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SLOW MOVEMENT OF RANDOM WALK IN RANDOM ENVIRONMENT ON A REGULAR TREE

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We consider a recurrent random walk in a random environment on a regular tree. Under suitable general assumptions concerning the distribution of the environment, we show that the walk exhibits an unusually slow movement: the order of magnitude of the walk in the first n steps is $(\log n)^3$.

1. Introduction. Let $\mathbb{T}$ be a rooted b-ary tree, with $b \ge 2$. Let $\omega := (\omega(x, y), x, y \in \mathbb{T})$ be a collection of nonnegative random variables such that $\sum_{y\in\mathbb{T}} \omega(x, y) = 1$ for any $x \in \mathbb{T}$. Given ω, we define a Markov chain $X := (X_n, n \ge 0)$ on $\mathbb{T}$ with $X_0 = e$ and

$$P_\omega(X_{n+1} = y | X_n = x) = \omega(x, y).$$

The process X is called a random walk in a random environment (or simply RWRE) on $\mathbb{T}$. (By informally taking $b = 1$, X would become a usual RWRE on the half-line $\mathbb{Z}_+$.)

We refer to page 106 of [19] for a list of motivations for studying tree-valued RWREs. For information on a close connection between tree-valued RWREs and Mandelbrot's multiplicative cascades, see [16].

We use $\mathbf{P}$ to denote the law of ω and the semiproduct measure $\mathbb{P}(\cdot) := \int P_\omega(\cdot)\mathbf{P}(d\omega)$ to denote the averaged over the environment.

Some basic notation for the tree is in order. Let e denote the root of $\mathbb{T}$. For any vertex $x \in \mathbb{T} \setminus \{e\}$, let $\overleftarrow{x}$ denote the parent of x. As such, each vertex $x \in \mathbb{T} \setminus \{e\}$ has one parent $\overleftarrow{x}$ and b children, whereas the root e has b children, but no parent. For any $x \in \mathbb{T}$, we use $|x|$ to denote the distance between x and the root e: thus, $|e| = 0$ and $|x| = |\overleftarrow{x}| + 1$.

We define

$$A(x) := \frac{\omega(\overleftarrow{x}, x)}{\omega(\overleftarrow{x}, \overleftleftarrows{x})}, \qquad x \in \mathbb{T}, |x| \ge 2, \tag{1.1}$$

where $\overleftleftarrows{x}$ denotes the parent of $\overleftarrow{x}$.

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Following Lyons and Pemantle [14], we assume throughout the paper that $(\omega(x,\bullet))_{x\in\mathbb{T}\backslash\{e\}}$ is a family of i.i.d. *nondegenerate* random vectors and that $(A(x), x\in\mathbb{T}, |x|\ge 2)$ are identically distributed. We also assume the existence of $\varepsilon_0>0$ such that $\omega(x,y)\ge\varepsilon_0$ if either $x=\overleftarrow{y}$ or $y=\overleftarrow{x}$, and $\omega(x,y)=0$ otherwise; in words, (X_n) is a nearest-neighbor walk satisfying an ellipticity condition.

Let A denote a generic random variable having the common distribution of $A(x)$ (for $|x|\ge 2$) defined in (1.1). Let

$$p := \inf_{t\in[0,1]} \mathbf{E}(A^t). \tag{1.2}$$

An important criterion of Lyons and Pemantle [14] says that with $\mathbb{P}$-probability one, the walk (X_n) is recurrent or transient, according to whether $p\le\frac{1}{b}$ or $p>\frac{1}{b}$. It is, moreover, positive recurrent if $p<\frac{1}{b}$. Later, Menshikov and Petritis [16] proved that the walk is null recurrent if $p=\frac{1}{b}$.

Throughout the paper, we write

$$X_n^* := \max_{0\le k\le n} |X_k|, \qquad n\ge 0.$$

In the positive recurrent case $p<\frac{1}{b}$, $\frac{X_n^*}{\log n}$ converges $\mathbb{P}$-almost surely to a constant $c\in(0,\infty)$ whose value is known; see [9].

The null recurrent case $p=\frac{1}{b}$ is more interesting. It turns out that the behavior of the walk depends also on the sign of $\psi'(1)$, where

$$\psi(t) := \log \mathbf{E}(A^t), \qquad t\ge 0. \tag{1.3}$$

In [9], we proved that if $p=\frac{1}{b}$ and $\psi'(1)<0$, then

$$\lim_{n\to\infty} \frac{\log X_n^*}{\log n} = 1-\frac{1}{\min\{\kappa,2\}}, \qquad \mathbb{P}\text{-a.s.}, \tag{1.4}$$

where $\kappa := \inf\{t>1 : \mathbf{E}(A^t)=\frac{1}{b}\}\in(1,\infty]$, with $\inf\varnothing:=\infty$.

The delicate case $p=\frac{1}{b}$ and $\psi'(1)\ge 0$ was left open and is studied in the present paper. See Figure 1.

We will see in Remark 2.3 that the case $\psi'(1)>0$ reduces to the case $\psi'(1)=0$ via a simple transformation of the distribution of the random environment. As pointed out by Biggins and Kyprianou [3] in the study of Mandelbrot's multiplicative cascades, the case $\psi'(1)=0$ is likely to be "both subtle and important."

The following theorem reveals an unusually slow regime for the walk.

THEOREM 1.1. *If $p=\frac{1}{b}$ and $\psi'(1)\ge 0$, then there exist constants $0<c_1\le c_2<\infty$ such that*

$$c_1 \le \liminf_{n\to\infty} \frac{X_n^*}{(\log n)^3} \le \limsup_{n\to\infty} \frac{X_n^*}{(\log n)^3} \le c_2, \qquad \mathbb{P}\text{-a.s.} \tag{1.5}$$

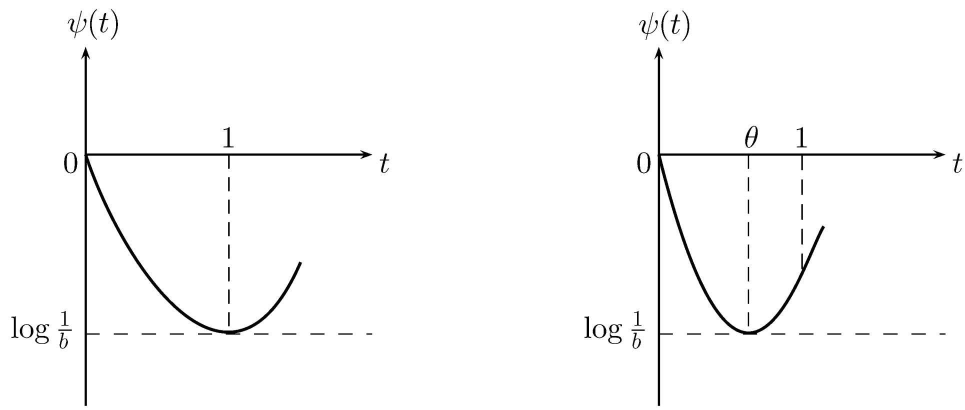

FIG. 1. *Case $\psi'(1) = 0$ and case $\psi'(1) > 0$ with θ defined as in* (2.5).

REMARK 1.2. (i) Theorem 1.1 is somewhat reminiscent of Sinai's result [21] on slow movement of recurrent one-dimensional RWRE, whereas (1.4) is a (weaker) analogue of the Kesten–Kozlov–Spitzer characterization [10] of subdiffusive behaviors of *transient* one-dimensional RWREs.

(ii) It is interesting to note that tree-valued RWREs possess both regimes (slow movement and subdiffusivity) in the recurrent case.

