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# FELLER PROCESSES ON NONLOCALLY COMPACT SPACES

BY TOMASZ SZAREK[1]

*University of Silesia*

We consider Feller processes on a complete separable metric space $X$ satisfying the ergodic condition of the form

$$\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}P^i(x,O)\right)>0 \qquad \text{for some } x\in X,$$

where $O$ is an arbitrary open neighborhood of some point $z\in X$ and $P$ is a transition function. It is shown that e-chains which satisfy the above condition admit an invariant probability measure. Some results on the stability of such processes are also presented.

**1. Introduction.** The theory of Feller processes is still being developed [3, 4, 11, 12, 17, 20, 22, 23], although these processes were the subject of several papers over thirty years ago (see [8–10, 21, 24, 26, 27]). In most of the literature, the state space is assumed compact, or at least locally compact, so that existence of an invariant measure is almost immediate. In the nonlocally compact case, this may be proved, in turn, if a strong form of Harris recurrence on some compact set holds (see [23]). However, this condition is rather hard to verify. It is easier to obtain ergodicity on some open sets which, unfortunately, are not precompact. Similar difficulties occur when we attempt to state the Doeblin condition (see [23]).

It seems that the nonlocally compact case has not yet been completely analyzed. In this note, we contribute to this effort. The work was motivated by the need to investigate the limit behavior of discrete Markov chains generated by iterated function systems [1, 6, 17, 19, 28] and stochastic differential equations on Hilbert spaces (see [5]). The utility of our method in proving the existence of an invariant measure for stochastic partial differential equations with an impulsive noise will be shown in [18].

Let $(X,\rho)$ be a complete and separable metric space and let $\mathbf{\Phi}=(\Phi_n)_{n\geq 1}$ be a discrete-time Markov chain on $X$. By $\mathcal{B}(X)$, we denote the space of all Borel sets. Let $P(x,A)$ be a transition function defined for $x\in X$ and $A\in\mathcal{B}(X)$. *Feller's property* means that the function $x\to P(x,U)$ is lower semicontinuous for all open sets $U$. Alternatively, we can say that

$$C(X)\ni f(\cdot)\to Pf(\cdot)=\int_X f(y)P(\cdot,dy)\in C(X),$$

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where $C(X)$ denotes the space of all bounded continuous functions on $X$.

We are interested in the existence of an invariant probability measure for $\mathbf{\Phi}$. A measure $\mu$ is called *invariant* if

$$\mu(A) = \mu P(A) = \int_X P(x, A)\mu(dx)$$

for $A \in \mathcal{B}(X)$.

Let $\mu$ be an arbitrary Borel measure. We define the *support of the measure* $\mu$ by setting

$$\operatorname{supp} \mu = \{x \in X : \mu(B(x, \varepsilon)) > 0 \text{ for every } \varepsilon > 0\}.$$

In order to establish the existence of an invariant measure and stability, we introduce the following condition:

($\mathcal{E}$) There exists $z \in X$ such that for every open set $O$ containing $z$,

$$\limsup_{n\to\infty} \left( \frac{1}{n} \sum_{i=1}^{n} P^i(x, O) \right) > 0 \qquad \text{for some } x \in X. \tag{1.1}$$

## 2. Existence of invariant measures.

PROPOSITION 2.1. *Let $P : X \times \mathcal{B}(X) \to [0, 1]$ be a transition function for a discrete-time Markov chain $\mathbf{\Phi}$ and assume that condition ($\mathcal{E}$) holds for some $z \in X$. If $\{P^n f : n \in \mathbb{N}\}$ is equicontinuous in $z$ for every Lipschitz continuous function $f$, then $\mathbf{\Phi}$ admits an invariant probability measure.*

PROOF. To finish the proof, it suffices to show that for every $\varepsilon > 0$, there exists a compact set $K \subset X$ such that

$$\liminf_{n\to\infty} P^n(z, K^\varepsilon) \geq 1 - \varepsilon, \tag{2.1}$$

where $K^\varepsilon = \{x \in X : \inf_{y\in K} \rho(x, y) < \varepsilon\}$. This, in conjunction with Theorem 2.2 in [7], tells us that the measures $\{P^n(z, \cdot) : n \in \mathbb{N}\}$ are tight. Therefore, the Cesaro averages are weakly precompact by the Prokhorov theorem (see [7]). Note that any weak limit of the Cesaro averages is invariant.

Assume, contrary to our claim, that (2.1) does not hold for some $\varepsilon > 0$. By Ulam's lemma (see [2]), there exist a sequence of compact sets $(K_i)_{i\geq1}$ and a sequence of integers $(q_i)_{i\geq1}$ satisfying

$$P^{q_i}(z, K_i) > \varepsilon$$

and

$$\min\{\rho(x, y) : x \in K_i, y \in K_j\} \geq \varepsilon/3 \qquad \text{for } i, j \in \mathbb{N}, i \neq j. \tag{2.2}$$

We first show that for every open set $O$ containing $z$ and $j \in \mathbb{N}$, there exist $y \in O$ and $i \geq j$ such that

$$P^{q_i}(y, K_i^{\varepsilon/12}) < \varepsilon/2.$$

On the contrary, suppose that there exist an open set $O'$ containing $z$ and $i_0 \in \mathbb{N}$ such that

$$\inf\{P^{q_i}(y, K_i^{\varepsilon/12}) : y \in O', i \geq i_0\} \geq \varepsilon/2. \tag{2.3}$$

Let $x \in X$ be such that condition (1.1) holds with $O'$ in place of $O$. Let $\alpha > 0$ be such that

$$\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n} P^i(x, O')\right) > \alpha.$$

By (2.2), (2.3) and the Chapman–Kolmogorov equation, we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} P^i\left(u_1, \bigcup_{j=i_0}^{N} K_j^{\varepsilon/12}\right) > (N - i_0)\alpha\varepsilon/2$$

for every $N \geq i_0$, which is impossible.

