ when $d = 2$ and $2 \le p \le \infty$ when $d = 1$) and $r(p)$ satisfies $2/r(p) = d(1/2 - 1/p)$.

For every $(r(p),p)$ admissible pair and T positive, we define the Banach spaces

$$Y^{(T,p)} = \mathrm{C}([0,T];\mathrm{L}^2) \cap \mathrm{L}^{r(p)}(0,T;\mathrm{L}^p)$$

and

$$X^{(T,p)} = \mathrm{C}([0,T];\mathrm{H}^1) \cap \mathrm{L}^{r(p)}(0,T;\mathrm{W}^{1,p}),$$

where the norms are the maxima of the norms in the two intersecting Banach spaces. The Schrödinger linear group is denoted by $(U(t))_{t\ge 0}$; it is defined on L^2 or on H^1. Let us recall the Strichartz inequalities (see [1]):

(i) there exists C positive such that for u_0 in L^2, T positive and $(r(p),p)$ an admissible pair

$$\|U(t)u_0\|_{Y^{(T,p)}} \le C\|u_0\|_{\mathrm{L}^2};$$

(ii) for every T positive, $(r(p),p)$ and $(r(q),q)$ admissible pairs, s and ρ such that $1/s + 1/r(q) = 1$ and $1/\rho + 1/q = 1$, there exists C positive such that, for f in $\mathrm{L}^s(0,T;\mathrm{L}^\rho)$,

$$\left\|\int_0^\cdot U(\cdot - s)f(s)\,ds\right\|_{Y^{(T,p)}} \le C\|f\|_{\mathrm{L}^s(0,T;\mathrm{L}^\rho)}.$$

Similar inequalities hold when the group is acting on H^1, replacing L^2 by H^1, $Y^{(T,p)}$ by $X^{(T,p)}$ and $\mathrm{L}^s(0,T;\mathrm{L}^\rho)$ by $\mathrm{L}^s(0,T;\mathrm{W}^{1,\rho})$.

It is known that, in the Hilbert space setting, only direct images of uncorrelated spacewise Wiener processes by Hilbert–Schmidt operators are well defined. However, when the semigroup has regularizing properties, the semigroup may act as a Hilbert–Schmidt operator and a white-in-space noise may be considered. It is not possible here since the Schrödinger group is an isometry on the Sobolev spaces based on L^2. The Wiener process W is thus defined as ΦW_c, where W_c is a cylindrical Wiener process on L^2 and Φ is Hilbert–Schmidt. $\Phi\Phi^*$ is then is the correlation operator of $W(1)$. It has finite trace.

We consider the following Cauchy problems:

$$\begin{cases} i\,du^{\varepsilon,u_0} = (\Delta u^{\varepsilon,u_0} + \lambda|u^{\varepsilon,u_0}|^{2\sigma}u^{\varepsilon,u_0} - i\alpha u^{\varepsilon,u_0})\,dt + \sqrt{\varepsilon}\,dW, \\ u^{\varepsilon,u_0}(0) = u_0, \end{cases} \tag{2.1}$$

with u_0 in L^2 and Φ in $\mathcal{L}_2^{0,0}$ or u_0 in H^1 and Φ in $\mathcal{L}_2^{0,1}$, and

$$\begin{cases} i\,du^{\varepsilon,u_0} = (\Delta u^{\varepsilon,u_0} + \lambda|u^{\varepsilon,u_0}|^{2\sigma}u^{\varepsilon,u_0} - i\alpha u^{\varepsilon,u_0})\,dt \\ \qquad\qquad + \sqrt{\varepsilon}u^{\varepsilon,u_0} \circ dW, \\ u^{\varepsilon,u_0}(0) = u_0, \end{cases} \tag{2.2}$$

with u_0 in H^1 and Φ in $\mathcal{L}_{2,\mathbb{R}}^{0,s}$, where $s > d/2+1$. When the noise is of multiplicative type, we may write the equation in terms of an Itô product,

$$\begin{aligned} i\,du^{\varepsilon,u_0} = &\big(\Delta u^{\varepsilon,u_0} + \lambda|u^{\varepsilon,u_0}|^{2\sigma}u^{\varepsilon,u_0} - i\alpha u^{\varepsilon,u_0} - (i\varepsilon/2)u^{\varepsilon,u_0}F_\Phi\big)\,dt \\ &+ \sqrt{\varepsilon}u^{\varepsilon,u_0}\,dW, \end{aligned}$$

where $F_\Phi(x) = \sum_{j\in\mathbb{N}}(\Phi e_j(x))^2$ for x in $\mathbb{R}^d$ and $(e_j)_{j\in\mathbb{N}}$ is a complete orthonormal system of L^2. We consider mild solutions, for example, the mild solution of (2.1) satisfies

$$\begin{aligned} u^{\varepsilon,u_0}(t) = U(t)u_0 - i\int_0^t U(t-s)\big(\lambda|u^{\varepsilon,u_0}(s)|^{2\sigma}u^{\varepsilon,u_0}(s) - i\alpha u^{\varepsilon,u_0}(s)\big)\,ds \\ - i\sqrt{\varepsilon}\int_0^t U(t-s)\,dW(s), \qquad t>0. \end{aligned}$$

The Cauchy problems are globally well-posed in L^2 and H^1 with the same arguments as in [5].

The main tools in this article are the sample paths LDP's for the solutions of the three Cauchy problems. They are uniform in the initial data. Unlike in [8, 15, 16], we use a Freidlin–Wentzell-type formulation of the upper and lower bounds of the LDP's. Indeed, it seems that the restriction in [16] that initial data must be in compact sets is a real limitation for stochastic NLS equations. The linear Schrödinger group is not compact due to the lack of smoothing effect and to the fact that we work on the whole space $\mathbb{R}^d$. This limitation disappears when we work with the Freidlin–Wentzell-type formulation; we may now obtain bounds for initial data in balls of L^2 or H^1 for ε small enough. It is well known that in metric spaces and for nonuniform LDP's, the two formulations are equivalent. A proof is given in the Appendix and we stress, in the multiplicative case, the slight differences with the proof of the result in [16].

We denote by $\mathbf{S}(u_0, h)$ the skeleton of equation (2.1) or (2.2), that is, the mild solution of the controlled equation

$$\begin{cases} i\left(\dfrac{du}{dt} + \alpha u\right) = \Delta u + \lambda |u|^{2\sigma} u + \Phi h, \\ u(0) = u_0, \end{cases}$$

where u_0 belongs to L^2 or H^1 in the additive case and the mild solution of

$$\begin{cases} i\left(\dfrac{du}{dt} + \alpha u\right) = \Delta u + \lambda |u|^{2\sigma} u + u\Phi h, \\ u(0) = u_0, \end{cases}$$

where u_0 belongs to H^1 in the multiplicative case.

The rate functions of the LDP's are always defined as

$$I_T^{u_0}(w) = (1/2) \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2) : \mathbf{S}(u_0,h)=w} \int_0^T \|h(s)\|_{\mathrm{L}^2}^2 \, ds.$$

For T and a positive, we denote by $K_T^{u_0}(a) = (I_T^{u_0})^{-1}([0, a])$ the sets

$$K_T^{u_0}(a) = \left\{ w \in \mathrm{C}([0, T]; \mathrm{L}^2) : w = \mathbf{S}(u_0, h), (1/2) \int_0^T \|h(s)\|_{\mathrm{L}^2}^2 \, ds \le a \right\}.$$

We also denote by $d_{\mathrm{C}([0,T];\mathrm{L}^2)}$ the usual distance between sets of $\mathrm{C}([0, T]; \mathrm{L}^2)$ and by $d_{\mathrm{C}([0,T];\mathrm{H}^1)}$ the distance between sets of $\mathrm{C}([0, T]; \mathrm{H}^1)$.

We write $\tilde{\mathbf{S}}(u_0, f)$ for the skeleton of equation (2.2), where we replace Φh by $\frac{\partial f}{\partial t}$, where f belongs to $\mathrm{H}_0^1(0, T; \mathrm{H}_{\mathbb{R}}^s)$, the subspace of $\mathrm{C}([0, T]; \mathrm{H}_{\mathbb{R}}^s)$ of functions that vanishes at zero and whose time derivative is square integrable. Also, C_a denotes the set

$$C_a = \left\{ f \in \mathrm{H}_0^1(0, T; \mathrm{H}_{\mathbb{R}}^s) : \frac{\partial f}{\partial t} \in \mathrm{im}\Phi, \right.$$
$$\left. I_T^W(f) = (1/2) \left\| \Phi^{-1}_{|(\ker \Phi)^\perp} \frac{\partial f}{\partial t} \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}^2 \le a \right\}$$

and $\mathcal{A}(d)$ the set $[2, \infty)$ when $d = 1$ or $d = 2$ and $[2, 2(3d-1)/(3(d-1)))$ when $d \ge 3$. The above I_T^W is the good rate function of the LDP for the Wiener process.

The uniform LDP with the Freidlin–Wentzell formulation that we will need in the sequel is then as follows. In the additive case, we consider the L^2 and H^1-topologies, while in the multiplicative case we consider the H^1-topology only. As has been explained previously, we do not consider the L^2-topology for multiplicative noises since then the L^2-norm remains invariant for the stochastic evolution.

THEOREM 2.1. *In the additive case and in* L^2, *we have, for every* a, ρ, T, δ *and* γ *positive*:

(i) *there exists* ε_0 *positive such that for every* ε *in* $(0, \varepsilon_0)$, u_0 *such that* $\|u_0\|_{\mathrm{L}^2} \le \rho$ *and* $\tilde{a}$ *in* $(0, a]$,

$$\mathbb{P}\big(d_{\mathrm{C}([0,T];\mathrm{L}^2)}(u^{\varepsilon,u_0}, K_T^{u_0}(\tilde{a})) \ge \delta\big) < \exp\big(-(\tilde{a}-\gamma)/\varepsilon\big);$$

(ii) *there exists* ε_0 *positive such that for every* ε *in* $(0, \varepsilon_0)$, u_0 *such that* $\|u_0\|_{\mathrm{L}^2} \le \rho$ *and* w *in* $K_T^{u_0}(a)$,

$$\mathbb{P}\big(\|u^{\varepsilon,u_0} - w\|_{\mathrm{C}([0,T];\mathrm{L}^2)} < \delta\big) > \exp\big(-\big(I_T^{u_0}(w) + \gamma\big)/\varepsilon\big).$$

In H^1, *the result holds for additive and multiplicative noises replacing in the above* $\|u_0\|_{\mathrm{L}^2}$ *by* $\|u_0\|_{\mathrm{H}^1}$ *and* $\mathrm{C}([0,T];\mathrm{L}^2)$ *by* $\mathrm{C}([0,T];\mathrm{H}^1)$.

The proof of this result is given in the Appendix.

REMARK 2.2. The extra condition "For every a positive and K compact in L^2, the set $K_T^K(a) = \bigcup_{u_0\in K} K_T^{u_0}(a)$ is a compact subset of $\mathrm{C}([0,T];\mathrm{L}^2)$" often appears to be part of a uniform LDP. It is not used in the following.

3. Exit from a domain of attraction in $\mathbf{L}^2$.

3.1. *Statement of the results.* In this section, we only consider the case of an additive noise. Recall that for the real multiplicative noise, the mass is decreasing and thus exit is impossible.

We may easily check that the mass $\mathbf{N}(\mathbf{S}(u_0, 0))$ of the solution of the deterministic equation satisfies

$$\mathbf{N}(\mathbf{S}(u_0,0)(t)) = \mathbf{N}(u_0)\exp(-2\alpha t). \tag{3.1}$$

With noise, though, the mass fluctuates around the deterministic decay. Recall how the Itô formula applies to the fluctuation of the mass (see [5] for a proof),

$$\begin{aligned}\mathbf{N}(u^{\varepsilon,u_0}(t)) - \mathbf{N}(u_0) = {} & -2\sqrt{\varepsilon}\,\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \overline{u}^{\varepsilon,u_0}\,dW\,dx \\ & - 2\alpha\|u^{\varepsilon,u_0}\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \varepsilon t\|\Phi\|^2_{\mathcal{L}_2^{0,0}}.\end{aligned} \tag{3.2}$$

We consider domains D which are bounded measurable subsets of L^2 containing 0 in its interior and invariant under the deterministic flow, that is

$$\forall u_0 \in D,\ \forall t \ge 0 \qquad \mathbf{S}(u_0,0)(t) \in D.$$

It is thus possible to consider balls. There exists R positive such that $D \subset B_R$.

We define by

$$\tau^{\varepsilon,u_0} = \inf\{t \ge 0 : u^{\varepsilon,u_0}(t) \in D^c\}$$

the first exit time of the process u^{ε,u_0} from the domain D.

