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CONTINUUM TREE LIMIT FOR THE RANGE OF RANDOM WALKS ON REGULAR TREES

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Let b be an integer greater than 1 and let $W^\varepsilon = (W_n^\varepsilon; n \geq 0)$ be a random walk on the b-ary rooted tree $\mathbb{U}_b$, starting at the root, going up (resp. down) with probability $1/2 + \varepsilon$ (resp. $1/2 - \varepsilon$), $\varepsilon \in (0, 1/2)$, and choosing direction $i \in \{1, \ldots, b\}$ when going up with probability a_i. Here $\mathbf{a} = (a_1, \ldots, a_b)$ stands for some nondegenerated fixed set of weights. We consider the range $\{W_n^\varepsilon; n \geq 0\}$ that is a subtree of $\mathbb{U}_b$. It corresponds to a unique random rooted ordered tree that we denote by τ_ε. We rescale the edges of τ_ε by a factor ε and we let ε go to 0: we prove that correlations due to frequent backtracking of the random walk only give rise to a deterministic phenomenon taken into account by a positive factor $\gamma(\mathbf{a})$. More precisely, we prove that τ_ε converges to a continuum random tree encoded by two independent Brownian motions with drift conditioned to stay positive and scaled in time by $\gamma(\mathbf{a})$. We actually state the result in the more general case of a random walk on a tree with an infinite number of branches at each node ($b = \infty$) and for a general set of weights $\mathbf{a} = (a_n, n \geq 0)$.

1. Introduction. Random walks on trees have been intensively studied by many authors having different motivations coming from group theory, discrete potential theory, statistical mechanics or genetics. We refer to [21] for a general introduction to random walks on infinite graphs and to [15] for a probabilistic approach more focused on trees. See also [14] for a survey of open problems concerning random walks on trees. In most of the papers about random walks on trees, given the treelike environment the transition probabilities of the random walk are fixed and one focuses on a certain range of questions: the speed of the random walk (see [18] for random walks on groups, [20] for random walks on periodic trees, [12] and [13] for random walks on Galton–Watson trees), large deviation principle for the distance-from-the-root process (see [8] for random walks on Galton–Watson trees), central-limit theorem for the distance-from-the-root process and the number of visited vertices (see [4] for the b-ary tree and [17] for the simple random walk on supercritical Galton–Watson trees). In this paper we consider a different problem; the transition probabilities are not fixed: we study, near criticality, transient random walks on the b-ary rooted tree and more generally on the ∞-ary tree, in a "diffusive" regime.

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Let us specify that we only consider ordered rooted trees that are formally defined as in [16]: Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of the nonnegative integers, set $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The ∞-ary tree is the set $\mathbb{U} = \{\varnothing\} \cup \bigcup_{n \geq 1} (\mathbb{N}^*)^n$ of the finite words written with positive integers by. Let $u \in \mathbb{U}$ be the word $u_1 \ldots u_n$, $u_i \in \mathbb{N}^*$. We denote the length of u by $|u| : |u| = n$. $|u|$ is viewed as the *height* of the vertex u in $\mathbb{U}$. Let $v = v_1 \ldots v_m \in \mathbb{U}$. Then the word uv stands for the concatenation of u and v: $uv = u_1 \ldots u_n v_1 \ldots v_m$. Observe that $\mathbb{U}$ is totally ordered by the *lexicographical order* denoted by $\leq$. A rooted ordered tree t is a subset of $\mathbb{U}$ satisfying the following conditions:

(i) $\varnothing \in t$ and $\varnothing$ is called the *root* of t.
(ii) If $v \in t$ and if $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in t$.
(iii) For every $u \in t$, there exists $k_u(t) \geq 0$ such that $uj \in t$ for every $1 \leq j \leq k_u(t)$.

We denote by $\mathbb{T}$ the set of ordered rooted trees. Let us mention that we sometimes see ordered rooted trees as family trees. So, we often use the genealogical terminology instead of the graph-theoretical one. All the random objects introduced in this paper are defined on an underlying probability space denoted by $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\varepsilon \in (0, 1/2)$ and let $\mathbf{a} = (a_n, n \geq 1)$ be some nondegenerated fixed set of weights, namely $\sum a_n = 1$ and $0 \leq a_n < 1, n \geq 1$.

We attach to the infinite tree $\mathbb{U}$ a cemetery point $\partial \notin \mathbb{U}$ situated at height (-1) and we view ∂ as the parent of the root $\varnothing$. Then, we let run a particle on $\mathbb{U} \cup \{\partial\}$ that evolves as follows:

(a) The particle starts at $\varnothing$ at time 0 and it stops when it reaches ∂.
(b) If at time n the particle is at vertex $v \in \mathbb{U}$, then it jumps down to the parent of v with probability $1/2 - \varepsilon$ and it goes up with probability $1/2 + \varepsilon$.
(c) When going up, the particle chooses direction $j \in \mathbb{N}^*$ and jumps to the vertex $vj \in \mathbb{U}$ with probability a_j.

The height of the particle evolving in $\mathbb{U} \cup \{\partial\}$ is then distributed as a random walk on $\mathbb{Z}$ started at 0, stopped when reaching state -1, and whose possible jumps are $(+1)$ with probability $1/2 + \varepsilon$ and (-1) with probability $1/2 - \varepsilon$. In this paper *we condition the particle to never reach* ∂ (observe that this conditioning is nonsingular). We denote by $W^\varepsilon = (W_n^\varepsilon; n \geq 0)$ the sequence of vertices in $\mathbb{U}$ visited by the conditioned particle.

We study the range $\{W_n^\varepsilon; n \geq 0\}$ when ε goes to zero. Observe that it is an ordered rooted subtree of $\mathbb{U}$. There exists a unique ordered rooted tree $\tau_\varepsilon \in \mathbb{T}$ corresponding to $\{W_n^\varepsilon; n \geq 0\}$ via a one-to-one map that fixes the root $\varnothing$, preserves adjacency and that is increasing with respect to the lexicographical order.

Since W^ε goes to infinity, τ_ε has one single infinite line of descent. Following Aldous' terminology introduced in [2] we call *sin-tree* such trees (see Section 2.1 for precise definitions). The distribution of τ_ε is not simple and it shows correlations due to frequent backtracking of the random walk (see comments in Section 2.3). However, Theorem 2.1, which is the main result of the paper, asserts that τ_ε converges in distribution to some continuum random tree. More precisely, think of τ_ε as a planar graph embedded in the clockwise oriented half-plane and suppose that its edges have length 1; consider a particle visiting continuously the edges of τ_ε at speed 1 from the left to the right, going backward as little as possible; we denote by $C_s(\tau_\varepsilon)$ the distance from the root of the particle at time s and we call the resulting process $C(\tau_\varepsilon) = (C_s(\tau_\varepsilon); s \geq 0)$ the *left contour process* of τ_ε. It is clear that the particle never reaches the part of τ_ε at the right hand of the infinite line of descent; observe, however, that $C(\tau_\varepsilon)$ completely encodes the left part of τ_ε. Denote by $C^\bullet(\tau_\varepsilon)$ the process corresponding to a particle visiting τ_ε from the right to the left. Thus, $(C(\tau_\varepsilon), C^\bullet(\tau_\varepsilon))$ completely encodes τ_ε (see Section 2.2 for more careful definitions and other encodings of sin-trees). Let D and $D^\bullet$ be two independent copies of the process $s \to B_s - 2s - 2\inf_{r \leq s}(B_r - 2r)$ where B is distributed as the standard linear Brownian motion started at 0. Theorem 2.1 asserts that the convergence

$$\left(\varepsilon C_{s/\varepsilon^2}(\tau_\varepsilon), \varepsilon C^\bullet_{s/\varepsilon^2}(\tau_\varepsilon)\right)_{s\geq 0} \underset{\varepsilon\to 0}{\longrightarrow} (2D_{\gamma s}, 2D^\bullet_{\gamma s})_{s\geq 0}$$

holds in distribution in $C([0,\infty), \mathbb{R}^2)$ endowed with the topology of uniform convergence on compact sets. We see that correlations in τ_ε only give rise to a deterministic phenomenon characterized by a constant $\gamma = \gamma(\mathbf{a})$ that is defined by

$$1/\gamma = \mathbf{E}[(1 + X_1 + X_1X_2 + X_1X_2X_3 + \cdots)^{-1}], \tag{1}$$

where $(X_n; n \geq 1)$ stands for a sequence of i.i.d. $\{a_n, n \geq 1\}$-valued random variables whose distribution is given by $\mathbf{P}(X_n = a_i) = \sum a_j$, the sum being taken over the j's such that $a_j = a_i$. Observe that if b is some integer greater than 1 and if $a_n = 0$ for all $n \geq b+1$, then the particle remains in the b-ary ordered rooted tree $\mathbb{U}_b = \{\varnothing\} \cup \bigcup_{n\geq 1}\{1, \dots, b\}^n$. More comments about this limit theorem are added before and after the statement of Theorem 2.1.

Before ending this section, let us give a short overview of the proof of the theorem: one part of the proof relies on a specific encoding of the range $\{W_n^\varepsilon; n \geq 0\}$ that can be explained as follows: Denote by $(|W_n^\varepsilon|; n \geq 0)$ the sequence of successive heights of the particle. It is obviously distributed as a random walk started at 0 whose possible jumps are $(+1)$ with probability $1/2 + \varepsilon$ and (-1) with probability $1/2 - \varepsilon$, conditioned to stay nonnegative. Then, the piecewise linear process

$$t \longrightarrow |W^\varepsilon_{\lfloor t \rfloor}| + (t - \lfloor t \rfloor)|W^\varepsilon_{\lfloor t \rfloor + 1}|$$

is the contour process of an infinite "fictive" tree denoted by $\overline{\tau}_\varepsilon$ whose distribution can be informally described as follows: $\overline{\tau}_\varepsilon$ has one infinite line of descent; at each vertex v on the infinite line of descent an independent random number with distribution μ of independent Galton–Watson trees with offspring distribution μ is attached at the left of the infinite line. Here, μ stands for the probability measure on $\mathbb{N}$ given by $\mu(k) = (1/2 + \varepsilon)(1/2 - \varepsilon)^k$, $k \geq 0$ (see Section 2 for precise definitions concerning trees and Lemma 3.1 for the details).

We then encode the walk $(W_n^\varepsilon; n \geq 0)$ by the tree $\overline{\tau}_\varepsilon$ and random marks $\overline{\mu}_u \in \mathbb{N}^*$, $u \in \overline{\tau}_\varepsilon$ that are defined as follows: Let $u \in \overline{\tau}_\varepsilon$ be distinct from the root $\varnothing$. Denote $\overleftarrow{u}$ its parent. By definition of the contour process the edge $(\overleftarrow{u}, u)$ corresponds to a unique upcrossing of the process $(|W_n^\varepsilon|; n \geq 0)$ between times $n(u)$ and $n(u) + 1$. Thus, there exists $j \in \mathbb{N}^*$ such that the word $W_{n(u)+1}^\varepsilon$ is written $W_{n(u)}^\varepsilon j$ and we set $\overline{\mu}_u = j$. Then, we easily check that conditional on $\overline{\tau}_\varepsilon$, the marks $\overline{\mu}_u$, $u \in \overline{\tau}_\varepsilon \setminus \{\varnothing\}$ are independent and distributed on $\mathbb{N}^*$ in accordance with $\mathbf{a}$ (see Section 3.1 for details). We get back the walk W^ε from the marked tree $\overline{\mathcal{T}_\varepsilon} = (\overline{\tau}_\varepsilon; (\overline{\mu}_u, u \in \overline{\tau}_\varepsilon))$, in the following way: consider $u \in \overline{\tau}_\varepsilon$, distinct from the root $\varnothing$ at height $|u| = n$; denote by $u_0 = \varnothing, u_1, \dots, u_n = u$ the ancestors of u listed in the genealogical order. Then we define the *track* of u, $\mathbf{Tr}_{\overline{\mathcal{T}_\varepsilon}}(u)$, by the word $\overline{\mu}_{u_1} \dots \overline{\mu}_{u_n} \in \mathbb{U}$ (observe that the mark of the root plays no role). Then,

$$W_{n(u)+1}^\varepsilon = \mathbf{Tr}_{\overline{\mathcal{T}_\varepsilon}}(u)$$

and thus

$$\mathbf{Tr}_{\overline{\mathcal{T}_\varepsilon}}(\overline{\tau}_\varepsilon) = \{W_n^\varepsilon; n \geq 0\}.$$

Taking the trace of $\overline{\tau}_\varepsilon$ has two distinct effects: the first one shuffles $\overline{\tau}_\varepsilon$ in the order of the marks in $\mathbb{N}^*$. The second one shrinks the tree because several edges of $\overline{\tau}_\varepsilon$ might correspond to the same vertex in $\mathbb{U}$.

Let us briefly explain how to deal with the shuffling effect of the tree: it is possible to reorder randomly the marked tree $\overline{\mathcal{T}_\varepsilon}$ into a new marked tree $\widetilde{\mathcal{T}}_\varepsilon = (\widetilde{\tau}_\varepsilon; (\widetilde{\mu}_u, u \in \widetilde{\tau}_\varepsilon))$ such that:

(a) $\widetilde{\tau}_\varepsilon$ has the same distribution as the tree obtained from $\overline{\tau}_\varepsilon$ by changing independently and uniformly at random the order of birth of brothers in $\overline{\tau}_\varepsilon$.

(b) If $u_1, u_2 \in \widetilde{\tau}_\varepsilon$ are such that $u_1 \leq u_2$, then

$$\mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon}(u_1) \leq \mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon}(u_2)$$

(see Section 3.1 for a precise definition). Thus, the shuffled tree $\widetilde{\tau}_\varepsilon$ has a simple distribution specified by Remark 3.1. Up to the shrinking effect, $\widetilde{\tau}_\varepsilon$ is close to τ_ε and if we denote by $(C(\widetilde{\tau}_\varepsilon), C^\bullet(\widetilde{\tau}_\varepsilon))$ the left and the right contour processes of $\widetilde{\tau}_\varepsilon$ we prove in Section 2.1 that

$$\left(\varepsilon C_{s/\varepsilon^2}(\widetilde{\tau}_\varepsilon), \varepsilon C^\bullet_{s/\varepsilon^2}(\widetilde{\tau}_\varepsilon)\right)_{s\geq 0} \xrightarrow[\varepsilon\to 0]{} (2D_s, 2D_s^\bullet)_{s\geq 0}, \tag{2}$$

in distribution in $C([0, \infty), \mathbb{R}^2)$ endowed with the topology of uniform convergence on compact sets. Denote by $\mathbf{d}$ the graph distance in $\mathbb{U}$. Informally speaking, (2) says that the metric space $(\widetilde{\tau}_\varepsilon, \varepsilon \cdot \mathbf{d})$ converges to some random metric space that is a continuum random tree encoded by D and $D^\bullet$ [see comment (c) after Theorem 2.1 for a more precise discussion of that point]. Next, observe that for any u_1 and u_2 in $\widetilde{\tau}_\varepsilon$

$$0 \leq \mathbf{d}(u_1, u_2) - \mathbf{d}\big(\mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon}(u_1), \mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon}(u_2)\big) \leq 2G(u_1, u_2)$$

where conditional on $u_1, u_2 \in \widetilde{\tau}_\varepsilon$, $G(u_1, u_2)$ is a random integer with a geometric distribution with parameter $q = \mathbf{P}(X_1 \neq X_2)$, which is a quantity that does not depend on ε. This gives an informal argument to explain why the limits of the metric spaces $(\tau_\varepsilon, \varepsilon \cdot \mathbf{d})$ and $(\widetilde{\tau}_\varepsilon, \varepsilon \cdot \mathbf{d})$ should be close and why tightness for the contour processes of τ_ε is not the difficult part of the proof: it is deduced from (2) by (now standard) arguments inspired from the proof of Theorem 20 in Aldous' paper [3].