(iii) We mention an important difference between Theorem 1.1 and Sinai's result. If $(Y_n, n \ge 0)$ is a recurrent *one-dimensional* RWRE, then Sinai's theorem says that $\frac{Y_n}{(\log n)^2}$ converges in distribution (under $\mathbb{P}$) to a nondegenerate limit law, whereas it is known (see [8]) that

$$\limsup_{n\to\infty} \frac{Y_n^*}{(\log n)^2} = \infty, \qquad \liminf_{n\to\infty} \frac{Y_n^*}{(\log n)^2} = 0, \qquad \mathbb{P}\text{-a.s.},$$

where $Y_n^* := \max_{0\le k\le n} |Y_k|$.

(iv) It is not clear to us whether $\frac{X_n^*}{(\log n)^3}$ converges $\mathbb{P}$-almost surely.

(v) We believe that $\frac{|X_n|}{(\log n)^3}$ would converge *in distribution* under $\mathbb{P}$.

In Section 2, we describe the method used to prove Theorem 1.1. In particular, we introduce an associated branching random walk and prove an almost sure result for this branching random walk (Theorem 2.2) which may be of independent interest. (The two theorems are related via Proposition 2.4.)

The organization of the proof of the theorems is described at the end of Section 2. Theorem 1.1 is proved in Section 6.

Throughout the paper, c (possibly with a subscript) denotes a finite and positive constant; we write $c(\omega)$ instead of c when the value of c depends on the environment ω.

2. An associated branching random walk. For any $m \geq 0$, let

$$\mathbb{T}_m := \{x \in \mathbb{T} : |x| = m\},$$

which stands for the mth generation of the tree. For any $n \geq 0$, let

$$\tau_n := \inf\{i \geq 1 : X_i \in \mathbb{T}_n\} = \inf\{i \geq 1 : |X_i| = n\},$$

the first hitting time of the walk at level n (whereas τ_0 is the first *return* time to the root). We write

$$\varrho_n := P_\omega\{\tau_n < \tau_0\}.$$

In words, ϱ_n denotes the (quenched) probability that the RWRE makes an excursion of height at least n.

An important step in the proof of Theorem 1.1 is the following estimate for ϱ_n, in the case $\psi'(1) = 0$.

THEOREM 2.1. *Assume that $p = \frac{1}{b}$ and $\psi'(1) = 0$.*

(i) *There exist constants $0 < c_3 \leq c_4 < \infty$ such that* **P**-*almost surely for all large n*,

$$e^{-c_4 n^{1/3}} \leq \varrho_n \leq e^{-c_3 n^{1/3}}. \tag{2.1}$$

(ii) *There exist constants $0 < c_5 \leq c_6 < \infty$ such that for all large n*,

$$e^{-c_6 n^{1/3}} \leq \mathbf{E}(\varrho_n) \leq e^{-c_5 n^{1/3}}. \tag{2.2}$$

It turns out that ϱ_n is closely related to a branching random walk. But let us first extend the definition of $A(x)$ to all $x \in \mathbb{T} \setminus \{e\}$.

For any $x \in \mathbb{T}$, let $\{x_i\}_{1 \leq i \leq b}$ denote the set of the children of x. In addition to the random variables $A(x)$ ($|x| \geq 2$) defined in (1.1), let $(A(e_i), 1 \leq i \leq b)$ be a random vector independent of $(\omega(x, y), |x| \geq 1, y \in \mathbb{T})$ and distributed as $(A(x_i), 1 \leq i \leq b)$ for any $x \in \mathbb{T}_m$ with $m \geq 1$. As such, $A(x)$ is well defined for all $x \in \mathbb{T} \setminus \{e\}$. [The values of ω at a finite number of vertices are of no particular interest. Our choice of $(A(e_i), 1 \leq i \leq b)$ allows us to make unified statements concerning $A(x)$, $V(x)$, etc., without having to distinguish whether $|x| = 1$ or $|x| \geq 2$.]

For any $x \in \mathbb{T} \setminus \{e\}$, the set of vertices on the shortest path relating e and x is denoted by $[\![e, x]\!]$; we also set $]\!]e, x]\!]$ to be $[\![e, x]\!] \setminus \{e\}$.

We now define the process $V = (V(x), x \in \mathbb{T})$ by $V(e) := 0$ and

$$V(x) := -\sum_{z \in]\!]e, x]\!]} \log A(z), \qquad x \in \mathbb{T} \setminus \{e\}.$$

It is clear that V only depends on the environment ω. In the literature, V is often referred to as a branching random walk; see, for example, [2].

We first state the main result of the section. Let

$$\overline{V}(x) := \max_{z \in]\!]e,x]\!]} V(z), \tag{2.3}$$

which stands for the maximum of V over the path $]\!]e, x]\!]$.

THEOREM 2.2. *If $p = \frac{1}{b}$ and $\psi'(1) \geq 0$, then there exist constants $0 < c_7 \leq c_8 < \infty$ such that*

$$c_7 \leq \liminf_{n\to\infty} \frac{1}{n^{1/3}} \min_{x\in\mathbb{T}_n} \overline{V}(x) \leq \limsup_{n\to\infty} \frac{1}{n^{1/3}} \min_{x\in\mathbb{T}_n} \overline{V}(x) \leq c_8, \qquad \mathbf{P}\text{-}a.s. \tag{2.4}$$

REMARK 2.3. (i) We cannot replace $\min_{x\in\mathbb{T}_n} \overline{V}(x)$ by $\min_{x\in\mathbb{T}_n} V(x)$ in Theorem 2.2; in fact, it is proved by McDiarmid [15] that there exists a constant c_9 such that $\mathbf{P}$-almost surely for all large n, we have $\min_{x\in\mathbb{T}_n} V(x) \leq c_9 \log n$.

(ii) If ($p = \frac{1}{b}$ and) $\psi'(1) < 0$, it is well known [1, 7, 11] that $\frac{1}{n}\min_{x\in\mathbb{T}_n} V(x)$ converges $\mathbf{P}$-almost surely to a (strictly) positive constant whose value is known; thus, $\min_{x\in\mathbb{T}_n} \overline{V}(x)$ grows linearly in this case.

(iii) Only the case $\psi'(1) = 0$ needs to be proven. Indeed, if ($p = \frac{1}{b}$ and) $\psi'(1) > 0$, then there exists a unique $0 < \theta < 1$ such that

$$\psi'(\theta) = 0, \qquad \mathbf{E}(A^\theta) = \frac{1}{b}. \tag{2.5}$$

We define $\widetilde{A} := A^\theta$, $\widetilde{p} := \inf_{t\in[0,1]} \mathbf{E}(\widetilde{A}^t)$ and $\widetilde{\psi}(t) := \log \mathbf{E}(\widetilde{A}^t)$, $t \geq 0$. Clearly, we have

$$\widetilde{p} = \frac{1}{b}, \qquad \widetilde{\psi}'(1) = 0.$$

Let $\widetilde{V}(x) := -\sum_{z\in]\!]e,x]\!]} \log \widetilde{A}(z)$. Then $V(x) = \frac{1}{\theta}\widetilde{V}(x)$, which leads us to the case $\psi'(1) = 0$.

The following result contains the promised relation between ϱ_n and V for recurrent RWRE on $\mathbb{T}$.