Simple information on the exit time is obtained as follows. The expectation of an integration via the Duhamel formula of the Itô decomposition, the process u^{ε,u_0} being stopped at the first exit time, gives $\mathbb{E}[\exp(-2\alpha\tau^{\varepsilon,u_0})] = 1 - 2\alpha R/(\varepsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}})$. Without damping, we obtain $\mathbb{E}[\tau^{\varepsilon,u_0}] = R/(\varepsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}})$. To obtain more precise information for small noises, we use LDP techniques.

Let us introduce

$$\overline{e} = \inf\{I_T^0(w) : w(T) \in \overline{D}^c,\ T > 0\}.$$

When ρ is positive and small enough, we set

$$e_\rho = \inf\{I_T^{u_0}(w) : \|u_0\|_{\mathrm{L}^2} \le \rho,\ w(T) \in (D_{-\rho})^c,\ T > 0\},$$

where $D_{-\rho} = D \setminus \mathcal{N}^0(\partial D, \rho)$ and ∂D is the boundary of ∂D in L^2. We then define

$$\underline{e} = \lim_{\rho \to 0} e_\rho.$$

In this section, we shall denote by $\|\Phi\|_c$ the norm of Φ as a bounded operator on L^2. Let us start with the following lemma.

LEMMA 3.1. $0 < \underline{e} \le \overline{e}$.

PROOF. It is clear that $\underline{e} \le \overline{e}$. Let us check that $\underline{e} > 0$. Let d denote the positive distance between 0 and ∂D. Take ρ sufficiently small so that the distance between B_ρ^0 and $(D_{-\rho})^c$ is larger than $d/2$. Multiplying the evolution equation by $-i\overline{\mathbf{S}(u_0, h)}$, taking the real part, integrating over space and using the Duhamel formula, we obtain

$$\begin{aligned}
&\mathbf{N}(\mathbf{S}(u_0, h)(T)) - \exp(-2\alpha T)\mathbf{N}(u_0) \\
&\quad = 2\int_0^T \exp\bigl(-2\alpha(T-s)\bigr)\mathfrak{Im}\biggl(\int_{\mathbb{R}^d} \overline{\mathbf{S}(u_0, h)}\Phi h\, dx\, ds\biggr).
\end{aligned}$$

If $\mathbf{S}(u_0, h)(T) \in (D_{-\rho})^c$ and corresponds to the first escape from D, then

$$\begin{aligned}
d/2 &\le 2\|\Phi\|_c \int_0^T \exp\bigl(-2\alpha(T-s)\bigr)\|\mathbf{S}(u_0, h)(s)\|_{\mathrm{L}^2}\|h(s)\|_{\mathrm{L}^2}\, ds \\
&\le 2R\|\Phi\|_c\biggl(\int_0^T \exp\bigl(-4\alpha(T-s)\bigr)\, ds\biggr)^{1/2}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)},
\end{aligned}$$

thus

$$\alpha d^2/(8R^2\|\Phi\|_c^2) \le \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}/2$$

and the result follows. $\square$

REMARK 3.2. We would expect $\underline{e}$ and $\overline{e}$ to be equal. We may check that it is enough to prove approximate controllability. The argument is difficult, however, since we are dealing with noises which are colored spacewise, since the Schrödinger group does not have global smoothing properties and because of the nonlinearity. If these two bounds were indeed equal, they would also correspond to

$$\begin{aligned}\mathcal{E}(D) &= (1/2)\inf\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} : \exists T > 0 : \mathbf{S}(0,h)(T) \in \partial D\} \\ &= \inf_{v\in\partial D} V(0,v),\end{aligned}$$

where the quasi-potential is defined as

$$V(u_0,u_f) = \inf\{I_T^{u_0}(w) : w \in \mathrm{C}([0,\infty);\mathrm{L}^2),\ w(0)=u_0,\ w(T)=u_f,\ T>0\}.$$

In this section, we prove the two following results. The first theorem characterizes the first exit time from the domain.

THEOREM 3.3. *For every u_0 in D and δ positive, there exists L positive such that*

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbb{P}\big(\tau^{\varepsilon,u_0} \notin \big(\exp((\underline{e}-\delta)/\varepsilon), \exp((\overline{e}+\delta)/\varepsilon)\big)\big) \le -L \tag{3.3}$$

and for every u_0 in D,

$$\underline{e} \le \underline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbb{E}(\tau^{\varepsilon,u_0}) \le \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbb{E}(\tau^{\varepsilon,u_0}) \le \overline{e}. \tag{3.4}$$

Moreover, for every δ positive, there exists L positive such that

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \sup_{u_0\in D} \mathbb{P}\big(\tau^{\varepsilon,u_0} \ge \exp((\overline{e}+\delta)/\varepsilon)\big) \le -L \tag{3.5}$$

and

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \sup_{u_0\in D} \mathbb{E}(\tau^{\varepsilon,u_0}) \le \overline{e}. \tag{3.6}$$

The second theorem formally characterizes the exit points. We shall define, for ρ positive and small enough, N a closed subset of ∂D,

$$e_{N,\rho} = \inf\{I_T^{u_0}(w) : \|u_0\|_{\mathrm{L}^2} \le \rho,\ w(T) \in (D \setminus \mathcal{N}^0(N,\rho))^c,\ T>0\}.$$

We then define

$$\underline{e}_N = \lim_{\rho\to 0} e_{N,\rho}.$$

Note that $e_\rho \le e_{N,\rho}$ and thus $\underline{e} \le \underline{e}_N$.

THEOREM 3.4. *If $\underline{e}_N > \overline{e}$, then, for every u_0 in D, there exists L positive such that*

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbb{P}\big(u^{\varepsilon,u_0}(\tau^{\varepsilon,u_0}) \in N\big) \le -L.$$

Thus, the probability of an escape from D via points of N such that $e_\rho \le e_{N,\rho}$ goes to zero exponentially fast with ε.

Supposing that we are able to solve the previous control problem, then, as the noise goes to zero, the probability of an exit via closed subsets of ∂D where the quasi-potential is not minimal goes to zero. As the expected exit time is finite, an exit occurs almost surely. It is exponentially more likely that it occurs via infima of the quasi-potential. When there are several infima, the exit measure is a probability measure on ∂D. When there is only one infimum, we may state the following corollary.

COROLLARY 3.5. *Assume that v^* in ∂D is such that, for every δ positive and $N = \{v \in \partial D : \|v - v^*\|_{\mathrm{L}^2} \ge \delta\}$, we have $\underline{e}_N > \overline{e}$. We then have*

$$\forall \delta > 0,\ \forall u_0 \in D,\ \exists L > 0 : \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbb{P}\big(\|u^{\varepsilon,u_0}(\tau^{\varepsilon,u_0}) - v^*\|_{\mathrm{L}^2} \ge \delta\big) \le -L.$$

3.2. *Preliminary lemmas.* Let us define

$$\sigma_\rho^{\varepsilon,u_0} = \inf\{t \ge 0 : u^{\varepsilon,u_0}(t) \in B_\rho^0 \cup D^c\},$$

where $B_\rho^0 \subset D$.

LEMMA 3.6. *For every ρ and L positive with $B_\rho^0 \subset D$, there exist T and ε_0 positive such that, for every u_0 in D and ε in $(0, \varepsilon_0)$,*

$$\mathbb{P}(\sigma_\rho^{\varepsilon,u_0} > T) \le \exp(-L/\varepsilon).$$

PROOF. The result is straightforward if u_0 belongs to B_ρ^0. Suppose, now, that u_0 belongs to $D \setminus B_\rho^0$. From equation (3.1), the bounded subsets of L^2 are uniformly attracted to zero by the flow of the deterministic equation. Thus, there exists a positive time T_1 such that, for every u_1 in the $\rho/8$-neighborhood of $D \setminus B_\rho^0$ and $t \ge T_1$, $\mathbf{S}(u_1, 0)(t) \in B_{\rho/8}^0$. We shall choose $\rho < 8$ and follow three steps.

Step 1. Let us first recall why there exists $M' = M'(T_1, R, \sigma, \alpha)$ such that

$$\sup_{u_1 \in \mathcal{N}^0(D\setminus B_\rho^0, \rho/8)} \|\mathbf{S}(u_1, 0)\|_{Y^{(T_1, 2\sigma+2)}} \le M'. \tag{3.7}$$

From the Strichartz inequalities, there exists C positive such that

$$\begin{aligned}\|\mathbf{S}(u_1, 0)\|_{Y^{(t,2\sigma+2)}} \le{}& C\|u_1\|_{\mathrm{L}^2} + C\big\||\mathbf{S}(u_1, 0)|^{2\sigma+1}\big\|_{\mathrm{L}^{r'}(0,t;\mathrm{L}^{s'})}\\ &+ C\alpha\|\mathbf{S}(u_1, 0)\|_{\mathrm{L}^1(0,t;\mathrm{L}^2)},\end{aligned}$$

where γ' and s' are such that $1/\gamma' + 1/r(\tilde{p}) = 1$ and $1/s' + 1/\tilde{p} = 1$, and $(r(\tilde{p}), \tilde{p})$ is an admissible pair. Note that the first term is smaller than $C(R+1)$. From the Hölder inequality, setting

$$\frac{2\sigma}{2\sigma+2} + \frac{1}{2\sigma+2} = \frac{1}{s'}, \qquad \frac{2\sigma}{\omega} + \frac{1}{r(2\sigma+2)} = \frac{1}{\gamma'},$$

we can write

$$\begin{aligned}&\| |\mathbf{S}(u_1,0)|^{2\sigma+1}\|_{\mathrm{L}^{\gamma'}(0,t;\mathrm{L}^{s'})}\\ &\qquad \le C\|\mathbf{S}(u_1,0)\|_{\mathrm{L}^{r(2\sigma+2)}(0,t;\mathrm{L}^{2\sigma+2})}\|\mathbf{S}(u_1,0)\|^{2\sigma}_{\mathrm{L}^{\omega}(0,t;\mathrm{L}^{2\sigma+2})}.\end{aligned}$$

It is easy to check that since $\sigma < 2/d$, we have $\omega < r(2\sigma+2)$. Thus, it follows that

$$\begin{aligned}\|\mathbf{S}(u_1,0)\|_{Y(t,2\sigma+2)} &\le C(R+1) + Ct^{(\omega r(2\sigma+2))/(r(2\sigma+2)-\omega)}\|\mathbf{S}(u_1,0)\|^{2\sigma+1}_{Y(t,2\sigma+2)}\\ &\quad + C\alpha\sqrt{t}\|\mathbf{S}(u_1,0)\|_{Y(t,2\sigma+2)}.\end{aligned}$$

The function $x \mapsto C(R+1) + Ct^{(\omega r(2\sigma+2))/(r(2\sigma+2)-\omega)}x^{2\sigma+1} + C\alpha\sqrt{t}x - x$ is positive on a neighborhood of zero. For $t_0 = t_0(R,\sigma,\alpha)$ small enough, the function has at least one zero. Also, the function goes to ∞ as x goes to ∞. Thus, denoting by $M(R,\sigma)$ the first zero of the above function, we obtain, by a classical argument, that $\|\mathbf{S}(u_1,0)\|_{Y(t_0,2\sigma+2)} \le M(R,\sigma)$ for every u_1 in $\mathcal{N}^0(D \setminus B^0_\rho, \rho/8)$.

Also, as for every t in $[0,T]$, $\mathbf{S}(u_1,0)(t)$ belongs to $\mathcal{N}^0(D \setminus B^0_\rho, \rho/8)$, repeating the previous argument, where u_1 is replaced by $\mathbf{S}(u_1,0)(t_0)$ and so on, we obtain

$$\sup_{u_1 \in \mathcal{N}^0(D\setminus B^0_\rho,\rho/8)} \|\mathbf{S}(u_1,0)\|_{Y(T_1,p)} \le M',$$

where $M' = \lceil T_1/t_0 \rceil M$, proving (3.7).