The technical point of the paper concerns the identification of the limiting tree by studying precisely the "shrinking effect" via explicit computations for the $\mathbb{U}$-indexed Markov process $(Z_v, v \in \mathbb{U})$ given by

$$Z_v = \#\{u \in \widetilde{\tau}_\varepsilon : \mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon}(u) = v\}.$$

This analysis is done in Propositions 3.3 and 3.4. More precisely, if we fix a real number $x > 0$ and if we remove from $\widetilde{\tau}_\varepsilon$ all the descendents of the unique vertex at height $\lfloor x/\varepsilon \rfloor$ on the infinite line of descent, we get a finite tree denoted by $\widetilde{\tau}_\varepsilon^x$. Let $\mathcal{U}$ be a uniform random variable in $[0, 1]$ independent of $\widetilde{\tau}_\varepsilon$. Denote by $U(\varepsilon)$ the vertex of $\widetilde{\tau}_\varepsilon^x$ coming in the $\lfloor U \# \widetilde{\tau}_\varepsilon^x \rfloor$th position in the lexicographical order. Set

$$\overline{V}(\varepsilon) = \sum_{\substack{v \in \mathbb{U} \\ v \leq \mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon}(U(\varepsilon))}} \mathbf{1}_{\{Z_v > 0\}}.$$

Then, we prove that

$$\varepsilon^2 \left(\overline{V}(\varepsilon) - \frac{1}{\gamma} \mathcal{U} \# \tilde{\tau}_\varepsilon^x \right) \underset{\varepsilon \to 0}{\longrightarrow} 0$$

in probability. This key result is stated more precisely in Lemma 3.7.

The paper is organized as follows: In Sections 2.1 and 2.2 we specify our notation and we define various encodings of trees and forests; Theorem 2.1 is stated in Section 2.3; Section 3 is devoted to its proof that relies on a certain combinatorial representation of the range $\{W_n^\varepsilon; n \geq 0\}$ given in Section 3.1 and on a technical estimate (Lemma 3.7) whose proof is postponed to Section 3.3 while the proof of Theorem 2.1 itself is done in Section 3.2.

2. Preliminaries and definitions.

2.1. *Trees, forests and sin-trees.* We first start with some notation. We define on $\mathbb{U}$ the *genealogical* order $\preccurlyeq$ by

$$\forall u, v \in \mathbb{U} \qquad u \preccurlyeq v \quad \Longleftrightarrow \quad \exists\, w \in \mathbb{U} : v = uw.$$

If $u \preccurlyeq v$, we say that u is an ancestor of v. If u is distinct from the root, it has a unique predecessor with respect to $\preccurlyeq$ that is called its parent and that is denoted by $\overleftarrow{u}$. We define the youngest common ancestor of u and v by the $\preccurlyeq$-maximal element $w \in \mathbb{U}$ such that $w \preccurlyeq u$ and $w \preccurlyeq v$ and we denote it by $u \wedge v$. We also define the distance between u and v by $\mathbf{d}(u,v) = |u| + |v| - 2|u \wedge v|$ and we use notation $[\![u, v]\!]$ for the shortest path between u and v. Let $t \in \mathbb{T}$ and $u \in t$. We define the tree t shifted at u by $\theta_u(t) = \{v \in \mathbb{U} : uv \in t\}$ and we denote by $[t]_u$ the tree t cut at the node $u : [t]_u := \{u\} \cup \{v \in t : v \wedge u \neq u\}$. Observe that $[t]_u \in \mathbb{T}$. For any $u_1, \dots, u_k \in t$ we also set $[t]_{u_1,\dots,u_k} := [t]_{u_1} \cap \cdots \cap [t]_{u_k}$ and

$$[t]_n = [t]_{\{u \in t\,:\,|u|=n\}} = \{u \in t : |u| \le n\}, \qquad n \ge 0.$$

Let us denote by $\mathcal{G}$ the σ-field on $\mathbb{T}$ generated by the sets $\{t \in \mathbb{T} : u \in t\}$, $u \in \mathbb{U}$ and let μ be a probability distribution on $\mathbb{N}$. We call *Galton–Watson tree* with offspring distribution μ [a GW(μ)-tree for short] any $(\mathcal{F}, \mathcal{G})$-measurable random variable τ whose distribution is characterized by the two following conditions:

(i) $\mathbf{P}(k_\varnothing(\tau) = i) = \mu(i)$, $i \ge 0$.

(ii) For every $i \ge 1$ such that $\mu(i) \neq 0$, the shifted trees $\theta_1(\tau), \dots, \theta_i(\tau)$ under $\mathbf{P}(\cdot \mid k_\varnothing(\tau) = i)$ are independent copies of τ under $\mathbf{P}$.

REMARK 2.1. Let $u_1, \dots, u_k \in \mathbb{U}$ such that $u_i \wedge u_j \notin \{u_1, \dots, u_k\}$, $1 \le i$, $j \le k$, and let τ be a GW(μ)-tree. Then, conditional on the event $\{u_1, \dots, u_k \in \tau\}$, $\theta_{u_1}(\tau), \dots, \theta_{u_k}(\tau)$ are i.i.d. GW(μ)-trees independent of $[\tau]_{u_1,\dots,u_k}$.

We often consider a forest (i.e., a sequence of trees) instead of a single tree. More precisely, we define the forest f associated with the sequence of trees $(t_l; l \ge 1)$ by the set

$$f = \{(-1, \varnothing)\} \cup \bigcup_{l \ge 1} \{(l, u), u \in t_l\}$$

and we denote by $\mathbb{F}$ the set of forests. Vertex $(-1, \varnothing)$ is viewed as a fictive root situated at generation -1. Let $u' = (l, u) \in f$ with $l \ge 1$; the height of u' is defined by $|u'| := |u|$ and its ancestor is defined by $(l, \varnothing)$. For convenience, we denote it by $\varnothing_l := (l, \varnothing)$. As already specified, all the ancestors $\varnothing_1, \varnothing_2, \dots$ are the descendants of $(-1, \varnothing)$ and are situated at generation 0. Most of the notation concerning trees extend to forests: The lexicographical order $\le$ is defined on f by taking first the individuals of t_1, next those of $t_2, \dots,$ and so on and leaving $(-1, \varnothing)$ unordered. The genealogical order $\preccurlyeq$ on f is defined tree by tree in an obvious way. Let $v' \in f$. The youngest common ancestor of u' and v' is then defined as the $\preccurlyeq$-maximal element of w' such that $w' \preccurlyeq u'$ and $w' \preccurlyeq v'$ and we keep denoting it by $u' \wedge v'$. The number of children of u' is $k_{u'}(f) := k_u(t_l)$ and the forest f shifted at u' is defined as the tree $\theta_{u'}(f) := \theta_u(t_l)$. We also define $[f]_{u'}$ as the forest $\{u'\} \cup \{v' \in f : v' \wedge u' \neq u'\}$ and we extend notation $[f]_{u'_1,\dots,u'_k}$ and $[f]_n$ in an obvious way. For convenience of notation, we often identify f with the sequence $(t_l; l \ge 1)$. When $(t_l; l \ge 1) = (t_1, \dots, t_k, \varnothing, \varnothing, \dots)$, we say that f is a finite forest with k elements and we abusively write $f = (t_1, \dots, t_k)$.

We define the set of sin-trees by

$$\mathbb{T}_{\sin} = \{t \in \mathbb{T} : \forall n \geq 0, \#\{v \in t : |v| = n \text{ and } \#\theta_v(t) = \infty\} = 1\}.$$

Let $t \in \mathbb{T}_{\sin}$. For any $n \geq 0$, we denote by $u_n^*(t)$ the unique individual u on the infinite line of descent [i.e., such that $\#\theta_u(t) = \infty$] situated at height n. Observe that $u_0^*(t) = \varnothing$. We use notation $\ell_\infty(t) = \{u_n^*(t); n \geq 0\}$ for the infinite line of descent of t and we denote by $(l_n(t); n \geq 1)$ the sequence of positive integers such that $u_n^*(t)$ is the word $l_1(t) \dots l_n(t) \in \mathbb{U}$. We also introduce the set of *sin-forests* $\mathbb{F}_{\sin}$ that is defined as the set of forests $f = (t_l; l \geq 1)$ such that all the trees t_l are finite except one sin-tree t_{l_0}. We extend to sin-forests notation u_n^* and l_n by setting $l_n(f) = l_n(t_{l_0})$, $u_n^*(f) = (l_0, u_n^*(t_{l_0}))$ and $u_0^*(f) = \varnothing_{l_0}$.

Next, we introduce a natural class of random sin-trees called *Galton–Watson trees with immigration* (GWI-trees for short). The distribution of a GWI-tree is characterized by:

(a) its *offspring distribution* μ on $\mathbb{N}$ that we suppose critical or subcritical: $\bar{\mu} = \sum_{k \geq 0} k\mu(k) \leq 1$;

(b) its *dispatching distribution* r defined on the first octant $\{(k,l) \in \mathbb{N}^* \times \mathbb{N}^* : 1 \leq l \leq k\}$ that prescribes the distribution of the number of immigrants and their positions with respect to the infinite line of descent.

More precisely, τ is a GWI(μ, r)-tree if it satisfies the two following conditions:

(i) The sequence $S = ((k_{u_n^*(\tau)}(\tau), l_{n+1}(\tau)); n \geq 0)$ is i.i.d. with distribution r.

(ii) Conditional on S, the trees $\theta_{u_n^*(\tau)i}(\tau)$ with $n \in \mathbb{N}$ and $1 \leq i \leq k_{u_n^*(\tau)}(\tau)$ with $i \neq l_{n+1}(\tau)$ are mutually independent GW(μ)-trees.

We define a GWI(μ, r)-forest with $l \geq 1$ elements by the forest $\varphi = (\tau, \tau_1, \dots, \tau_{l-1})$ where the τ_i's are i.i.d. GW(μ)-trees independent of the GWI(μ, r)-tree τ. It will be sometimes convenient to insert τ at random in the sequence $(\tau_1, \dots, \tau_{l-1})$ but unless otherwise specified the random sin-tree in a random sin-forest occupies the first row.

The word "immigration" comes from the following obvious observation: Let φ be a GWI(μ, r)-forest with $l + 1$ elements. Set for any $n \geq 0$, $Z_n(\varphi) = \#\{u \in \varphi : |u| = n\} - 1$. Then the process $(Z_n(\varphi); n \geq 0)$ is a *Galton–Watson process with immigration* started at state l, with offspring distribution μ and *immigration distribution* ν given by

$$\nu(k) = \sum_{1 \leq j \leq k+1} r(k+1, j), \qquad k \geq 0.$$

Recall that a Galton–Watson process with immigration $(Z_n(\varphi); n \geq 0)$ is an $\mathbb{N}$-valued Markov chain whose transition probabilities are characterized by

$$\mathbf{E}[x^{Z_{n+m}(\varphi)} | Z_m(\varphi)] = f_n(x)^{Z_m(\varphi)} g(f_{n-1}(x)) g(f_{n-2}(x)) \cdots g(f_0(x)), \tag{3}$$

where f (resp. g) stands for the generating function of μ (resp. ν) and where f_n is recursively defined by $f_n = f_{n-1} \circ f$, $n \geq 1$ and $f_0 = \mathrm{Id}$.

We conclude this section by giving an elementary result on the so-called GW(μ)-*size-biased trees* that are GWI(μ, r)-trees with dispatching distribution of the form $r(k, j) = \mu(k)/\bar{\mu}$, $1 \leq j \leq k$. Size-biased trees arise naturally by conditioning critical or subcritical GW-trees on nonextinction: see [1, 2, 9] or [11] for related results. The term "size-biased" can be justified by the following elementary result needed in Section 3.3: Let φ be a random forest corresponding to a sequence of l independent GW(μ)-trees and let $\varphi_\flat$ be a GWI(μ, r)-forest with l elements where r is taken as above and where the position of the unique random sin-tree in $\varphi_\flat$ is picked uniformly at random among the l possible choices. Check that for any nonnegative measurable functional G on $\mathbb{F} \times \mathbb{U}$:

$$\mathbf{E}\left[\sum_{u \in \varphi} G([\varphi]_u, u)\right] = \sum_{n \geq 0} l \bar{\mu}^n \mathbf{E}\left[G\left([\varphi_\flat]_{u_n^*(\varphi_\flat)}, u_n^*(\varphi_\flat)\right)\right] \tag{4}$$

and in particular $d\mathbf{P}([\varphi_\flat]_n \in \cdot)/d\mathbf{P}([\varphi]_n \in \cdot) = Z_n(\varphi)/l\bar{\mu}^n$.

2.2. *The encoding of sin-trees.* The purpose of the paper is to provide a limit theorem for τ_ε thanks to its encoding by two contour processes as briefly explained in the Introduction. It will be convenient to introduce two additional encoding processes: namely, the *height process* (also called *exploration process*) and a certain kind of random walk.

Encoding of finite trees and forests. Let $t \in \mathbb{T}$ be a finite tree and let $u_0 = \varnothing < u_1 < \cdots < u_{\#t-1}$ be the vertices of t listed in the lexicographical order. We define the *height process* of t by $H_n(t) = |u_n|$, $0 \leq n < \#t$. $H(t)$ clearly characterizes the tree t.

We also encode t by its contour process which is informally defined as follows: think of t as a graph embedded in the clockwise oriented half-plane with unit length edges; let run a particle starting at the root at time 0 that explores t from the left to the right moving continuously along each edge at unit speed until it comes back to its starting point. In this evolution, each edge is crossed twice and the total amount of time needed to explore the tree is thus $2(\#t - 1)$. The contour process $C(t) = (C_s(t); 0 \leq s \leq 2(\#t - 1))$ is defined as the distance-from-the-root process of the particle at time $s \in [0, 2(\#t - 1)]$. More precisely, $C(t)$ can be recovered from the height process by the following transform: Set $b_n = 2n - H_n(t)$ for $0 \leq n < \#t$ and $b_{\#t} = 2(\#t - 1)$. Then observe that

$$C_s(t) = \begin{cases} H_n(t) - s + b_n, & \text{if } s \in [b_n, b_{n+1} - 1) \text{ and } n < \#t - 1, \\ s - b_{n+1} + H_{n+1}(t), & \text{if } s \in [b_{n+1} - 1, b_{n+1}] \text{ and } n < \#t - 1, \\ H_{\#t-1}(t) - s + b_{\#t-1}, & \text{if } s \in [b_{\#t-1}, b_{\#t}]. \end{cases} \tag{5}$$

See Figure 1. We also need to encode t in a third way by a path $V(t) = (V_n(t); 0 \leq n \leq \#t)$ that is defined by $V_{n+1}(t) = V_n(t) + k_{u_n}(t) - 1$ and $V_0(t) = 0$. $V(t)$ is

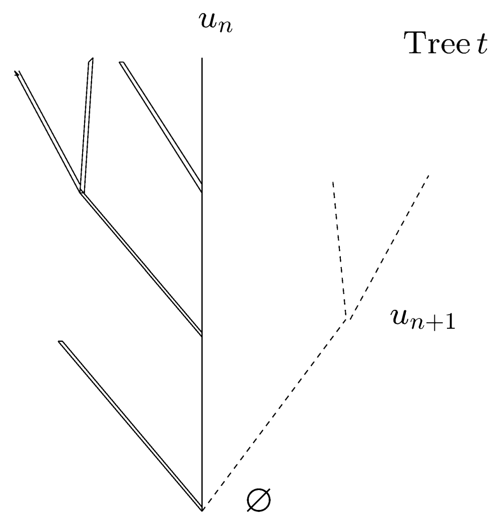

FIG. 1. *When the particle reaches the vertex u_n for the first time, then the double-line edges have been visited two times* [*the total number of such edges is $n - H_n(t)$*], *the one-line edges have been visited one time* [*the total number of such edges is equal to $H_n(t)$*] *and the dashed-line edges have not been visited. Then the total amount of time needed to reach u_n is $b_n = 2(n - H_n(t)) + H_n(t) = 2n - H_n(t)$.*

sometimes called the Lukaciewicz path associated with t. It is clear that we can reconstruct t from $V(t)$. Observe that the jumps of $V(t)$ are ≥ -1. Moreover, $V_n(t) \geq 0$ for any $0 \leq n < \#t$ and $V_{\#t}(t) = -1$. We recall from [10] without proof the following formula that allows to write the height process as a functional of $V(t)$:

$$H_n(t) = \#\left\{0 \leq j < n : V_j(t) = \inf_{j \leq k \leq n} V_k(t)\right\}, \qquad 0 \leq n < \#t. \tag{6}$$

REMARK 2.2. If τ is a critical or subcritical GW(μ)-tree, then it is clear from our definition that $V(\tau)$ is a random walk started at 0 that is stopped at -1 and whose jump distribution is given by $\rho(k) = \mu(k+1)$, $k \geq -1$. However, neither $H(\tau)$ nor $C(\tau)$ is Markov process except for the geometric case: $\mu(k) = (1-p)p^k$ with $p \in (0, 1/2]$. In this case, $C(\varphi)$ is distributed as a random walk killed at -1 and whose possible jumps are $(+1)$ with probability p and (-1) with probability $1-p$ [more precisely, it is the restriction of the first $T_{-1} - 1$ steps of a random walk killed at the reaching time of level (-1)].