PROPOSITION 2.4. *If (X_n) is recurrent, then there exists a constant $c_{10} > 0$ such that for any $n \geq 1$,*

$$\varrho_n \geq \frac{c_{10}}{n} \exp\Big(-\min_{x\in\mathbb{T}_n} \overline{V}(x)\Big). \tag{2.6}$$

PROOF. For any $x \in \mathbb{T}$, let

$$T(x) := \inf\{i \geq 0 : X_i = x\}, \tag{2.7}$$

which is the first hitting time of the walk at vertex x. By definition, $\tau_n = \min_{x\in\mathbb{T}_n} T(x)$ for $n \ge 1$. Therefore,

$$\varrho_n \ge \max_{x\in\mathbb{T}_n} P_\omega\{T(x) < \tau_0\}. \tag{2.8}$$

We now compute the (quenched) probability $P_\omega\{T(x) < \tau_0\}$. We fix $x \in \mathbb{T}_n$ and define a random sequence $(\sigma_j)_{j\ge 0}$ by $\sigma_0 := 0$ and

$$\sigma_j := \inf\{k > \sigma_{j-1} : X_k \in [\![e, x]\!] \setminus \{X_{\sigma_{j-1}}\}\}, \qquad j \ge 1.$$

(Of course, the sequence depends on x.) Let

$$Z_k := X_{\sigma_k}, \qquad k \ge 0. \tag{2.9}$$

In words, $Z = (Z_k, k \ge 0)$ is the restriction of X to the path $[\![e, x]\!]$; that is, it is almost the original walk, except that we remove excursions away from $[\![e, x]\!]$. Clearly, Z is a one-dimensional RWRE with (writing $[\![e, x]\!] = \{e =: x^{(0)}, x^{(1)}, \dots, x^{(n)} := x\}$)

$$P_\omega\{Z_{k+1} = x^{(i+1)} | Z_k = x^{(i)}\} = \frac{A(x^{(i+1)})}{1 + A(x^{(i+1)})},$$
$$P_\omega\{Z_{k+1} = x^{(i-1)} | Z_k = x^{(i)}\} = \frac{1}{1 + A(x^{(i+1)})}$$

for all $1 \le i \le n-1$. We observe that

$$\begin{aligned} P_\omega\{T(x) < \tau_0\} &= \omega(e, x^{(1)}) P_\omega\{Z \text{ hits } x^{(n)} \text{ before hitting } e | Z_0 = x^{(1)}\} \\ &= \omega(e, x^{(1)}) \frac{e^{V(x^{(1)})}}{\sum_{z\in]\!] e,x]\!]} e^{V(z)}}, \end{aligned}$$

the second identity following from a general formula ([22], formula (2.1.4)) for the exit problem for one-dimensional RWREs. By the ellipticity condition, there exists a constant $c_{11} > 0$ such that $\omega(e, x^{(1)}) e^{V(x^{(1)})} \ge c_{11}$. Substituting this estimate into (2.8) yields

$$\varrho_n \ge \max_{x\in\mathbb{T}_n} \frac{c_{11}}{\sum_{y\in]\!] e,x]\!]} e^{V(y)}},$$

completing the proof of Proposition 2.4. $\square$

The proof of the theorems is organized as follows.

- Section 3: Theorem 2.2, upper bound.
- Section 4: Theorem 2.1 (by means of the upper bound in Theorem 2.2; this is the technical part of the paper).
- Section 5: Theorem 2.2, lower bound (by means of the upper bound in Theorem 2.1).
- Section 6: Theorem 1.1.

3. Proof of Theorem 2.2: upper bound. Throughout this section, we assume that $p = \frac{1}{b}$ and $\psi'(1) = 0$.

Let

$$B(x) := \prod_{y \in \rrbracket e, x \rrbracket} A(y), \qquad x \in \mathbb{T} \setminus \{e\}. \tag{3.1}$$

We start by recalling a change-of-probability formula from [2]; see also [6] and [4].

FACT 3.1 ([2]). For any $n \geq 1$ and any positive measurable function G,

$$\sum_{x \in \mathbb{T}_n} \mathbf{E}\big[B(x) G\big(B(z), z \in \rrbracket e, x\rrbracket\big)\big] = \mathbf{E}[G(e^{S_i}, 1 \leq i \leq n)], \tag{3.2}$$

where S_n is the sum of n i.i.d. centered random variables whose common distribution is determined by

$$\mathbf{E}[g(S_1)] = b\mathbf{E}[Ag(\log A)]$$

for any positive measurable function g.

The fact that S_1 is centered is a consequence of the assumption $\psi'(1) = 0$. We note that in (3.2), the value of $\mathbf{E}[B(x)G(B(z), z \in \rrbracket e, x\rrbracket)]$ is the same for all $x \in \mathbb{T}_n$.

We now have all of the ingredients needed for the proof of the upper bound in Theorem 2.2.

Proof of Theorem 2.2: *upper bound.* By Remark 2.3, only the case $\psi'(1) = 0$ needs to be treated. We assume in the rest of the section that ($p = \frac{1}{b}$ and) $\psi'(1) = 0$. The proof borrows some ideas of Bramson [5] concerning branching Brownian motions. Let

$$E_m := \Big\{x \in \mathbb{T}_m : \max_{z \in \rrbracket e, x\rrbracket} |V(z)| \leq m^{1/3}\Big\}.$$

We first estimate $\mathbf{E}[\# E_m]$:

$$\mathbf{E}[\# E_m] = \sum_{x \in \mathbb{T}_m} \mathbf{P}\Big\{\max_{z \in \rrbracket e, x\rrbracket} |V(z)| \leq m^{1/3}\Big\}.$$

By assumption, for any given $x \in \mathbb{T}_m$, $(V(z), z \in \rrbracket e, x\rrbracket)$ is the set of the first m partial sums of i.i.d. random variables whose common distribution is A. By (3.2), this leads to:

$$\mathbf{E}[\# E_m] = \mathbf{E}\big(e^{-S_m} \mathbf{1}_{\{\max_{1 \leq i \leq m} |S_i| \leq m^{1/3}\}}\big) \geq \mathbf{P}\Big\{\max_{1 \leq i \leq m} |S_i| \leq m^{1/3}, S_m \leq 0\Big\}.$$

The probability on the right-hand side is a "small deviation" probability with an unimportant condition on the terminal value. By a general result of Mogul'skii [17], we have, for all sufficiently large m (say $m \geq m_0$),

$$\mathbf{E}[\#E_m] \geq \exp(-c_{12}m^{1/3}).$$

We now estimate the second moment of $\#E_m$. For any pair of vertices x and y, we write $x < y$ if x is an ancestor of y, and $x \leq y$ if x is either y itself or an ancestor of y. Then

$$\begin{aligned}
&\mathbf{E}[(\#E_m)^2] - \mathbf{E}[\#E_m] \\
&\quad = \sum_{u,v \in \mathbb{T}_m, u \neq v} \mathbf{P}\{u \in E_m, v \in E_m\} \\
&\quad = \sum_{j=0}^{m-1} \sum_{z \in \mathbb{T}_j} \sum_{x \in \mathbb{T}_{j+1}: z < x} \sum_{y \in \mathbb{T}_{j+1} \setminus \{x\}: z < y} \sum_{u \in \mathbb{T}_m: x \leq u} \sum_{v \in \mathbb{T}_m: y \leq v} \mathbf{P}\{u \in E_m, v \in E_m\}.
\end{aligned}$$

In words, z is the youngest common ancestor of u and v, while x and y are distinct children of z at generation $j+1$. If $j = m-1$, we have $x = u$ and $y = v$, otherwise x is an ancestor of u and y of v.

