The Annals of Probability
2006, Vol. 34, No. 4, 1601–1607
DOI: 10.1214/009117906000000142

ON THE SECOND MOMENT OF THE NUMBER OF CROSSINGS BY A STATIONARY GAUSSIAN PROCESS

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Cramér and Leadbetter introduced in 1967 the sufficient condition

$$\frac{r''(s) - r''(0)}{s} \in L^1([0, \delta], dx), \qquad \delta > 0,$$

to have a finite variance of the number of zeros of a centered stationary Gaussian process with twice differentiable covariance function r. This condition is known as the Geman condition, since Geman proved in 1972 that it was also a necessary condition. Up to now no such criterion was known for counts of crossings of a level other than the mean. This paper shows that the Geman condition is still sufficient and necessary to have a finite variance of the number of any fixed level crossings. For the generalization to the number of a curve crossings, a condition on the curve has to be added to the Geman condition.

1. Introduction and main result.

Let $X = \{X_t,\ t \in \mathbb{R}\}$ be a centered stationary Gaussian process. Its correlation function r is supposed to be twice differentiable and to satisfy on $[0, \delta]$, with $\delta > 0$,

$$\begin{aligned} & r(\tau) = 1 + \frac{r''(0)}{2}\tau^2 + \theta(\tau) \\ & \text{with } \theta(\tau) > 0, \frac{\theta(\tau)}{\tau^2} \to 0, \frac{\theta'(\tau)}{\tau} \to 0, \theta''(\tau) \to 0, \text{ as } \tau \to 0. \end{aligned} \tag{1}$$

The nonnegative function L defined by $\theta''(\tau) := \tau L(\tau)$ will be referred to as the Geman function.

Let us consider a continuous differentiable real function ψ and let us define, as in [2], the number of crossings of the function ψ by the process X on an interval $[0, t]$ ($t \in \mathbb{R}$), as the random variable $N_t^\psi = N_t(\psi) = \#\{s \le t : X_s = \psi_s\}$.

The number N_t^ψ of ψ-crossings by X can also be seen as the number of zero crossings $N_t^Y(0)$ by the nonstationary (but stationary in the sense of the covariance) Gaussian process $Y = \{Y_s,\ s \in \mathbb{R}\}$, with $Y_s := X_s - \psi_s$, that is, $N_t^\psi = N_t^Y(0)$.

Received December 2004; revised June 2005.

[1]Supported in part by the Project No. 97003647 “Modelaje Estocástico Aplicado” of the Agenda Petróleo of FONACIT Venezuela.

AMS 2000 subject classifications. Primary 60G15; secondary 60G10, 60G70.

Key words and phrases. Crossings, Gaussian processes, Geman condition, Hermite polynomials, level curve, spectral moment.

Regarding the moments of the number of crossings by X, one of the most well-known first results was obtained by Rice [9] for a given level x, namely

$$\mathbb{E}[N_t(x)] = te^{-x^2/2}\sqrt{-r''(0)}/\pi.$$

This equality was proved two decades later by Itô [7] and Ylvisaker [11], providing a necessary and sufficient condition to have a finite mean number of crossings:

$$\mathbb{E}[N_t(x)] < \infty \quad \Longleftrightarrow \quad -r''(0) < \infty.$$

Also in the 1960s, following on the work of Cramér, generalization to curve crossings and higher-order moments for $N_t(\cdot)$ were considered in a series of papers by Cramér and Leadbetter [2] and Ylvisaker [12].

Moreover, Cramér and Leadbetter [2] provided an explicit formula for the second factorial moment of the number of zeros of the process X, and proposed a sufficient condition on the correlation function of X in order to have the random variable $N_t(0)$ belonging to $L^2(\Omega)$, namely

$$\text{If } L(t) := \frac{r''(t) - r''(0)}{t} \in L^1([0,\delta], dx) \qquad \text{then } \mathbb{E}[N_t^2(0)] < \infty.$$

Geman [6] proved that this condition was not only sufficient but also necessary:

$$\mathbb{E}[N_t^2(0)] < \infty \quad \Longleftrightarrow \quad L(t) \in L^1([0,\delta], dx) \qquad \text{(Geman condition)}. \tag{2}$$

This condition held only when choosing the level as the mean of the process.