We will now define by induction a sequence of Lipschitz continuous functions $(\tilde{f}_n)_{n\geq 1}$, a sequence of points $(y_n)_{n\geq 1}$, $y_n \to z$ as $n \to \infty$ and three increasing sequences of integers $(i_n)_{n\geq 1}$, $(k_n)_{n\geq 1}$, $(m_n)_{n\geq 1}$, $i_{n+1} > k_n > i_n$ for $n \in \mathbb{N}$, such that

$$\tilde{f}_{n|K_{i_n}} = 1 \quad \text{and} \quad 0 \leq \tilde{f}_n \leq \mathbf{1}_{K_{i_n}^{\varepsilon/12}}, \tag{2.4}$$

$$\left|P^{m_n}\left(\sum_{i=1}^{n}\tilde{f}_i\right)(z) - P^{m_n}\left(\sum_{i=1}^{n}\tilde{f}_i\right)(y_n)\right| > \varepsilon/4 \tag{2.5}$$

and

$$P^{m_n}\left(u, \bigcup_{i=k_n}^{\infty} K_i^{\varepsilon/12}\right) < \varepsilon/16 \qquad \text{for } u = z, y_n, n \in \mathbb{N}. \tag{2.6}$$

Let $n = 1$. From what has already been proved, it follows that there exist $y_1 \in B(z, 1)$ and $i_1 \in \mathbb{N}$ such that

$$P^{q_{i_1}}(y_1, K_{i_1}^{\varepsilon/12}) < \varepsilon/2.$$

Set $m_1 = q_{i_1}$ and let $k_1 > i_1$ be such that

$$P^{m_1}\left(u, \bigcup_{i=k_1}^{\infty} K_i^{\varepsilon/12}\right) < \varepsilon/16 \qquad \text{for } u = z, y_1.$$

Let $\tilde{f}_1$ be an arbitrary Lipschitz continuous function satisfying

$$\tilde{f}_{1|K_{i_1}} = 1 \quad \text{and} \quad 0 \le \tilde{f}_1 \le \mathbf{1}_{K_{i_1}^{\varepsilon/12}}. \tag{2.7}$$

Thus,

$$|P^{m_1}\tilde{f}_1(z) - P^{m_1}\tilde{f}_1(y_1)| \ge P^{m_1}(z, K_{i_1}) - P^{m_1}(y_1, K_{i_1}^{\varepsilon/12}) > \varepsilon/2.$$

If $n \ge 2$ is fixed and $\tilde{f}_1, \dots, \tilde{f}_{n-1}$, $y_1, \dots, y_{n-1}$, $i_1, \dots, i_{n-1}$, $k_1, \dots, k_{n-1}$, $m_1, \dots, m_{n-1}$ are given, we choose $\sigma < n^{-1}$ such that

$$\left| P^m \left( \sum_{i=1}^{n-1} \tilde{f}_i \right)(z) - P^m \left( \sum_{i=1}^{n-1} \tilde{f}_i \right)(y) \right| < \varepsilon/8 \tag{2.8}$$

for $y \in B(z, \sigma)$ and $m \in \mathbb{N}$. Similarly to the first part, we may choose $y_n \in B(z, \sigma)$ and $i_n > k_{n-1}$ such that

$$P^{q_{i_n}}(y_n, K_{i_n}^{\varepsilon/12}) < \varepsilon/2.$$

Set $m_n = q_{i_n}$ and let $\tilde{f}_n$ be an arbitrary Lipschitz continuous function satisfying condition (2.4). Let $k_n > i_n$ be such that

$$P^{m_n}\left(u, \bigcup_{i=k_n}^{\infty} K_i^{\varepsilon/12}\right) < \varepsilon/16 \qquad \text{for } u = z, y_n.$$

From this, (2.8) and the definition of $\tilde{f}_n$, we have

$$\begin{aligned} &\left| P^{m_n}\left(\sum_{i=1}^{n} \tilde{f}_i\right)(z) - P^{m_n}\left(\sum_{i=1}^{n} \tilde{f}_i\right)(y_n) \right| \\ &\qquad \ge |P^{m_n}\tilde{f}_n(z) - P^{m_n}\tilde{f}_n(y_n)| \\ &\qquad\quad - \left| P^{m_n}\left(\sum_{i=1}^{n-1} \tilde{f}_i\right)(z) - P^{m_n}\left(\sum_{i=1}^{n-1} \tilde{f}_n\right)(y_n) \right| \\ &\qquad > \varepsilon/2 - \varepsilon/8 > \varepsilon/4. \end{aligned}$$

We now define $f = \sum_{i=1}^{\infty} \tilde{f}_i$. Without loss of generality we may assume that all the functions $\tilde{f}_i$ have the same Lipschitz constant. By (2.2) and (2.4), $f$ is a Lipschitz continuous function and $\|f\|_\infty \le 1$. Finally, by (2.5) and (2.6), we have

$$|P^{m_n} f(z) - P^{m_n} f(y_n)| > \varepsilon/8 \qquad \text{for } n \in \mathbb{N}$$

and since $y_n \to z$ as $n \to \infty$, this contradicts the assumption that $\{P^n f : n \in \mathbb{N}\}$ is equicontinuous in $z$. $\square$

The Markov transition function $P$ is called *equicontinuous* if for $f \in C_b(X)$ the sequence of functions $\{P^n f : n \in N\}$ is equicontinuous on compact sets. Recall that by $C_b(X)$ we denote the space of all bounded continuous functions with a bounded support.

A Markov chain which possesses an equicontinuous Markov transition function will be called an *e-chain*.

REMARK. The concept of e-chains appears in [15, 16, 24, 26, 27]. It is, of course, clear that the condition appearing in the definition of an e-chain is equivalent to equicontinuity of $\{P^n f : n \in \mathbb{N}\}$, $f \in C_b(X)$, in every point $x \in X$.

In Proposition 2.1, we assumed that equicontinuity holds for all Lipschitz continuous functions. We now introduce a condition which allows restriction to the case of all Lipschitz continuous functions with a bounded support.