Fix $z \in \mathbb{T}_j$ and let x and y be a pair of distinct children of z. Let $u \in \mathbb{T}_m$ and $v \in \mathbb{T}_m$ be such that $x \leq u$ and $y \leq v$. Then

$$\begin{aligned}
&\mathbf{P}\{u \in E_m, v \in E_m\} \\
&\qquad \leq \mathbf{P}\Big\{\max_{r \in]\!]e,z]\!]} |V(r)| \leq m^{1/3}\Big\} \times \Big(\mathbf{P}\Big\{\max_{r \in]\!]z,x]\!]} |V(r) - V(z)| \leq 2m^{1/3}\Big\}\Big)^2.
\end{aligned}$$

We have, by (3.2),

$$\mathbf{P}\Big\{\max_{r \in]\!]e,z]\!]} |V(r)| \leq m^{1/3}\Big\} = b^{-j}\mathbf{E}\big[e^{-S_j}\mathbf{1}_{\{\max_{1 \leq i \leq j} |S_i| \leq m^{1/3}\}}\big] \leq b^{-j}e^{m^{1/3}},$$

and, similarly, $\mathbf{P}\{\max_{r \in]\!]z,x]\!]} |V(r) - V(z)| \leq 2m^{1/3}\} \leq b^{-(m-j)}e^{2m^{1/3}}$. Therefore,

$$\begin{aligned}
&\mathbf{E}[(\#E_m)^2] - \mathbf{E}[\#E_m] \\
&\quad \leq \sum_{j=0}^{m-1} \sum_{z \in \mathbb{T}_j} \sum_{x \in \mathbb{T}_{j+1}: z < x} \sum_{y \in \mathbb{T}_{j+1} \setminus \{x\}: z < y} \sum_{u \in \mathbb{T}_m: x \leq u} \sum_{v \in \mathbb{T}_m: y \leq v} b^{j-2m}e^{5m^{1/3}} \\
&\quad = \sum_{j=0}^{m-1} \sum_{z \in \mathbb{T}_j} b(b-1)b^{m-j-1}b^{m-j-1}b^{j-2m}e^{5m^{1/3}} \\
&\quad = \frac{b-1}{b} m e^{5m^{1/3}}.
\end{aligned}$$

Recall that $\mathbf{E}[\#E_m] \geq \exp(-c_{12}m^{1/3})$ for $m \geq m_0$. Therefore, for $m \geq m_0$,

$$\frac{\mathbf{E}[(\#E_m)^2]}{\{\mathbf{E}[\#E_m]\}^2} \leq \frac{b-1}{b} m e^{(5+2c_{12})m^{1/3}} + e^{c_{12}m^{1/3}} \leq e^{c_{13}m^{1/3}}.$$

By the Cauchy–Schwarz inequality, for $m \geq m_0$,

$$\mathbf{P}\{E_m \neq \varnothing\} = \mathbf{P}\{\#E_m > 0\} \geq \frac{\{\mathbf{E}[\#E_m]\}^2}{\mathbf{E}[(\#E_m)^2]} \geq e^{-c_{13}m^{1/3}}.$$

A fortiori, for $m \geq m_0$,

$$\mathbf{P}\{\exists x \in \mathbb{T}_m, \overline{V}(x) \leq m^{1/3}\} \geq e^{-c_{13}m^{1/3}},$$

which implies that

$$\mathbf{P}\Big\{\min_{x \in \mathbb{T}_m} \overline{V}(x) > m^{1/3}\Big\} \leq 1 - e^{-c_{13}m^{1/3}} \leq \exp(-e^{-c_{13}m^{1/3}}).$$

Let $n > m$. By the ellipticity condition stated in the introduction, there exists a constant $c_{14} > 0$ such that $\max_{z \in]\!]e,y]\!]} V(z) \leq c_{14}(n-m)$ for any $y \in \mathbb{T}_{n-m}$. Accordingly, for $m \geq m_0$,

$$\begin{aligned}
&\mathbf{P}\Big\{\min_{x \in \mathbb{T}_n} \overline{V}(x) > m^{1/3} + c_{14}(n-m)\Big\} \\
&\quad \leq \mathbf{P}\Big\{\min_{y \in T_{n-m}} \min_{x \in \mathbb{T}_n : y < x} \max_{r \in]\!]y,x]\!]} [V(r) - V(y)] > m^{1/3}\Big\} \\
&\quad = \Big(\mathbf{P}\Big\{\min_{s \in \mathbb{T}_m} \overline{V}(s) > m^{1/3}\Big\}\Big)^{b^{n-m}} \\
&\quad \leq \exp(-b^{n-m} e^{-c_{13}m^{1/3}}).
\end{aligned}$$

We now choose $m = m(n) := n - \lfloor c_{15} n^{1/3} \rfloor$, where the constant c_{15} is sufficiently large such that $\sum_n \exp(-b^{n-m} e^{-c_{13}m^{1/3}}) < \infty$. Then, by the Borel–Cantelli lemma,

$$\limsup_{n \to \infty} \frac{1}{n^{1/3}} \min_{x \in \mathbb{T}_n} \overline{V}(x) \leq 1 + c_{14}c_{15}, \qquad \mathbf{P}\text{-a.s.},$$

yielding the desired upper bound in Theorem 2.2.

4. Proof of Theorem 2.1. Throughout this section, we assume that $p = \frac{1}{b}$ and $\psi'(1) = 0$.

Proof of Theorem 2.1: *lower bound.* The estimate $\varrho_n \geq e^{-c_4 n^{1/3}}$ (**P**-almost surely for all large n) follows immediately from the upper bound in Theorem 2.2 (proved in Section 3) by means of Proposition 2.4, with any constant $c_4 > c_8$. By Fatou's lemma, we have $\liminf_{n \to \infty} e^{c_4 n^{1/3}} \mathbf{E}(\varrho_n) \geq 1$.

We now introduce the important "additive martingale" M_n; in particular, the lower tail behavior of M_n is studied in Lemma 4.1, by means of another martingale called the "multiplicative martingale." The upper bound in Theorem 2.1 will then be proved, based on the asymptotics of M_n and on the lower bound which was just proven.

Let $B(x) := \prod_{y\in]\!]e,x]\!]} A(y)$ (for $x \in \mathbb{T}\setminus\{e\}$), as in (3.1), and let

$$M_n := \sum_{x\in\mathbb{T}_n} B(x), \qquad n \ge 1. \tag{4.1}$$

When $\mathbf{E}(A) = \frac{1}{b}$ [which is the case if $p = \frac{1}{b}$ and $\psi'(1) = 0$], the process $(M_n, n \ge 1)$ is a martingale, and is referred to as an associated *additive martingale*.

It is more convenient to study the behavior of M_n by means of another martingale. It is known (see [12]) that under the assumptions $p = \frac{1}{b}$ and $\psi'(1) = 0$, there is a unique nontrivial function $\varphi^* : \mathbb{R}_+ \to (0, 1]$ such that

$$\varphi^*(t) = \mathbf{E}\left\{\prod_{i=1}^{b} \varphi^*(tA(e_i))\right\}, \qquad t \ge 0. \tag{4.2}$$

(By nontrivial, we mean that φ^* is not identically 1.) Let

$$M_n^* := \prod_{x\in\mathbb{T}_n} \varphi^*(B(x)), \qquad n \ge 1.$$

The process $(M_n^*, n \ge 1)$ is also a martingale [12]. Following Neveu [18], we call M_n^* an associated *multiplicative martingale*.

Since the martingale M_n^* takes values in (0, 1], it converges almost surely (when $n \to \infty$) to, say, M_∞^*, and $\mathbf{E}(M_\infty^*) = 1$. It is proved by Liu [12] that $\mathbf{E}\{(M_\infty^*)^t\} = \varphi^*(t)$ for any $t \ge 0$.