Generalizing this result to any given level x and to some differentiable curve ψ has been subject to some investigation and nice papers, such as the ones of Cuzick [4, 5] proposing sufficient conditions. But to get necessary conditions remained an open problem for many years. The solution of this problem is enunciated in the following theorem.

THEOREM.

(1) *For any given level x, we have*

$$\mathbb{E}[N_t^2(x)] < \infty \quad \Longleftrightarrow \quad \exists\,\delta > 0, L(t) = \frac{r''(t) - r''(0)}{t} \in L^1([0,\delta], dx)$$

(*Geman condition*).

(2) *Suppose that the continuous differentiable real function ψ is such that*

$$\exists\,\delta > 0 \qquad \int_0^\delta \frac{\gamma(s)}{s}\,ds < \infty \qquad \text{where } \gamma(\cdot) \text{ is the modulus of continuity of } \dot{\psi}. \tag{3}$$

Then

$$\mathbb{E}[N_t^2(\psi)] < \infty \quad \Longleftrightarrow \quad L(t) \in L^1([0,\delta], dx).$$

REMARK. This smooth condition on ψ is satisfied by a large class of functions which includes in particular functions whose derivatives are Hölder.

Finally let us mention the work of Belyaev [1] and Cuzick [3–5] who proposed some sufficient conditions to have the finiteness of the kth (factorial) moments for the number of crossings for $k \geq 2$. When $k \geq 3$, the difficult problem of finding necessary conditions when considering levels other than the mean is still open.

2. Proof. Generalizing the formula of Cramér and Leadbetter ([2], page 209) concerning the zero crossings, the second factorial moment M_2^ψ of the number of ψ-crossings can be expressed as

$$M_2^\psi = \int_0^t \int_0^t \int_{R^2} |\dot{x}_1 - \dot{\psi}_{t_1}||\dot{x}_2 - \dot{\psi}_{t_2}| \times p_{t_1,t_2}(\psi_{t_1}, \dot{x}_1, \psi_{t_2}, \dot{x}_2)\, d\dot{x}_1\, d\dot{x}_2\, dt_1\, dt_2, \tag{4}$$

where $p_{t_1,t_2}(x_1, \dot{x}_1, x_2, \dot{x}_2)$ is the density of the vector $(X_{t_1}, \dot{X}_{t_1}, X_{t_2}, \dot{X}_{t_2})$ that is supposed nonsingular for all $t_1 \neq t_2$. The formula holds whether M_2^ψ is finite or not.

We also have

$$M_2^\psi = 2\int_0^t \int_{t_1}^t p_{t_1,t_2}(\psi_{t_1}, \psi_{t_2}) \times \mathbb{E}\big[|\dot{X}_{t_1} - \dot{\psi}_{t_1}||\dot{X}_{t_2} - \dot{\psi}_{t_2}| \big| X_{t_1} = \psi_{t_1}, X_{t_2} = \psi_{t_2}\big]\, dt_2\, dt_1, \tag{5}$$

where $p_{t_1,t_2}(x_1, x_2)$ is the density of (X_{t_1}, X_{t_2}).

From now on, let us put $t_2 = t_1 + \tau$, $\tau > 0$.

The method used to prove that the Geman condition keeps being the sufficient and necessary condition to have M_2^ψ finite can be sketched into three steps.

The first one consists in using the following regression model to compute the expectation in M_2^ψ:

$$\begin{aligned} \dot{X}_{t_1} &= \zeta + \alpha_1(\tau) X_{t_1} + \alpha_2(\tau) X_{t_1+\tau}, \\ \dot{X}_{t_1+\tau} &= \zeta^* - \beta_1(\tau) X_{t_1} - \beta_2(\tau) X_{t_1+\tau}, \end{aligned} \tag{R}$$

where (ζ, ζ^*) is jointly Gaussian such that

$$\operatorname{Var}(\zeta) = \operatorname{Var}(\zeta^*) := \sigma^2(\tau) = -r''(0) - \frac{r'^2(\tau)}{1 - r^2(\tau)}, \tag{6}$$