A continuous function $V : X \to [0, \infty)$ is called a *Lyapunov function* if

$$\lim_{\rho(x,x_0)\to\infty} V(x) = \infty$$

for some $x_0 \in X$.

THEOREM 2.2. *Let $\mathbf{\Phi}$ be an e-chain such that condition $(\mathcal{E})$ holds, and let $P : X \times \mathcal{B}(X) \to [0, 1]$ be its transition function. If there exist a Lyapunov function $V : X \to [0, \infty)$ and $\lambda < 1$, $b < \infty$, $R < \infty$, $x_0 \in X$ such that*

$$PV(x) \le \lambda V(x) + b\mathbf{1}_{B(x_0,R)}(x) \qquad \text{for } x \in X, \tag{2.9}$$

*then $\mathbf{\Phi}$ admits at least one invariant probability measure.*

PROOF. Observe that (2.9) implies that $\mathbf{\Phi}$ is *bounded in probability*, that is, for $x \in X$ and $\varepsilon > 0$, there exists a bounded Borel set $C \subset X$ such that $P^n(x, C) \ge 1 - \varepsilon$ for $n \in \mathbb{N}$ (see [17]). If we assume, contrary to our claim, that $\mathbf{\Phi}$ does not admit an invariant probability measure, the same conclusion as in the proof of Proposition 2.1 can be drawn for some Lipschitz continuous function with bounded support. $\square$

As an illustration of the power of Proposition 2.1, we have the following example:

EXAMPLE (Jump process). We consider a jump process connected with an iterated function system. A similar process on $\mathbb{R}^n$ was considered in [25]. Let $(\Omega, \mathcal{F}, \mathrm{Prob})$ be a probability space and let $(\tau_n)_{n\ge0}$ be a sequence of random variables $\tau_n : \Omega \to \mathbb{R}_+$ with $\tau_0 = 0$ and such that $\Delta\tau_n = \tau_n - \tau_{n-1}$, $n \ge 1$, are independent and have the same density $\gamma e^{-\gamma t}$. Let $(S(t))_{t\ge0}$ be a continuous semigroup on $X$. We have also given a sequence of continuous transformations $w_i : X \to X$, $i = 1, \dots, N$, and a probabilistic vector $(p_1(x), \dots, p_N(x))$,

$p_i(x) \geq 0$, $\sum_{i=1}^N p_i(x) = 1$ for $x \in X$. The pair $(w_1, \dots, w_N; p_1, \dots, p_N)$ is called an *iterated function system*.

We now define the $X$-valued Markov chain $\Phi = (\Phi_n)_{n\geq 1}$ in the following way. We choose $x \in X$ and let $\xi_1 = S(\tau_1)(x)$. We randomly select from the set $\{1, \dots, N\}$ an integer $i_1$. The probability that $i_1 = k$ is equal to $p_k(\xi_1)$. Set $\Phi_1 = w_{i_1}(\xi_1)$.

Let $\Phi_1, \dots, \Phi_{n-1}$, $n \geq 2$, be given. Assuming that $\Delta\tau_n = \tau_n - \tau_{n-1}$ is *independent of* $\Phi_1, \dots, \Phi_{n-1}$, we define $\xi_n = S(\Delta\tau_n)(\Phi_{n-1})$. Further, we randomly choose $i_n$ from the set $\{1, \dots, N\}$ in such a way that the probability of the event $\{i_n = k\}$ is equal to $p_k(\xi_n)$. Finally, we define $\Phi_n = w_{i_n}(\xi_n)$.

We will assume that there exists $r \in (0, 1)$ such that

$$\sum_{i=1}^N p_i(x)\rho\bigl(w_i(x), w_i(y)\bigr) \leq r\rho(x, y) \qquad \text{for } x, y \in X. \tag{2.10}$$

Moreover, there exist $a > 0$ such that

$$\sum_{i=1}^N |p_i(x) - p_i(y)| \leq a\rho(x, y) \qquad \text{for } x, y \in X \tag{2.11}$$

and $\kappa \geq 0$ such that

$$\rho\bigl(S(t)(x), S(t)(y)\bigr) \leq e^{\kappa t}\rho(x, y) \qquad \text{for } x, y \in X \text{ and } t \geq 0. \tag{2.12}$$

We will assume that a semigroup $(S(t))_{t\geq 0}$ admits a global attractor. Recall that a compact set $\mathcal{K} \subset X$ is called a *global attractor* if it is invariant and attracting for $(S(t))_{t\geq 0}$, that is, $S(t)\mathcal{K} = \mathcal{K}$ for every $t \geq 0$, and for every bounded ball $B$ and open set $U$, $\mathcal{K} \subset U$, there exists $t_* > 0$ such that $S(t)B \subset U$ for $t \geq t_*$.

PROPOSITION 2.3. *Assume that conditions* (2.10)–(2.12) *hold and that*

$$r + \kappa/\gamma < 1. \tag{2.13}$$

*If $(S(t))_{t\geq 0}$ has a global attractor, then $\Phi$ admits an invariant probability measure.*

PROOF. It is easily seen that $\Phi$ is a Markov chain. Analysis similar to that in [14] (see also [13, 19]) shows that its transition function must be of the form

$$P(x, A) = \sum_{i=1}^N \int_0^\infty \gamma e^{-\gamma t} p_i(S(t)(x))\mathbf{1}_A(w_i(S(t)(x)))\,dt \tag{2.14}$$

for $x \in X$ and $A \in \mathcal{B}(X)$. Then

$$Pf(x) = \sum_{i=1}^N \int_0^\infty \gamma e^{-\gamma t} p_i(S(t)(x)) f(w_i(S(t)(x)))\,dt$$

for every $f \in C(X)$ and $x \in X$.