Recall that for some $0 < \alpha < 1$,

$$\log\left(\frac{1}{\varphi^*(t)}\right) \sim t \log\left(\frac{1}{t}\right), \qquad t \to 0, \tag{4.3}$$

$$\log\left(\frac{1}{\varphi^*(s)}\right) \ge c_{16} s^\alpha, \qquad s \ge 1; \tag{4.4}$$

see [12], Theorem 2.5, for (4.3), and [13], Theorem 2.5, for (4.4).

LEMMA 4.1. *Assume that $p = \frac{1}{b}$ and $\psi'(1) = 0$. For any $\chi > 1/2$, there exists $\delta > 0$ such that for all sufficiently large n,*

$$\mathbf{P}\{M_n < n^{-\chi}\} \le \exp(-n^\delta). \tag{4.5}$$

PROOF. Let $K > 0$ be such that $\mathbf{P}\{M_\infty^* > e^{-K}\} > 0$. Then $\varphi^*(t) = \mathbf{E}\{(M_\infty^*)^t\} \ge \mathbf{P}\{M_\infty^* > e^{-K}\} e^{-Kt}$ for all $t > 0$. Thus, there exists $c_{17} > 0$ such that for all $t \ge 1$, $\varphi^*(t) \ge e^{-c_{17} t}$.

Let $\varepsilon > 0$. By (4.3) and (4.4), there exists a constant c_{18} such that

$$\log\left(\frac{1}{M_n^*}\right) = \sum_{x\in\mathbb{T}_n} \log\left(\frac{1}{\varphi^*(B(x))}\right) \le c_{18}(J_{1,n} + J_{2,n} + J_{3,n}),$$

where

$$J_{1,n} := \sum_{x\in\mathbb{T}_n} B(x)\left(\log\frac{1}{B(x)}\right)\mathbf{1}_{\{B(x)<\exp(-n^{(1/2)+\varepsilon})\}},$$

$$J_{2,n} := \sum_{x\in\mathbb{T}_n} B(x)\left(\log\frac{e}{B(x)}\right)\mathbf{1}_{\{\exp(-n^{(1/2)+\varepsilon})\le B(x)\le 1\}},$$

$$J_{3,n} := \sum_{x\in\mathbb{T}_n} B(x)\mathbf{1}_{\{B(x)>1\}}.$$

Clearly, $J_{3,n} \le \sum_{x\in\mathbb{T}_n} B(x) = M_n$, whereas $J_{2,n} \le (n^{(1/2)+\varepsilon} + 1)M_n$. Hence, $J_{2,n} + J_{3,n} \le (n^{(1/2)+\varepsilon} + 2)M_n \le 2n^{(1/2)+\varepsilon}M_n$ (for $n \ge 4$). Accordingly, for $n \ge 4$,

$$n^{(1/2)+\varepsilon}M_n \ge \frac{1}{2c_{18}}\log\left(\frac{1}{M_n^*}\right) - \frac{1}{2}J_{1,n}. \tag{4.6}$$

We now estimate the tail probability of M_n^*. Let $\lambda \ge 1$ and $z > 0$. By Chebyshev's inequality,

$$\mathbf{P}\left\{\log\left(\frac{1}{M_n^*}\right) < z\right\} \le e^{\lambda z}\mathbf{E}\{(M_n^*)^\lambda\}.$$

Since M_n^* is a bounded martingale, $\mathbf{E}\{(M_n^*)^\lambda\} \le \mathbf{E}\{(M_\infty^*)^\lambda\} = \varphi^*(\lambda)$. Therefore,

$$\mathbf{P}\left\{\log\left(\frac{1}{M_n^*}\right) < z\right\} \le e^{\lambda z}\varphi^*(\lambda).$$

Choosing $z := 4c_{18}n^{-\varepsilon}$ and $\lambda := n^\varepsilon$, it follows from (4.4) that

$$\mathbf{P}\left\{\log\left(\frac{1}{M_n^*}\right) < 4c_{18}n^{-\varepsilon}\right\} \le \exp(4c_{18} - c_{16}n^{\varepsilon\alpha}).$$

Substituting this into (4.6) yields that for $n \ge 4$,

$$\mathbf{P}\left\{n^{(1/2)+\varepsilon}M_n + \tfrac{1}{2}J_{1,n} < 2n^{-\varepsilon}\right\} \le \exp(4c_{18} - c_{16}n^{\varepsilon\alpha}). \tag{4.7}$$

We note that $J_{1,n} \ge 0$. By (3.2),

$$\mathbf{E}(J_{1,n}) = \mathbf{E}\left\{(-S_n)\mathbf{1}_{\{S_n<-n^{(1/2)+\varepsilon}\}}\right\}.$$

Recall that S_n is the sum of n i.i.d. bounded centered random variables. It follows that for all sufficiently large n,

$$\mathbf{E}(J_{1,n}) \le \exp(-c_{19}n^{2\varepsilon}).$$

By (4.7) and Chebyshev's inequality,

$$\begin{aligned}\mathbf{P}\{n^{(1/2)+\varepsilon}M_n < n^{-\varepsilon}\} &\le \mathbf{P}\Big\{n^{(1/2)+\varepsilon}M_n + \frac{1}{2}J_{1,n} < 2n^{-\varepsilon}\Big\} + \mathbf{P}\{J_{1,n} \ge 2n^{-\varepsilon}\}\\ &\le \exp(4c_{18} - c_{16}n^{\varepsilon\alpha}) + \frac{n^{\varepsilon}}{2}\exp(-c_{19}n^{2\varepsilon}),\end{aligned}$$

from which (4.5) follows. □

We have now all of the ingredients needed for the proof of the upper bound in Theorem 2.1.

Proof of Theorem 2.1: *upper bound.* We only need to prove the upper bound in (2.2), namely, that there exists c_5 such that for all large n,

$$\mathbf{E}(\varrho_n) \le e^{-c_5 n^{1/3}}. \tag{4.8}$$

If (4.8) holds, then the upper bound in (2.1) follows by an application of Chebyshev's inequality and the Borel–Cantelli lemma.

It remains to prove (4.8). For any $x \in \mathbb{T} \setminus \{e\}$, we define

$$\beta_n(x) := P_\omega\{\text{starting from } x, \text{ the RWRE hits } \mathbb{T}_n \text{ before hitting } \overleftarrow{x}\},$$

where, as before, $\overleftarrow{x}$ is the parent of x. In the notation of (2.7),

$$\beta_n(x) = P_\omega\{T_n < T(\overleftarrow{x}) | X_0 = x\},$$

where $T_n := \min_{x\in\mathbb{T}_n} T(x)$. Clearly, $\beta_n(x) = 1$ if $x \in \mathbb{T}_n$.

Recall that for any $x \in \mathbb{T}$, $\{x_i\}_{1\le i\le b}$ is the set of children of x. By the Markov property, if $1 \le |x| \le n-1$, then

$$\beta_n(x) = \sum_{i=1}^{b} \omega(x, x_i) P_\omega\{T_n < T(\overleftarrow{x}) | X_0 = x_i\}.$$

Consider the event $\{T_n < T(\overleftarrow{x})\}$ when the walk starts from x_i. There are two possible situations: either (i) $T_n < T(x)$ [which happens with probability $\beta_n(x_i)$, by definition] or (ii) $T_n > T(x)$ and after hitting x for the first time, the walk hits $\mathbb{T}_n$ before hitting $\overleftarrow{x}$. By the strong Markov property, $P_\omega\{T_n < T(\overleftarrow{x}) | X_0 = x_i\} = \beta_n(x_i) + [1 - \beta_n(x_i)]\beta_n(x)$. Therefore,

$$\begin{aligned}\beta_n(x) &= \sum_{i=1}^{b} \omega(x, x_i)\beta_n(x_i) + \beta_n(x)\sum_{i=1}^{b} \omega(x, x_i)[1 - \beta_n(x_i)]\\ &= \sum_{i=1}^{b} \omega(x, x_i)\beta_n(x_i) + \beta_n(x)[1 - \omega(x, \overleftarrow{x})] - \beta_n(x)\sum_{i=1}^{b} \omega(x, x_i)\beta_n(x_i),\end{aligned}$$

from which it follows that

$$\beta_n(x) = \frac{\sum_{i=1}^b A(x_i)\beta_n(x_i)}{1+\sum_{i=1}^b A(x_i)\beta_n(x_i)}, \qquad 1 \le |x| \le n-1. \tag{4.9}$$

Together with condition $\beta_n(x) = 1$ (for $x \in \mathbb{T}_n$), these equations determine the value of $\beta_n(x)$ for all $x \in \mathbb{T}$ such that $1 \le |x| \le n$.