The previous definition of V and of the height process can be easily extended to a forest $f = (t_l; l \geq 1)$ of finite trees as follows: Since all the trees t_l are finite, it is possible to list all the vertices of f but $(-1, \varnothing)$ in the lexicographical order: $u_0 = \varnothing_1 < u_1 < \cdots$, and so on. We then simply define the height process of f by

$H_n(f)=|u_n|$ and $V(f)$ by $V_{n+1}(f)=V_n(f)+k_{u_n}(f)-1$ with $V_0(f)=0$. Set $n_p=\#t_1+\cdots+\#t_p$ and $n_0=0$ and observe that

$$H_{n_p+k}(f)=H_k(t_{p+1}) \quad \text{and} \quad V_{n_p+k}(f)=V_k(t_{p+1})-p, \qquad 0\le k<\#t_{p+1},\ p\ge 0.$$

We thus see that the height process of f is the concatenation of the height processes of the trees composing f. Moreover, the nth visited vertex u_n is in t_p iff $p=1-\inf_{0\le k\le n}V_k(f)$. Then, it is easy to check that (6) remains true for every $n\ge 0$ when $H(t)$ and $V(t)$ are replaced by respectively $H(f)$ and $V(f)$.

Encodings of sin-trees. Let $t\in\mathbb{T}_{\text{sin}}$. A particle visiting t in the lexicographical order never reaches the part of t at the right hand of the infinite line of descent. So we need two height processes or equivalently two contour processes to encode t. More precisely, the left part of t is the set $\{u\in t:\exists\, v\in\ell_\infty(t)\text{ s.t. } u\le v\}$. It can be listed in a lexicographically increasing sequence of individuals denoted by $\varnothing=u_0<u_1<\cdots$. We simply define the *left height process* of t by $H_n(t)=|u_n|$, $n\ge 0$. $H(t)$ completely encodes the left part of t. To encode the right part we consider the "mirror image" $t^\bullet$ of t. More precisely, let $v\in t$ be the word $c_1c_2\dots c_n$. For any $j\le n$, denote by $v_j:=c_1\dots c_j$ the jth ancestor of v with $v_0=\varnothing$. Set $c_j^\bullet=k_{v_{j-1}}(t)-c_j+1$ and $v^\bullet=c_1^\bullet\dots c_n^\bullet$. We then define $t^\bullet$ as $\{v^\bullet, v\in t\}$ and we define the *right height process* of t as $H^\bullet(t):=H(t^\bullet)$.

REMARK 2.3. Observe that τ and $\tau^\bullet$ have the same distribution if τ is a GW(μ)-tree. This is no longer the case if τ is a GWI(μ,r)-tree unless $r(k,m)=r(k,k-m+1)$.

We now give a decomposition of $H(t)$ and $H^\bullet(t)$ along $\ell_\infty(t)$ that is well suited to GWI-trees and that is used in Section 3.2: Recall that $(u_n;n\ge 0)$ stands for the sequence of vertices of the left part of t listed in the lexicographical order. Let us consider the set $\{u^*_{n-1}(t)i;\,1\le i<l_n(t);\,n\ge 1\}$ of individuals at the left hand of $\ell_\infty(t)$ having a brother on $\ell_\infty(t)$. To avoid trivialities, we assume that this set is not empty and we denote by $v_1<v_2<\cdots$ the (possibly finite) sequence of its elements listed in the lexicographical order.

The forest $f(t)=(\theta_{v_1}(t),\theta_{v_2}(t),\dots)$ is then composed of the bushes rooted at the left hand of $\ell_\infty(t)$ taken in the lexicographical order of their roots. Define $L_n(t):=(l_1(t)-1)+\cdots+(l_n(t)-1)$, $n\ge 1$, with $L_0(t)=0$ and consider the pth individual of $f(t)$ with respect to the lexicographical order on $f(t)$; check that the corresponding bush is rooted in t at height

$$\boldsymbol{\alpha}(p)=\inf\Big\{k\ge 0: L_k(t)\ge 1-\inf_{j\le p}V_j(f(t))\Big\}.$$

Thus the corresponding individual in t is $u_{\mathbf{n}(p)}$ where $\mathbf{n}(p)$ is given by

$$\mathbf{n}(p)=p+\boldsymbol{\alpha}(p) \tag{7}$$

[note that the first individual of $f(t)$ is labelled by 0]. Conversely, let us consider u_n that is the nth individual of the left part of t with respect to the lexicographical order on t. Set $\mathbf{p}(n) = \#\{k < n : u_k \notin \ell_\infty(t)\}$ that is the number of individuals coming before u_n and not belonging to $\ell_\infty(t)$. Then

$$\mathbf{p}(n) = \inf\{p \geq 0 : \mathbf{n}(p) \geq n\} \tag{8}$$

and the desired decomposition follows:

$$H_n(t) = n - \mathbf{p}(n) + H_{\mathbf{p}(n)}(f(t)). \tag{9}$$

Since $n - \mathbf{p}(n) = \#\{0 \leq k < n : u_k \in \ell_\infty(t)\}$, we also get

$$\boldsymbol{\alpha}\big(\mathbf{p}(n) - 1\big) \leq n - \mathbf{p}(n) \leq \boldsymbol{\alpha}(\mathbf{p}(n)). \tag{10}$$

Observe that if $u_n \notin \ell_\infty(t)$, then $n - \mathbf{p}(n) = \boldsymbol{\alpha}(\mathbf{p}(n))$. The proofs of these identities follow from simple counting arguments and they are left to the reader (see Figure 2). Similar formulas hold for $H^\bullet(t)$ taking $t^\bullet$ instead of t in (7), (8), (9) and (10).

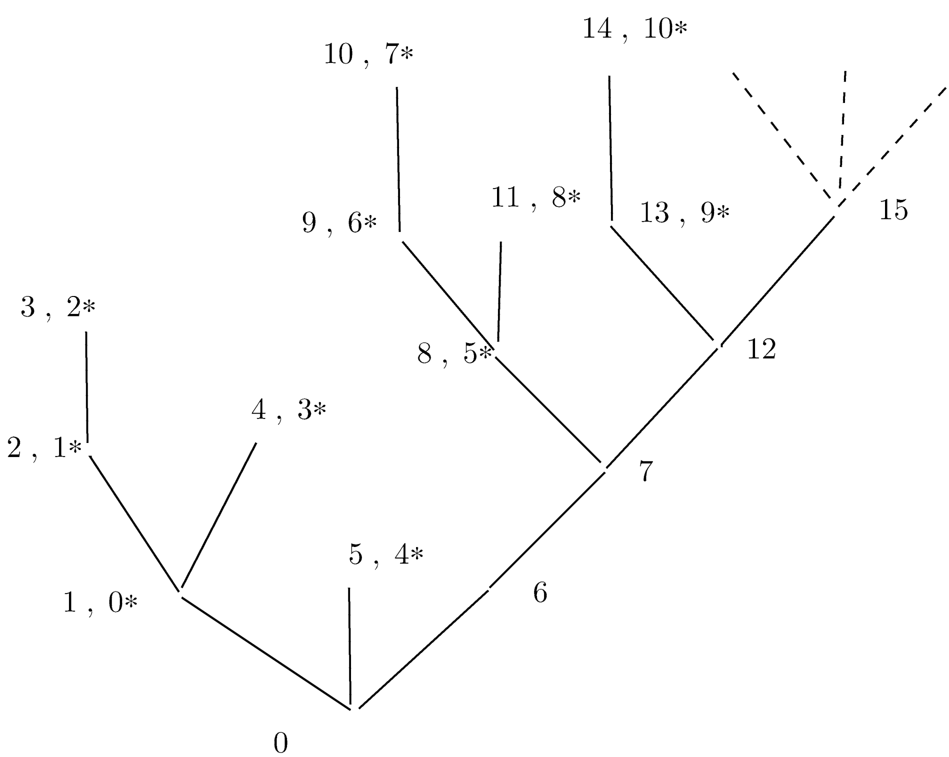

FIG. 2. *The left part of a sin-tree t. The individuals which are not on $\ell_\infty(t)$ have two labels: the first one is their row in the lexicographical order on t and the second one (tagged with a star) corresponds to their row in $f(t)$; individuals of $\ell_\infty(t)$ have only one label corresponding to their row in t.*

REMARK 2.4. The latter decomposition is particularly useful when we consider a GWI(μ, r)-tree τ: In this case $(f(\tau), f(\tau^\bullet))$ is independent of $(L(\tau), L(\tau^\bullet))$, $f(\tau)$ and $f(\tau^\bullet)$ are mutually independent and $f(\tau)$ [resp. $f(\tau^\bullet)$] is a forest of i.i.d. GW(μ)-trees if for some $k \geq 2$ we have $r(k,2) + \cdots + r(k,k) \neq 0$ [resp. $r(k,k-1) + \cdots + r(k,1) \neq 0$]; it is otherwise an empty forest. Moreover, the process $(L(\tau), L(\tau^\bullet))$ is an $\mathbb{N} \times \mathbb{N}$-valued random walk whose jump distribution is given by

$$\mathbf{P}\big(L_{n+1}(\tau) - L_n(\tau) = m; L_{n+1}(\tau^\bullet) - L_n(\tau^\bullet) = m'\big) = r(m + m' + 1, m + 1).$$

We next define the left contour process of the sin-tree t denoted by $C(t)$ as the distance-from-the-root process of a particle starting at the root and moving clockwise on t viewed as a planar graph embedded in the oriented half-plane with edges of unit length. We define $C^\bullet(t)$ as the contour process corresponding to the anticlockwise journey and we can also write $C(t^\bullet) = C^\bullet(t)$. More precisely, $C(t)$ [resp. $C^\bullet(t)$] can be recovered from $H(t)$ [resp. $H^\bullet(t)$] through (5) that still holds for sin-trees [note that in that case the sequence $(b_n; n \geq 0)$ is infinite].

It will be sometimes convenient to approximate a sin-tree t by the finite tree $[t]_{u_n^*(t)}$ with n large. The formula connecting the contour processes of t and $[t]_{u_n^*(t)}$ is given as follows: Set $\sigma_n(t) = \#\{u \in t : u < u_n^*(t)\}$ and $\sigma_n(t^\bullet) = \#\{u \in t^\bullet : u < u_n^*(t^\bullet)\}$. We get $\sigma_n(t) + \sigma_n(t^\bullet) = \#[t]_{u_n^*(t)} + n - 1$ since the individuals of $[\![\varnothing, u_{n-1}^*(t)]\!]$ have been counted twice. Check that

$$\begin{aligned} \sigma_n(t) &= \sup\{k \geq 0 : H_k(t) \leq n\}, \\ 2\sigma_n(t) - n &= \sup\{s \geq 0 : C_s(t) \leq n\}, \end{aligned} \tag{11}$$

with similar formulas for $t^\bullet$. Thus we get

$$\begin{aligned} C_s(t) &= C_s\big([t]_{u_n^*(t)}\big) && \text{if } s \in [0, 2\sigma_n(t) - n], \\ C_s^\bullet(t) &= C_{2(\#[t]_{u_n^*(t)} - 1) - s}\big([t]_{u_n^*(t)}\big) && \text{if } s \in [0, 2\sigma_n(t^\bullet) - n]. \end{aligned} \tag{12}$$

(Observe that a similar formula is not available for height processes.)

2.3. *Statement of the main result.* For convenience of notation, we set $d = 1/2 - \varepsilon$ and $u = 1/2 + \varepsilon$. Recall that $\tau_\varepsilon \in \mathbb{T}$ denotes the random ordered rooted tree associated with the range of the random walk W^ε in $\mathbb{U}$. First observe that the process $(|W_n^\varepsilon|; n \geq 0)$ giving the distance from the root of the particle performing the random walk does contain an important part of the information concerning τ_ε. Moreover, this process is simply distributed as the post-infimum path of a random walk whose possible jumps are $+1$ with probability u and -1 with probability d. Recall that $(B_s; s \geq 0)$ stands for the linear Brownian motion and set for any $y \in \mathbb{R}$, $B_s^{(y)} = B_s + ys$ and $I_s^{(y)} = \inf_{u \leq s} B_u^{(y)}$. Standard arguments imply

$$\big(\varepsilon\big|W^\varepsilon_{\lfloor s/\varepsilon^2 \rfloor}\big|; s \geq 0\big) \xrightarrow[\varepsilon \to 0]{\text{(d)}} \big(B^{(2)}_{s+g} - I^{(2)}_\infty; s \geq 0\big),$$

where we have set $g=\inf\{s\geq 0: B_s^{(2)}=I_\infty^{(2)}\}$. Notation $\overset{\text{(d)}}{\to}$ stands for the convergence in distribution in the appropriate space of right-continuous functions with left limits endowed with Skorohod topology. We also use notation $\overset{\text{(fd)}}{\to}$ for the convergence in distribution of all finite-dimensional marginals.

This result turns out to provide the right scaling for τ_ε though the connection between $(|W_n^\varepsilon|; n\geq 0)$ and τ_ε is nontrivial and the distribution of τ_ε is not simple; for instance, we can check that τ_ε and $\tau_\varepsilon^\bullet$ might not have the same distribution. Take the binary case $\mathbf{a}=(a,1-a,0,0,\dots)$ for some $a\in(0,1)$. Define the set $A\subset\mathbb{T}$ by $A=\{t\in\mathbb{T}: k_\varnothing(t)=2, k_1(t)=0, k_2(t)>0\}$. Then it follows from simple arguments discussed in Section 3.1 that

$$\mathbf{P}(\tau_\varepsilon\in A)=\frac{du^2a(1-a)}{(u+da)(u+d^2a)},$$

$$\mathbf{P}(\tau_\varepsilon^\bullet\in A)=\frac{du^2a(1-a)}{(u+d(1-a))(u+d^2(1-a))}.$$

Thus, except for $a=1/2$, $\mathbf{P}(\tau_\varepsilon\in A)\neq\mathbf{P}(\tau_\varepsilon^\bullet\in A)$. Actually, when ε goes to zero, the particle backtracks more and more often causing correlations. However, Theorem 2.1 asserts that the correlations only give rise to a deterministic phenomenon that is taken into account by the coefficient $\gamma=\gamma(\mathbf{a})$ given by (1).

THEOREM 2.1. *Let D and $D^\bullet$ be two independent copies of $B^{(-2)}-2I^{(-2)}$. Then*:

(i) $(\varepsilon C_{s/\varepsilon^2}(\tau_\varepsilon), \varepsilon C^\bullet_{s/\varepsilon^2}(\tau_\varepsilon))_{s\geq 0}\xrightarrow[\varepsilon\to 0]{\text{(d)}}(2D_{\gamma s}, 2D^\bullet_{\gamma s})_{s\geq 0}$,

(ii) $(\varepsilon H_{\lfloor s/2\varepsilon^2\rfloor}(\tau_\varepsilon), \varepsilon H^\bullet_{\lfloor s/2\varepsilon^2\rfloor}(\tau_\varepsilon))_{s\geq 0}\xrightarrow[\varepsilon\to 0]{\text{(d)}}(2D_{\gamma s}, 2D^\bullet_{\gamma s})_{s\geq 0}$.

Let us make some comments. (a) The limit of the height and the contour processes are the same up to the multiplicative time constant 2. This comes from the fact that vertices are visited once by the height process while the edges are crossed exactly twice by the contour process.

(b) The definition of γ through expectation (1) is only for practical reasons. We have not found a simpler expression except for the case $a_1=\cdots=a_b=1/b$, where b is an integer greater than 1. In that case the X_i's are deterministic and $\gamma=1-1/b$.