Step 2. Let us now prove that for T large enough (to be specified later) and larger than T_1, we have

$$\begin{aligned}\mathcal{T}_\rho &= \{w \in \mathrm{C}([0,T];\mathrm{L}^2) : \forall t \in [0,T],\ w(t) \in \mathcal{N}^0(D \setminus B^0_\rho, \rho/8)\}\\ &\subset K^{u_0}_T(2L)^c.\end{aligned} \tag{3.8}$$

Since $K^{u_0}_T(2L)$ is included in the image of $\mathbf{S}(u_0,\cdot)$, it suffices to consider w in $\mathcal{T}_\rho$ such that $w = \mathbf{S}(u_0,h)$ for some h in $\mathrm{L}^2(0,T;\mathrm{L}^2)$. For h such that $\mathbf{S}(u_0,h)$ belongs to $\mathcal{T}_\rho$, we have

$$\|\mathbf{S}(u_0,h) - \mathbf{S}(u_0,0)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)} \ge \|\mathbf{S}(u_0,h)(T_1) - \mathbf{S}(u_0,0)(T_1)\|_{\mathrm{L}^2} \ge 3\rho/4.$$

Thus, for the admissible pair $(r(2\sigma+2), 2\sigma+2)$, we have

$$\|\mathbf{S}(u_0,h) - \mathbf{S}(u_0,0)\|_{Y(T_1,2\sigma+2)} \ge 3\rho/4. \tag{3.9}$$

Denote by $\mathbf{S}^{M'+1}$ the skeleton corresponding to the following control problem:

$$\begin{cases} i\left(\dfrac{du}{dt}+\alpha u\right)=\Delta u+\lambda\theta\left(\dfrac{\|u\|_{Y^{(t,2\sigma+2)}}}{M'+1}\right)|u|^{2\sigma}u+\Phi h, \\ u(0)=u_1, \end{cases}$$

where θ is a C^∞ function with compact support, such that $\theta(x)=0$ if $x\geq 2$ and $\theta(x)=1$ if $0\leq x\leq 1$. (3.9) then implies that

$$\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\|_{Y^{(T_1,2\sigma+2)}}\geq 3\rho/4.$$

We shall now split the interval $[0,T_1]$ into many parts. We shall here denote by $Y^{s,t,2\sigma+2}$, for $s<t$, the space $Y^{t,2\sigma+2}$ on the interval $[s,t]$. Applying the Strichartz inequalities on a small interval $[0,t]$ with the computations in the proof of Lemma 3.3 in [4], we obtain

$$\begin{aligned}\|\mathbf{S}^{M'+1}&(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\|_{Y^{(t,2\sigma+2)}}\\ &\leq C\alpha\sqrt{t}\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\|_{Y^{(t,2\sigma+2)}}\\ &\quad+C_{M'+1}t^{1-d\sigma/2}\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\|_{Y^{(t,2\sigma+2)}}\\ &\quad+C\sqrt{t}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)},\end{aligned}$$

where $C_{M'+1}$ is a constant which depends on $M'+1$. Now, take t_1 small enough so that $C_{M'+1}t_1^{1-d\sigma d/2}+C\alpha\sqrt{t_1}\leq 1/2$. We then obtain

$$\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\|_{Y^{(t_1,2\sigma+2)}}\leq 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)}.$$

In the case where $2t_1<T_1$, let us see how such an inequality propagates on $[t_1,2t_1]$. We now have two different initial data, $\mathbf{S}^{M'+1}(u_0,h)(t_1)$ and $\mathbf{S}^{M'+1}(u_0,0)(t_1)$. We similarly obtain

$$\begin{aligned}\|\mathbf{S}^{M'+1}&(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\|_{Y^{(t_1,2t_1,2\sigma+2)}}\\ &\leq 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)}\\ &\quad+2\|\mathbf{S}^{M'+1}(u_0,h)(t_1)-\mathbf{S}^{M'+1}(u_0,0)(t_1)\|_{\mathrm{L}^2}\\ &\leq 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)}\\ &\quad+2\|\mathbf{S}^{M'+1}(u_0,h)(t_1)-\mathbf{S}^{M'+1}(u_0,0)(t_1)\|_{Y^{(0,t_1,2\sigma+2)}}.\end{aligned}$$

Then, iterating on each interval of the form $[kt_1,(k+1)t_1]$ for k in $\{1,\ldots,\lfloor T_1/t_1-1\rfloor\}$, the remaining term can be treated similarly and, using the triangle inequality, we obtain that

$$\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\|_{Y^{(T_1,2\sigma+2)}}\leq 2^{\lceil T_1/t_1\rceil+1}C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)}.$$

We may then conclude that

$$\|h\|^2_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)}/2\geq M'',$$

where $M'' = \rho^2/(8C(t_1, T_1)\|\Phi\|_c^2)$ and $C(t_1, T_1)$ is a constant which depends only on t_1 and T_1. Note that we have (used for later purposes) that $3\rho/2 > \rho/2$.

Similarly replacing $[0, T_1]$ by $[T_1, 2T_1]$ and u_0 by $\mathbf{S}(u_0, h)(T_1)$ and $\mathbf{S}(u_0, 0)(T_1)$, respectively, in (3.9), the inequality still holds true. Thus, thanks to the inverse triangle inequality, we obtain the following on $[T_1, 2T_1]$:

$$\begin{aligned}
&\|\mathbf{S}^{M'+1}(u_0, h) - \mathbf{S}^{M'+1}(u_0, 0)\|_{Y(T_1,2T_1,2\sigma+2)} \\
&\quad = \|\mathbf{S}^{M'+1}(\mathbf{S}^{M'+1}(u_0, h)(T_1), h) - \mathbf{S}^{M'+1}(\mathbf{S}^{M'+1}(u_0, 0)(T_1), 0)\|_{Y(0,T_1,2\sigma+2)} \\
&\quad \geq 3\rho/4.
\end{aligned}$$

Thus, from the inverse triangle inequality, along with the fact that for both $\mathbf{S}^{M'+1}(u_0, h)(T_1)$ and $\mathbf{S}^{M'+1}(u_0, 0)(T_1)$ as initial data, the deterministic solutions belong to the ball $B^0_{\rho/8}$, we obtain

$$\|\mathbf{S}^{M'+1}(\mathbf{S}^{M'+1}(u_0, h)(T_1), h) - \mathbf{S}^{M'+1}(\mathbf{S}^{M'+1}(u_0, h)(T_1), 0)\|_{Y(0,T_1,2\sigma+2)} \geq \rho/2.$$

We finally obtain the same lower bound

$$\|h\|^2_{\mathrm{L}^2(T_1,2T_1;\mathrm{L}^2)}/2 \geq M''$$

as before.

Iterating the argument, we obtain, if $T > 2T_1$, that

$$\|h\|^2_{\mathrm{L}^2(0,2T_1;\mathrm{L}^2)}/2 = \|h\|^2_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)}/2 + \|h\|^2_{\mathrm{L}^2(T_1,2T_1;\mathrm{L}^2)}/2 \geq 2M''.$$

Thus, for j positive and $T > jT_1$, we obtain, iterating the above argument, that

$$\|h\|^2_{\mathrm{L}^2(0,jT_1;\mathrm{L}^2)}/2 \geq jM''.$$

The result (3.8) is obtained for $T = jT_1$, where j is such that $jM'' > 2L$.

Step 3. We may now conclude from part (i) of Theorem 2.1 since

$$\begin{aligned}
\mathbb{P}(\sigma_\rho^{\varepsilon,u_0} > T) &= \mathbb{P}\big(\forall t \in [0, T],\ u^{\varepsilon,u_0}(t) \in D \setminus B^0_\rho\big) \\
&= \mathbb{P}\big(d_{\mathrm{C}([0,T];\mathrm{L}^2)}(u^{\varepsilon,u_0}, \mathcal{T}^c_\rho) > \rho/8\big), \\
&\leq \mathbb{P}\big(d_{\mathrm{C}([0,T];\mathrm{L}^2)}(u^{\varepsilon,u_0}, K_T^{u_0}(2L)) \geq \rho/8\big),
\end{aligned}$$

taking $a = 2L$, $\rho = R$ where $D \subset B_R$, $\delta = \rho/8$ and $\gamma = L$.

Note that if $\rho \geq 8$, we should replace $R+1$ by $R+\rho/8$ and $M'+1$ by $M'+\rho/8$. In any case, we will use the lemma for small ρ. $\square$

LEMMA 3.7. *For every ρ positive such that $B^0_\rho \subset D$ and u_0 is in D, there exists L positive such that*

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbb{P}\big(u^{\varepsilon,u_0}(\sigma_\rho^{\varepsilon,u_0}) \in \partial D\big) \leq -L.$$

PROOF. Take ρ positive satisfying the assumptions of the lemma and take u_0 in D. When u_0 belongs to B_ρ^0 the result is straightforward. Suppose, now, that u_0 belongs to $D \setminus B_\rho^0$. Letting T be defined as

$$T = \inf\{t \geq 0 : \mathbf{S}(u_0, 0)(t) \in B_{\rho/2}^0\},$$

then since $\mathbf{S}(u_0, 0)([0, T])$ is a compact subset of D, the distance d between $\mathbf{S}(u_0, 0)([0, T])$ and D^c is well defined and positive. The conclusion then follows from the fact that

$$\mathbb{P}\big(u^{\varepsilon,u_0}(\sigma_\rho^{\varepsilon,u_0}) \in \partial D\big) \leq \mathbb{P}\big(\|u^{\varepsilon,u_0} - \mathbf{S}(u_0, 0)\|_{\mathrm{C}([0,T];\mathrm{L}^2)} \geq (\rho \wedge d)/2\big),$$

the LDP and the fact that, from the compactness of the sets $K_T^{u_0}(a)$ for a positive, we have

$$\inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2) : \|\mathbf{S}(u_0,h) - \mathbf{S}(u_0,0)\|_{\mathrm{C}([0,T];\mathrm{L}^2)} \geq (\rho \wedge d)/2} \|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}^2 > 0.$$

We have used the fact that the upper bound of the LDP in the Freidlin–Wentzell formulation implies the classical upper bound. Note that this is a well-known result for nonuniform LDP's. Indeed, we do not need a uniform LDP in this proof. $\square$

The following lemma replaces Lemma 5.7.23 in [10]. Indeed, the case of a stochastic PDE is more intricate than that of an SDE since the linear group is only strongly and not uniformly continuous. However, it is possible to prove that, when acting on bounded sets of H^1, the group on L^2 is uniformly continuous. We shall proceed in a different manner though in order to keep working in L^2.

LEMMA 3.8. *For every ρ and L positive such that $B_{2\rho}^0 \subset D$, there exists $T(L, \rho) < \infty$ such that*

$$\overline{\lim_{\varepsilon \to 0}}\, \varepsilon \log \sup_{u_0 \in S_\rho^0} \mathbb{P}\bigg(\sup_{t \in [0, T(L,\rho)]} \big(\mathbf{N}(u^{\varepsilon,u_0}(t)) - \mathbf{N}(u_0)\big) \geq 3\rho^2\bigg) \leq -L.$$

PROOF. Take L and ρ positive. Note that for every ε in $(0, \varepsilon_0)$ where $\varepsilon_0 = \rho^2/\|\Phi\|_{\mathcal{L}_2^{0,0}}^2$, for $T(L, \rho) \leq 1$, we have $\varepsilon T(L, \rho)\|\Phi\|_{\mathcal{L}_2^{0,0}}^2 < \rho^2$. Thus, from equation (3.2), we know that it is enough to prove that there exists $T(L, \rho) \leq 1$ such that for ε_1 small enough, $\varepsilon_1 < \varepsilon_0$, and for all $\varepsilon < \varepsilon_0$,

$$\varepsilon \log \sup_{u_0 \in S_\rho^0} \mathbb{P}\bigg(\sup_{t \in [0, T(L,\rho)]} \bigg(-2\sqrt{\varepsilon}\,\mathfrak{Im} \int_{\mathbb{R}^d} \int_0^t \overline{u}^{\varepsilon,u_0,\tau}\, dW\, dx\bigg) \geq 2\rho^2\bigg) \leq -L,$$

where $u^{\varepsilon,u_0,\tau}$ is the process u^{ε,u_0} stopped at $\tau_{S_{2\rho}^0}^{\varepsilon,u_0}$, the first time when u^{ε,u_0} hits $S_{2\rho}^0$. Setting $Z(t) = \mathfrak{Im} \int_{\mathbb{R}^d} \int_0^t \overline{u}^{\varepsilon,u_0,\tau}\, dW\, dx$, it is enough to show that

$$\varepsilon \log \sup_{u_0 \in S_\rho^0} \mathbb{P}\bigg(\sup_{t \in [0, T(L,\rho)]} |Z(t)| \geq \rho^2/\sqrt{\varepsilon}\bigg) \leq -L$$

and thus to show exponential tail estimates for the process $Z(t)$. Our proof now closely follows that of Theorem 2.1 in [21]. We introduce the function $f_l(x) = \sqrt{1 + lx^2}$, where l is a positive parameter. We now apply the Itô formula to $f_l(Z(t))$ and the process decomposes into $1 + E_l(t) + R_l(t)$, where

$$E_l(t) = \int_0^t \frac{2lZ(t)}{\sqrt{1 + lZ(t)^2}} \, dZ(t) - (1/2) \int_0^t \left(\frac{2lZ(t)}{\sqrt{1 + lZ(t)^2}} \right)^2 d\langle Z\rangle_t$$

and

$$R_l(t) = (1/2) \int_0^t \left(\frac{2lZ(t)}{\sqrt{1 + lZ(t)^2}} \right)^2 d\langle Z\rangle_t + \int_0^t \frac{l}{(1 + lZ(t)^2)^{3/2)}} \, d\langle Z\rangle_t.$$