We introduce the random variable

$$\beta_n(e) := \frac{\sum_{i=1}^b A(e_i)\beta_n(e_i)}{1+\sum_{i=1}^b A(e_i)\beta_n(e_i)}. \tag{4.10}$$

The value of $\beta_n(e)$ for given ω is of no importance, but the distribution of $\beta_n(e)$, which is identical to that of $\beta_{n+1}(e_1)$, plays a certain role at several points in the proof. For example, for $1 \le |x| < n$, the random variables $\beta_n(x)$ and $\beta_{n-|x|}(e)$ have the same distribution; in particular, $\mathbf{E}[\beta_n(x)] = \mathbf{E}[\beta_{n-|x|}(e)]$. In the rest of this section, we make frequent use of this property without further mention. We also make the trivial observation that for $1 \le |x| < n$, $\beta_n(x)$ depends only on those $A(y)$ such that $|x|+1 \le |y| \le n$ and x is an ancestor of y.

Recall that $\varrho_n = P_\omega\{\tau_n < \tau_0\}$. Therefore,

$$\varrho_n = \sum_{i=1}^b \omega(e, e_i)\beta_n(e_i). \tag{4.11}$$

In particular,

$$\mathbf{E}(\varrho_n) = \mathbf{E}[\beta_n(e_i)] = \mathbf{E}[\beta_{n-1}(e)] \qquad \forall 1 \le i \le b. \tag{4.12}$$

Let $a_j := \mathbf{E}(\varrho_{j^3+1}) = \mathbf{E}[\beta_{j^3}(e)]$, $j = 0, 1, 2, \ldots, \lfloor n^{1/3} \rfloor$. Clearly, $a_0 = 1$ and $j \mapsto a_j$ is nonincreasing for $0 \le j \le \lfloor n^{1/3} \rfloor$. We look for an upper bound for $a_{\lfloor n^{1/3} \rfloor}$.

Let $m > \Delta \ge 1$ be integers. Let $1 \le i \le b$ and let $(e_{ij}, 1 \le j \le b)$ be the set of children of e_i. By (4.9), we have

$$\beta_m(e_i) \le \sum_{j=1}^b A(e_{ij})\beta_m(e_{ij}).$$

Iterating the same argument, we arrive at

$$\beta_m(e_i) \le \sum_{y \in \mathbb{T}_\Delta : y < e_i} \left(\prod_{z\,:\, e_i < z, z \le y} A(z) \right) \beta_m(y) = \sum_{y \in \mathbb{T}_\Delta : y < e_i} \frac{B(y)}{A(e_i)} \beta_m(y).$$

By (4.10), this yields

$$\beta_m(e) \le \frac{\sum_{i=1}^b \sum_{y \in \mathbb{T}_\Delta : y < e_i} B(y)\beta_m(y)}{1+\sum_{i=1}^b \sum_{y \in \mathbb{T}_\Delta : y < e_i} B(y)\beta_m(y)} = \frac{\sum_{y \in \mathbb{T}_\Delta} B(y)\beta_m(y)}{1+\sum_{y \in \mathbb{T}_\Delta} B(y)\beta_m(y)}.$$

Fix n and $0 \le j \le \lfloor n^{1/3} \rfloor - 1$. Let

$$\Delta = \Delta(j) := (j+1)^3 - j^3 = 3j^2 + 3j + 1.$$

Then

$$a_{j+1} = \mathbf{E}[\beta_{(j+1)^3}(e)] \le \mathbf{E}\left(\frac{\sum_{y \in \mathbb{T}_\Delta} B(y) \beta_{(j+1)^3}(y)}{1 + \sum_{y \in \mathbb{T}_\Delta} B(y) \beta_{(j+1)^3}(y)} \right).$$

We note that $(\beta_{(j+1)^3}(y), y \in \mathbb{T}_\Delta)$ is a collection of i.i.d. random variables distributed as $\beta_{j^3}(e)$ and independent of $(B(y), y \in \mathbb{T}_\Delta)$.

Let $(\xi(x), x \in \mathbb{T})$ be i.i.d. random variables distributed as $\beta_{j^3}(e)$, independent of all other random variables and processes. Let

$$N_m := \sum_{x \in \mathbb{T}_m} B(x) \xi(x), \qquad m \ge 1.$$

The last inequality can be written as

$$a_{j+1} \le \mathbf{E}\left(\frac{N_\Delta}{1 + N_\Delta} \right). \tag{4.13}$$

By definition,

$$\mathbf{E}\left(\frac{N_\Delta}{1 + N_\Delta} \right) = \sum_{x \in \mathbb{T}_\Delta} \mathbf{E}\left(\frac{B(x)\xi(x)}{1 + N_\Delta} \right) = \sum_{x \in \mathbb{T}_\Delta} \mathbf{E}\{B(x)\xi(x)e^{-YN_\Delta}\}, \tag{4.14}$$

where Y is an exponential random variable of parameter 1, independent of everything else.

Let us fix $x \in \mathbb{T}_\Delta$, and estimate $\mathbf{E}\{B(x)\xi(x)e^{-YN_\Delta}\}$. Since $N_m = \sum_{x \in \mathbb{T}_m} B(x)\xi(x)$ (for any $m \ge 1$), we have

$$N_\Delta \ge B(\overleftarrow{x}) A(y) \xi(y)$$

for any $y \in \mathbb{T}_\Delta \setminus \{x\}$ such that $\overleftarrow{y} = \overleftarrow{x}$. Note that by the ellipticity condition, $A(y) \ge c > 0$ for some constant c. Accordingly,

$$\begin{aligned} \mathbf{E}\{B(x)\xi(x)e^{-YN_\Delta}\} &\le \mathbf{E}\{B(x)\xi(x)e^{-cYB(\overleftarrow{x})\xi(y)}\} \\ &= \mathbf{E}\{\xi(x)\}\mathbf{E}\{B(x)e^{-cYB(\overleftarrow{x})\xi(y)}\}. \end{aligned}$$