(c) The continuum random sin-tree whose $2D_{(\gamma\cdot)}$ and $2D^\bullet_{(\gamma\cdot)}$ are respectively the left and the right height processes can be defined as follows: To any real s corresponds a vertex in the tree at height $H_s=\mathbf{1}_{(-\infty,0)}(s)2D_{-\gamma s}+\mathbf{1}_{[0,\infty)}(s)2D^\bullet_{\gamma s}$. Let $s\leq s'$. The youngest common ancestor of the vertices corresponding to s and s' is situated at height

$$m(s,s')=\inf\{H_u; u\in I(s,s')\},$$

where $I(s,s')$ is taken as $[s,s']$ if $0 \notin [s,s']$ and as $\mathbb{R} \setminus [s,s']$ otherwise. Thus, the distance between the vertices corresponding to s and s' is

$$\mathbf{d}(s,s') = H_s + H_{s'} - 2m(s,s').$$

We say that s and s' are equivalent if they correspond to the same vertex in the tree, that is, $\mathbf{d}(s,s') = 0$ that is denoted by $s \sim s'$. We formally define the continuum random sin-tree as the quotient set $T = \mathbb{R}/\sim$. Then $\mathbf{d}$ induces a metric on T that makes it be a (random) Polish space.

We can show that the metric space $(T, \mathbf{d})$ is an $\mathbb{R}$-tree (see [6] for related results). Due to the Brownian nature of H, all fractal dimensions of T are a.s. equal to 2. A point $\sigma \in T$ is called a branching point if the open set $T \setminus \{\sigma\}$ has more than two connected components and it corresponds to times at which H reaches a local minimum. Since all the local minima of H are distinct, all the branching points are binary, that is, $T \setminus \{\sigma\}$ has *three* connected components.

(d) Observe that the limiting tree T is symmetric since D and $D^{\bullet}$ have the same distribution. A heuristic explanation is the following: arguments discussed in Section 3.2 imply that an unbalanced set of weights $\mathbf{a}$ breaks the symmetry of τ_ε only if τ_ε has branching points of order ≥ 3 which does not happen to the limiting tree T that is binary.

3. Proof of the main result.

3.1. *Combinatorial results.* In this section ε is fixed and for convenience of notation we drop the corresponding subscript in the random variables. Thus, we write W and τ instead of W^ε and τ_ε. As explained in the Introduction, the linear interpolation of the process $(|W_n|; n \geq 0)$ can be viewed as the left contour process of a (fictive) GWI-tree denoted by $\overline{\tau}$ and whose distribution is given by the following lemma.

LEMMA 3.1. *The linear interpolation of* $(|W_n|; n \geq 0)$ *is distributed as the left contour process of a GWI*(μ, r)*-tree where* $\mu(k) = ud^k$, $r(k,k) = \mu(k-1)$ *and* $r(k,m) = 0$, $1 \leq m < k$, $k \geq 0$.

PROOF. Let $(\xi_n; n \geq 1)$ be i.i.d. such that $\mathbf{P}(\xi_n = 1) = u$ and $\mathbf{P}(\xi_n = -1) = d$. Set $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$ and define T_{-1} as $T_{-1} := \inf\{n \geq 0 : S_n = -1\}$ (with the convention $\inf \varnothing = \infty$). Since the random walk $S = (S_n; n \geq 0)$ a.s. drifts to $+\infty$, $P(T_{-1} = \infty) > 0$. By definition of W, $(|W_n|; n \geq 0)$ has the same distribution as S under $\mathbf{P}(\cdot | T_{-1} = \infty)$.

Let us denote by $(T_i^{(0)}; i \geq 0)$ the passage times to state 0: $T_0^{(0)} = 0$ and $T_{i+1}^{(0)} = \inf\{n > T_i^{(0)} : S_n = 0\}$, with the convention $\inf \varnothing = \infty$. Set

$$K = \sup\{i \geq 0 : T_i^{(0)} < \infty\} < \infty \qquad \text{a.s.}$$

We denote by $\mathcal{E}_1, \dots, \mathcal{E}_K, \mathcal{E}_{K+1}$ the excursions of S away from 0 defined by

$$\mathcal{E}_i = \big(S_{T^{(0)}_{i-1}+n}; 0 \le n \le \zeta_i := T^{(0)}_i - T^{(0)}_{i-1}\big), \qquad 1 \le i \le K,$$

and by $\mathcal{E}_{K+1} = (S_{T^{(0)}_K+n}; n \ge 0)$. We first consider the tree τ whose contour process is the linear interpolation of $(S_n; 0 \le n \le T_{-1} - 1)$ under $\mathbf{P}(\cdot|T_{-1} < \infty)$:

CLAIM. *τ is a GW(μ)-tree.*

PROOF. If L stands for the number of children of the root of τ, then $L+1$ is also the number of times S visits 0 before $T_{-1} : L = \sup\{i \ge 0 : T^{(0)}_i < T_{-1}\}$. By applying the Markov property at the stopping times $T^{(0)}_i$'s, we show that $\mathbf{P}(L = 0) = d$ and that for any $l \ge 1$, conditional on the event $\{L = l; T_{-1} < \infty\}$:

(a) $\mathcal{E}_1, \dots, \mathcal{E}_l$ are i.i.d. and they are distributed as $\mathcal{E}_1$ under $\mathbf{P}(\cdot|\mathcal{E}_1(1) = 1; T^{(0)}_1 < \infty)$. Moreover, the Markov property at time 1 implies that

(b) $(\mathcal{E}_1(n+1) - 1; 0 \le n \le T^{(0)}_1 - 2)$ under $\mathbf{P}(\cdot|\mathcal{E}_1(1) = 1; T^{(0)}_1 < \infty)$ has the same distribution as $(S_n; 0 \le n \le T_{-1} - 1)$ under $\mathbf{P}(\cdot|T_{-1} < \infty)$.

Now observe that the contour processes of the subtrees $\theta_1\tau, \dots, \theta_L\tau$ are the linear interpolations of $(\mathcal{E}_i(n+1) - 1; 0 \le n \le \zeta_i - 2)$, $1 \le i \le L$. We deduce from (a) and (b) that τ satisfies the two conditions of the definition of a GW-tree; its distribution is then the distribution of L under $\mathbf{P}(\cdot|T_{-1} < \infty)$, which can be computed as follows: Observe first that

$$\{L = l; T_{-1} < \infty\} \\ = \big\{\mathcal{E}_1(1) = 1; T^{(0)}_1 < \infty; \dots; \mathcal{E}_l(1) = 1; T^{(0)}_l < \infty; \mathcal{E}_{l+1}(1) = -1\big\}.$$

Then, by (a):

$$\mathbf{P}(L = l; T_{-1} < \infty) = d\mathbf{P}\big(\mathcal{E}_1(1) = 1; T^{(0)}_1 < \infty\big)^k.$$

But (b) implies that $\mathbf{P}(\mathcal{E}_1(1) = 1; T^{(0)}_1 < \infty) = u\mathbf{P}(T_{-1} < \infty)$. Thus,

$$\mathbf{P}(L = l; T_{-1} < \infty) = d\big(u\mathbf{P}(T_{-1} < \infty)\big)^k$$

and by summing over l we get $\mathbf{P}(T_{-1} < \infty) = d/(1 - u\mathbf{P}(T_{-1} < \infty))$ which implies that $\mathbf{P}(T_{-1} < \infty) = d/u$. Finally we get

$$\mathbf{P}(L = l|T_{-1} < \infty) = ud^k = \mu(k),$$

which achieves the proof of the claim. $\square$

Let us achieve the proof of the lemma: we now consider the tree $\overline{\tau}$ whose contour process is the linear interpolation of $(|W_n|; n \ge 0)$. To simplify notation, we identify this process to S under $\mathbf{P}(\cdot|T_{-1}=\infty)$. Then $K+1$ is the number of children of the ancestor of $\overline{\tau}$. First, observe that

$$\begin{aligned}(13)\qquad &\{K=k;\, T_{-1}=\infty\}\\ &\quad =\{\mathcal{E}_1(1)=1;\, T_1^{(0)}<\infty;\ldots;\mathcal{E}_k(1)=1;\, T_k^{(0)}<\infty;\, T_{k+1}^{(0)}=\infty\}.\end{aligned}$$

By applying the Markov property, we then show that conditional on $\{K=k;\, T_{-1}=\infty\}$:

(1) $\mathcal{E}_1,\ldots,\mathcal{E}_{k+1}$ are independent;
(2) $\mathcal{E}_1,\ldots,\mathcal{E}_k$ are distributed as $\mathcal{E}_1$ under $\mathbf{P}(\cdot|\mathcal{E}_1(1)=1;\, T_1^{(0)}<\infty)$;
(3) $\mathcal{E}_{k+1}$ is distributed as $\mathcal{E}_1$ under $\mathbf{P}(\cdot|T_1^{(0)}=\infty)$.

Now, by applying the Markov property at time 1 we see that

$$(14)\quad \big(\mathcal{E}_1(n+1)-1;\, n\geq 0\big)\quad \text{under}\quad \mathbf{P}\big(\cdot|T_1^{(0)}=\infty\big)\overset{\text{(law)}}{=} S \text{ under } \mathbf{P}(\cdot|T_{-1}=\infty).$$

Observe that the contour processes of the subtrees $\theta_1\overline{\tau},\ldots,\theta_{K+1}\overline{\tau}$ are the linear interpolations of $(\mathcal{E}_i(n+1)-1;\, 0\leq n\leq \zeta_i-2)$, $1\leq i\leq K+1$. Deduce from (b), from the previous claim and from (14) that conditional on $\{K=k;\, T_{-1}=\infty\}$, the subtrees $\theta_1\overline{\tau},\ldots,\theta_k\overline{\tau}$ are k independent GW(μ)-trees and that $\theta_{k+1}\overline{\tau}$ is distributed as $\overline{\tau}$. It implies that $\overline{\tau}$ satisfies the two conditions of the definition of GWI-trees. Since the infinite subtree is $\theta_{k+1}\overline{\tau}$, $\overline{\tau}$ is a GWI(μ, r)-tree with

$$\begin{aligned} r(k+1,m)&=0, & 1\leq m<k+1,\\ r(k+1,k+1)&=\mathbf{P}(K=k|T_{-1}=\infty), & k\geq 0,\end{aligned}$$

which can be computed as follows: Deduce from (13) and the Markov property

$$\mathbf{P}(K=k;\, T_{-1}=\infty)=\mathbf{P}\big(\mathcal{E}_1(1)=1;\, T_1^{(0)}<\infty\big)^k\mathbf{P}\big(T_1^{(0)}=\infty\big).$$

Now observe that $\mathbf{P}(T_1^{(0)}=\infty)=u\mathbf{P}(T_{-1}=\infty)$ and that $\mathbf{P}(\mathcal{E}_1(1)=1;\, T_1^{(0)}<\infty)=d$. Thus $\mathbf{P}(K=k|T_{-1}=\infty)=\mu(k)$, $k\geq 0$, which achieves the proof of the lemma. $\square$

Observe that $\overline{\tau}$ is completely asymmetric, that is, it has no vertices at the right hand of its infinite line of descent. Note also that its immigration distribution ν is equal to μ. In what follows, we explain how to recover the full range $\{W_n;\, n\geq 0\}$ from $\overline{\tau}$. To that end we need to label $\overline{\tau}$ by random marks in $\mathbb{N}^*$ as explained in the Introduction. Let us introduce some notation: the set $T=(t;\,(m_u, u\in t))$ is an $\mathbb{N}^*$-marked tree T if $t\in\mathbb{T}$ and if $m_u\in\mathbb{N}^*$, $u\in t$. The m_u's are the marks of T. The set of $\mathbb{N}^*$-marked trees is denoted by $\mathbb{T}_{\mathbb{N}^*}$. We define the *track* of T as the mapping $\mathbf{Tr}_T: t\to\mathbb{U}$ defined as follows: Let $u\in t$; if we denote by $u_0=\varnothing\preccurlyeq u_1\preccurlyeq\cdots\preccurlyeq u_n=u$ the ancestors of u, then we define $\mathbf{Tr}_T(u)$ as the word $m_{u_1}\ldots m_{u_n}\in\mathbb{U}$, with the convention $\mathbf{Tr}_T(\varnothing)=\varnothing$ (observe that $m_\varnothing$ plays no role in the definition of $\mathbf{Tr}_T$).

Similarly we define marked forests as sets of the form $F = (f; (m_u, u \in f))$ where $f \in \mathbb{F}$ and $m_u \in \mathbb{N}^*$. The set of marked forests is denoted by $\mathbb{F}_{\mathbb{N}^*}$. We define the track $\mathbf{Tr}_F$ of F exactly as we have defined the track of marked trees and we set for any $u \in F$

$$\theta_u(F) = \big(\theta_u(f); \big(m_{uv}, v \in \theta_u(f)\big)\big) \quad \text{and} \quad [F]_u = \big([f]_u; (m_v, v \in [f]_u)\big).$$

Since the linear interpolation of the process $(|W_n|; n \geq 0)$ is the distance-from-the-root process of a (fictive) particle exploring continuously $\overline{\tau}$ at unit speed from left to right $\overline{\tau}$, we can associate with each vertex $u \in \overline{\tau} \setminus \{\varnothing\}$ a unique time $n(u) \in \mathbb{N}$ such that the (fictive) particle climbs the edge $(\overleftarrow{u}, u)$ between times $n(u)$ and $n(u) + 1$. Since $|W_{n(u)+1}| = 1 + |W_{n(u)}|$, we can find $\overline{\mu}_u \in \mathbb{N}^*$ such that the word $W_{n(u)+1}$ is written $W_{n(u)}\overline{\mu}_u \in \mathbb{U}$. We then define the random marked tree $\overline{\mathcal{T}}$ as

$$\overline{\mathcal{T}} = \big(\overline{\tau}; (\overline{\mu}_u, u \in \overline{\tau})\big),$$

where the mark of the root $\overline{\mu}_\varnothing$ is taken independent of W and distributed on $\mathbb{N}^*$ in accordance with the set of weights $\mathbf{a}$: $\mathbf{P}(\overline{\mu}_\varnothing = i) = a_i$, $i \in \mathbb{N}^*$. The distribution of $\overline{\mathcal{T}}$ is described by an elementary lemma whose proof is left to the reader.

LEMMA 3.2. *Conditional on $\overline{\tau}$, the marks $(\overline{\mu}_u, u \in \overline{\tau})$ are independent and distributed in accordance with* $\mathbf{a}$. *Moreover,*

$$\mathbf{Tr}_{\overline{\mathcal{T}}}(\overline{\tau}) = \{W_n; n \geq 0\}. \tag{15}$$

As already explained in the Introduction, to take the track of $\overline{\tau}$ is a procedure that can be broken up in two distinct subprocedures: The first one "shuffles" $\overline{\tau}$ by putting its edges in a certain random order. The second one "shrinks" $\overline{\tau}$ by identifying some successive edges with respect to the new random order. Let us first specify what we mean by *shuffling*: Let $t \in \mathbb{T}$; we say that $p = (p_u, u \in t)$ is a permutation of t if each p_u is a permutation of the (possibly empty) set $\{1, \dots, k_u(t)\}$. Let $u \in t$ be the word $c_1 \dots c_n$. We denote by $u_k = c_1 \dots c_k$ the kth ancestor of u. We define the word u^p by $p_{u_0}(c_1) \dots p_{u_{n-1}}(c_n) \in \mathbb{U}$ if $u \neq \varnothing$ and by $\varnothing$ otherwise. We set $t^p = \{u^p; u \in t\}$. Now, pick uniformly at random a permutation π of t among the $\prod_{u \in t} k_u(t)!$ possible ones. We define the *shuffling of* t as the random tree $\mathbf{Sh}(t) := t^\pi$.

REMARK 3.1. Shuffling a GW-tree does not change its distribution. It is also easy to check that $\mathbf{Sh}(\overline{\tau})$ is a GWI(μ, r')-tree with r' given by $r'(k, j) = ud^{k-1}/k$, $1 \leq j \leq k$.