Let $L \geq a\gamma(\kappa - \gamma(1+r))^{-1}$ and let $f$ be a Lipschitz continuous function with the Lipschitz constant $L$. If $\|f\|_\infty \leq 1$, then $\|Pf\|_\infty \leq 1$ and

$$\begin{aligned}
&|Pf(x) - Pf(y)| \\
&\quad \leq \sum_{i=1}^{N} \int_0^\infty \gamma e^{-\gamma t} p_i(S(t)(x)) |f(w_i(S(t)(x))) - f(w_i(S(t)(y)))| \, dt \\
&\qquad + \sum_{i=1}^{N} \int_0^\infty \gamma e^{-\gamma t} |p_i(S(t)(x)) - p_i(S(t)(y))| \, dt \\
&\quad \leq Lr\left(\int_0^\infty \gamma e^{-\gamma t + \kappa t} \, dt\right) \rho(x, y) + a\left(\int_0^\infty \gamma e^{-\gamma t + \kappa t} \, dt\right) \rho(x, y) \\
&\quad \leq L\rho(x, y) \qquad \text{for } x, y \in X.
\end{aligned}$$

From this, and the fact that $P$ is linear, it follows that $\{P^n f : n \in \mathbb{N}\}$ is equicontinuous in any $x \in X$ for an arbitrary Lipschitz continuous function $f$. Let $x_0 \in X$ and set $V(x) = \rho(x, x_0)$ for $x \in X$. An easy computation shows that

$$PV(x) \leq r\gamma(\gamma - \kappa)^{-1} V(x) + N\tilde{b} \qquad \text{for } x \in X,$$

where $\tilde{b} = \sup_{t \geq 0, 1 \leq i \leq N} \rho(w_i(S(t)(x_0)), x_0) < \infty$, by the fact that $(S(t))_{t \geq 0}$ has a global attractor. Set $\lambda_0 = r\gamma(\gamma - \kappa)^{-1}$. By (2.13), we have $\lambda_0 < 1$. Let $\lambda \in (\lambda_0, 1)$. Since $V$ is a Lyapunov function, there exists $R > 0$ such that condition (2.9) holds with $b = N\tilde{b}$. Hence, $\Phi$ is bounded in probability (see [23]). Fix $x \in X$ and let $C \subset X$ be a bounded Borel set such that $P^n(x, C) > 1/2$. Let $\mathcal{K} \subset X$ be an attractor for $(S(t))_{t \geq 0}$ and let $K = \bigcup_{i=1}^N w_i(\mathcal{K})$. Since $w_i$, $i = 1, \dots, N$, are continuous, the set $K \subset X$ is compact. Further, from (2.14), and the fact that $\mathcal{K}$ was a global attractor, it follows that for every open set $U$, $K \subset U$, there exists a positive constant $\beta$ such that

$$P(y, U) \geq \beta \qquad \text{for } y \in C.$$

Together with the Chapman–Kolmogorov equation, this gives

$$\liminf_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} P^i(x, O) \right) > \beta/2.$$

Since $K$ is compact, we see that there exists $z \in K$ such that condition (1.1) holds for every open neighborhood $U$ of $z$. Thus, $\Phi$ has an invariant measure by Proposition 2.1. $\square$

## 3. Stability results.

THEOREM 3.1. *Let* $\mathbf{\Phi}$ *be an e-chain. Let* $P : X \times \mathcal{B}(X) \to [0, 1]$ *be its transition function and assume that there exists* $z \in X$ *such that for every open set* $O$

*containing* $z$,

$$\liminf_{n\to\infty} P^n(x, O) > 0 \qquad \text{for } x \in X. \tag{3.1}$$

*Let*

$$\mathcal{Z} = \overline{\bigcup_{n=1}^{\infty} \operatorname{supp} P^n(z, \cdot)}.$$

*If there exist a Lyapunov function* $V : X \to [0, \infty)$ *and* $\lambda < 1$, $b < \infty$, $R < \infty$, $x_0 \in X$ *such that* (2.9) *holds*, *then* $\mathbf{\Phi}$ *admits a unique invariant probability measure* $\mu_*$ *supported on* $\mathcal{Z}$. *Moreover*,

$$\mu P^n \xrightarrow{\mathrm{w}} \mu_* \qquad \text{as } n \to \infty$$

*for every probability measure* $\mu$ *such that* $\operatorname{supp} \mu \subset \mathcal{Z}$.

PROOF. Since (3.1) implies (1.1), from Theorem 2.2 it follows that $\mathbf{\Phi}$ has an invariant probability measure, say $\mu_*$. It may be obtained (see [7, 30]) as any weak limit of the Cesaro averages of $(P^n(z, \cdot))_{n\ge 1}$. Therefore, we may assume that $\operatorname{supp} \mu_* \subset \mathcal{Z}$.

Let us denote by $\Delta(x_1, x_2; f; \varepsilon)$ for $x_1, x_2 \in X$, $f \in C_b(X)$, $\varepsilon > 0$ the set of all $\alpha \in (0, 1]$ such that there exist probability measures $\mu_1, \mu_2$ and an integer $m$ satisfying

$$P^m(x_i, \cdot) \ge \alpha \mu_i(\cdot) \qquad \text{for } i = 1, 2 \tag{3.2}$$

and

$$\left| \int_X f(y) \mu_1 P^n(dy) - \int_X f(y) \mu_2 P^n(dy) \right| \le \varepsilon \qquad \text{for } n \in \mathbb{N}. \tag{3.3}$$

We claim that $\sup \Delta(x_1, x_2; f; \varepsilon) = 1$ for $x_1, x_2 \in \mathcal{Z}$, $f \in C_b(X)$ and $\varepsilon > 0$. Fix $x_1, x_2 \in \mathcal{Z}$, $f \in C_b(X)$ and $\varepsilon > 0$. By the Chapman–Kolmogorov equation, we easily obtain that

$$\liminf_{n\to\infty} P^n(x, O_i) > 0 \qquad \text{for } x \in X,$$

where $O_i$ is an arbitrary open set containing $x_i$, $i = 1, 2$. Now, from the proof of Proposition 2.1, it follows that the families $\{P^n(x_i, \cdot) : n \in \mathbb{N}\}$, $i = 1, 2$, are weakly precompact (see also Theorem 2.2 in [7]). Let $\sigma > 0$ be such that

$$|P^n f(z) - P^n f(y)| \le \varepsilon \qquad \text{for } y \in B(z, \sigma) \text{ and } n \in \mathbb{N}. \tag{3.4}$$