Recall that $\mathbf{E}\{\xi(x)\} = \mathbf{E}\{\beta_{j^3}(e)\} = a_j$ and that $\xi(y)$ is distributed as $\beta_{j^3}(e)$, independent of $(B(x), Y, B(\overleftarrow{x}))$. At this stage, it is convenient to recall the following inequality (see [9] for an elementary proof): if $\mathbf{E}(A) = \frac{1}{b}$ [which is guaranteed by the assumptions $p = \frac{1}{b}$ and $\psi'(1) = 0$], then

$$\mathbf{E}\left\{ \exp\left(-t \frac{\beta_k(e)}{\mathbf{E}[\beta_k(e)]} \right) \right\} \le \mathbf{E}\{e^{-tM_k}\} \qquad \forall k \ge 1, \forall t \ge 0,$$

where M_k is defined in (4.1). As a consequence,

$$\mathbf{E}\{B(x)\xi(x)e^{-YN_\Delta}\} \le a_j \mathbf{E}\big\{B(x)e^{-cYB(\overleftarrow{x})a_j\widetilde{M}_{j^3}}\big\},$$

where $\widetilde{M}_{j^3}$ is distributed as M_{j^3} and is independent of everything else. Since $A(x) = \frac{B(x)}{B(\overleftarrow{x})}$ is independent of $B(\overleftarrow{x})$ (and Y and $\widetilde{M}_{j^3}$), with $\mathbf{E}\{A(x)\} = \frac{1}{b}$, this yields

$$\mathbf{E}\{B(x)\xi(x)e^{-YN_\Delta}\} \le \frac{a_j}{b}\mathbf{E}\big\{B(\overleftarrow{x})e^{-ca_jYB(\overleftarrow{x})\widetilde{M}_{j^3}}\big\}.$$

Substituting this into (4.14), we see that

$$\begin{aligned}\mathbf{E}\bigg(\frac{N_\Delta}{1+N_\Delta}\bigg) &\le a_j \sum_{u\in\mathbb{T}_{\Delta-1}} \mathbf{E}\big\{B(u)e^{-ca_jYB(u)\widetilde{M}_{j^3}}\big\} \\ &= a_j\mathbf{E}\{\exp(-ca_jYe^{S_{\Delta-1}}\widetilde{M}_{j^3})\},\end{aligned}$$

the last identity being a consequence of (3.2), the random variables Y, $S_{\Delta-1}$ and $\widetilde{M}_{j^3}$ being independent. By (4.13), $a_{j+1} \le \mathbf{E}(\frac{N_\Delta}{1+N_\Delta})$. Thus,

$$a_{j+1} \le a_j\mathbf{E}\{\exp(-ca_jYe^{S_{\Delta-1}}\widetilde{M}_{j^3})\}.$$

As a consequence,

$$a_{\lfloor n^{1/3}\rfloor} \le \prod_{j=0}^{\lfloor n^{1/3}\rfloor-1} \mathbf{E}\{\exp(-ca_jYe^{S_{\Delta-1}}\widetilde{M}_{j^3})\}.$$

We claim that for any collection of nonnegative random variables $(\eta_j, 0 \le j \le n)$ and $\lambda \ge 0$,

$$\prod_{j=0}^{n}\mathbf{E}(e^{-\eta_j}) \le e^{-\lambda} + \prod_{j=0}^{n}\mathbf{P}\{\eta_j < \lambda\}.$$

Indeed, without loss of generality, we can assume that the η_j are independent; then

$$\begin{aligned}\prod_{j=0}^{n}\mathbf{E}(e^{-\eta_j}) &\le \mathbf{E}(e^{-\max_{0\le j\le n}\eta_j}) \\ &\le e^{-\lambda} + \mathbf{P}\Big\{\max_{0\le j\le n}\eta_j < \lambda\Big\} \\ &= e^{-\lambda} + \prod_{j=0}^{n}\mathbf{P}\{\eta_j < \lambda\},\end{aligned}$$

as claimed.

We have thus proved that

$$a_{\lfloor n^{1/3}\rfloor} \le e^{-n} + \prod_{j=0}^{\lfloor n^{1/3}\rfloor -1} \mathbf{P}\{ca_j Y e^{S_{\Delta-1}} \widetilde{M}_{j^3} < n\}.$$

Recall that $a_j = \mathbf{E}(\varrho_{j^3+1})$. By the lower bound in Theorem 2.1 which we have proved, we have $a_j \ge \exp(-c_6 j)$ for $j \ge j_0$. Hence, for $j_0 \le j \le \lfloor n^{1/3}\rfloor - 1$,

$$\begin{aligned}&\mathbf{P}\{ca_j Y e^{S_{\Delta-1}} \widetilde{M}_{j^3} \ge n\}\\&\quad \ge \mathbf{P}\{Y \ge 1\}\mathbf{P}\left\{\widetilde{M}_{j^3} \ge \frac{1}{j^3}\right\}\mathbf{P}\left\{S_{\Delta-1} \ge c_6 j + \log\left(\frac{j^3 n}{c}\right)\right\}.\end{aligned}$$

Of course, $\mathbf{P}\{Y \ge 1\} = e^{-1}$ and by (4.5), $\mathbf{P}\{\widetilde{M}_{j^3} \ge \frac{1}{j^3}\} = \mathbf{P}\{M_{j^3} \ge \frac{1}{j^3}\} \ge \frac{1}{2}$ for all large j. On the other hand, since $\Delta - 1 \ge 3j^2$, we have $\mathbf{P}\{S_{\Delta-1} \ge c_6 j + \log(\frac{j^3 n}{c})\} \ge c_{20} > 0$ for large n and all $j \ge \log n$. We have thus proved that for large n and some constant $c_{21} \in (0,1)$,

$$a_{\lfloor n^{1/3}\rfloor} \le e^{-n} + \prod_{j=\lceil \log n\rceil}^{\lfloor n^{1/3}\rfloor -1} (1 - c_{21}) \le \exp(-c_{22} n^{1/3}).$$

Since $a_{\lfloor n^{1/3}\rfloor} = \mathbf{E}(\varrho_{\lfloor n^{1/3}\rfloor^3+1}) \ge \mathbf{E}(\varrho_{n+1})$, this yields (4.8) and thus the upper bound in Theorem 2.1.

5. Proof of Theorem 2.2: lower bound. Without loss of generality (see Remark 2.3), we can assume that $\psi'(1) = 0$. In this case, the lower bound in Theorem 2.2 follows from the upper bound in Theorem 2.1 (proven in the previous section) by means of Proposition 2.4, with $c_7 := c_3$.

6. Proof of Theorem 1.1. For the sake of clarity, Theorem 1.1 is proved in two distinct parts.

6.1. *Upper bound.* We first assume that $\psi'(1) = 0$. By Theorem 2.1, $P_\omega\{\tau_n < \tau_0\} = \varrho_n \le \exp(-c_3 n^{1/3})$ $\mathbf{P}$-almost surely for all large n. Hence, by writing $L(\tau_n) := \#\{1 \le i \le \tau_n : X_i = e\}$, we obtain that $\mathbf{P}$-almost surely for all large n and any $j \ge 1$,

$$P_\omega\{L(\tau_n) \ge j\} = [P_\omega\{\tau_n > \tau_0\}]^j \ge [1 - e^{-c_3 n^{1/3}}]^j,$$

which, by the Borel–Cantelli lemma, implies that for any constant $c_{23} < c_3$ and $\mathbb{P}$-almost surely all sufficiently large n,

$$L(\tau_n) \ge e^{c_{23} n^{1/3}}.$$

Since $\{L(\tau_n) \ge j\} \subset \{X^*_{2j} < n\}$, we obtain the desired upper bound in Theorem 1.1 [case $\psi'(1) = 0$], with $c_2 := 1/(c_3)^3$.