We would like to shuffle an $\mathbb{N}^*$-marked tree $T = (t; (m_u, \in t))$ in accordance with the order of its marks in $\mathbb{N}^*$: for any permutation p of t, set $T^p = (t^p; (m_{u^p}, u \in t))$ and observe that

$$\mathbf{Tr}_{T^p}(t^p) = \mathbf{Tr}_T(t). \tag{16}$$

Let $\pi(T) = (\pi_u, u \in t)$ be a random permutation of t such that the π_u's are mutually independent and π_u is picked uniformly at random among the permutations σ of $\{1, \dots, k_u(t)\}$ satisfying

$$m_{u\sigma(1)} \leq m_{u\sigma(2)} \leq \cdots \leq m_{u\sigma(k_u(t))}.$$

We define the shuffling of T as $\mathbf{Sh}(T) := T^{\pi(T)}$. By definition the mapping $\mathbf{Tr}_{\mathbf{Sh}(T)} : t^{\pi(T)} \to \mathbb{U}$ is increasing with respect to the lexicographical order:

$$\forall u, v \in t^{\pi(T)} \qquad u \leq v \quad \Longrightarrow \quad \mathbf{Tr}_{\mathbf{Sh}(T)}(u) \leq \mathbf{Tr}_{\mathbf{Sh}(T)}(v). \tag{17}$$

Observe that if any brothers in T have distinct marks, then $\pi(T)$ is deterministic. Thus, $t^{\pi(T)}$ has clearly not the same distribution as $\mathbf{Sh}(t)$. However, when the marks m_u, $u \in t$, are i.i.d. random variables, we can easily check that $t^{\pi(T)}$ is distributed as $\mathbf{Sh}(t)$. Thus, if we set

$$\widetilde{\mathcal{T}} = \mathbf{Sh}(\overline{\mathcal{T}}) := \big(\widetilde{\tau}; (\widetilde{\mu}_u, u \in \widetilde{\tau})\big),$$

then we deduce from the previous observation that

$$\mathbf{Tr}_{\widetilde{\mathcal{T}}}(\widetilde{\tau}) = \{W_n; n \geq 0\}, \qquad \widetilde{\tau} \overset{\text{(law)}}{=} \mathbf{Sh}(\overline{\tau}), \tag{18}$$

and that

$$\forall u, v \in \widetilde{\tau} \qquad u \leq v \quad \Longrightarrow \quad \mathbf{Tr}_{\widetilde{\mathcal{T}}}(u) \leq \mathbf{Tr}_{\widetilde{\mathcal{T}}}(v). \tag{19}$$

So, we first obtain τ by shuffling the GWI-tree $\overline{\tau}$ and then by identifying the edges of the resulting marked tree that have the same random marks. We now give estimates in Propositions 3.3 and 3.4 on how much this edge identification does shrink $\widetilde{\tau}$. Let us introduce some notation: with any marked forest $F = (f; (m_u, u \in f))$ we associate a collection $(Z_v(F); v \in \mathbb{U})$ of integers defined by

$$Z_v(F) = \#\{u \in f : \mathbf{Tr}_F(u) = v\}.$$

Some key estimates in the proof of Theorem 2.1 rely on a precise computation of the law of the $Z_v(F)$'s when F is distributed as a GW-forest or a GWI-forest. *From now until the end of the paper* all the GW or GWI-forests that we consider share the same offspring distribution $\mu(k) = ud^k$, $k \geq 0$. We set for any $i \in \mathbb{N}^*$ and for any $x \in [0, 1]$

$$f(x) := \sum_{k \geq 0} ud^k x^k = \frac{u}{1 - dx} \quad \text{and} \quad f_i(x) := f(1 - a_i + a_i x).$$

For any $v = m_1 \dots m_n \in \mathbb{U}$ we also define

$$f_v := f_{m_1} \circ \cdots \circ f_{m_n} \quad \text{and} \quad a_v := a_{m_1} \dots a_{m_n},$$

with $f_\varnothing = \mathrm{Id}$ and $a_\varnothing = 1$. *We adopt the following convention:* to simplify notation, we do not distinguish constants in inequalities and we denote them in a generic way by a symbol $K_{\alpha,\beta,\dots}$ meaning that we bound by a positive constant that only depends on parameters $\alpha, \beta, \dots,$ and so on.

We first describe the law of $(Z_v(\mathcal{F}); v \in \mathbb{U})$ with $\mathcal{F} = (\varphi; (\mu_u, u \in \varphi))$, where $\varphi = (\tau_1, \dots, \tau_l)$ is a forest of l i.i.d. GW(μ)-trees and where conditional on φ the marks $(\mu_u, u \in \varphi)$ are taken mutually independent and distributed in accordance with $\mathbf{a}$.

PROPOSITION 3.3.

(i) *For any* $v, w \in \mathbb{U}$,

$$\mathbf{E}\big[x^{Z_{vw}(\mathcal{F})} | Z_v(\mathcal{F})\big] = f_w(x)^{Z_v(\mathcal{F})}.$$

(ii) *Moreover for any* $v = m_1 \dots m_n \in \mathbb{U}$,

$$1 - f_v(1-x) = \frac{x}{A(v)x + B(v)} \qquad \textit{with } 1/B(v) = a_v(d/u)^n$$

and

$$A(v) = 1 + \frac{u}{d}\frac{1}{a_{m_1}} + \left(\frac{u}{d}\right)^2 \frac{1}{a_{m_1}a_{m_2}} + \cdots + \left(\frac{u}{d}\right)^{n-1} \frac{1}{a_{m_1} \dots a_{m_{n-1}}}.$$

(iii) *For any positive integer* p,

$$\mathbf{E}\left[\sum_{v \in \mathbb{U}} Z_v(\mathcal{F})^p\right] \le K_{\mathbf{a},p} \frac{l^p}{1 - d/u}.$$

PROOF. We first show (i) whose proof reduces to the "$l = 1$" case by an immediate independence argument. Let us take $\mathcal{F} = \mathcal{T}_1 = (\tau_1; (\mu_u, u \in \tau_1))$ and $v \in \mathbb{U}$. Consider the set $\mathcal{L}_v$ of the vertices $u \in \tau_1$ satisfying $\mathbf{Tr}_{\mathcal{T}_1}(u) = v$. We denote by $u_1 < \cdots < u_{Z_v(\mathcal{T}_1)}$ the elements of $\mathcal{L}_v$ listed in the lexicographical order. As a consequence of Remark 2.1, we see that conditional on $\mathcal{L}_v$ the marked trees $(\theta_{u_i}(\mathcal{T}_1), 1 \le i \le Z_v(\mathcal{T}_1))$ are i.i.d. marked trees distributed as $\mathcal{T}_1$. Observe next that for any $w \in \mathbb{U}$

$$Z_{vw}(\mathcal{T}_1) = \sum_{i=1}^{Z_v(\mathcal{T}_1)} \#\big\{u \in \theta_{u_i}(\mathcal{T}_1) : \mathbf{Tr}_{\theta_{u_i}(\mathcal{T}_1)}(u) = w\big\}.$$

So we get

$$\mathbf{E}\big[x^{Z_{vw}(\mathcal{T}_1)} | Z_v(\mathcal{T}_1)\big] = \mathbf{E}\big[x^{Z_w(\mathcal{T}_1)}\big]^{Z_v(\mathcal{T}_1)}.$$

Then it remains to prove $\mathbf{E}[x^{Z_w(\mathcal{T}_1)}] = f_w(x)$, which follows from iterating the previous identity and from the easy observation $\mathbf{E}[x^{Z_i(\mathcal{T}_1)}] = f_i(x)$, $i \ge 1$.

The proof of (ii) is a simple recurrence. Let us prove (iii): For any positive integer p and any $v = m_1 \dots m_n \in \mathbb{U}$, we deduce from (ii) the following inequality:

$$f_v^{(p)}(1) = p! \frac{1}{B(v)} \left(\frac{A(v)}{B(v)} \right)^{p-1} \tag{20}$$

$$\leq p! a_v \left(\frac{d}{u} \right)^{|v|} (1 - a_+)^{1-p}, \tag{21}$$

where we have set $a_+ = \max_{i \geq 1} a_i < 1$. For any integer i we denote by $(x)_i$ the factorial polynomial $x(x-1)\dots(x-i+1)$ [with the convention: $(x)_0 = 1$]. Check recursively that for any $l, p \geq 1$ and any $h \in C^\infty(\mathbb{R}, \mathbb{R})$,

$$\frac{d^p h^l}{dx^p} = \sum_{j=1}^{p} (l)_j h(x)^{l-j} Q_{j,p}\big(h'(x), \dots, h^{(p)}(x)\big), \tag{22}$$

where the $Q_{j,p}$'s are j-homogeneous polynomials with $\mathbb{N}$-valued coefficients that only depend on j and p. Deduce from (21) that for any $v \in \mathbb{U}$,

$$\mathbf{E}[(Z_v(\mathcal{F}))_p] = \frac{d^p f_v^l}{dx^p}(1) \tag{23}$$

$$= \sum_{j=1}^{p} (l)_j Q_{j,p}\big(f_v'(1), \dots, f_v^{(p)}(1)\big) \tag{24}$$

$$\leq \sum_{j=1}^{p} (l)_j a_v^j \left(\frac{d}{u} \right)^{j|v|} Q_{j,p}\big(1!, 2!(1-a_+)^{-1}, \dots, p!(1-a_+)^{1-p}\big) \tag{25}$$

$$\leq K_{\mathbf{a},p} l^p a_v \left(\frac{d}{u} \right)^{|v|}. \tag{26}$$

Then, by an easy argument,

$$\mathbf{E}[Z_v(\mathcal{F})^p] \leq K_{\mathbf{a},p} l^p a_v \left(\frac{d}{u} \right)^{|v|} \tag{27}$$

which implies (iii) by the following observation:

$$\sum_{v \in \mathbb{U}} a_v \left(\frac{d}{u} \right)^{|v|} = \sum_{n \geq 0} \left(\frac{d}{u} \right)^n \sum_{m_1, \dots, m_n \geq 1} a_{m_1} \cdots a_{m_n} = \frac{1}{1 - d/u}. \tag{28}$$

□

We need similar results for GWI-forests. Let r be some fixed repartition probability measure. We denote by ν the corresponding immigration distribution given by $\nu(k-1) = \sum_{1 \leq j \leq k} r(k, j)$, $k \geq 1$. For any $x \in [0, 1]$ and any $i \in \mathbb{N}^*$ we write

$$g(x) := \sum_{k \geq 0} \nu(k) x^k \quad \text{and} \quad g_i(x) := g(1 - a_i + a_i x).$$

Let $\mathcal{F}_0 = (\varphi_0; (\mu_u, u \in \varphi_0))$ be a random marked GWI-forest whose distribution is characterized as follows: $\varphi_0 = (\tau_0, \tau_1, \dots, \tau_l)$, the τ_i's are mutually independent, $\tau_1, \dots, \tau_l$ are i.i.d. GW(μ)-trees, τ_0 is a GWI(μ, r)-tree and conditional on φ_0 the marks μ_u are i.i.d. random variables distributed in accordance with **a**. For convenience of notation, we set

$$u_n^* = u_n^*(\varphi_0) \quad \text{and} \quad v_n^* = \mathbf{Tr}_{\mathcal{F}_0}(u_n^*), \qquad n \geq 0.$$

We also set $\mathbf{Sp} = \{v_n^* i, i \in \mathbb{N}^* \setminus \{\mu_{u_n^*}\}, n \geq 0\}$ and we define $\mathcal{S}$ as the σ-field generated by the random variables $(\mu_{u_n^*}; n \geq 0)$ and $(Z_w(\mathcal{F}_0); w \in \mathbf{Sp})$.

PROPOSITION 3.4.

(i) *Conditional on $\mathcal{S}$, the collection of the $\mathbb{U}$-indexed processes $((Z_{wv}(\mathcal{F}_0); v \in \mathbb{U}); w \in \mathbf{Sp})$ are mutually independent. Moreover, for any $w \in \mathbf{Sp}$, the process $(Z_{wv}(\mathcal{F}_0); v \in \mathbb{U})$ only depends on $\mathcal{S}$ through $Z_w(\mathcal{F}_0)$. More precisely,*

$$\big(Z_{wv}(\mathcal{F}_0); v \in \mathbb{U}\big) \quad \textit{under} \quad \boldsymbol{P}\big(\cdot | w \in \mathbf{Sp}; Z_w(\mathcal{F}_0) = l\big) \overset{(law)}{=} \big(Z_v(\mathcal{F}); v \in \mathbb{U}\big)$$

where $\mathcal{F} = (\varphi; (\mu_u, u \in \varphi))$, where φ is a sequence of l i.i.d. GW(μ)-trees and where conditional on φ the marks $(\mu_u, u \in \varphi)$ are i.i.d. distributed in accordance with **a**.

(ii) *For any $p \geq 1$, any $n \geq 0$,*

$$\mathbf{E}\big[Z_{v_n^*}(\mathcal{F}_0)^p\big] \leq K_{\mathbf{a},p}(l+1)^p \max_{0 \leq j \leq p} g^{(j)}(1)^p,$$

and for any $i \in \mathbb{N}^*$,

$$\mathbf{E}\big[Z_{v_n^* i}(\mathcal{F}_0)^p\big] \leq K_{\mathbf{a},p} a_i (l+1)^p \max_{0 \leq j \leq p} g^{(j)}(1)^p,$$

with the convention $g^{(0)} = g$.

(iii) *For any $p \geq 1$ and any $n \geq 0$,*

$$\mathbf{E}\Bigg[\sum_{v \in \mathbb{U}} Z_v\big([\mathcal{F}_0]_{u_n^*}\big)^p\Bigg] \leq K_{\mathbf{a},p} \frac{n+1}{1 - d/u}(l+1)^p \max_{0 \leq j \leq p} g^{(j)}(1)^p.$$

PROOF. Set for any $w \in \mathbf{Sp}$, $\mathcal{L}_w = \{u \in \varphi_0 : \mathbf{Tr}_{\mathcal{F}_0}(u) = w\}$. Then by definition, $Z_w(\mathcal{F}_0) = \#\mathcal{L}_w$. Check that

$$\forall u \neq u' \in \bigcup_{w \in \mathbf{Sp}} \mathcal{L}_w \qquad u, u' \notin \mathcal{L}_\infty(\varphi_0) \quad \text{and} \quad u \wedge u' \notin \{u, u'\}.$$

These two observations combined with Remark 2.1 imply that conditional on $\mathcal{S}$ the marked trees $\theta_u(\mathcal{F}_0)$, $u \in \bigcup_{w \in \mathbf{Sp}} \mathcal{L}_w$ are i.i.d. marked GW(μ)-trees with independent marks distributed in accordance with **a**. This implies (i) thanks to the following equality valid for any $w \in \mathbf{Sp}$ and any $v \in \mathbb{U}$:

$$Z_{wv}(\mathcal{F}_0) = \sum_{u \in \mathcal{L}_w} \#\big\{u' \in \theta_u(\mathcal{F}_0) : \mathbf{Tr}_{\theta_u(\mathcal{F}_0)}(u') = v\big\}.$$

Let us prove (ii): Suppose that the word u_n^* is written $l_1 \dots l_n \in \mathbb{U}$ for some nonnegative integers $l_1, \dots, l_n$. Consider $u \in \varphi_0$ such that $\mathbf{Tr}_{\mathcal{F}_0}(u) = v_n^*$. There are three cases:

(i) If $|u \wedge u_n^*| = n$, then $u = u_n^*$.

(ii) If $|u \wedge u_n^*| = -1$, then $u \wedge u_n^*$ is the fictive root $(-1, \varnothing)$. Thus, the ancestor $\varnothing_0$ of the sin-tree τ_0 is not an ancestor of u. It implies

$$\#\{u \in \varphi_0 : |u \wedge u_n^*| = -1 \text{ and } \mathbf{Tr}_{\mathcal{F}_0}(u) = v_n^*\} = \sum_{j=1}^{l} Z_{v_n^*}(\theta_{\varnothing_j}(\mathcal{F}_0)).$$

(iii) If $u \wedge u_n^* = u_k^*$ with $0 \le k < n$, we can find some $j \in \{1, \dots, k_{u_k^*}(\varphi_0)\}$ with $j \ne l_{k+1}$ and some $u' \in \mathbb{U}$ such that

$$u = u_k^* j u', \qquad \mu_{u_k^* j} = \mu_{u_{k+1}^*}, \qquad \mathbf{Tr}_{\theta_{u_k^* j}(\mathcal{F}_0)}(u') = w_{k+1}^*,$$

where $w_{k+1}^* \in \mathbb{U}$ stands for the word $\mu_{u_{k+2}^*} \dots \mu_{u_n^*} \in \mathbb{U}$, with the convention $w_n^* = \varnothing$.