$$\rho(\tau) := \frac{\operatorname{Cov}(\zeta, \zeta^*)}{\sigma^2(\tau)} = \frac{-r''(\tau)(1 - r^2(\tau)) - r'^2(\tau) r(\tau)}{-r''(0)(1 - r^2(\tau)) - r'^2(\tau)}, \tag{7}$$

and where

$$\alpha_1 = \alpha_1(\tau) = \frac{r'(\tau)r(\tau)}{1 - r^2(\tau)}; \qquad \alpha_2 = \alpha_2(\tau) = -\frac{r'(\tau)}{1 - r^2(\tau)}$$

$$\beta_1 = \beta_1(\tau) = \alpha_2(\tau); \qquad \beta_2 = \beta_2(\tau) = \alpha_1(\tau).$$

In the second step, the expectation, formulated in terms of ζ and ζ^*, will be expand into Hermite polynomials. Recall that the Hermite polynomials $(H_n)_{n\ge 0}$, defined by $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$, constitute a complete orthogonal system in the Hilbert space $L^2(\mathbb{R}, \varphi(u)\,du)$, φ denoting the standard normal density.

Finally, this Hermite expansion will allow us to find, in an easier way, lower and upper bounds for M_2^ψ. Nevertheless, it will required a fine study in the neighborhood of 0, on one hand on the correlation function r of X and its derivatives, showing in particular the close relation between the existence of the Geman function L and the existence of $r^{(iv)}(0)$, on the other hand, on the correlation function ρ of the r.v. ζ and ζ^* of the model (R). It will be presented in the two first lemmas below. Moreover, since the bounds will be expressed in terms of the variance $\sigma^2(\tau)$ of the r.v. ζ (or ζ^*), an interesting lemma (see Lemma 3 below) will show that the behavior of L is closely related to the behavior of $\sigma^2(\tau)$.

LEMMA 1.

(i) *If* $r^{(iv)}(0) = +\infty$, *then* $\lim_{\tau\to 0} \frac{L(\tau)}{\tau} = +\infty$.

(ii) *If* $r^{(iv)}(0) < +\infty$, *then* $\lim_{\tau\to 0} \frac{L(\tau)}{\tau} = \frac{r^{(iv)}(0)}{2}$.

LEMMA 2. *For* τ *belonging to a neighborhood of* 0:

(i) $|\frac{r'(\tau)}{\sigma(\tau)}|$ *is bounded*;

(ii) $\rho(\tau) \le 0$.

LEMMA 3. *For* τ *belonging to a neighborhood of* 0:

(i) $\frac{\sigma^2(\tau)}{\tau} \le L(\tau) \le (2 + C)\frac{\sigma^2(\tau)}{\tau}$, *with* $C \ge 0$;

(ii) *For* $\delta > 0$, $\int_0^\delta \frac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}}\,d\tau < \infty \Leftrightarrow \int_0^\delta L(\tau)\,d\tau < \infty$ (*Geman condition*).

The proofs of the lemmas are given in [8].

To illustrate the method, we will present the complete proof when considering a fixed level x. For the case of curve-crossings, you can refer to [8].

So suppose $\dot\psi_s = 0$ and $\psi_s \equiv x, \forall s$.

Let C be a positive constant which may vary from equation to equation. By using the regression (R), M_2^x can be written as

$$M_2^x = 2\int_0^t (t - \tau) p_\tau(x, x)\sigma^2(\tau) A(m, \rho, \tau)\,d\tau,$$

where

$$A(m,\rho,\tau) := \mathbb{E}\left|\left(\frac{\zeta}{\sigma(\tau)} + \frac{r'(\tau)}{(1+r(\tau))\sigma(\tau)}x\right)\left(\frac{\zeta^*}{\sigma(\tau)} - \frac{r'(\tau)}{(1+r(\tau))\sigma(\tau)}x\right)\right|,$$

and $p_\tau(x,x) := p_{0,\tau}(x,x)$.