To treat the case $\psi'(1) > 0$, we first consider an RWRE $(Y_k, k \ge 0)$ on the half-line $\mathbb{Z}_+$ with a reflecting barrier at the origin. We write $T_Y(y) := \inf\{k \ge 0 : Y_k = y\}$ for $y \in \mathbb{Z}_+ \setminus \{0\}$. Then

$$P_\omega\{T_Y(y) \le m\} = \sum_{i=1}^m P_\omega\{T_Y(y) = i\} \le \sum_{i=1}^m P_\omega\{Y_i = y\} = \sum_{i=1}^m \omega^i(0, y),$$

where, by an abuse of notation, we use $\omega(\cdot, \cdot)$ to also denote the transition matrix of (Y_k). Since (Y_k) is reversible, we have $\omega^i(0, y) = \frac{\pi(y)}{\pi(0)} \omega^i(y, 0)$, where π is an invariant measure. Accordingly,

$$P_\omega\{T_Y(y) \le m\} \le \sum_{i=1}^m \frac{\pi(y)}{\pi(0)} \omega^i(y, 0) \le m \frac{\pi(y)}{\pi(0)}.$$

As a consequence, for any $n \ge 1$,

$$P_\omega\{T_Y(n) \le m\} \le \min_{1 \le y \le n} P_\omega\{T_Y(y) \le m\} \le m \min_{1 \le y \le n} \frac{\pi(y)}{\pi(0)}.$$

It is easy to compute π: we can take $\pi(0) = 1$ and

$$\pi(y) := \sum_{z=1}^y \log \frac{\omega(z, z-1)}{\omega(z, z+1)}, \qquad y \in \mathbb{Z}_+ \setminus \{0\}.$$

Therefore, for $n \ge 1$,

$$P_\omega\{T_Y(n) \le m\} \le m \min_{y \in \rrbracket e, x \rrbracket} A(y) = m e^{-\overline{V}(x)}, \tag{6.1}$$

where $\overline{V}(x)$ is defined in (2.3).

We now return to the study of X, the RWRE on $\mathbb{T}$. Fix $x \in \mathbb{T}_n$. Let $Z = (Z_k, k \ge 0)$ be the restriction of X to the path $\llbracket e, x \rrbracket$, as in (2.9). Let $T_Z(x) := \inf\{k \ge 0 : Z_k = x\}$. By (6.1), we have $P_\omega\{T_Z(x) \le m\} \le m e^{-\overline{V}(x)}$. It follows from the trivial inequality $T(x) \ge T_Z(x)$ that

$$P_\omega\{\tau_n \le m\} \le \sum_{x \in \mathbb{T}_n} P_\omega\{T(x) \le m\} \le \sum_{x \in \mathbb{T}_n} P_\omega\{T_Z(x) \le m\} \le m \sum_{x \in \mathbb{T}_n} e^{-\overline{V}(x)}.$$

Since $\psi'(1) > 0$, we can consider $0 < \theta < 1$, as in (2.5). Then

$$\sum_{x \in \mathbb{T}_n} e^{-\overline{V}(x)} \le \exp\Big(-(1-\theta) \min_{x \in \mathbb{T}_n} \overline{V}(x)\Big) \sum_{x \in \mathbb{T}_n} e^{-\theta V(x)}.$$

Since $\mathbf{E}(A^\theta) = 1$, it is easily seen that $\sum_{x \in \mathbb{T}_n} e^{-\theta V(x)}$ is a positive martingale. In particular, $\sup_{n \ge 1} \sum_{x \in \mathbb{T}_n} e^{-\theta V(x)} < \infty$ $\mathbf{P}$-almost surely. On the other hand, according to Theorem 2.2, we have $\min_{x \in \mathbb{T}_n} \overline{V}(x) \ge c_7 n^{1/3}$ $\mathbf{P}$-almost surely for all large n. Therefore, for any constant $c_{24} < (1-\theta)c_7$, we have

$$\sum_n P_\omega\{\tau_n \le e^{c_{24}n^{1/3}}\} < \infty, \qquad \mathbf{P}\text{-a.s.},$$

from which the upper bound in Theorem 1.1 [case $\psi'(1) > 0$] follows readily, with $c_2 := 1/[(1-\theta)c_7]^3$.

6.2. *Lower bound.* By means of the Markov property, one can easily obtain a recurrence relation for $E_\omega(\tau_n)$, from which it follows that for $n \ge 1$,

$$E_\omega(\tau_n) = \frac{\gamma_n(e)}{\varrho_n}, \tag{6.2}$$

where ϱ_n and $\gamma_n(e)$ are defined as follows: $\beta_n(x) = 1$ and $\gamma_n(x) = 0$ (for $x \in \mathbb{T}_n$), and

$$\beta_n(x) = \frac{\sum_{i=1}^b A(x_i)\beta_n(x_i)}{1 + \sum_{i=1}^b A(x_i)\beta_n(x_i)},$$

$$\gamma_n(x) = \frac{[1/\omega(x, \overleftarrow{x})] + \sum_{i=1}^b A(x_i)\gamma_n(x_i)}{1 + \sum_{i=1}^b A(x_i)\beta_n(x_i)}, \qquad 1 \le |x| \le n,$$

and $\varrho_n := \sum_{i=1}^b \omega(e, e_i)\beta_n(e_i)$, $\gamma_n(e) := \sum_{i=1}^b \omega(e, e_i)\gamma_n(e_i)$, see [20] for more details. As a matter of fact, $\beta_n(x)$ (for $1 \le |x| \le n$) is the same as the one introduced in (4.9) and ϱ_n can also be expressed as $P_\omega\{\tau_n < \tau_0\}$.

We claim that

$$\sup_{n\ge 1} \frac{\gamma_n(e)}{n} < \infty, \qquad \mathbf{P}\text{-a.s.} \tag{6.3}$$

By admitting (6.3) for the moment, we are able to prove the lower bound in Theorem 1.1. Indeed, in view of (the lower bound in) Theorem 2.1 and (6.2), we have $E_\omega(\tau_n) \le c_{25}(\omega) n \exp(c_4 n^{1/3})$ $\mathbf{P}$-almost surely for all large n. It follows from Chebyshev's inequality and the Borel–Cantelli lemma that $\mathbb{P}$-almost surely for all sufficiently large n, $\tau_n \le c_{25}(\omega) n^3 \exp(c_4 n^{1/3})$, which yields

$$\liminf_{n\to\infty} \frac{X_n^*}{(\log n)^3} \ge \frac{1}{(c_4)^3}, \qquad \mathbb{P}\text{-a.s.}$$

This is the desired lower bound in Theorem 1.1.

It remains to prove (6.3). By the ellipticity condition, $\frac{1}{\omega(x, \overleftarrow{x})} \le c_{26}$, so

$$\gamma_n(x) \le c_{26} + \sum_{i=1}^b A(x_i)\gamma_n(x_i).$$

Iterating the inequality, we obtain

$$\gamma_n(e) \le c_{26}\left(1 + \sum_{j=1}^{n-1} \sum_{x\in\mathbb{T}_j} \prod_{y\in]\!]e_i,x]\!]} A(y)\right) = c_{26}\left(1 + \sum_{j=1}^{n-1} M_j\right), \qquad n \ge 2,$$

M_j having already been introduced in (4.1).

There exists $0 < \theta \le 1$ such that $\mathbf{E}(A^\theta) = \frac{1}{b}$: indeed, if $p = \frac{1}{b}$ and $\psi'(1) = 0$, then we simply take $\theta = 1$, whereas if $p = \frac{1}{b}$ and $\psi'(1) > 0$, then we take $0 < \theta < 1$, as in (2.5). We have

$$M_j^\theta \le \sum_{x\in\mathbb{T}_j} \prod_{y\in]\!]e_i,x]\!]} A(y)^\theta.$$

Since $j \mapsto \sum_{x\in\mathbb{T}_j} \prod_{y\in]\!]e_i,x]\!]} A(y)^\theta$ is a positive martingale, we have $\sup_{j\ge 1} M_j < \infty$ $\mathbf{P}$-almost surely. This yields (6.3) and thus completes the proof of the lower bound in Theorem 1.1.

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