By (3.1) there exist $m \in \mathbb{N}$ and $\tilde{\alpha} > 0$ such that

$$P^m\big(x_i, B(z, \sigma)\big) \ge \tilde{\alpha} \qquad \text{for } i = 1, 2.$$

Define

$$\tilde{\mu}_i(\cdot) = \frac{P^m(x_i, B(z, \sigma) \cap \cdot)}{P^m(x_i, B(z, \sigma))} \qquad \text{for } i = 1, 2, \tag{3.5}$$

and observe that condition (3.2) is satisfied with $\tilde{\mu}_i$ in place of $\mu_i$ and $\tilde{\alpha}$ in place of $\alpha$. Moreover, from (3.4), it follows that (3.3) holds with $\tilde{\mu}_i$ in place of $\mu_i$. Hence, $\Delta(x_1, x_2; f; \varepsilon) \neq \varnothing$. Set $\alpha_0 = \sup \Delta(x_1, x_2; f; \varepsilon)$. Suppose, contrary to our claim, that $\alpha_0 < 1$. Let $(\alpha_n)_{n\geq 1}$ be such that $\alpha_n \to \alpha_0$ as $n \to \infty$ and $\alpha_n \in \Delta(x_1, x_2; f; \varepsilon)$ for $n \in \mathbb{N}$. Let $\mu_i^n$, $i = 1, 2$, and $m_n$ satisfy (3.2) with $\alpha_n$ in place of $\alpha$. Since $\{P^n(x_i, \cdot) : n \in \mathbb{N}\}$, $i = 1, 2$, are tight, $\{P^{m_n}(x_i, \cdot) - \alpha_n \mu_i^n : n \in \mathbb{N}\}$, $i = 1, 2$, are weakly precompact. Therefore, without loss of generality, we may assume that $(P^{m_n}(x_i, \cdot) - \alpha_n \mu_i^n)_{n\geq 1}$, $i = 1, 2$, converge to some measures $\tilde{\mu}_1, \tilde{\mu}_2$, respectively. Choose $y_1 \in \operatorname{supp} \tilde{\mu}_1$ and $y_2 \in \operatorname{supp} \tilde{\mu}_2$. From (3.1), it follows that there exist $m \in \mathbb{N}$ and $\gamma > 0$ such that

$$P^m\big(y_i, B(z, \sigma)\big) \geq \gamma \qquad \text{for } i = 1, 2.$$

By Feller's property, there exists $r > 0$ such that

$$P^m\big(y, B(z, \sigma)\big) \geq \gamma/2 \qquad \text{for } y \in B(y_i, r), i = 1, 2.$$

Set

$$s_0 = \min\{\tilde{\mu}_1\big(B(y_1, r)\big), \tilde{\mu}_2\big(B(y_2, r)\big)\}$$

and observe that $s_0 > 0$. By the Alexandrov theorem (see [2]), we may choose $k \in \mathbb{N}$ such that

$$P^{m_k}\big(x_i, B(y_i, r)\big) - \alpha_k \mu_i^k\big(B(y_i, r)\big) > s_0/2 \qquad \text{for } i = 1, 2.$$

Let $k \in \mathbb{N}$ be such that

$$\alpha_k + s_0\gamma/4 > \alpha_0. \tag{3.6}$$

Then, by the Chapman–Kolmogorov equation [see also (3.5)], we obtain that there exist probability measures $\hat{\mu}_i$ with $\operatorname{supp} \hat{\mu}_i \subset B(z, \sigma)$, $i = 1, 2$, such that

$$P^{m_k+m}(x_i, \cdot) - \alpha_k \mu_i^k P^m \geq s_0\gamma \hat{\mu}_i/4.$$

Set

$$\mu_i = (\alpha_k + s_0\gamma/4)^{-1}(\alpha_k \mu_i^k P^m + s_0\gamma \hat{\mu}_i/4) \qquad \text{for } i = 1, 2.$$

Since $\operatorname{supp} \hat{\mu}_i \subset B(z, \sigma)$ for $i = 1, 2$, from (3.4) it follows that $\mu_i$, $i = 1, 2$, satisfy (3.3). Finally, observe that $\mu_i$, $i = 1, 2$, satisfy condition (3.2) with $m_k + m$ in place of $m$ and $\alpha = \alpha_k + s_0\gamma/4$. Hence, $\alpha_k + s_0\gamma/4 \in \Delta(x_1, x_2; f; \varepsilon)$, which contradicts the definition of $\alpha_0$, by (3.6).

We have proved that

$$\lim_{n\to\infty} \left| \int_X f(y)\mu_1 P^n(dy) - \int_X f(y)\mu_2 P^n(dy) \right| = 0$$

for all point measures $\mu_1, \mu_2$ supported on $\mathcal{Z}$ and for every $f \in C_b(X)$. Since linear combinations of point measures are dense in the space of all measures equipped with the weak topology, the above convergence holds for all probability measures $\mu_1, \mu_2$ supported on $\mathcal{Z}$ and for every $f \in C_b(X)$. Since $\mathbf{\Phi}$ is bounded in probability, the above convergence is also satisfied for every $f \in C(X)$. From this, it follows that $\mu_*$ is a unique invariant measure supported on $\mathcal{Z}$ and that

$$\mu P^n \xrightarrow{\mathrm{w}} \mu_* \qquad \text{as } n \to \infty$$

for every probability measure $\mu$ such that $\operatorname{supp} \mu \subset \mathcal{Z}$, completing the proof. $\square$

A point $x \in X$ is called *reachable* if for every open set $O$ containing $x$,

$$\sum_{n=1}^{\infty} P^n(y, O) > 0 \qquad \text{for every } y \in X.$$

The chain $\mathbf{\Phi}$ is called *open set irreducible* if every point is reachable.