Moreover, given $(e_j)_{j\in\mathbb{N}}$, a complete orthonormal system on L^2,

$$\langle Z(t)\rangle = \int_0^t \sum_{j\in\mathbb{N}} (u^{\varepsilon,u_0,\tau}, -i\Phi e_j)^2_{\mathrm{L}^2}(s) \, ds,$$

and we prove with the Hölder inequality that $|R_l(t)| \le 12l\rho^2 \|\Phi\|^2_{\mathcal{L}_2^{0,0}} t$ for every u_0 in D. We may thus write

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,T(L,\rho)]} |Z(t)| \ge \rho^2/\sqrt{\varepsilon} \Big) \\
&\quad = \mathbb{P}\Big(\sup_{t\in[0,T(L,\rho)]} \exp(f_l(Z(t))) \ge \exp\big(f_l\big(\rho^2/\sqrt{\varepsilon}\big)\big) \Big) \\
&\quad \le \mathbb{P}\Big(\sup_{t\in[0,T(L,\rho)]} \exp(E_l(t)) \\
&\qquad\quad \ge \exp\big(f_l\big(\rho^2/\sqrt{\varepsilon}\big) - 1 - 12l\rho^2 \|\Phi\|^2_{\mathcal{L}_2^{0,0}} T(L,\rho)\big) \Big).
\end{aligned}$$

The Novikov condition is also satisfied and $E_l(t)$ is such that $(\exp(E_l(t)))_{t\in\mathbb{R}^+}$ is a uniformly integrable martingale. The exponential tail estimates follow from the Doob inequality, optimizing the parameter l. We may then write

$$\sup_{u_0\in S_\rho^0} \mathbb{P}\Big(\sup_{t\in[0,T(L,\rho)]} |Z(t)| \ge \rho^2/\sqrt{\varepsilon} \Big) \le 3\exp\left(-\frac{\rho^2}{48\varepsilon \|\Phi\|^2_{\mathcal{L}_2^{0,0}} T(L,\rho)} \right).$$

We now conclude by setting $T(L,\rho) = \rho^2/(50\|\Phi\|^2_{\mathcal{L}_2^{0,0}} L)$ and choosing $\varepsilon_1 < \varepsilon_0$ small enough. □

3.3. *Proofs of Theorem* 3.3 *and Theorem* 3.4. We first prove Theorem 3.3.

PROOF OF THEOREM 3.3. Let us first prove (3.6) and then deduce (3.5). Fix δ positive and choose h and T_1 such that $\mathbf{S}(0,h)(T_1) \in \overline{D}^c$ and

$$I_{T_1}^0(\mathbf{S}(0,h)) = (1/2)\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \le \overline{e} + \delta/5.$$

Let d_0 denote the positive distance between $\mathbf{S}(0,h)(T_1)$ and $\overline{D}$. With similar arguments as in [5] or with a truncation argument, we may prove that the skeleton is continuous with respect to the initial datum for the L^2-topology. Thus, there exists ρ positive, a function of h which has been fixed, such that if u_0 belongs to B_ρ^0, then

$$\|\mathbf{S}(u_0,h)-\mathbf{S}(0,h)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)}<d_0/2.$$

We may assume that ρ is such that $B_\rho^0\subset D$. From the triangle inequality and part (ii) of Theorem 2.1, there exists ε_1 positive such that for all ε in $(0,\varepsilon_1)$ and u_0 in B_ρ^0,

$$\begin{aligned}\mathbb{P}(\tau^{\varepsilon,u_0}<T_1)&\geq\mathbb{P}\big(\|u^{\varepsilon,u_0}-\mathbf{S}(0,h)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)}<d_0\big)\\&\geq\mathbb{P}\big(\|u^{\varepsilon,u_0}-\mathbf{S}(u_0,h)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)}<d_0/2\big)\\&\geq\exp\biggl(-\biggl(I_{T_1}^{u_0}(\mathbf{S}(u_0,h))+\frac{\delta}{5}\biggr)\Big/\varepsilon\biggr).\end{aligned}$$

From Lemma 3.6, there exists T_2 and ε_2 positive such that for all ε in $(0,\varepsilon_2)$,

$$\inf_{u_0\in D}\mathbb{P}(\sigma_\rho^{\varepsilon,u_0}\leq T_2)\geq 1/2.$$

Thus, for $T=T_1+T_2$, from the strong Markov property we obtain that for all $\varepsilon<\varepsilon_3<\varepsilon_1\wedge\varepsilon_2$,

$$\begin{aligned}q=\inf_{u_0\in D}\mathbb{P}(\tau^{\varepsilon,u_0}\leq T)&\geq\inf_{u_0\in D}\mathbb{P}(\sigma_\rho^{\varepsilon,u_0}\leq T_2)\inf_{u_0\in B_\rho^0}\mathbb{P}(\tau^{\varepsilon,u_0}\leq T_1)\\&\geq(1/2)\exp\big(-\big(I_{T_1}^{u_0}(\mathbf{S}(u_0,h))+\delta/5\big)/\varepsilon\big)\\&\geq\exp\big(-\big(I_{T_1}^{u_0}(\mathbf{S}(u_0,h))+2\delta/5\big)/\varepsilon\big).\end{aligned}$$

Thus, for any $k\geq 1$, we have

$$\begin{aligned}\mathbb{P}\big(\tau^{\varepsilon,u_0}>(k+1)T\big)&=\big[1-\mathbb{P}\big(\tau^{\varepsilon,u_0}\leq(k+1)T\,|\,\tau^{\varepsilon,u_0}>kT\big)\big]\mathbb{P}(\tau^{\varepsilon,u_0}>kT)\\&\leq(1-q)\mathbb{P}(\tau^{\varepsilon,u_0}>kT)\\&\leq(1-q)^k.\end{aligned}$$

We may now compute, since $I_{T_1}^{u_0}(\mathbf{S}(u_0,h))=I_{T_1}^{0}(\mathbf{S}(0,h))=(1/2)\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$,

$$\begin{aligned}\sup_{u_0\in D}\mathbb{E}(\tau^{\varepsilon,u_0})&=\sup_{u_0\in D}\int_0^\infty\mathbb{P}(\tau^{\varepsilon,u_0}>t)\,dt\\&\leq T\Biggl[1+\sum_{k=1}^\infty\sup_{x\in D}\mathbb{P}(\tau^{\varepsilon,u_0}>kT)\Biggr]\\&\leq T/q\\&\leq T\exp\big((\overline{e}+3\delta/5)/\varepsilon\big).\end{aligned}$$

This implies that there exists ε_4 small enough such that for ε in $(0, \varepsilon_4)$,

$$\sup_{u_0 \in D} \mathbb{E}(\tau^{\varepsilon,u_0}) \leq \exp\bigl((\overline{e} + 4\delta/5)/\varepsilon\bigr). \tag{3.10}$$

Thus, the Chebyshev inequality gives that

$$\sup_{u_0 \in D} \mathbb{P}\bigl(\tau^{\varepsilon,u_0} \geq \exp\bigl((\overline{e} + \delta)/\varepsilon\bigr)\bigr) \leq \exp\bigl(-(\overline{e} + \delta)/\varepsilon\bigr) \sup_{u_0 \in D} \mathbb{E}(\tau^{\varepsilon,u_0}),$$

in other words,

$$\sup_{u_0 \in D} \mathbb{P}\bigl(\tau^{\varepsilon,u_0} \geq \exp\bigl((\overline{e} + \delta)/\varepsilon\bigr)\bigr) \leq \exp\bigl(-\delta/(5\varepsilon)\bigr). \tag{3.11}$$

Relations (3.10) and (3.11) imply (3.6) and (3.5).

Let us now prove the lower bound on τ^{ε,u_0}. Take δ positive. Recall that we have proven that $\underline{e} > 0$. Take ρ positive and small enough so that $\underline{e} - \delta/4 \leq e_\rho$ and $B^0_{2\rho} \subset D$. We define the following sequences of stopping times, for $\theta_0 = 0$ and k in $\mathbb{N}$,

$$\tau_k = \inf\{t \geq \theta_k : u^{\varepsilon,u_0}(t) \in B^0_\rho \cup D^c\},$$
$$\theta_{k+1} = \inf\{t > \tau_k : u^{\varepsilon,u_0}(t) \in S^0_{2\rho}\},$$

where $\theta_{k+1} = \infty$ if $u^{\varepsilon,u_0}(\tau_k) \in \partial D$. Fix $T_1 = T(\underline{e} - 3\delta/4, \rho)$ given in Lemma 3.8. We know that there exists ε_1 positive such that for all ε in $(0, \varepsilon_1)$, for all $k \geq 1$ and u_0 in D,

$$\mathbb{P}(\theta_k - \tau_{k-1} \leq T_1) \leq \exp\bigl(-(\underline{e} - 3\delta/4)/\varepsilon\bigr).$$

For u_0 in D and an m in $\mathbb{N}^*$, we have

$$\begin{aligned}
\mathbb{P}(\tau^{\varepsilon,u_0} \leq mT_1) &\leq \mathbb{P}(\tau^{\varepsilon,u_0} = \tau_0) + \sum_{k=1}^{m} \mathbb{P}(\tau^{\varepsilon,u_0} = \tau_k) \\
&\quad + \mathbb{P}(\exists k \in \{1, \dots, m\} : \theta_k - \tau_{k-1} \leq T_1) \\
&= \mathbb{P}(\tau^{\varepsilon,u_0} = \tau_0) + \sum_{k=1}^{m} \mathbb{P}(\tau^{\varepsilon,u_0} = \tau_k) \\
&\quad + \sum_{k=1}^{m} \mathbb{P}(\theta_k - \tau_{k-1} \leq T_1).
\end{aligned} \tag{3.12}$$

In other words, the escape before mT_1 can occur either as an escape without passing in the small ball B^0_ρ (if u_0 belongs to $D \setminus B^0_\rho$) or as an escape with k in $\{1, \dots, m\}$ significant fluctuations off B^0_ρ, that is, crossing $S^0_{2\rho}$, or at least one of the first m transitions between S^0_ρ and $S^0_{2\rho}$ happens in less than T_1. The latter is known to be arbitrarily small. Let us prove that the remaining probabilities are small enough for small ε.

For every $k \geq 1$ and T_2 positive, we may write

$$\mathbb{P}(\tau^{\varepsilon,u_0} = \tau_k) \leq \mathbb{P}(\tau^{\varepsilon,u_0} \leq T_2; \tau^{\varepsilon,u_0} = \tau_k) + \mathbb{P}(\sigma_\rho^{\varepsilon,u_0} > T_2).$$

Fix T_2 as in Lemma 3.6, with $L = \underline{e} - 3\delta/4$. Thus, there exists ε_2 small enough so that for ε in $(0, \varepsilon_2)$,

$$\mathbb{P}(\sigma_\rho^{\varepsilon,u_0} > T_2) \leq \exp\bigl(-(\underline{e} - 3\delta/4)/\varepsilon\bigr).$$

Also, from part (i) of Theorem 2.1, we obtain that there exists ε_3 positive such that for every u_1 in B_ρ^0 and ε in $(0, \varepsilon_3)$,

$$\begin{aligned}\mathbb{P}(\tau^{\varepsilon,u_1} \leq T_2) &\leq \mathbb{P}\bigl(d_{\mathrm{C}([0,T_2];\mathrm{L}^2)}\bigl(u^{\varepsilon,u_1}, K_{T_2}^{u_1}(e_\rho - \delta/4)\bigr) \geq \rho\bigr)\\ &\leq \exp\bigl(-(e_\rho - \delta/2)/\varepsilon\bigr)\\ &\leq \exp\bigl(-(\underline{e} - 3\delta/4)/\varepsilon\bigr).\end{aligned}$$

Thus, the above bound holds for $\mathbb{P}(\tau^{\varepsilon,u_0} \leq T_2; \tau^{\varepsilon,u_0} = \tau_k)$, replacing u_1 by $u^{\varepsilon,u_0}(\tau_{k-1})$, since, as $k \geq 1$, $u^{\varepsilon,u_0}(\tau_{k-1})$ belongs to B_ρ^0 and $\tau_k - \tau_{k-1} \leq T_2$, and by using the Markov property. The inequality (3.12) gives that, for all ε in $(0, \varepsilon_0)$, where $\varepsilon_0 = \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$,

$$\mathbb{P}(\tau^{\varepsilon,u_0} \leq mT_1) \leq \mathbb{P}\bigl(u^{\varepsilon,u_0}(\sigma_\rho^{\varepsilon,u_0}) \in \partial D\bigr) + 3m \exp\bigl(-(\underline{e} - 3\delta/4)/\varepsilon\bigr).$$

Fix $m = \lceil (1/T_1) \exp((\underline{e} - \delta)/\varepsilon) \rceil$. Then, for all ε in $(0, \varepsilon_0)$,

$$\begin{aligned}\mathbb{P}\bigl(\tau^{\varepsilon,u_0} \leq \exp\bigl((\underline{e} - \delta)/\varepsilon\bigr)\bigr) &\leq \mathbb{P}(\tau^{\varepsilon,u_0} \leq mT_1)\\ &\leq \mathbb{P}\bigl(u^{\varepsilon,u_0}(\sigma_\rho^{\varepsilon,u_0}) \in \partial D\bigr) + (3/T_1) \exp\bigl(-\delta/(4\varepsilon)\bigr).\end{aligned}$$

We may now conclude by using Lemma 3.7 and obtain the expected lower bound on $\mathbb{E}(\tau^{\varepsilon,u_0})$ from the Chebyshev inequality. $\square$

Let us now prove Theorem 3.4.