Now, set for any $0 \le k < n$,

$$E_{k+1} = \{j \in \{1, \dots, k_{u_k^*}(\varphi_0)\} : j \ne l_{k+1} \text{ and } \mu_{u_k^* j} = \mu_{u_{k+1}^*}\}.$$

The combination of the three preceding cases implies that

$$\begin{aligned} Z_{v_n^*}(\mathcal{F}_0) = 1 &+ \sum_{j=1}^{l} Z_{v_n^*}(\theta_{\varnothing_j}(\mathcal{F}_0)) \\ &+ \sum_{k=0}^{n-1} \sum_{j \in E_{k+1}} \#\{u' \in \theta_{u_k^* j}(\varphi_0) : \mathbf{Tr}_{\theta_{u_k^* j}(\mathcal{F}_0)}(u') = w_{k+1}^*\}. \end{aligned}$$

Set $\kappa_k = \#E_k$, $1 \le k \le n$, $\kappa_0 = l$ and $w_0^* = v_n^*$. Then, by (i) we get

$$\mathbf{E}[r^{Z_{v_n^*}(\mathcal{F}_0)} | (\kappa_{k+1}, \mu_{u_{k+1}^*})_{0 \le k \le n-1}] = r f_{v_n^*}(r)^l \prod_{k=0}^{n-1} f_{w_{k+1}^*}(r)^{\kappa_{k+1}} \tag{29}$$

$$= r \prod_{k=0}^{n} f_{w_k^*}(r)^{\kappa_k}. \tag{30}$$

It also follows from the previous observations that $\kappa_1, \dots, \kappa_n$ are mutually independent with the same distribution specified by

$$\mathbf{E}[x^{\kappa_1}] = \mathbf{E}[g_{\mu_{u_0^*}}(x)] = \sum_{i \in \mathbb{N}^*} a_i g_{a_i}(x). \tag{31}$$

From Proposition 3.3(ii) we get a.s.

$$f_{w_k^*}(1+z) = 1 + \frac{z}{B(w_k^*)}\left(1 - \frac{A(w_k^*)}{B(w_k^*)} z\right)^{-1}$$

and since

$$\frac{A(w_k^*)}{B(w_k^*)} \leq \left(\frac{d}{u}a_+\right)^{|w_k^*|} + \cdots + \frac{d}{u}a_+ \leq (1-a_+)^{-1},$$

$f_{w_k^*}(1+z)$ has a.s. a power series expansion with a radius of convergence greater than $1-a_+>0$. Then, for any $|z|<1-a_+$ we can write

$$f_{w_k^*}(1+z)^{\kappa_k} = 1 + \sum_{p\geq 1} D_p^{(k)} z^p \qquad \text{with } D_p^{(k)} = \frac{1}{p!}\frac{d^p f_{w_k^*}^{\kappa_k}}{dz^p}(1).$$

Deduce from (26)

$$0 \leq \frac{1}{p!}\frac{d^p f_{w_k^*}^{\kappa_k}}{dz^p}(1) \leq K_{p,\mathbf{a}}\kappa_k^p a_{w_k^*}\left(\frac{d}{u}\right)^{|w_k^*|} \tag{32}$$

$$\leq K_{p,\mathbf{a}}\kappa_k^p a_+^{n-k}. \tag{33}$$

Then observe that

$$\prod_{k=0}^{n} f_{w_k^*}(1+z)^{\kappa_k} = 1 + \sum_{p\geq 1} D_p z^p, \qquad |z|<1-a_+,$$

where

$$D_p = \sum_{\mathcal{P}\subset\{0,\dots,n\}} \sum_{\substack{\sum_{k\in\mathcal{P}} q_k = p\\ q_k\geq 1}} \prod_{k\in\mathcal{P}} D_{q_k}^{(k)}.$$

Set $D_0=1$ and deduce from (30)

$$\mathbf{E}\big[\big(Z_{v_n^*}(\mathcal{F}_0)\big)_p \big| \big(\kappa_k,\mu_{u_k^*}\big)_{0\leq k\leq n}\big] = p!(D_p + D_{p-1}). \tag{34}$$

Use (33) and the independence of the κ_i's to get

$$\mathbf{E}[D_p] \leq \sum_{\mathcal{P}\subset\{0,\dots,n\}} \sum_{\substack{\sum_{k\in\mathcal{P}} q_k = p\\ q_k\geq 1}} \prod_{k\in\mathcal{P}} K_{q_k,\mathbf{a}}\mathbf{E}[\kappa_k^{q_k}]a_+^{n-k}.$$

If $\mathcal{P}\subset\{0,\dots,n\}$ and $\sum_{k\in\mathcal{P}} q_k = p$ with $q_k\geq 1$, $k\in\mathcal{P}$, then $\#\mathcal{P}\leq p$ and $q_k\leq p$ for any $k\in\mathcal{P}$. Thus,

$$\prod_{k\in\mathcal{P}} \mathbf{E}[\kappa_k^{q_k}] \leq (l+1)^p(1\vee \mathbf{E}[\kappa_1^p])^p$$

since $\kappa_1,\dots,\kappa_n$ are identically distributed and $\kappa_0=l$. Deduce from (31)

$$1\vee \mathbf{E}[\kappa_1^p] \leq K_{\mathbf{a},p}\max\big(1, g'(1),\dots,g^{(p)}(1)\big).$$

Thus,

$$\mathbf{E}[D_p] \leq K_{\mathbf{a},p}(l+1)^p \max_{0\leq j\leq p} g^{(j)}(1)^p \tag{35}$$

since

$$\sum_{\mathcal{P}\subset\{0,\dots,n\}}\sum_{\substack{\sum_{k\in\mathcal{P}}q_k=p\\ q_k\geq 1}}\prod_{k\in\mathcal{P}}a_+^{n-k}\leq K_p(1-a_+)^{-p}.$$

Then by (34) and an easy argument

$$\mathbf{E}\big[Z_{v_n^*}(\mathcal{F}_0)^p\big]\leq K_{\mathbf{a},p}(l+1)^p\max_{0\leq j\leq p}g^{(j)}(1)^p. \tag{36}$$

To achieve the proof of (ii), we set $\mathcal{L}_{v_n^*}=\{u\in\varphi_0\setminus\{u_n^*\}:\mathbf{Tr}_{\mathcal{F}_0}(u)=v_n^*\}$. Recall that any $u\in\mathcal{L}_{v_n^*}$ has offspring distribution μ and that u_n^* has offspring distribution ν. Since $Z_{v_n^*}=1+\#\mathcal{L}_{v_n^*}$, we get for any $i\in\mathbb{N}^*$,

$$\mathbf{E}\big[x^{Z_{v_n^*i}}|\mathcal{L}_{v_n^*},\mu_{u_{n+1}^*}\big]=x^{\mathbf{1}_{\{\mu_{u_{n+1}^*}=i\}}}f_i(x)^{Z_{v_n^*}-1}g_i(x).$$

Thus,

$$\mathbf{E}\big[x^{Z_{v_n^*i}}|Z_{v_n^*}=k+1\big]=(1-a_i+a_ix)f_i(x)^kg_i(x).$$

By differentiating p times at $x=1$ we get

$$\mathbf{E}\big[(Z_{v_{ni}^*})_p|Z_{v_n^*}=k+1\big]=\frac{d^pf_i^kg_i}{dx^p}(1)+pa_i\frac{d^{p-1}f_i^kg_i}{dx^{p-1}}(1).$$

Now observe that for any $q\geq 0$, $g_i^{(q)}(1)=a_i^qg^{(q)}(1)$ and

$$\frac{d^qf_i^k}{dx^q}(1)=(a_id/u)^q(k+q-1)_q,$$

by a simple computation. Thus,

$$\begin{aligned}\frac{d^pf_i^kg_i}{dx^p}(1)&=\sum_{q=0}^p\frac{p!}{q!(p-q)!}a_i^{p-q}g^{(p-q)}(1)(a_id/u)^q(k+q-1)_q\\&\leq K_pa_i^p(k+p)_p\max_{1\leq j\leq p}g^{(j)}(1).\end{aligned}$$

Consequently,

$$\mathbf{E}\big[(Z_{v_{ni}^*})_p|Z_{v_n^*}\big]\leq K_pa_i^p\big(Z_{v_n^*}-1+p\big)_p\max_{1\leq j\leq p}g^{(j)}(1).$$

Since $p\geq 1$ and by (36) we get

$$\mathbf{E}\big[(Z_{v_{ni}^*})_p\big]\leq K_pa_i(l+1)^p\max_{1\leq j\leq p}g^{(j)}(1)^p$$

this easily implies the second inequality of Proposition 3.4(ii).

We now prove (iii): First observe that the decomposition

$$\sum_{v\in\mathbb{U}}Z_v\big([\mathcal{F}_0]_{u_n^*}\big)^p=e_1+e_2+e_3$$

holds with

$$e_1 = \sum_{\substack{w \in \mathbf{Sp} \\ |w| \le n}} \sum_{v \in \mathbb{U}} Z_{wv}([\mathcal{F}_0]_{u_n^*})^p,$$

$$e_2 = \sum_{0 \le k < n} Z_{v_k^*}([\mathcal{F}_0]_{u_n^*})^p,$$

$$e_3 = \sum_{v \in \mathbb{U}} Z_{v_n^* v}([\mathcal{F}_0]_{u_n^*})^p.$$

Note for any $v \in \mathbb{U}$ and for any $w \in \mathbf{Sp}$ such that $|w| \le n$ that $Z_{wv}([\mathcal{F}_0]_{u_n^*}) = Z_{wv}(\mathcal{F}_0)$. Then by Proposition 3.3(i) and Proposition 3.4(iii)

$$\begin{aligned} \mathbf{E}[e_1 | \mathcal{S}] &\le K_{\mathbf{a},p} \sum_{\substack{w \in \mathbf{Sp} \\ |w| \le n}} \frac{Z_w(\mathcal{F}_0)^p}{1 - d/u} \\ &\le K_{\mathbf{a},p} (1 - d/u)^{-1} \sum_{k=0}^{n} \sum_{i \in \mathbb{N}^*} Z_{v_k^* i}(\mathcal{F}_0)^p. \end{aligned} \tag{37}$$

We then deduce from the second inequality of Proposition 3.4(ii)

$$\mathbf{E}[e_1] \le K_{\mathbf{a},p} (l+1)^p \max_{0 \le j \le p} g^{(j)}(1)^p \frac{n+1}{1 - d/u}. \tag{38}$$

Observe next that $Z_{v_k^*}([\mathcal{F}_0]_{u_n^*}) = Z_{v_k^*}(\mathcal{F}_0)$. Then by the first inequality of (ii), we get

$$\mathbf{E}[e_2] \le K_{\mathbf{a},p} n (l+1)^p \max_{0 \le j \le p} g^{(j)}(1)^p. \tag{39}$$

To bound $\mathbf{E}[e_3]$, note that conditional on $Z_{v_n^*}([\mathcal{F}_0]_{u_n^*}) = l$, the process $(Z_{v_n^* v}([\mathcal{F}_0]_{u_n^*}); v \in \mathbb{U})$ is distributed as $(Z_v(\mathcal{F}); v \in \mathbb{U})$ where $\mathcal{F} = (\varphi; (\mu_u, u \in \varphi))$, where φ is a sequence of l independent GW(μ)-trees and where conditional on φ the marks $(\mu_u, u \in \varphi)$ are i.i.d. random variables distributed in accordance with $\mathbf{a}$. Thus, by Proposition 3.3:

$$\mathbf{E}\big[e_3 | Z_{v_n^*}\big([\mathcal{F}_0]_{u_n^*}\big)\big] \le K_{\mathbf{a},p} \frac{Z_{v_n^*}([\mathcal{F}_0]_{u_n^*})^p}{1 - d/u}.$$

Now observe that $Z_{v_n^*}(\mathcal{F}_0) = Z_{v_n^*}([\mathcal{F}_0]_{u_n^*})$ and use (36) to get

$$\mathbf{E}[e_3] \le K_{\mathbf{a},p} \max_{0 \le j \le p} g^{(j)}(1)^p \frac{(l+1)^p}{1 - d/u}. \tag{40}$$

Then, (iv) follows by adding (38), (39) and (40). $\square$

3.2. *Proof of Theorem* 2.1. Let us first explain why Theorem 2.1 reduces to a convergence for finite trees: we restore ε in the random variables $\overline{\mathcal{T}}_\varepsilon = (\overline{\tau}_\varepsilon; (\overline{\mu}_u, u \in \overline{\tau}_\varepsilon))$ and $\widetilde{\mathcal{T}}_\varepsilon = \mathbf{Sh}(\overline{\mathcal{T}}_\varepsilon) = (\widetilde{\tau}_\varepsilon; (\widetilde{\mu}_u, u \in \widetilde{\tau}_\varepsilon))$. For any positive real number x we set $x_\varepsilon = \lfloor x/\varepsilon \rfloor$ and we define $\widetilde{\zeta}_{x,\varepsilon} = \sup\{n \geq 0 : |W_n^\varepsilon| \leq x_\varepsilon\}$. As explained in the Introduction, we associate a unique finite ordered rooted tree τ_ε^x with the subtree $\{W_n^\varepsilon; 0 \leq n \leq \widetilde{\zeta}_{x,\varepsilon}\} \subset \mathbb{U}$. Observe that in general $\tau_\varepsilon^x \neq [\tau_\varepsilon]_{u^*_{x_\varepsilon}(\tau_\varepsilon)}$; however, τ_ε^x and τ_ε coincide up to level x_ε:

$$[\tau_\varepsilon^x]_{x_\varepsilon} = [\tau_\varepsilon]_{x_\varepsilon}. \tag{41}$$

The following proposition asserts that the convergence of τ_ε is equivalent to the convergence of the τ_ε^x's for all $x > 0$. For convenience of notation, we set $\zeta_{x,\varepsilon} = 2\varepsilon^2 \# \tau_\varepsilon^x$ and

$$H_s(x,\varepsilon) = \varepsilon \mathbf{1}_{[0,\zeta_{x,\varepsilon})}(s) H_{\lfloor s/2\varepsilon^2 \rfloor}(\tau_\varepsilon^x) \quad \text{and} \quad C_s(x,\varepsilon) = \varepsilon \mathbf{1}_{[0,\zeta_{x,\varepsilon}-2\varepsilon^2]}(s) C_{s/\varepsilon^2}(\tau_\varepsilon^x).$$

We also define the limiting process by

$$D_s^{(x)} = \mathbf{1}_{[0,\sigma_x]}(s) D_s + \mathbf{1}_{[\sigma_x,\infty)}(s) D^\bullet_{(\zeta_x - s)_+},$$

where $\zeta_x = \sigma_x + \sigma_x^\bullet$ with σ_x (resp. $\sigma_x^\bullet$) $= \sup\{s \geq 0 : D_s \text{ (resp.} D_s^\bullet) \leq x\}$.

PROPOSITION 3.5. *Theorem* 2.1 *is implied by either of the following equivalent convergences*:

(i) $\forall x > 0\ C(x,\varepsilon) \xrightarrow[\varepsilon \to 0]{\text{(d)}} (2D^{(x)}_{\gamma s}; s \geq 0)$,

(ii) $\forall x > 0\ H(x,\varepsilon) \xrightarrow[\varepsilon \to 0]{\text{(d)}} (2D^{(x)}_{\gamma s}; s \geq 0)$.