As a consequence of Theorem 3.1 and the above definition, we obtain the following theorem:

THEOREM 3.2. *Let $\mathbf{\Phi}$ be an open set irreducible e-chain. Let $P : X \times \mathcal{B}(X) \to [0,1]$ be its transition function and assume that there exists $z \in X$ such that for every open set $O$ containing $z$, condition (3.1) holds. If there exist a Lyapunov function $V : X \to [0, \infty)$ and $\lambda < 1$, $b < \infty$, $R < \infty$, $x_0 \in X$ such that (2.9) holds, then $\mathbf{\Phi}$ admits a unique invariant probability measure $\mu_*$. Moreover,*

$$\mu P^n \xrightarrow{\mathrm{w}} \mu_* \qquad \text{as } n \to \infty$$

*for every probability measure $\mu$.*

PROOF. It suffices to note that

$$\overline{\bigcup_{n=1}^{\infty} \operatorname{supp} P^n(z, \cdot)} = X.$$

$\square$

THEOREM 3.3. *Let $\mathbf{\Phi}$ be an e-chain. Let $P : X \times \mathcal{B}(X) \to [0,1]$ be its transition function and assume that there exists $z \in X$ such that for every open set $O$ containing $z$, there exists $\alpha > 0$ satisfying*

$$\liminf_{n \to \infty} P^n(x, O) \geq \alpha \qquad \text{for } x \in X. \tag{3.7}$$

*If there exist a Lyapunov function $V : X \to [0, \infty)$ and $\lambda < 1$, $b < \infty$, $R < \infty$, $x_0 \in X$ such that (2.9) holds, then $\mathbf{\Phi}$ admits a unique invariant probability measure $\mu_*$. Moreover,*

$$\mu P^n \xrightarrow{\mathrm{w}} \mu_* \qquad \text{as } n \to \infty \tag{3.8}$$

*for every probability measure $\mu$.*

PROOF. The existence of an invariant measure $\mu_*$ follows from Theorem 2.2. Fix $\varepsilon > 0, x_1, x_2 \in X$ and $f \in C_b(X)$. By equicontinuity of $\{P^n f : n \in \mathbb{N}\}$ in $z \in X$, we choose $r > 0$ such that

$$|P^n f(z) - P^n f(x)| < \varepsilon/4 \qquad \text{for } x \in B(z, r) \text{ and } n \in \mathbb{N}. \tag{3.9}$$

Let $\alpha > 0$ be such that (3.7) holds with $O = B(z, r)$. Then, by Fatou's lemma, we have

$$\liminf_{n\to\infty} \mu P^n(O) \geq \alpha \tag{3.10}$$

for every probability measure $\mu$. Let $k \in \mathbb{N}$ be such that $4(1 - \alpha/2)^k \|f\|_\infty \leq \varepsilon$. Further, from the Lasota–Yorke theorem (see Theorem 4.1 in [19]) and (3.10), it follows that there exist integers $n_1, \dots, n_k$ and probability measures $\nu_1^i, \dots, \nu_k^i, \mu_k^i$ such that $\operatorname{supp} \nu_j^i \subset O$, $j = 1, \dots, k$, and

$$\begin{aligned} P^{n_1+\cdots+n_k}(x_i, \cdot) &= \frac{\alpha}{2}\nu_1^i P^{n_2+\cdots+n_k} + \frac{\alpha}{2}\left(1 - \frac{\alpha}{2}\right)\nu_2^i P^{n_3+\cdots+n_k} \\ &\quad + \cdots + \frac{\alpha}{2}\left(1 - \frac{\alpha}{2}\right)^{k-1} \nu_k^i + \left(1 - \frac{\alpha}{2}\right)^k \mu_k^i \qquad \text{for } i = 1, 2. \end{aligned}$$

Then, by the Markov property, we obtain

$$\begin{aligned} P^n(x_i, \cdot) &= \frac{\alpha}{2}\nu_1^i P^{n-n_1} + \frac{\alpha}{2}\left(1 - \frac{\alpha}{2}\right)\nu_2^i P^{n-n_1-n_2} \\ &\quad + \cdots + \frac{\alpha}{2}\left(1 - \frac{\alpha}{2}\right)^{k-1} \nu_k^i P^{n-n_1-\cdots-n_k} + \left(1 - \frac{\alpha}{2}\right)^k \mu_k^i P^{n-n_1-\cdots-n_k} \end{aligned}$$

for $i = 1, 2$ and $n \geq n_1 + \cdots + n_k$. From (3.9), we have

$$\begin{aligned} &\left| \int_X f(y)\nu_j^1 P^n(dy) - \int_X f(y)\nu_j^2 P^n(dy) \right| \\ &\quad = \left| \int_X P^n f(y)\nu_j^1(dy) - \int_X P^n f(y)\nu_j^2(dy) \right| \leq \varepsilon/2 \end{aligned}$$

for $j = 1, \dots, k$. By the definition of $k$, we then obtain

$$\begin{aligned} &|P^n f(x_1) - P^n f(x_2)| \\ &\quad = \left| \int_X f(y) P^n(x_1, dy) - \int_X f(y) P^n(x_2, dy) \right| \\ &\quad < \varepsilon/2 + 2\|f\|_\infty (1 - \alpha/2)^k = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ and $f \in C_b(X)$ were arbitrary, and since linear combinations of point measures are dense in the space of all measures equipped with the weak topology, we have

$$\lim_{n\to\infty} \left| \int_X f(y)\mu_1 P^n(dy) - \int_X f(y)\mu_2 P^n(dy) \right| = 0$$

for all probability measures $\mu_1, \mu_2$ and for arbitrary $f \in C_b(X)$. Since $\mathbf{\Phi}$ is bounded in probability, the above condition is also satisfied for every $f \in C(X)$. On the other hand, from the above condition, it follows that $\mu_*$ is a unique invariant measure and

$$\mu P^n \xrightarrow{\mathrm{w}} \mu_* \qquad \text{as } n \to \infty$$

for every probability measure $\mu$, completing the proof. $\square$

As an immediate consequence of this theorem, we obtain the following result, due to Stettner (see [29]):