PROOF OF THEOREM 3.4. Let N be a closed subset of ∂D. When $\underline{e}_N = \infty$, we shall replace, in the proof that follows, $\underline{e}_N$ by an increasing sequence of positive numbers. Take δ such that $0 < \delta < (\underline{e}_N - \overline{e})/3$, ρ positive such that $\underline{e}_N - \delta/3 \leq e_{N,\rho}$ and $B_{2\rho}^0 \subset D$. Define the same sequences of stopping times $(\tau_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ as in the proof of Theorem 3.3.

Take $L = \underline{e}_N - \delta$ and T_1 and $T_2 = T(L, \rho)$ as in Lemmas 3.6 and 3.8. Thanks to Lemma 3.6 and the uniform LDP, with a computation similar to the one following inequality (3.12), we obtain that for ε_0 small enough and

$\varepsilon \le \varepsilon_0$,

$$
\begin{aligned}
&\sup_{u_0 \in S^0_{2\rho}} \mathbb{P}\big(u^{\varepsilon,u_0}(\sigma^{\varepsilon,u_0}_\rho) \in N\big) \\
&\quad \le \sup_{u_0 \in S^0_{2\rho}} \mathbb{P}\big(u^{\varepsilon,u_0}(\sigma^{\varepsilon,u_0}_\rho) \in N, \sigma^{\varepsilon,u_0}_\rho \le T_1\big) + \sup_{u_0 \in S^0_{2\rho}} \mathbb{P}(\sigma^{\varepsilon,u_0}_\rho > T_1) \\
&\quad \le \sup_{u_0 \in B^0_{2\rho}} \mathbb{P}\big(d_{\mathrm{C}([0,T_1];\mathrm{L}^2)}\big(u^{\varepsilon,u_0}, K^{u_0}_{T_1}(e_{N,\rho} - \delta/3)\big) \ge \rho\big) \\
&\qquad + \sup_{u_0 \in D} \mathbb{P}\big(\sigma^{\varepsilon,u_0}_\rho > T_1\big) \\
&\quad \le 2\exp\big(-(\underline{e}_N - \delta)/\varepsilon\big).
\end{aligned}
$$

Possibly choosing ε_0 smaller, we may assume that for every positive integer l and every $\varepsilon \le \varepsilon_0$,

$$
\begin{aligned}
\sup_{u_0 \in D} \mathbb{P}(\tau_l \le lT_2) &\le l \sup_{u_0 \in S^0_\rho} \mathbb{P}\Big(\sup_{t \in [0,T_2]} \big(\mathbf{N}(u^{\varepsilon,u_0}(t)) - \mathbf{N}(u_0)\big) \ge \rho\Big) \\
&\le l\exp\big(-(\underline{e}_N - \delta)/\varepsilon\big).
\end{aligned}
$$

Thus, if u_0 belongs to B^0_ρ, then

$$
\begin{aligned}
\mathbb{P}\big(u^{\varepsilon,u_0}(\tau^{\varepsilon,u_0}) \in N\big) &\le \mathbb{P}(\tau^{\varepsilon,u_0} > \tau_l) \\
&\quad + \sum_{k=1}^{l} \mathbb{P}\big(u^{\varepsilon,u_0}(\tau^{\varepsilon,u_0}) \in N, \tau^{\varepsilon,u_0} = \tau_k\big) \\
&\le \mathbb{P}(\tau^{\varepsilon,u_0} > lT_2) + \mathbb{P}(\tau_l \le lT_2) \\
&\quad + l \sup_{u_0 \in S^0_{2\rho}} \mathbb{P}\big(u^{\varepsilon,u_0}(\sigma^{\varepsilon,u_0}_\rho) \in N\big) \\
&\le \mathbb{P}(\tau^{\varepsilon,u_0} > lT_2) + 3l\exp\big(-(\underline{e}_N - \delta)/\varepsilon\big).
\end{aligned}
$$

Now, taking $l = \lceil (1/T_2)\exp((\overline{e} + \delta)/\varepsilon)\rceil$ and using the upper bound (3.11), possibly choosing ε_0 smaller, we obtain that for $\varepsilon \le \varepsilon_0$,

$$
\begin{aligned}
&\sup_{u_0 \in B^0_\rho} \mathbb{P}\big(u^{\varepsilon,u_0}(\tau^{\varepsilon,u_0}) \in N\big) \\
&\quad \le \exp\big(-\delta/(5\varepsilon)\big) + (4/T_2)\exp\big(-(\underline{e}_N - \overline{e} + 2\delta)/\varepsilon\big) \\
&\quad \le \exp\big(-\delta/(5\varepsilon)\big) + (4/T_2)\exp(-\delta/\varepsilon).
\end{aligned}
$$

Finally, when u_0 is any function in D, we conclude the proof by using

$$
\mathbb{P}\big(u^{\varepsilon,u_0}(\tau^{\varepsilon,u_0}) \in N\big) \le \mathbb{P}\big(u^{\varepsilon,u_0}(\sigma^{\varepsilon,u_0}_\rho) \in \partial D\big) + \sup_{u_0 \in B^0_\rho} \mathbb{P}\big(u^{\varepsilon,u_0}(\tau^{\varepsilon,u_0}) \in N\big)
$$

and Lemma 3.7. $\square$

REMARK 3.9. It is proposed in [22] to introduce control elements to reduce or enhance exponentially the expected exit time or to act on the exiting points, for a limited cost. We could then optimize these external fields. However, the problem is computationally involved since the optimal control problem requires double optimization.

4. Exit from a domain of attraction in H^1.

4.1. *Preliminaries.* We now consider a measurable bounded subset D of H^1, invariant under the flow of the deterministic equation and which contains zero in its interior. We choose R such that $D \subset B_R^1$. We consider both (2.1) and (2.2), where the noise is either of additive or of multiplicative type. In this section, we are interested in both the fluctuation of the L^2-norm and that of the L^2-norm of the gradient. The Hamiltonian and a modified Hamiltonian are thus of particular interest. We first distinguish the case where the nonlinearity is defocusing ($\lambda = -1$) where the Hamiltonian takes nonnegative values from the case where the nonlinearity is focusing ($\lambda = 1$) where the Hamiltonian may take negative values.

We may prove—see, for example, [18]—that

$$\frac{d}{dt}\mathbf{H}(\mathbf{S}(u_0, 0)(t)) + 2\alpha\Psi(\mathbf{S}(u_0, 0)) = 0,$$

where $\mathbf{S}(u_0, 0)$ is the solution of the deterministic weakly damped nonlinear Schrödinger equation with initial datum u_0 in H^1 and

$$\Psi(\mathbf{S}(u_0, 0)) = \|\nabla\mathbf{S}(u_0, 0)\|_{\mathrm{L}^2}^2/2 - \lambda\int_{\mathbb{R}^d} |\mathbf{S}(u_0, 0)(x)|^{2\sigma+2}\,dx/2.$$

Thus, when the nonlinearity is defocusing, we have

$$0 \le \mathbf{H}(\mathbf{S}(u_0, 0)(t)) \le \mathbf{H}(u_0)\exp(-2\alpha t). \tag{4.1}$$

As in [9], we consider in the focusing case, a modified Hamiltonian denoted by $\tilde{\mathbf{H}}(u)$, defined for u in H^1 by

$$\tilde{\mathbf{H}}(u) = \mathbf{H}(u) + \beta(\sigma, d)C\|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)},$$

where the constant C is that of the third inequality in the following sequence of inequalities, where we use the Gagliardo–Nirenberg inequality

$$\|u\|_{\mathrm{L}^{2\sigma+2}}^{2\sigma+2}/(2\sigma+2) \le C\|u\|_{\mathrm{L}^2}^{2\sigma+2-\sigma d}\|\nabla u\|_{\mathrm{L}^2}^{\sigma d} \le \|\nabla u\|_{\mathrm{L}^2}^2/4 + C\|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)}$$

and $\beta(\sigma, d) = \frac{2\sigma(2-\sigma d)}{(\sigma+2)(2-\sigma d)+2\sigma(4\sigma+3)} \vee 2$. When evaluated at the deterministic solution, the modified Hamiltonian satisfies

$$0 \le \tilde{\mathbf{H}}(\mathbf{S}(u_0, 0)(t)) \le \tilde{\mathbf{H}}(u_0)\exp\left(-2\alpha\frac{3(\sigma+1)}{4\sigma+3}t\right). \tag{4.2}$$

Also, when the nonlinearity is defocusing, we now have, for every β positive,

$$0 \le \tilde{\mathbf{H}}(\mathbf{S}(u_0, 0)(t)) \le \tilde{\mathbf{H}}(u_0) \exp(-2\alpha t). \tag{4.3}$$

From the Sobolev inequalities, for ρ positive, the sets

$$\tilde{\mathbf{H}}_\rho = \{u \in \mathrm{H}^1 : \tilde{\mathbf{H}}(u) = \rho\} = \tilde{\mathbf{H}}^{-1}(\{\rho\}), \qquad \rho > 0,$$

are closed subsets of H^1 and

$$\tilde{\mathbf{H}}_{<\rho} = \{u \in \mathrm{H}^1 : \tilde{\mathbf{H}}(u) < \rho\} = \tilde{\mathbf{H}}^{-1}([0, \rho)), \qquad \rho > 0,$$

are open subsets of H^1.

Also, $\tilde{\mathbf{H}}$ is such that

$$\begin{aligned} &\|\nabla u\|_{\mathrm{L}^2}^2/2 + \beta C \|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)} \\ &\quad \le \tilde{\mathbf{H}}(u) \le 3\|\nabla u\|_{\mathrm{L}^2}^2/4 + (\beta + 1) C \|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)} \end{aligned} \tag{4.4}$$

when the nonlinearity is defocusing and

$$\begin{aligned} &\|\nabla u\|_{\mathrm{L}^2}^2/4 + C \|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)} \le \tilde{\mathbf{H}}(u) \\ &\quad \le \|\nabla u\|_{\mathrm{L}^2}^2/2 + \beta(\sigma, d) C \|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)} \end{aligned} \tag{4.5}$$

when it is focusing. Thus, the sets $\tilde{\mathbf{H}}_{<\rho}$ for ρ positive are bounded in H^1 and a bounded set in H^1 is bounded for $\tilde{\mathbf{H}}$. Note that the domain of attraction D may be a domain of the form $\tilde{\mathbf{H}}_{<\rho}$.

We no longer distinguish the focusing and defocusing cases and take the same value of β, that is, $\beta(\sigma, d)$. Also, to simplify the notation, we now sometimes drop the dependence of the solution on ε and u_0.

The fluctuation of $\tilde{\mathbf{H}}(u^{\varepsilon,u_0}(t))$ is of particular interest. We have the following result when the noise is of additive type.

PROPOSITION 4.1. *When u denotes the solution of equation* (2.1) *and $(e_j)_{j\in\mathbb{N}}$ is a complete orthonormal system on L^2, the following decomposition holds*:

$$\begin{aligned} \tilde{\mathbf{H}}(u(t)) = \tilde{\mathbf{H}}(u_0) &- 2\alpha \int_0^t \Psi(u(s))\, ds \\ &- 2\beta C\big(1 + 2\sigma/(2-\sigma d)\big)\alpha \int_0^t \|u(s)\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)}\, ds \\ &+ \sqrt{\varepsilon}\bigg(\mathfrak{Im} \int_{\mathbb{R}^d} \int_0^t \nabla \overline{u}(s) \nabla\, dW(s)\, dx \\ &\qquad - \lambda \mathfrak{Im} \int_{\mathbb{R}^d} \int_0^t |u(s)|^{2\sigma} \overline{u}(s)\, dW(s)\, dx \\ &\qquad + 2\beta C\big(1 + 2\sigma/(2-\sigma d)\big) \end{aligned}$$

$$
\begin{aligned}
&\qquad\times \mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \|u(s)\|_{\mathrm{L}^2}^{4\sigma/(2-\sigma d)}\overline{u}(s)\,dW(s)\,dx\Big)\\
&-(\lambda\varepsilon/2)\sum_{j\in\mathbb{N}}\int_0^t\int_{\mathbb{R}^d}[|u(s)|^{2\sigma}|\Phi e_j|^2\\
&\qquad\qquad + 2\sigma|u(s)|^{2\sigma-2}(\mathfrak{Re}(\overline{u}(s)\Phi e_j))^2]\,dx\,ds\\
&+(\varepsilon/2)\|\nabla\Phi\|_{\mathcal{L}_2^{0,0}}^2 t\\
&+\varepsilon\beta C\big(1+2\sigma/(2-\sigma d)\big)\|\Phi\|_{\mathcal{L}_2^{0,0}}^2\int_0^t \|u(s)\|_{\mathrm{L}^2}^{4\sigma/(2-\sigma d)}\,ds\\
&+\varepsilon\beta C\big(4\sigma/(2-\sigma d)\big)\big(1+2\sigma/(2-\sigma d)\big)\\
&\quad\times\sum_{j\in\mathbb{N}}\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2(2\sigma/(2-\sigma d)-1)}\left(\mathfrak{Re}\int_{\mathbb{R}^d}\overline{u}(s)\Phi e_j\,dx\right)^2 ds.
\end{aligned}
$$

PROOF. The result follows from the Itô formula. The main difficulty is in justifying the computations. We may proceed as in [5]. □

Also, when the noise is of multiplicative type, we obtain the following proposition.