PROOF. The proof of (ii) $\Longrightarrow$ (i) can be copied from the proof of Theorem 2.4.1 of [5]. It relies on formula (5) that makes the contour process of a finite ordered rooted tree an explicit functional of the corresponding height process. Since (5) also holds for contour processes of sin-trees, similar arguments work to show that Theorem 2.1(ii) implies Theorem 2.1(i). Let us prove that Proposition 3.5(i) implies Proposition 3.5(ii): Recall from (5) that

$$H_n(\tau_\varepsilon^x) = C_{2n - H_n(\tau_\varepsilon^x)}(\tau_\varepsilon^x).$$

So, if we denote by $S(\varepsilon)$ the maximal height of τ_ε^x we get

$$\sup_{n < \#\tau_\varepsilon^x} |H_n(\tau_\varepsilon^x) - C_{2n}(\tau_\varepsilon^x)| \leq \max_{|n-n'| \leq S(\varepsilon)} |C_n(\tau_\varepsilon^x) - C_{n'}(\tau_\varepsilon^x)|,$$

which implies after scaling

$$\sup_{s \leq \zeta_{x,\varepsilon}} |H_s(x,\varepsilon) - C_s(x,\varepsilon)| \leq \max_{|s-s'| \leq \varepsilon^2 S(\varepsilon)} |C_s(x,\varepsilon) - C_{s'}(x,\varepsilon)|.$$

Proposition 3.5(i) implies that $\varepsilon S(\varepsilon)$ converges in distribution to the supremum of $D^{(x)}$ that is a.s. finite. Thus, the right member of the latter inequality converges to zero in probability and Proposition 3.5(ii) follows. A similar argument shows that Theorem 2.1(i) implies Theorem 2.1(ii). Now, the proof will be achieved if we show that Proposition 3.5(ii) implies Theorem 2.1(ii): Assume that Proposition 3.5(ii) is true and deduce from (41) that

$$(42)\quad \big(\varepsilon H_{\lfloor s\wedge \mathbf{e}_{x,\varepsilon}/2\varepsilon^2\rfloor}(\tau_\varepsilon), \varepsilon H^{\bullet}_{\lfloor s\wedge \mathbf{e}^{\bullet}_{x,\varepsilon}/2\varepsilon^2\rfloor}(\tau_\varepsilon)\big)_{s\geq 0} \xrightarrow[\varepsilon\to 0]{(\mathrm{d})} \big(2D_{\gamma(s\wedge \mathbf{e}_x)}, 2D^{\bullet}_{\gamma(s\wedge \mathbf{e}^{\bullet}_x)}\big)_{s\geq 0},$$

where $\mathbf{e}_{x,\varepsilon}=\inf\{n\geq 0 : H_n(\tau^x_\varepsilon)\geq x_\varepsilon\}$ and $\mathbf{e}_x=\inf\{s\geq 0: D_s\geq x\}$ with similar definitions for $\mathbf{e}^{\bullet}_{x,\varepsilon}$ and $\mathbf{e}^{\bullet}_x$. Observe that Proposition 3.5(ii) implies for any $x>0$ that $(\mathbf{e}_{x,\varepsilon},\mathbf{e}^{\bullet}_{x,\varepsilon})$ converges in distribution to $(\mathbf{e}_x,\mathbf{e}^{\bullet}_x)$. Since $\mathbf{e}_x$ and $\mathbf{e}^{\bullet}_x$ a.s. go to infinity with x, we then get for any $M>0$,

$$\lim_{x\to\infty}\limsup_{\varepsilon\to 0}\mathbf{P}(e_{x,\varepsilon}\leq M; e^{\bullet}_{x,\varepsilon}\leq M)=0,$$

which implies Theorem 2.1(ii) by (42) and by standard arguments. □

We define $\overline{\tau}^x_\varepsilon=[\overline{\tau}_\varepsilon]_{u^*_{x_\varepsilon}(\overline{\tau}_\varepsilon)}$ and $\widetilde{\tau}^x_\varepsilon=[\widetilde{\tau}_\varepsilon]_{u^*_{x_\varepsilon}(\widetilde{\tau}_\varepsilon)}$ and we also set

$$\overline{\mathcal{T}}^x_\varepsilon=[\overline{\mathcal{T}}_\varepsilon]_{u^*_{x_\varepsilon}(\overline{\tau}_\varepsilon)}=\big(\overline{\tau}^x_\varepsilon; (\overline{\mu}_u, u\in\overline{\tau}^x_\varepsilon)\big),$$
$$\widetilde{\mathcal{T}}^x_\varepsilon=[\widetilde{\mathcal{T}}_\varepsilon]_{u^*_{x_\varepsilon}(\widetilde{\tau}_\varepsilon)}=\big(\widetilde{\tau}^x_\varepsilon; (\widetilde{\mu}_u, u\in\widetilde{\tau}^x_\varepsilon)\big).$$

By definition, $\#\widetilde{\tau}^x_\varepsilon=\#\overline{\tau}^x_\varepsilon=\widetilde{\zeta}_{x,\varepsilon}$. Deduce from (18) and (19)

$$(43)\qquad \mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon}(\widetilde{\tau}^x_\varepsilon)=\{W^\varepsilon_n; 0\leq n\leq\widetilde{\zeta}_{x,\varepsilon}\},\qquad \widetilde{\tau}^x_\varepsilon\overset{(\mathrm{law})}{=}\mathbf{Sh}(\overline{\tau}^x_\varepsilon)$$

and

$$(44)\qquad \forall u,v\in\widetilde{\tau}^x_\varepsilon\qquad u\leq v\quad\Longrightarrow\quad \mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon}(u)\leq\mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon}(v).$$

By Proposition 3.5, Theorem 2.1 reduces to prove that for any $x>0$:

$$(45)\qquad H(x,\varepsilon)\xrightarrow[\varepsilon\to 0]{(\mathrm{d})}\big(2D^{(x)}_{\gamma s}; s\geq 0\big).$$

The first step of the proof of (45) is a limit theorem for $\widetilde{\tau}^x_\varepsilon$: let us set for any $s\in[0,\infty)$

$$\widetilde{H}_s(x,\varepsilon)=\varepsilon\mathbf{1}_{[0,2\varepsilon^2\#\widetilde{\tau}^x_\varepsilon)}(s)H_{\lfloor s/2\varepsilon^2\rfloor}(\widetilde{\tau}^x_\varepsilon),$$
$$\widetilde{C}_s(x,\varepsilon)=\varepsilon\mathbf{1}_{[0,2\varepsilon^2(\#\widetilde{\tau}^x_\varepsilon-1)]}(s)C_{s/\varepsilon^2}(\widetilde{\tau}^x_\varepsilon).$$

LEMMA 3.6.

(i) $\forall x>0\ \widetilde{C}(x,\varepsilon)\xrightarrow[\varepsilon\to 0]{(\mathrm{d})}(2D^{(x)}_s; s\geq 0)$,

(ii) $\forall x>0\ \widetilde{H}(x,\varepsilon)\xrightarrow[\varepsilon\to 0]{(\mathrm{d})}(2D^{(x)}_s; s\geq 0)$.

PROOF. Deduce from (9)

$$H_n(\widetilde{\tau}_\varepsilon) = n - \mathbf{p}_\varepsilon(n) + H_{\mathbf{p}_\varepsilon(n)}(f(\widetilde{\tau}_\varepsilon)), \qquad n \geq 0. \tag{46}$$

Recall that $f(\widetilde{\tau}_\varepsilon)$ stands for the forest composed by the bushes rooted at the left hand of the infinite line of descent of $\widetilde{\tau}_\varepsilon$ and that $\mathbf{p}_\varepsilon(n)$ is given by $\mathbf{p}_\varepsilon(n) = \inf\{p \geq 0 : \mathbf{n}_\varepsilon(p) \geq n\}$ where

$$\mathbf{n}_\varepsilon(p) = p + \inf\Big\{k \geq 0 : L_k(\widetilde{\tau}_\varepsilon) > -\inf_{j \leq p} W_j(f(\widetilde{\tau}_\varepsilon))\Big\}$$

with $L_n(\widetilde{\tau}_\varepsilon) = (l_1(\widetilde{\tau}_\varepsilon) - 1) + \cdots + (l_n(\widetilde{\tau}_\varepsilon) - 1)$, $n \geq 1$, and $L_0(\widetilde{\tau}_\varepsilon) = 0$. By Remark 3.1, $\widetilde{\tau}_\varepsilon$ is a GWI(μ, r')-tree with $r'(k,l) = ud^{k-1}/k$, $1 \leq l \leq k$. Thus by Remark 2.4:

(a) the two forests $(f(\widetilde{\tau}_\varepsilon), f(\widetilde{\tau}_\varepsilon^\bullet))$ are independent of $(L(\widetilde{\tau}_\varepsilon), L(\widetilde{\tau}_\varepsilon^\bullet))$;
(b) $f(\widetilde{\tau}_\varepsilon)$ and $f(\widetilde{\tau}_\varepsilon^\bullet)$ are two mutually independent sequences of i.i.d. GW(μ)-trees;
(c) $(L(\widetilde{\tau}_\varepsilon), L(\widetilde{\tau}_\varepsilon^\bullet))$ is an $\mathbb{N} \times \mathbb{N}$-valued random walk whose jump distribution is given by

$$\mathbf{P}\big(L_{n+1}(\widetilde{\tau}_\varepsilon) - L_n(\widetilde{\tau}_\varepsilon) = l; L_{n+1}(\widetilde{\tau}_\varepsilon^\bullet) - L_n(\widetilde{\tau}_\varepsilon^\bullet) = l'\big) = \frac{1}{l + l' + 1} ud^{l+l'}.$$

Check first that $\mathbf{E}[L_n(\widetilde{\tau}_\varepsilon)] = \mathbf{E}[L_n(\widetilde{\tau}_\varepsilon^\bullet)] = nd/2u$, which implies

$$\big(\varepsilon L_{s/\varepsilon}(\widetilde{\tau}_\varepsilon), \varepsilon L_{s/\varepsilon}(\widetilde{\tau}_\varepsilon^\bullet)\big)_{s\geq 0} \xrightarrow[\varepsilon\to 0]{\text{(d)}} (s/2, s/2)_{s\geq 0}. \tag{47}$$

Next, we need to prove the joint convergence of $(\varepsilon H_{\lfloor s/2\varepsilon^2\rfloor}(\widetilde{\tau}_\varepsilon), \varepsilon V_{\lfloor s/2\varepsilon^2\rfloor}(f(\widetilde{\tau}_\varepsilon)))$: We know from Remark 2.2 that $(V_p(f(\widetilde{\tau}_\varepsilon)); p \geq 0)$ is a random walk with jump distribution given by $\rho(k) = ud^{k+1}$, $k \geq -1$. An elementary computation implies for any $\lambda \in \mathbb{R}$ that

$$\mathbf{E}\big[\exp\big(i\lambda\varepsilon V_{\lfloor s/2\varepsilon^2\rfloor}(f(\widetilde{\tau}_\varepsilon))\big)\big] = \exp\bigg(-\frac{s\lambda^2}{2} - 2i\lambda s\bigg) + o(1)$$

and by standard arguments

$$\big(\varepsilon V_{\lfloor s/2\varepsilon^2\rfloor}(f(\widetilde{\tau}_\varepsilon))\big)_{s\geq 0} \xrightarrow[\varepsilon\to 0]{\text{(d)}} B^{(-2)} \tag{48}$$

(see, e.g., Theorem 2.7 of [19]). We then use Theorem 2.3.1 of [5] that asserts that under (48) the joint convergence

$$\big(\varepsilon H_{\lfloor s/2\varepsilon^2\rfloor}(f(\widetilde{\tau}_\varepsilon)), \varepsilon V_{\lfloor s/2\varepsilon^2\rfloor}(f(\widetilde{\tau}_\varepsilon))\big)_{s\geq 0} \xrightarrow[\varepsilon\to 0]{\text{(d)}} \big(2\big(B^{(-2)} - I^{(-2)}\big), B^{(-2)}\big) \tag{49}$$

holds provided that for any $\delta > 0$,

$$\liminf_{\varepsilon\to 0} \big(f_{\lfloor \delta/\varepsilon\rfloor}(0)\big)^{\lfloor 1/\varepsilon\rfloor} > 0 \tag{50}$$

(recall that f_n is recursively defined by $f_n = f_{n-1} \circ f$). Check that

$$f_n(x) = \frac{u}{d} \frac{1-(u/d)^n - x(1-(u/d)^{n-1})}{1-(u/d)^{n+1} - x(1-(u/d)^n)}.$$

Then,

$$\lim_{\varepsilon \to 0} \big(f_{\lfloor \delta/\varepsilon \rfloor}(0)\big)^{\lfloor 1/\varepsilon \rfloor} = \exp\left(-\frac{4}{e^{4\delta}-1}\right) > 0$$

and (49) follows from (50). Recall notation α from Section 2.2 and observe that

$$\varepsilon \boldsymbol{\alpha}(\lfloor s/2\varepsilon^2 \rfloor) = \inf\left\{ s' \geq 0 : \varepsilon L_{\lfloor s'/\varepsilon \rfloor}(\widetilde{\tau}_\varepsilon) > -\inf_{r \leq \lfloor s/2\varepsilon^2 \rfloor} \varepsilon V_{\lfloor r/2\varepsilon^2 \rfloor}(f(\widetilde{\tau}_\varepsilon)) \right\}.$$

Deduce from (7) that

$$2\varepsilon^2 \mathbf{n}_\varepsilon(\lfloor s/2\varepsilon^2 \rfloor) = 2\varepsilon^2 \lfloor s/2\varepsilon^2 \rfloor + 2\varepsilon^2 \boldsymbol{\alpha}(\lfloor s/2\varepsilon^2 \rfloor).$$

Then by (47) and (49)

$$\big(\varepsilon \boldsymbol{\alpha}(\lfloor s/2\varepsilon^2 \rfloor)\big)_{s \geq 0} \xrightarrow[\varepsilon \to 0]{\text{(d)}} -2I^{(-2)} \quad \text{and} \quad \big(2\varepsilon^2 \mathbf{n}_\varepsilon(\lfloor s/2\varepsilon \rfloor); s \geq 0\big) \stackrel{\text{(d)}}{\to} (s; s \geq 0).$$

Thus, $(2\varepsilon^2 \mathbf{p}_\varepsilon(\lfloor s/2\varepsilon \rfloor); s \geq 0) \stackrel{\text{(d)}}{\to} (s; s \geq 0)$ and (10) combined with the convergence of $\mathbf{p}_\varepsilon$ and (46) imply

$$\big(\varepsilon H_{\lfloor s/2\varepsilon^2 \rfloor}(\widetilde{\tau}_\varepsilon)\big)_{s \geq 0} \xrightarrow[\varepsilon \to 0]{\text{(d)}} \big(2B_s^{(-2)} - 4I_s^{(-2)}\big)_{s \geq 0} = 2D.$$

The joint convergence (47) combined with the independence of $f(\widetilde{\tau}_\varepsilon)$ and $f(\widetilde{\tau}_\varepsilon^\bullet)$ also implies

$$\big(\varepsilon H_{\lfloor \cdot/2\varepsilon^2 \rfloor}(\widetilde{\tau}_\varepsilon), \varepsilon H_{\lfloor \cdot/2\varepsilon^2 \rfloor}(\widetilde{\tau}_\varepsilon^\bullet)\big)_{s \geq 0} \xrightarrow[\varepsilon \to 0]{\text{(d)}} (2D, 2D^\bullet).$$

Use (5) and arguments similar to those used in the proof of Theorem 2.4.1 of [5] to get

$$\big(\varepsilon C_{s/\varepsilon^2}(\widetilde{\tau}_\varepsilon), \varepsilon C^\bullet_{s/\varepsilon^2}(\widetilde{\tau}_\varepsilon)\big)_{s \geq 0} \xrightarrow[\varepsilon \to 0]{\text{(d)}} (2D, 2D^\bullet). \tag{51}$$

Set $\widetilde{\sigma}_{x,\varepsilon} = \sup\{s \geq 0 : C_s(\widetilde{\tau}_\varepsilon) \leq x_\varepsilon\}$ and define $\widetilde{\sigma}^\bullet_{x,\varepsilon}$ in a similar way. Recall notation σ_x, $\sigma_x^\bullet$ and $D^{(x)}$ introduced before Proposition 3.5 and deduce from (51) that

$$(\widetilde{\sigma}_{x,\varepsilon}, \widetilde{\sigma}^\bullet_{x,\varepsilon}) \xrightarrow[\varepsilon \to 0]{\text{(d)}} (\sigma_x, \sigma_x^\bullet).$$

It easily implies Lemma 3.6(i) by (12). Then, argue exactly as in the proof of Proposition 3.5 to deduce Lemma 3.6(ii) from Lemma 3.6(i). □

We now have to prove Proposition 3.5(ii). In one part of the proof we adapt Aldous' approach (Theorem 20 of [3]) and we get estimates for the tree τ_ε^x reduced at certain random times. The main technical difficulty is Lemma 3.7 that asserts that these random times are asymptotically uniformly distributed. Let us first define these random times: Let $(\mathcal{U}_i; i \geq 1)$ be a sequence of i.i.d. random variables independent of W^ε and uniformly distributed on $(0, 1)$. Let $u_0 = \varnothing < u_1 < \cdots < u_{\#\tilde{\tau}_\varepsilon^x - 1}$ be the vertices of $\tilde{\tau}_\varepsilon^x$ listed in the lexicographical order. We set

$$U_i(x, \varepsilon) = u_{\lfloor \mathcal{U}_i \#\tilde{\tau}_\varepsilon^x \rfloor} \quad \text{and} \quad V_i(x, \varepsilon) = \mathbf{Tr}_{\tilde{\mathcal{T}}_\varepsilon^x}(U_i(x, \varepsilon)) \in \mathbb{U}.$$

Then $V_i(x, \varepsilon) \in \{W_n^\varepsilon; 0 \leq n \leq \tilde{\zeta}_{x,\varepsilon}\}$ and the row of the corresponding vertex in τ_ε^x is given by

$$\overline{V}_i(x, \varepsilon) = \sum_{\substack{v \in \mathbb{U} \\ v \leq V_i(x,\varepsilon)}} \mathbf{1}_{\{Z_v(\tilde{\mathcal{T}}_\varepsilon^x) > 0\}}.$$

The key argument is the following lemma that is proved in the next section.