COROLLARY 3.4. *Assume that*:

(S1) *for every $\varepsilon > 0$ and every compact set $K \subset X$, there exists a compact set $W \subset X$ such that*

$$\inf_{x \in K} P^n(x, W) \geq 1 - \varepsilon \qquad \textit{for } n \in \mathbb{N};$$

(S2) *for every $f \in C_b(X)$, the functions $\{P^n f : n = 1, 2, \dots\}$ are equicontinuous on compact subsets of $X$*;

(S3) *for every open set $O \subset X$ and every $x \in X$,*

$$P(x, O) > 0;$$

(S4) *there exist $\eta > 0$ and a compact set $L \subset X$ such that for every compact set $W \subset X$,*

$$\inf_{x \in W} P^n(x, L) \geq \eta \qquad \textit{for some } n \in \mathbb{N}.$$

*Then there exists a unique invariant measure $\mu_*$ for $\mathbf{\Phi}$, and $P^n(x, \cdot)$ converges weakly to $\mu_*$.*

**4. A counterexample.** In the last section, we shall define a discrete-time Markov–Feller chain which satisfies condition $(\mathcal{E})$ but which does not have an invariant measure.

Let $(\Omega, \mathcal{F}, \mathrm{Prob})$ be a probability space and let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Define $\mathbf{x} : \mathbb{N} \times \mathbb{N} \times \overline{\mathbb{N}} \to l^\infty$ by the following:

$$\mathbf{x}(i, j, k) = (i, \overbrace{0, \dots, 0}^{j\text{-times}}, 2^{-k}, \dots).$$

It is easy to see that $X = \mathbf{x}(\mathbb{N} \times \mathbb{N} \times \overline{\mathbb{N}})$ is a closed subset of $l^\infty$. Consider the discrete-time Markov chain $\mathbf{\Phi} = (\Phi_n)_{n \geq 1}$ defined by the formula

$$\Phi_n = \mathbf{x}(\zeta_n, \eta_n, \xi_n) \qquad \text{for } n \in \mathbb{N},$$

where $\zeta_n, \eta_n : \Omega \to \mathbb{N}$ and $\xi_n : \Omega \to \overline{\mathbb{N}}$ are Markov chains satisfying

$$\begin{aligned}&\mathrm{Prob}(\zeta_{n+1} = i, \eta_{n+1} = j, \xi_{n+1} = k | \zeta_n = i_0, \eta_n = j_0, \xi_n = k_0)\\&\quad = \begin{cases} p_1(i_0, k_0) & \text{for } i = 1, j = j_0 + 1, k = 1; \\ p_2(k_0) & \text{for } i = i_0, j = j_0 + 1, k = k_0 + 1; \\ 1 - p_1(i_0, k_0) - p_2(k_0) & \text{for } i = i_0 + 1, j = j_0 + 1, k = k_0. \end{cases}\end{aligned}$$

Moreover, we assume that $p_2(k) = k^{-4}$ for $k \in \mathbb{N}$, $p_1(i, k) = 1 - p_2(k)$ for $k < i!$ and $p_1(i, k) = p_2(k)$ for $k \geq i!$. Further, $p_1(i, \infty) = p_2(\infty) = 0$.

To show that $\mathbf{\Phi}$ satisfies Feller's property, fix $f \in C(X)$ and $x_0 \in X$. Let $x_n \to x_0$ as $n \to \infty$. Without loss of generality we may assume that $x_n = \mathbf{x}(i, j_n, k_n)$, $x_0 = \mathbf{x}(i, 1, \infty)$ and $k_n \to \infty$ as $n \to \infty$. Then

$$\begin{aligned} Pf(x_n) &= p_1(i, k_n) f\big(\mathbf{x}(1, j_n + 1, 1)\big) + p_2(k_n) f\big(\mathbf{x}(i, j_n + 1, k_n + 1)\big) \\ &\qquad + \big(1 - p_1(i, k_n) - p_2(k_n)\big) f\big((i + 1, j_n + 1, k_n)\big) \\ &\underset{n\to\infty}{\to} f\big((i + 1, 1, \infty)\big) \\ &= Pf(x_0). \end{aligned}$$

Now let $x = \mathbf{x}(i_0, j_0, k_0)$ be such that $k_0 \neq \infty$. We will show that there exists $\vartheta > 0$ such that

$$P^n(x, U_0) \geq \vartheta \qquad \text{for } n \in \mathbb{N}, \tag{4.1}$$

where $U_0 = \{\mathbf{x}(i, j, k) : i = k = 1, j \in \mathbb{N}\}$. Since $p_2(k) = k^{-4}$ for $k \in \mathbb{N}$, $p_1(i, k) = 1 - p_2(k)$ for $k < i!$ and $p_1(i, k) = p_2(k)$ for $k \geq i!$, we can easily check that

$$\sup_{n\in\mathbb{N}} \mathrm{E}[\xi_n | \zeta_0 = i_0, \eta_0 = j_0, \xi_0 = k_0] < \infty.$$

Chebyshev's inequality now shows that there exists $M_0 > i_0$ such that

$$\inf_{n\in\mathbb{N}} \mathrm{Prob}(\xi_n \leq M_0 | \zeta_0 = i_0, \eta_0 = j_0, \xi_0 = k_0) > 0.$$

From this, and the fact that $p_1(i, k) = 1 - p_2(k)$ for $i < k!$, we obtain

$$\gamma = \inf_{n\in\mathbb{N}} \mathrm{Prob}(\zeta_n \leq M_0!, \xi_n \leq M_0 | \zeta_0 = i_0, \eta_0 = j_0, \xi_0 = k_0) > 0.$$

By the Markov property, we have

$$P^n(x, U_0) \geq \gamma \cdot \min_{1\leq i\leq M_0!, 1\leq k\leq M_0} p_1(i, k) \qquad \text{for } n \in \mathbb{N},$$

which shows that condition (4.1) holds with

$$\vartheta = \gamma \cdot \min_{1\leq i\leq M_0!, 1\leq k\leq M_0} p_1(i, k).$$

Let $z = (1, 0, \ldots)$. Fix an open set $U$ such that $z \in U$. Let $r > 0$ be such that $B(z, r) \subset U$. Choose $k \in \mathbb{N}$ such that $x(1, j, k) \in B(z, r)$ for $j \in \mathbb{N}$. Then, by the Markov property, we obtain

$$P^{n+k}(x, U) \geq \vartheta p_2(1) \cdots p_2(k) \qquad \text{for } n \in \mathbb{N},$$

which gives condition ($\mathcal{E}$).