PROPOSITION 4.2. *When u denotes the solution of equation (2.2) and $(e_j)_{j\in\mathbb{N}}$ is a complete orthonormal system on* L^2, *the following decomposition holds*:

$$
\begin{aligned}
\tilde{\mathbf{H}}(u(t)) = \tilde{\mathbf{H}}(u_0) &- 2\alpha\int_0^t \Psi(u(s))\,ds\\
&- 2\beta C\big(1+2\sigma/(2-\sigma d)\big)\alpha\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)}\,ds\\
&+\sqrt{\varepsilon}\,\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t u(s)\nabla\overline{u}(s)\nabla\,dW(s)\,dx\\
&+(\varepsilon/2)\sum_{j\in\mathbb{N}}\int_0^t\int_{\mathbb{R}^d}|u(s)|^2|\nabla\Phi e_j|^2\,dx\,ds.
\end{aligned}
$$

The first exit time τ^{ε,u_0} from the domain D in H^1 is defined as in Section 2. We also define

$$\overline{e} = \inf\{I_T^0(w) : w(T)\in\overline{D}^c,\ T>0\}$$

and, for ρ positive and small enough,

$$e_\rho = \inf\{I_T^{u_0}(w) : \tilde{\mathbf{H}}(u_0)\le\rho,\ w(T)\in(D_{-\rho})^c,\ T>0\},$$

where $D_{-\rho} = D \setminus \mathcal{N}^1(\partial D, \rho)$. We then set

$$\underline{e} = \lim_{\rho \to 0} e_\rho.$$

Also, for ρ positive and small enough and N a closed subset of the boundary of D, we define

$$e_{N,\rho} = \inf\{I_T^{u_0}(w) : \tilde{\mathbf{H}}(u_0) \le \rho,\ w(T) \in (D \setminus \mathcal{N}^1(N, \rho))^c,\ T > 0\}$$

and

$$\underline{e}_N = \lim_{\rho \to 0} e_{N,\rho}.$$

We also finally introduce

$$\sigma_\rho^{\varepsilon,u_0} = \inf\{t \ge 0 : u^{\varepsilon,u_0}(t) \in \tilde{\mathbf{H}}_{<\rho} \cup D^c\},$$

where $\tilde{\mathbf{H}}_{<\rho} \subset D$.

Again, we have the following inequalities.

LEMMA 4.3. $0 < \underline{e} \le \overline{e}$.

PROOF. We need only prove the first inequality. Integrating the equation describing the evolution of $\tilde{\mathbf{H}}(\mathbf{S}(u_0, h)(t))$ via the Duhamel formula, where the skeleton is that of the equation with an additive noise, we obtain

$$\begin{aligned}
&\tilde{\mathbf{H}}(\mathbf{S}(u_0, h)(T)) - \exp\Bigl(-2\alpha \frac{3(\sigma+1)}{4\sigma+3} T\Bigr)\tilde{\mathbf{H}}(u_0) \\
&\quad \le \int_0^T \exp\Bigl(-2\alpha \frac{3(\sigma+1)}{4\sigma+3}(T-s)\Bigr) \\
&\qquad\quad \times \biggl[\mathfrak{Im} \int_{\mathbb{R}^d} (\nabla \mathbf{S}(u_0, h) \nabla \overline{\Phi h})(s, x)\, dx \\
&\qquad\qquad - \lambda \mathfrak{Im} \int_{\mathbb{R}^d} (|\mathbf{S}(u_0, h)|^{2\sigma} \mathbf{S}(u_0, h) \overline{\Phi h})(s, x)\, dx \\
&\qquad\qquad - 2C\beta\bigl(1 + 2\sigma/(2 - \sigma d)\bigr) \mathfrak{Im} \int_{\mathbb{R}^d} (\mathbf{S}(u_0, h) \overline{\Phi h})(s, x)\, dx\biggr] ds,
\end{aligned}$$

with a focusing or defocusing nonlinearity. Let d denote the positive distance between 0 and ∂D. Take ρ such that the distance between B_ρ^1 and $(D_{-\rho})^c$ is larger than $d/2$. We then have, from the Sobolev injection of H^1 into $\mathrm{L}^{2\sigma+2}$,

$$
\begin{aligned}
d/2 \leq \int_0^T & \exp\Big(-2\alpha \frac{3(\sigma+1)}{4\sigma+3}(T-s)\Big) \\
& \times \big[R\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{H}^1)}\|h\|_{\mathrm{L}^2} \\
& \quad + CR^{2\sigma+1}\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{H}^1)}\|h\|_{\mathrm{L}^2} \\
& \quad + 2C\beta\big(1+2\sigma/(2-\sigma d)\big)R\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}\|h\|_{\mathrm{L}^2}\big]\,ds.
\end{aligned}
$$

We conclude as in Lemma 3.1 and use the fact that from the choice of β, the complement of a ball is included in the complement of a set $\tilde{\mathbf{H}}_{<a}$. In the case of the skeleton of the equation with a multiplicative noise, it is enough to replace the bracketed term in the right-hand side of the above formula by $\mathfrak{Im}\int_{\mathbb{R}^d}(\nabla\mathbf{S}(u_0,h)\overline{\mathbf{S}(u_0,h)\nabla\Phi h})(s,x)\,dx$. Recall that we can proceed as in the additive case since we have required that Φ belongs to $\mathcal{L}^{0,s}_{2,\mathbb{R}}$, where $s > d/2+1$. In particular, Φ belongs to $\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty})$. $\square$

4.2. *Statement of the results.* The theorems of Section 2 still hold for a domain of attraction in H^1 and a noise of additive or multiplicative type.

THEOREM 4.4. *For every u_0 in D and δ positive, there exists L positive such that*

$$\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbb{P}\big(\tau^{\varepsilon,u_0}\notin\big(\exp\big((\underline{e}-\delta)/\varepsilon\big),\exp\big((\overline{e}+\delta)/\varepsilon\big)\big)\big)\leq -L, \tag{4.6}$$

and for every u_0 in D,

$$\underline{e}\leq\underline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbb{E}(\tau^{\varepsilon,u_0})\leq\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbb{E}(\tau^{\varepsilon,u_0})\leq\overline{e}. \tag{4.7}$$

Moreover, for every δ positive, there exists L positive such that

$$\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\sup_{u_0\in D}\mathbb{P}\big(\tau^{\varepsilon,u_0}\geq\exp\big((\overline{e}+\delta)/\varepsilon\big)\big)\leq -L \tag{4.8}$$

and

$$\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\sup_{u_0\in D}\mathbb{E}(\tau^{\varepsilon,u_0})\leq\overline{e}. \tag{4.9}$$

REMARK 4.5. Again, the control argument to prove that $\underline{e}=\overline{e}$ seems difficult. It should be even more difficult for multiplicative noises.

THEOREM 4.6. *If $\underline{e}_N>\overline{e}$, then for every u_0 in D, there exists L positive such that*

$$\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbb{P}\big(u^{\varepsilon,u_0}(\tau^{\varepsilon,u_0})\in N\big)\leq -L.$$

Again, we may deduce the corollary

COROLLARY 4.7. *Assume that v^* in ∂D is such that for every δ positive and $N = \{v \in \partial D : \|v - v^*\|_{\mathrm{L}^2} \geq \delta\}$, we have $\underline{e}_N > \overline{e}$. Then,*

$$\forall \delta > 0,\ \forall u_0 \in D,\ \exists L > 0 : \overline{\lim_{\varepsilon \to 0}}\, \varepsilon \log \mathbb{P}\big(\|u^{\varepsilon,u_0}(\tau^{\varepsilon,u_0}) - v^*\|_{\mathrm{L}^2} \geq \delta\big) \leq -L.$$

4.3. *Proof of the results.* The proofs of these results still rely on three lemmas and the uniform LDP. Let us now state the lemmas for both a noise of additive and of multiplicative type.

LEMMA 4.8. *For every ρ and L positive with $\tilde{\mathbf{H}}_{<\rho} \subset D$, there exist T and ε_0 positive such that for every u_0 in D and ε in $(0, \varepsilon_0)$,*

$$\mathbb{P}(\sigma_\rho^{\varepsilon,u_0} > T) \leq \exp(-L/\varepsilon).$$

PROOF. We proceed as in the proof of Lemma 3.6.

Let d denote the positive distance between 0 and $D \setminus \tilde{\mathbf{H}}_{<\rho}$. Take α positive such that $\alpha\rho < d$. The domain D is uniformly attracted to 0, thus there exists a time T_1 such that for every initial datum u_1 in $\mathcal{N}^1(D \setminus \tilde{\mathbf{H}}_{<\rho}, \alpha\rho/8)$, for $t \geq T_1$, $\mathbf{S}(u_1, 0)(t)$ belongs to $B^1_{\alpha\rho/8}$.

We could also prove (see [5]) that there exists a constant M' which depends on T_1, R, σ and α such that

$$\sup_{u_1 \in \mathcal{N}^1(D \setminus \tilde{\mathbf{H}}_{<\rho}, \alpha\rho/8)} \|\mathbf{S}(u_1, 0)\|_{X^{(T_1, 2\sigma+2)}} \leq M'. \tag{4.10}$$

Step 2, corresponding to that of Lemma 3.6, in the proof of the additive case uses the truncation argument and an upper bound similar to that in [5] derived from the Strichartz inequalities on smaller intervals. We shall also replace $\rho/8$ by $\alpha\rho/8$ in the proof of Lemma 3.6.

In Step 2 for the multiplicative case, we also introduce the truncation in front of the term $u\Phi h$ in the controlled PDE.

The end of the proof is identical to that of Lemma 3.6. The LDP is the LDP in $\mathrm{C}([0, T]; \mathrm{H}^1)$ for additive or multiplicative noises. □

LEMMA 4.9. *For every ρ positive such that $\tilde{\mathbf{H}}_\rho \subset D$ and u_0 in D, there exists L positive such that*

$$\overline{\lim_{\varepsilon \to 0}}\, \varepsilon \log \mathbb{P}\big(u^{\varepsilon,u_0}(\sigma_\rho^{\varepsilon,u_0}) \in \partial D\big) \leq -L.$$

PROOF. It is the same proof as for Lemma 3.7. We have only to replace $B^0_{\rho/2}$ by any ball in H^1 centered at 0 and included in $\tilde{\mathbf{H}}_{<\rho}$ and to use the LDP in $\mathrm{C}([0, T]; \mathrm{H}^1)$. □

LEMMA 4.10. *For every ρ and L positive such that $\tilde{\mathbf{H}}_{2\rho} \subset D$, there exists $T(L,\rho) < \infty$ such that*

$$\overline{\lim_{\varepsilon \to 0}}\, \varepsilon \log \sup_{u_0 \in \tilde{\mathbf{H}}_\rho} \mathbb{P}\Bigl(\sup_{t \in [0,T(L,\rho)]} \bigl(\tilde{\mathbf{H}}(u^{\varepsilon,u_0}(t)) - \tilde{\mathbf{H}}(u_0)\bigr) \geq \rho \Bigr) \leq -L.$$

PROOF. Integrating the Itô differential relation using the Duhamel formula allows for the removal of the drift term that does not originate from the bracket. Indeed, the event

$$\Bigl\{ \sup_{t \in [0,T(L,\rho)]} \bigl(\tilde{\mathbf{H}}(u^{\varepsilon,u_0}(t)) - \tilde{\mathbf{H}}(u_0)\bigr) \geq \rho \Bigr\}$$

is included in

$$\Bigl\{ \sup_{t \in [0,T(L,\rho)]} \Bigl(\tilde{\mathbf{H}}(u^{\varepsilon,u_0}(t)) - \exp\Bigl(-2\alpha\Bigl(\frac{3(\sigma+1)}{4\sigma+3}\Bigr)T(L,\rho)\Bigr)\tilde{\mathbf{H}}(u_0)\Bigr) \geq \rho \Bigr\}.$$