Note that

$$(8)\quad M_2^x \geq M_2^{x,\delta} := 2\int_0^\delta (t-\tau)p_\tau(x,x)\sigma^2(\tau)A(m,\rho,\tau)\,d\tau, \qquad \delta \in [0,\tau].$$

Now, by using Mehler's formula (see, e.g., [10]), we have

$$A(m,\rho,\tau) = \sum_{k=0}^{\infty} a_k(m)a_k(-m)k!\rho^k(\tau) \qquad \text{where } m = m(\tau) := \frac{r'(\tau)x}{(1+r(\tau))\sigma(\tau)},$$

$|m| = |m(\tau)|$ being bounded because of (i) of Lemma 2, and $a_k(m)$ are the Hermite coefficients of the function $|\cdot - m|$, given by

$$\begin{aligned} a_0(m) &= \mathbb{E}|Z-m| \qquad Z \text{ being a standard Gaussian r.v.} \\ &= m[2\Phi(m)-1] + \sqrt{\frac{2}{\pi}}e^{-m^2/2}, \\ a_1(m) &= \big(1-2\Phi(m)\big) = -\sqrt{\frac{2}{\pi}}\int_0^m e^{-u^2/2}\,du \end{aligned}$$

and

$$a_l(m) = \sqrt{\frac{2}{\pi}}\frac{1}{l!}H_{l-2}(m)e^{-m^2/2}, \qquad l \geq 2.$$

Let us show that $M_2^x < \infty$ under the Geman condition.

Since by Cauchy–Schwarz inequality

$$|A(m,\rho,\tau)| \leq \sum_{k=0}^{\infty}|a_k(m)a_k(-m)|k! \leq \big(\mathbb{E}[(Y-m)^2]\mathbb{E}[(Y+m)^2]\big)^{1/2},$$

with Y a standard normal r.v., there follows

$$M_2^x \leq I_2 := 2\int_0^t (t-\tau)p_\tau(x,x)\sigma^2(\tau)\big(a_0(m)a_0(-m) + 1 + m^2\big)\,d\tau.$$

Hence, m^2 being bounded, we obtain $I_2 \leq C\int_0^t (t-\tau)p_\tau(x,x)\sigma^2(\tau)\,d\tau$.

The study of this last integral reduces to the one on $[0,\delta]$ because of the uniform continuity outside of a neighborhood of 0, so we can conclude that it is finite if $L \in L^1[0,\delta]$, by using Lemma 3(ii).

Let us look now at the reverse implication.

Suppose that $M_2^x < \infty$, and so, via (8), that $M_2^{x,\delta} < \infty$.

Let us compute $A(m, \rho, \tau)$ and bound it below.

By using the parity of the Hermite polynomials and the sign of ρ given in (ii) of Lemma 2, we obtain

$$\begin{aligned}
A(m, \rho, \tau) &= a_0^2(m) + |\rho(\tau)| a_1^2(m) + \sum_{k=1}^{\infty} a_{2k}^2(m)(2k)!\rho^{2k}(\tau) \\
&\quad + |\rho| \sum_{k=1}^{\infty} a_{2k+1}^2(m)(2k+1)!\rho^{2k}(\tau) \\
&\geq a_0^2(m) = \left(-m a_1(m) + \sqrt{\frac{2}{\pi}} e^{-m^2/2}\right)^2 \\
&\geq \frac{2}{\pi} e^{-m^2} \geq C \qquad (\text{since } |m| < \infty).
\end{aligned}$$

Hence

$$M_2^{x,\delta} \geq C \int_0^{\delta} (t-\tau) p_{\tau}(x,x) \sigma^2(\tau)\, d\tau \geq C \int_0^{\delta} (t-\tau) \frac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}}\, d\tau.$$

An application of Lemma 3(ii), yields that $M_2^{x,\delta} < \infty$ implies the Geman condition.

The proof of the general case follows the same approach. It requires also to use Taylor formula for ψ and to introduce the modulus of continuity of $\dot{\psi}$ to express the expectation in the integrand of M_2^{ψ} into two terms, one on which will be applied the described method, the other related to the modulus of continuity of $\dot{\psi}$, which is bounded thanks to the condition (3) of the theorem (for more details, see [8]).

Acknowledgments. J. León is grateful to the SAMOS-MATISSE (Univ. Paris 1) for their invitation in October 2004. We wish to thank the referees for some very useful comments.

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