Finally, it is obvious that $\mathbf{\Phi}$ does not admit an invariant measure since $\lim_{n\to\infty}\eta_n = \infty$ almost surely. $\square$

## REFERENCES


[1] BARNSLEY, M. F., DEMKO, S. G., ELTON, J. H. and GERONIMO, J. S. (1988). Invariant measures for Markov processes arising from iterated function systems with place-dependent probabilities. *Ann. Inst. H. Poincaré Probab. Statist.* **24** 367–394. MR0971099

[2] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[3] BOROVKOV, V. (1991). Ergodicity condition for Markov chains that are not connected with irreducibility in the sense of Harris. *Sibirsk. Math. Zh.* **32** 6–19. MR1142065

[4] COSTA, O. L. V. and DUFOUR, F. (2001). Necessary and sufficient conditions for non-singular invariant probability measures for Feller–Markov chains. *Statist. Probab. Lett.* **53** 47–57. MR1843340

[5] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions*. Cambridge Univ. Press. MR1207136

[6] DIACONIS, P. and FREEDMAN, D. (1999). Iterated random functions. *SIAM Rev.* **41** 41–76. MR1669737

[7] ETHIER, S. N. and KURTZ, T. G. (1985). *Markov Processes*: *Characterization and Convergence*. Wiley, New York. MR0838085

[8] FOGUEL, S. R. (1969). *The Ergodic Theory of Markov Processes*. *Van Nostrand Mathematical Studies* **21**. Van Nostrand Reinhold Co., New York. MR0261686

[9] FOGUEL, S. R. (1962). Existence of invariant measures for Markov processes. *Proc. Amer. Math. Soc.* **13** 833–838. MR0145590

[10] FOGUEL, S. R. (1966). Existence of invariant measures for Markov processes II. *Proc. Amer. Math. Soc.* **17** 387–389. MR0193673

[11] HERNANDEZ-LERMA, O. and LASSERRE, J.-B. (1998). Existence and uniqueness of fixed points for Markov operators and Markov processes. *Proc. London Math. Soc.* **76** 711–736. MR1620504

[12] HERNANDEZ-LERMA, O. and LASSERRE, J.-B. (1998). Ergodic theorems and ergodic decomposition for Markov chains. *Acta Appl. Math.* **54** 99–119. MR1656784

[13] HORBACZ, K. (2004). Random dynamical systems with jumps. *J. Appl. Probab.* **41** 890–910. MR2074830

[14] HORBACZ, K., MYJAK, J. and SZAREK, T. (2005). On stability of some general random dynamical system. *J. Statist. Physics* **119** 35–60. MR2144506

[15] JAMISON, B. (1964). Asymptotic behavior of successive iterates of continuous functions under a Markov operator. *J. Math. Anal. Appl.* **9** 203–214. MR0169040

[16] JAMISON, B. (1965). Ergodic decomposition induced by certain Markov operators. *Trans. Amer. Math. Soc.* **117** 451–468. MR0207041

[17] JARNER, S. and TWEEDIE, R. (2001). Locally contracting iterated random functions and stability of Markov chains. *J. Appl. Probab.* **38** 494–507. MR1834756

[18] LASOTA, A. and SZAREK, T. (2006). Lower bound technique in the theory of a stochastic differential equation. *J. Differential Equations*. To appear.

[19] LASOTA, A. and YORKE, J. (1994). Lower bounded technique for Markov operators and iterated function systems. *Random Comput. Dynam.* **2** 41–77. MR1265226

[20] LASSERRE, J.-B. (1997). Invariant probabilities for Markov chains on a metric space. *Statist. Probab. Lett.* **34** 259–265. MR1458020

[21] LIN, M. (1970). Conservative Markov processes on a topological space. *Israel J. Math.* **8** 165–186. MR0265559

[22] LU, G. and MUKHERJEA, A. (1997). Invariant measures and Markov chains with random transition probabilities. *Probab. Math. Statist.* **17** 115–138. MR1455613

[23] MEYN, S. P. and TWEEDIE, R. L. (1993). *Markov Chains and Stochastic Stability*. Springer, London. MR1287609

[24] ROSENBLATT, M. (1965). Equicontinuous Markov operators. *Teor. Verojatnost. i Primenen.* **9** 205–222. MR0171318

[25] PICHÓR, K. and RUDNICKI, R. (2000). Continuous Markov semigroups and stability of transport equations. *J. Math. Anal. Appl.* **249** 668–685. MR1781248

[26] SINE, R. (1974). Convergence theorems for weakly almost periodic Markov operators. *Israel J. Math.* **19** 246–255. MR0372159

[27] SINE, R. (1975). On local uniform mean convergence for Markov operators. *Pacific J. Math.* **60** 247–252. MR0397882

[28] STEINSALTZ, D. (1999). Locally contractive iterated function systems. *Ann. Probab.* **27** 1952–1979. MR1742896

[29] STETTNER, Ł. (1994). Remarks on ergodic conditions for Markov processes on Polish spaces. *Bull. Polish Acad. Sci. Math.* **42** 103–114. MR1810695

[30] SZAREK, T. (2000). The stability of Markov operators on Polish spaces. *Studia Math.* **143** 145–152. MR1813364

UNIVERSITY OF SILESIA
BANKOWA 14
40-007 KATOWICE
POLAND
E-MAIL: szarek@intertele.pl