Then, setting $c(\sigma) = \frac{3(\sigma+1)}{4\sigma+3}$ and $m(\sigma,d) = 1 + 2\sigma/(2-\sigma d)$, and dropping the exponents ε and u_0 to have more concise formulas, we obtain, in the additive case,

$$\begin{aligned}
&\tilde{\mathbf{H}}(u(t)) - \exp(-2\alpha c(\sigma)t)\tilde{\mathbf{H}}(u_0) \\
&\quad \leq \sqrt{\varepsilon}\Bigl(\mathfrak{Im} \int_{\mathbb{R}^d} \int_0^t \exp\bigl(-2\alpha c(\sigma)(t-s)\bigr) \nabla \overline{u}(s) \nabla\, dW(s)\, dx \\
&\qquad\quad - \lambda \mathfrak{Im} \int_{\mathbb{R}^d} \int_0^t \exp\bigl(-2\alpha c(\sigma)(t-s)\bigr) |u(s)|^{2\sigma} \overline{u}(s)\, dW(s)\, dx \\
&\qquad\quad + 2\beta C m(\sigma,d) \mathfrak{Im} \int_{\mathbb{R}^d} \int_0^t \exp\bigl(-2\alpha c(\sigma)(t-s)\bigr) \\
&\qquad\qquad\qquad\qquad \times \|u(s)\|_{\mathrm{L}^2}^{4\sigma/(2-\sigma d)} \overline{u}(s)\, dW(s)\, dx \Bigr) \\
&\qquad\quad - (\lambda\varepsilon/2) \sum_{j \in \mathbb{N}} \int_0^t \exp\bigl(-2\alpha c(\sigma)(t-s)\bigr) \\
&\qquad\qquad\quad \times \int_{\mathbb{R}^d} [|u(s)|^{2\sigma} |\Phi e_j|^2 \\
&\qquad\qquad\qquad\qquad + 2\sigma |u(s)|^{2\sigma-2} (\mathfrak{Re}(\overline{u}(s)\Phi e_j))^2]\, dx\, ds \\
&\qquad\quad + \bigl(\varepsilon/(4\alpha c(\sigma))\bigr)\bigl(1 - \exp(-2\alpha c(\sigma)t)\bigr) \|\nabla\Phi\|_{\mathcal{L}_2^{0,0}}^2 \\
&\qquad\quad + \varepsilon\beta C m(\sigma,d) \Bigl(\|\Phi\|_{\mathcal{L}_2^{0,0}}^2 \int_0^t \exp\bigl(-2\alpha c(\sigma)(t-s)\bigr) \|u(s)\|_{\mathrm{L}^2}^{4\sigma/(2-\sigma d)}\, ds \\
&\qquad\qquad\quad + \bigl(4\sigma/(2-\sigma d)\bigr)
\end{aligned}$$

$$\times \sum_{j\in\mathbb{N}} \int_0^t \exp\bigl(-2\alpha c(\sigma)(t-s)\bigr) \times \|u(s)\|_{\mathrm{L}^2}^{2(2\sigma/(2-\sigma d)-1)} \times \left(\mathfrak{Re}\int_{\mathbb{R}^d} \overline{u}(s)\Phi e_j dx\right)^2 ds\biggr).$$

We again use a localization argument and replace the process u by the process u^τ stopped at the first exit time from $\tilde{\mathbf{H}}_{<2\rho}$. We use (4.4) and (4.5) and obtain

$$\|u^\tau\|_{\mathrm{H}^1}^2 \le 8\rho + \bigl(2\rho/(C\sigma)\bigr)^{1/(1+2\sigma/(2-\sigma d))}.$$

We denote the right-hand side of the above by $b(\rho,\sigma,d)$.

From the Hölder inequality, along with the Sobolev injection of H^1 into $\mathrm{L}^{2\sigma+2}$, we obtain the following upper bound for the drift:

$$\begin{aligned}&\bigl(\varepsilon/(4\alpha c(\sigma))\bigr)[(1+2\sigma)c(1,2\sigma+2)^{2\sigma+2}\|\Phi\|_{\mathcal{L}_2^{0,1}}^2 b(\rho,\sigma,d)^{2\sigma} + \|\nabla\Phi\|_{\mathcal{L}_2^{0,0}}^2]\\ &\quad + m(\sigma,d)\bigl(\varepsilon\beta C/(2\alpha c(\sigma))\bigr)\bigl(1+4\sigma/(2-\sigma d)\bigr)\|\Phi\|_{\mathcal{L}_2^{0,0}}^2 b(\rho,\sigma,d)^{4\sigma/(2-\sigma d)},\end{aligned}$$

where we denote by $c(1,2\sigma+2)$ the norm of the continuous injection of H^1 into $\mathrm{L}^{2\sigma+2}$.

Thus, choosing ε small enough, it is enough to show the result for the stochastic integral replacing ρ by $\rho/2$. Also, it is enough to show the result for each of the three stochastic integrals replacing $\rho/2$ by $\rho/6$. With the same one-parameter families and similar computations as in the proof of Lemma 3.8, we know that it is enough to obtain upper bounds of the brackets of the stochastic integrals

$$\begin{aligned}Z_1(t) &= \mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp(2\alpha c(\sigma)s)\nabla\overline{u^\tau}(s)\nabla\, dW(s)\, dx,\\ Z_2(t) &= \mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp(2\alpha c(\sigma)s)|u^\tau(s)|^{2\sigma}\overline{u^\tau}(s)\, dW(s)\, dx,\\ Z_3(t) &= 2\beta C m(\sigma,d)\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp(2\alpha c(\sigma)s)\\ &\qquad\qquad \times \|u^\tau(s)\|_{\mathrm{L}^2}^{4\sigma/(2-\sigma d)}\overline{u^\tau}(s)\, dW(s)\, dx.\end{aligned}$$

We then obtain

$$\begin{aligned}d\langle Z_1\rangle_t &\le \exp(4\alpha c(\sigma)t)\sum_{j\in\mathbb{N}}(\nabla u^\tau(t), -i\nabla\Phi e_j)_{\mathrm{L}^2}^2\, dt,\\ d\langle Z_2\rangle_t &\le \exp(4\alpha c(\sigma)t)\sum_{j\in\mathbb{N}}(|u^\tau(t)|^{2\sigma}u^\tau(t), -i\Phi e_j)_{\mathrm{L}^2}^2\, dt,\end{aligned}$$

$$d\langle Z_3\rangle_t \le 4\beta^2 C^2 m(\sigma,d)^2 \exp(4\alpha c(\sigma)t)\|u^\tau(t)\|_{\mathrm{L}^2}^{8\sigma/(2-\sigma d)} \times \sum_{j\in\mathbb{N}} (u^\tau(t), -i\Phi e_j)_{\mathrm{L}^2}^2\, dt.$$

Using the Hölder inequality and, for Z_2, the continuous Sobolev injection of H^1 into $\mathrm{L}^{2\sigma+2}$, we obtain

$$d\langle Z_1\rangle_t \le \exp(4\alpha c(\sigma)t)\|\Phi\|_{\mathcal{L}_2^{0,1}}^2 b(\rho,\sigma,d)\, dt,$$
$$d\langle Z_2\rangle_t \le \exp(4\alpha c(\sigma)t) c(1, 2\sigma+2)^{2(2\sigma+2)}\|\Phi\|_{\mathcal{L}_2^{0,1}}^2 b(\rho,\sigma,d)^{2\sigma+1}\, dt,$$
$$d\langle Z_3\rangle_t \le 4\beta^2 C^2 m(\sigma,d)^2 \exp(4\alpha c(\sigma)t) b(\rho,\sigma,d)^{(1+4\sigma/(2-\sigma d))}\|\Phi\|_{\mathcal{L}_2^{0,1}}^2\, dt.$$

We can then bound each of the three remainders $(R_l^i(t))_{i=1,2,3}$ similarly to what was done in the proof of Lemma 3.8, using the inequality $R_l^i(t) \le 3l\int_0^t d\langle Z_i\rangle_t$.

We conclude that it is possible to choose $T(L,\rho)$ equal to

$$\frac{1}{4\alpha c(\sigma)}\log\left(\frac{\alpha c(\sigma)\rho^2}{90 b(\rho,\sigma,d)\|\Phi\|_{\mathcal{L}_2^{0,1}}^2 \max(1, c(1,2\sigma+2)^{2(2\sigma+1)} b(\rho,\sigma,d)^{2\sigma}, 4\beta^2 C^2 m(\sigma,d)^2 b(\rho,\sigma,d)^{4\sigma/(2-\sigma d)})}\right).$$

When the noise is of multiplicative type, we obtain

$$\begin{aligned}\tilde{\mathbf{H}}(u(t)) &- \exp(-2\alpha c(\sigma)t)\tilde{\mathbf{H}}(u_0)\\ &\le \sqrt{\varepsilon}\,\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp\bigl(-2\alpha c(\sigma)(t-s)\bigr) u(s)\nabla\overline{u}(s)\nabla\, dW(s)\, dx\\ &\quad + (\varepsilon/2)\sum_{j\in\mathbb{N}}\int_0^t \exp\bigl(-2\alpha c(\sigma)(t-s)\bigr)\int_{\mathbb{R}^d}|u(s)|^2|\nabla\Phi e_j|^2\, dx\, ds.\end{aligned}$$

Again, we use a localization argument and consider the process u stopped at the exit from $\tilde{\mathbf{H}}_{2\rho}$. As Φ is Hilbert–Schmidt from L^2 into $\mathrm{H}^s_{\mathbb{R}}$, the second term of the right-hand side is less than $\frac{\varepsilon}{4\alpha c(\sigma)}\|\Phi\|_{\mathcal{L}_2^{0,s}}^2 b(\rho,\sigma,d)$ and, for ε small enough, it is enough to prove the result for the stochastic integral replacing ρ by $\rho/2$. We know that it is enough to obtain an upper bound of the bracket of

$$Z(t) = \mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp(2\alpha c(\sigma)s) u^\tau(s)\nabla\overline{u}^\tau(s)\nabla\, dW(s)\, dx.$$

We obtain

$$d\langle Z\rangle_t \le \exp(4\alpha c(\sigma)t)\sum_{j\in\mathbb{N}}(\nabla u^\tau(t), -iu^\tau(t)\nabla\Phi e_j)_{\mathrm{L}^2}^2\, dt.$$

Denoting by $c(s,\infty)$ the norm of the Sobolev injection of $\mathrm{H}^s_{\mathbb{R}}$ into $\mathrm{W}^{1,\infty}_{\mathbb{R}}$, we deduce that

$$d\langle Z\rangle_t \le \exp(4\alpha c(\sigma)t) c(s,\infty)^2\|\Phi\|_{\mathcal{L}_2^{0,s}}^2 b(\rho,\sigma,d)^2\, dt.$$

Finally, we conclude that we may choose

$$T(L,\rho)=\frac{1}{4\alpha c(\sigma)}\log\left(\frac{\alpha c(\sigma)\rho^2}{10b(\rho,\sigma,d)^2c(s,\infty)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}}L}\right). \qquad \square$$

We may now prove Theorems 4.6 and 4.7.

ELEMENTS OF THE PROOF OF THEOREM 4.6. There is no difference in the proof of the upper bound on τ^{ε,u_0}. Let us thus focus on the lower bound. Take δ positive. Since $\underline{e}>0$, we now choose ρ positive such that $\underline{e}-\delta/4\le e_\rho$, $\tilde{\mathbf{H}}_{2\rho}\subset D$ and $\tilde{\mathbf{H}}_{2\rho}\subset D^c_{-\rho}$. We define the sequences of stopping times $\theta_0=0$ and, for k in $\mathbb{N}$,

$$\tau_k=\inf\{t\ge\theta_k:u^{\varepsilon,u_0}(t)\in\tilde{\mathbf{H}}_{<\rho}\cup D^c\},$$
$$\theta_{k+1}=\inf\{t>\tau_k:u^{\varepsilon,u_0}(t)\in\tilde{\mathbf{H}}_{2\rho}\},$$

where $\theta_{k+1}=\infty$ if $u^{\varepsilon,u_0}(\tau_k)\in\partial D$. Let us fix $T_1=T(\underline{e}-3\delta/4,\rho)$, given by Lemma 4.10. We now use the fact that for u_0 in D and m a positive integer,

$$\mathbb{P}(\tau^{\varepsilon,u_0}\le mT_1)\le\mathbb{P}(\tau^{\varepsilon,u_0}=\tau_0)+\sum_{k=1}^m\mathbb{P}(\tau^{\varepsilon,u_0}=\tau_k)+\sum_{k=1}^m\mathbb{P}(\theta_k-\tau_{k-1}\le T_1) \tag{4.11}$$

and conclude as in the proof of Theorem 3.3. $\square$

We may check that the proof of Theorem 3.4 also applies to Theorem 4.6. The LDP's are those in H^1 and the sequences of stopping times are those defined above.

REMARK 4.11. In [12], reaction-diffusion equations perturbed by an additive white noise are considered. When the space dimension is larger than one, the case where the vector field can be decomposed into a gradient and a second field which is orthogonal is treated. The quasi-potential is then equal to the potential at the endpoint. It again involves a control argument. In our case, since we consider colored noises and nonlinear equations, the orthogonality is lost for the geometry of the reproducing kernel Hilbert space of the law of $W(1)$. We thus obtain extra commutator terms. Under suitable assumptions on the space correlations of the noise, going to zero, it is possible that we obtain a nontrivial minimization problem. Recall that solitary waves are solutions of variational problems where we minimize the Hamiltonian for fixed levels of the mass.

APPENDIX: PROOF OF THEOREM 2.1

The following lemma is at the core of the proof of the uniform LDP's. It is often called the Azencott lemma or the Freidlin–Wentzell inequality. The differences with the result of [16] are that here, the initial data are the same for the random process and the skeleton and that the "for every ρ positive" precedes the "there exists ε_0 and γ positive." We shall only stress the differences in the proof.

LEMMA A.1. *For every a, L, T, δ and ρ positive, f in C_a and p in $\mathcal{A}(d)$, there exist ε_0 and γ positive such that for every ε in $(0, \varepsilon_0)$, $\|u_0\|_{\mathrm{H}^1} \le \rho$,*

$$\varepsilon \log \mathbb{P}\big(\|u^{\varepsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\|_{X^{(T,p)}} \ge \delta;\ \|\sqrt{\varepsilon} W - f\|_{\mathrm{C}([0,T];\mathrm{H}^s_{\mathbb{R}})} < \gamma\big) \le -L.$$

ELEMENTS OF THE PROOF. There are still three steps in the proof of this result. The first step is a change of measure to center the process around f. It uses the Girsanov theorem and is the same as in [16].

The second step is a reduction to estimates for the stochastic convolution. It strongly involves the Strichartz inequalities, but differs slightly from that in [16]. The truncation argument must hold for all $\|u_0\|_{\mathrm{H}^1} \le \rho$. Thus, we use the fact that there exists $M = M(T, \rho, \sigma)$ positive such that

$$\sup_{u_1 \in B^1_\rho} \|\tilde{\mathbf{S}}(u_1, f)\|_{X^{(T,p)}} \le M.$$

The proof of this fact follows from the computations in [5]; we have recalled the arguments in L^2 in the proof of Lemma 3.6. The result in H^1 is again used in the proof of Lemma 4.8. As the initial data are the same for the random process and the skeleton, the remainder of the argument does not require restrictions on ρ.

The third step corresponds to estimates for the stochastic convolution. It is the same as in [16].

The extra damping term in the drift is easily treated using the Strichartz inequalities. $\square$

ELEMENTS OF THE PROOF OF THEOREM 2.1. Let us start with the case of an additive noise. Recall that, in that case, the mild solution of the stochastic equation could be written as a function of the perturbation in the convolution form. Let $v^{u_0}(Z)$ denote the solution of

$$\begin{cases} i\dfrac{\partial v}{\partial t} - \big(\Delta v + |v - iZ|^{2\sigma}(v - iZ) - i\alpha(v - iZ)\big) = 0, \\ v(0) = u_0 \end{cases}$$

or, equivalently, a fixed point of the functional $\mathcal{F}_Z$ such that

$$\begin{aligned}\mathcal{F}_Z(v)(t) = U(t)u_0 &- i\lambda \int_0^t U(t-s)\big(|(v - iZ)(s)|^{2\sigma}(v - iZ)(s)\big)\,ds \\ &- \alpha \int_0^t U(t-s)(v - iZ)(s)\,ds,\end{aligned}$$

where Z belongs to $\mathrm{C}([0,T];\mathrm{L}^2)$ (resp. $\mathrm{C}([0,T];\mathrm{H}^1)$). If u^{ε,u_0} is defined as $u^{\varepsilon,u_0} = v^{u_0}(Z^\varepsilon) - iZ^\varepsilon$, where Z^ε is the stochastic convolution $Z^\varepsilon(t) = \sqrt{\varepsilon}\int_0^t U(t-s)\,dW(s)$, then u^{ε,u_0} is a solution of the stochastic equation. Consequently, if $\mathcal{G}(\cdot,u_0)$ denotes the mapping from $\mathrm{C}([0,T];\mathrm{L}^2)$ (resp. $\mathrm{C}([0,T];\mathrm{H}^1)$) to $\mathrm{C}([0,T];\mathrm{L}^2)$ (resp. $\mathrm{C}([0,T];\mathrm{H}^1)$) defined by $\mathcal{G}(Z,u_0) = v^{u_0}(Z) - iZ$, then we obtain $u^{\varepsilon,u_0} = \mathcal{G}(Z^\varepsilon,u_0)$. We may also check with arguments, similar to those of [5, 15], involving the Strichartz inequalities that the mapping $\mathcal{G}$ is equicontinuous in its first arguments for second arguments in bounded sets of L^2 (resp. H^1). The result now follows from Proposition 5 in [23].

Let us now consider the case of a multiplicative noise. Initial data belong to H^1 and we consider paths in H^1. The proof is very close to that in [16].

The main tool is again the Azencott lemma or the almost continuity of the Itô map. We need a slightly different result from that in [16].

Let us see how the above lemma implies (i) and (ii).

We start with the upper bound (i). Take a, ρ, T and δ positive. Take $L > a$. For $\tilde{a}$ in $(0,a]$, we define

$$A_{\tilde{a}}^{u_0} = \{v \in \mathrm{C}([0,T];\mathrm{H}^1) : d_{\mathrm{C}([0,T];\mathrm{H}^1)}(v, K_T^{u_0}(\tilde{a})) \ge \delta\}.$$

Note that we have $A_a^{u_0} \subset A_{\tilde{a}}^{u_0}$ and $C_{\tilde{a}} \subset C_a$. Take $\tilde{a} \in (0,a]$ and f such that $I_T^W(f) < \tilde{a}$.

We shall now apply the Azencott lemma and choose $p = 2$. We obtain $\varepsilon_{\rho,f,\delta}$ and $\gamma_{\rho,f,\delta}$ positive such that for every $\varepsilon \le \varepsilon_{\rho,f,\delta}$ and u_0 such that $\|u_0\|_{\mathrm{H}^1} \le \rho$,

$$\varepsilon \log \mathbb{P}\big(\|u^{\varepsilon,u_0} - \tilde{\mathbf{S}}(u_0,f)\|_{X^{(T,p)}} \ge \delta;\ \|\sqrt{\varepsilon}W - f\|_{\mathrm{C}([0,T];\mathrm{H}^s_{\mathbb{R}})} < \gamma_{\rho,f,\delta}\big) \le -L.$$

Let us denote by $O_{\rho,f,\delta}$ the set $O_{\rho,f,\delta} = B_{\mathrm{C}([0,T];\mathrm{H}^s_{\mathbb{R}})}(f,\gamma_{\rho,f,\delta})$. The family $(O_{\rho,f,\delta})_{f\in C_a}$ is a covering by open sets of the compact set C_a, thus there exists a finite subcovering of the form $\bigcup_{i=1}^N O_{\rho,f_i,\delta}$. We can now write

$$\begin{aligned}
&\mathbb{P}(u^{\varepsilon,u_0} \in A_{\tilde{a}}^{u_0}) \\
&\quad \le \mathbb{P}\left(\{u^{\varepsilon,u_0} \in A_{\tilde{a}}^{u_0}\} \cap \left\{\sqrt{\varepsilon}W \in \bigcup_{i=1}^N O_{\rho,f_i,\delta}\right\}\right) \\
&\qquad + \mathbb{P}\left(\sqrt{\varepsilon}W \notin \bigcup_{i=1}^N O_{\rho,f_i,\delta}\right) \\
&\quad \le \sum_{i=1}^N \mathbb{P}\big(\{u^{\varepsilon,u_0} \in A_{\tilde{a}}^{u_0}\} \cap \{\sqrt{\varepsilon}W \in O_{\rho,f_i,\delta}\}\big) \\
&\qquad + \mathbb{P}\big(\sqrt{\varepsilon}W \notin C_a\big)
\end{aligned}$$

$$\leq \sum_{i=1}^{N} \mathbb{P}\big(\{\|u^{\varepsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\|_{X^{(T,p)}} \geq \delta\} \cap \{\sqrt{\varepsilon} W \in O_{\rho,f_i,\delta}\}\big)$$
$$+ \exp(-a/\varepsilon)$$

for $\varepsilon \leq \varepsilon_0$, for some positive ε_0. We have used the fact that

$$d_{\mathrm{C}([0,T];\mathrm{H}^1)}(\tilde{\mathbf{S}}(u_0, f), A_{\tilde{a}}^{u_0}) \geq \delta,$$

which is a consequence of the definition of the sets $A_{\tilde{a}}^{u_0}$.

As a consequence, for $\varepsilon \leq \varepsilon_0 \wedge (\min_{i=1,\dots,N} \varepsilon_{u_0,f_i})$, we obtain, for u_0 in B_ρ^1, that

$$\mathbb{P}(u^{\varepsilon,u_0} \in A_{\tilde{a}}^{u_0}) \leq N \exp(-L/\varepsilon) + \exp(-a/\varepsilon)$$

and, for ε_1 small enough, for every $\varepsilon \in (0, \varepsilon_1)$, that

$$\varepsilon \log \mathbb{P}(u^{\varepsilon,u_0} \in A_{\tilde{a}}^{u_0}) \leq \varepsilon \log 2 + (\varepsilon \log N - L) \vee (-a).$$

If ε_1 is also chosen such that $\varepsilon_1 < \frac{\gamma}{\log(2)} \wedge \frac{L-a}{\log(N)}$, we obtain

$$\varepsilon \log \mathbb{P}(u^{\varepsilon,u_0} \in A_{\tilde{a}}^{u_0}) \leq -\tilde{a} - \gamma,$$

which holds for every u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$.

We now consider the lower bound (ii). Take a, ρ, T and δ positive. The continuity of $\tilde{\mathbf{S}}(u_0, \cdot)$, to be proven as in [16], along with the compactness of C_a, give that for u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$ and w in $K_T^{u_0}(a)$, there exists f such that $w = \tilde{\mathbf{S}}(u_0, f)$ and $I_T^{u_0}(w) = I_T^W(f)$. Take $L > I^{u_0}(w)$. Choose $\varepsilon_{\rho,f,\delta}$ positive and $O_{\rho,f,\delta}$, the ball centered at f of radius $\gamma_{\rho,f,\delta}$ as defined previously, such that, for every $\varepsilon \leq \varepsilon_{\rho,f,\delta}$ and u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$,

$$\varepsilon \log \mathbb{P}\big(\|u^{\varepsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\|_{X^{(T,p)}} \geq \delta; \|\sqrt{\varepsilon} W - f\|_{\mathrm{C}([0,T];\mathrm{H}^s_{\mathbb{R}})} < \gamma_{\rho,f,\delta}\big) \leq -L.$$

We obtain

$$\begin{aligned}
&\exp\big(-I_T^W(f)/\varepsilon\big) \\
&\quad \leq \mathbb{P}\big(\sqrt{\varepsilon} W \in O_{\rho,f,\delta}\big) \\
&\quad \leq \mathbb{P}\big(\{\|u^{\varepsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\|_{X^{(T,p)}} \geq \delta\} \cap \{\sqrt{\varepsilon} W \in O_{\rho,f,\delta}\}\big) \\
&\quad\quad + \mathbb{P}\big(\|u^{\varepsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\|_{X^{(T,p)}} < \delta\big).
\end{aligned}$$

Thus, for $\varepsilon \leq \varepsilon_{\rho,f,\delta}$ and every u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$,

$$-I^{u_0}(w) \leq \varepsilon \log 2 + \big(\varepsilon \log \mathbb{P}\big(\|u^{\varepsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\|_{X^{(T,p)}} < \delta\big)\big) \vee (-L)$$

and for ε_1 small enough and such that $\varepsilon_1 \log(2) < \gamma$, for every positive ε such that $\varepsilon < \varepsilon_1$ and every u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$,

$$-I^{u_0}(w) - \gamma \leq \varepsilon \log \mathbb{P}\big(\|u^{\varepsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\|_{X^{(T,p)}} < \delta\big).$$

This completes the proof of (i) and (ii). $\square$

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