LEMMA 3.7. *For any $i \geq 1$, the following convergence holds in probability*:

$$\varepsilon^2 \left(\overline{V}_i(x, \varepsilon) - \frac{1}{\gamma} \mathcal{U}_i \#\tilde{\tau}_\varepsilon^x \right) \xrightarrow[\varepsilon \to 0]{} 0.$$

From now until the end of the section we assume that Lemma 3.7 is true and we prove Proposition 3.5(ii): Fix $x > 0$ and set for any $\delta > 0$:

$$\omega\big(H(x, \varepsilon), \delta\big) = \sup\{|H_s(x, \varepsilon) - H_{s'}(x, \varepsilon)|; |s - s'| \leq \delta\}.$$

We first prove tightness for $H(x, \varepsilon)$, $\varepsilon > 0$: By a standard criterion (see, e.g., Corollary 3.7.4 of [7]) we only need to prove

$$\text{(T1)} \qquad \lim_{M \to \infty} \liminf_{\varepsilon \to 0} \mathbf{P}\left(\sup_{s \geq 0} H_s(x, \varepsilon) \leq M \right) = 1$$

and for any $\eta > 0$,

$$\text{(T2)} \qquad \lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \mathbf{P}\big(\omega\big(H(x, \varepsilon), \delta\big) > \eta\big) = 0.$$

PROOF OF (T1). Note that the mapping **Tr** preserves height. So, we get

$$\sup_{s \geq 0} H_s(x, \varepsilon) = \varepsilon \sup\{\mathbf{Tr}_{\tilde{\mathcal{T}}_\varepsilon^x}(u) : u \in \tilde{\tau}_\varepsilon^x\} = \sup_{0 \leq s < 2\varepsilon^2 \#\tilde{\tau}_\varepsilon^x} \varepsilon H_{\lfloor s/2\varepsilon^2 \rfloor}(\tilde{\tau}_\varepsilon^x)$$

which is a tight family of random variables by Lemma 3.6. □

PROOF OF (T2). Let k be a positive integer and let p be a permutation of $\{1, \ldots, k\}$ such that $V_{p(1)}(x, \varepsilon) \leq \cdots \leq V_{p(k)}(x, \varepsilon)$ in $\mathbb{U}$. It implies

$$\overline{V}_{p(0)}(x, \varepsilon) \leq \overline{V}_{p(1)}(x, \varepsilon) \leq \cdots \leq \overline{V}_{p(k)}(x, \varepsilon) \leq \overline{V}_{p(k+1)}(x, \varepsilon)$$

where we set $0 = \overline{V}_{p(0)}(x, \varepsilon)$ and $\#\tau_\varepsilon^x = \overline{V}_{p(k+1)}(x, \varepsilon)$. We first need to get an upper bound for the quantities q_i defined for any $0 \le i \le k$ by

$$q_i = \sup\{|H_n(\tau_\varepsilon^x) - H_{\overline{V}_{p(i)}(x,\varepsilon)}(\tau_\varepsilon^x)|; \overline{V}_{p(i)}(x, \varepsilon) \le n \le \overline{V}_{p(i+1)}(x, \varepsilon)\}.$$

Observe that q_i can be rewritten

$$q_i = \sup\{||v| - |V_{p(i)}(x, \varepsilon)||; v \in \mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon^x}(\widetilde{\tau}_\varepsilon^x) \text{ and } V_{p(i)}(x, \varepsilon) \le v \le V_{p(i+1)}(x, \varepsilon)\}. \tag{52}$$

Set

$$w_0(k, x, \varepsilon) = \max_{0 \le i \le k} q_i, \qquad w_1(k, x, \varepsilon) = \max_{0 \le i \le k} ||V_{p(i+1)}(x, \varepsilon)| - |V_{p(i)}(x, \varepsilon)||$$

and

$$\Delta(k, x, \varepsilon) = \max_{v \in \mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon^x}(\widetilde{\tau}_\varepsilon^x)} \mathbf{d}(v, \{\varnothing, V_1(x, \varepsilon), \dots, V_k(x, \varepsilon)\}).$$

Equation (52) easily implies

$$w_0(k, x, \varepsilon) \le w_1(k, x, \varepsilon) + \Delta(k, x, \varepsilon). \tag{53}$$

Since **Tr** preserves height, we get for any $i \ge 1$,

$$|V_i(x, \varepsilon)| = |U_i(x, \varepsilon)| = H_{\lfloor \mathcal{U}_i \#\widetilde{\tau}_\varepsilon^x \rfloor}(\widetilde{\tau}_\varepsilon^x).$$

Then by Lemma 3.6 we get the following convergence in distribution:

$$\varepsilon w_1(k, x, \varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \max_{0 \le i \le k} |D^{(x)}_{\mathcal{U}_{(i+1)}\zeta_x} - D^{(x)}_{\mathcal{U}_{(i)}\zeta_x}|,$$

where $0 = \mathcal{U}_{(0)} \le \mathcal{U}_{(1)} \le \cdots \le \mathcal{U}_{(k)} \le \mathcal{U}_{(k+1)} = 1$ denotes the increasing reordering of $\{0, 1, \mathcal{U}_1, \dots, \mathcal{U}_k\}$. Thus,

$$\forall \eta > 0 \qquad \lim_{k \to \infty} \limsup_{\varepsilon \to 0} \mathbf{P}(\varepsilon w_1(k, x, \varepsilon) > \eta) = 0. \tag{54}$$

We next want to prove

$$\forall \eta > 0 \qquad \lim_{k \to \infty} \limsup_{\varepsilon \to 0} \mathbf{P}(\varepsilon \Delta(k, x, \varepsilon) > \eta) = 0. \tag{55}$$

To that end, observe that for any $u, u' \in \widetilde{\tau}_\varepsilon^x$, $\mathbf{d}(\mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon^x}(u), \mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon^x}(u')) \le \mathbf{d}(u, u')$. Then if we set

$$\Delta'(k, x, \varepsilon) = \max_{u \in \widetilde{\tau}_\varepsilon^x} \mathbf{d}(u; \{\varnothing, U_1(x, \varepsilon), \dots, U_k(x, \varepsilon)\}),$$

we get

$$\Delta(k, x, \varepsilon) \le \Delta'(k, x, \varepsilon) \tag{56}$$

and we control $\Delta'(k, x, \varepsilon)$ thanks to Lemma 3.6 (the following argument is directly inspired from the proof of Theorem 20 of [3]): With any $l \in \{0, \dots, \#\widetilde{\tau}_\varepsilon^x - 1\}$ we associate the index $i(l) \in \{0, \dots, k+1\}$ such that $\mathcal{U}_{i(l)}$ is the smallest element $y \in \{0, 1, \mathcal{U}_1, \dots, \mathcal{U}_k\}$ such that $l \le \lfloor y \# \widetilde{\tau}_\varepsilon^x \rfloor$. Check that

$$\Delta'(k, x, \varepsilon) \le \max_{0 \le l < \# \widetilde{\tau}_\varepsilon^x} \Bigl(H_l(\widetilde{\tau}_\varepsilon^x) + H_{\lfloor \mathcal{U}_{i(l)} \# \widetilde{\tau}_\varepsilon^x \rfloor}(\widetilde{\tau}_\varepsilon^x) - 2 \inf_{l \le j \le \lfloor \mathcal{U}_{i(l)} \# \widetilde{\tau}_\varepsilon^x \rfloor} H_j(\widetilde{\tau}_\varepsilon^x) \Bigr).$$

Lemma 3.6 implies that the right member of the previous inequality converges in distribution to

$$\sup_{0 \le s \le \zeta_x} \Bigl(D_s^{(x)} + D_{\mathcal{U}_{i(s)}\zeta_x}^{(x)} - 2 \inf_{s \le r \le \mathcal{U}_{i(s)}\zeta_x} D_r^{(x)} \Bigr), \tag{57}$$

where we denote by $\mathcal{U}_{i(s)}\zeta_x$ the smallest element $y \in \{\zeta_x, \mathcal{U}_1\zeta_x, \dots, \mathcal{U}_k\zeta_x\}$ such that $s \le y$ (recall that ζ_x stands for the lifetime of the process $D^{(x)}$ as defined before Proposition 3.5). We easily check that (57) converges to 0 in probability when k goes to infinity since

$$\sup_{0 \le s \le \zeta_x} (\zeta_x \mathcal{U}_{i(s)} - s) \le \max_{0 \le i \le k} \mathcal{U}_{(i+1)} - \mathcal{U}_{(i)} \underset{k \to \infty}{\longrightarrow} 0$$

in probability. Thus, it implies (55) by (56). Finally, as a consequence of (53), (54) and (55) we get

$$\forall \eta > 0 \qquad \lim_{k \to \infty} \limsup_{\varepsilon \to 0} \mathbf{P}\bigl(\varepsilon w_0(k, x, \varepsilon) > \eta\bigr) = 0. \tag{58}$$

Then, check that on the event

$$E(k, x, \varepsilon, \delta) = \Bigl\{ \min_{0 \le i \le k} \varepsilon^2 \bigl(\overline{V}_{p(i+1)}(x, \varepsilon) - \overline{V}_{p(i)}(x, \varepsilon) \bigr) > \delta \Bigr\}$$

the following inequality holds a.s.:

$$\omega\bigl(H(x, \varepsilon), \delta\bigr) \le 3 w_0(k, x, \varepsilon). \tag{59}$$

Use Lemma 3.7 to get

$$\min_{0 \le i \le k} \varepsilon^2 \bigl(\overline{V}_{p(i)}(x, \varepsilon) - \overline{V}_{p(i+1)}(x, \varepsilon) \bigr) \underset{\varepsilon \to 0}{\longrightarrow} \frac{\zeta_x}{2\gamma} \min_{0 \le i \le k} \bigl(\mathcal{U}_{(i+1)} - \mathcal{U}_{(i)} \bigr)$$

in distribution. Thus,

$$\forall k \ge 1 \qquad \lim_{\delta \to 0} \liminf_{\varepsilon \to 0} \mathbf{P}(E(k, x, \varepsilon, \delta)) = 1.$$

Easy arguments combined with (58) and (59) achieve the proof of (T2) and at the same time the tightness for $H(x, \varepsilon)$, $\varepsilon > 0$. $\square$

It remains to prove that $(2D_{\gamma s}^{(x)}; s \ge 0)$ is the only possible weak limit for the processes $H(x, \varepsilon)$, $\varepsilon > 0$. Tightness for the $H(x, \varepsilon)$'s, $\varepsilon > 0$, and Lemma 3.6 imply that the joint distributions of $(H(x, \varepsilon), 2\varepsilon^2 \# \widetilde{\tau}_\varepsilon^x)$, $\varepsilon > 0$, are tight. Assume that along a subsequence $\varepsilon_p \to 0$ the joint convergence

$$\bigl(H(x, \varepsilon_p), 2\varepsilon_p^2 \# \widetilde{\tau}_{\varepsilon_p}^x\bigr) \overset{\text{(d)}}{\underset{p \to \infty}{\longrightarrow}} (H', \zeta')$$

holds for some continuous process H' and some positive random variable ζ'. Lemma 3.7 implies

$$\big(H(x,\varepsilon_p); 2\varepsilon_p^2 \# \widetilde{\tau}^x_{\varepsilon_p}; 2\varepsilon_p^2 \overline{V}_1(x,\varepsilon_p), \dots, 2\varepsilon_p^2 \overline{V}_k(x,\varepsilon_p)\big)$$
$$\xrightarrow[\varepsilon\to 0]{\text{(d)}} (H'; \zeta'; \mathcal{U}_1\zeta'/\gamma, \dots, \mathcal{U}_k\zeta'/\gamma),$$

where the $\mathcal{U}_i$'s are chosen independent of (H', ζ'). Since **Tr** preserves height, we get for any $i \geq 1$

$$H_{\overline{V}_i(x,\varepsilon_p)}(\tau^x_\varepsilon) = |V_i(x,\varepsilon_p)| = |U_i(x,\varepsilon_p)| = H_{\lfloor \mathcal{U}_i \# \widetilde{\tau}^x_{\varepsilon_p} \rfloor}(\widetilde{\tau}^x_{\varepsilon_p}).$$

Then Lemmas 3.6 and 3.7 imply for any $k \geq 1$

$$\big(H_{2\varepsilon_p^2 \overline{V}_1(x,\varepsilon_p)}(x,\varepsilon_p), \dots, H_{2\varepsilon_p^2 \overline{V}_k(x,\varepsilon_p)}(x,\varepsilon_p);$$
$$2\varepsilon_p^2 \# \widetilde{\tau}^x_{\varepsilon_p}; 2\varepsilon_p^2 \overline{V}_1(x,\varepsilon_p), \dots, 2\varepsilon_p^2 \overline{V}_k(x,\varepsilon_p)\big)$$
$$\xrightarrow[\varepsilon\to 0]{} \big(2D^{(x)}_{\mathcal{U}_1\zeta_x}, \dots, 2D^{(x)}_{\mathcal{U}_k\zeta_x}; \zeta_x; \mathcal{U}_1\zeta_x/\gamma, \dots, \mathcal{U}_k\zeta_x/\gamma\big)$$

in distribution. Consequently,

$$\big(H'_{\mathcal{U}_1\zeta'/\gamma}, \dots, H'_{\mathcal{U}_k\zeta'/\gamma}; \zeta'; \mathcal{U}_1\zeta'/\gamma, \dots, \mathcal{U}_k\zeta'/\gamma\big)$$
$$\overset{\text{(law)}}{=} \big(2D^{(x)}_{\mathcal{U}_1\zeta_x}, \dots, 2D^{(x)}_{\mathcal{U}_k\zeta_x}; \zeta_x; \mathcal{U}_1\zeta_x/\gamma, \dots, \mathcal{U}_k\zeta_x/\gamma\big).$$

It implies $(H'_s; s \geq 0) \overset{\text{(law)}}{=} (2D^{(x)}_{\gamma s}; s \geq 0)$, which achieves the proof of (45).

3.3. *Proof of Lemma* 3.7. We introduce the notation

$$\overline{U}_i(x,\varepsilon) = \sum_{\substack{v\in\mathbb{U} \\ v \leq V_i(x,\varepsilon)}} Z_v(\widetilde{\mathcal{T}}^x_\varepsilon)$$

and we first prove the following convergence in probability:

$$\varepsilon^2\big(\overline{U}_i(x,\varepsilon) - \mathcal{U}_i \# \widetilde{\tau}^x_\varepsilon\big) \xrightarrow[\varepsilon\to 0]{} 0. \tag{60}$$

PROOF. Deduce from (44):

$$\big\{u \in \widetilde{\tau}^x_\varepsilon : \mathbf{Tr}_{\widetilde{\mathcal{T}}^x_\varepsilon}(u) < V_i(x,\varepsilon)\big\} \subset \{u \in \widetilde{\tau}^x_\varepsilon : u \leq U_i(x,\varepsilon)\}$$
$$\subset \big\{u \in \widetilde{\tau}^x_\varepsilon : \mathbf{Tr}_{\widetilde{\mathcal{T}}^x_\varepsilon}(u) \leq V_i(x,\varepsilon)\big\}$$

which implies

$$0 \leq \overline{U}_i(x,\varepsilon) - \lfloor \mathcal{U}_i \# \widetilde{\tau}^x_\varepsilon \rfloor \leq Z_{V_i(x,\varepsilon)}(\widetilde{\mathcal{T}}^x_\varepsilon). \tag{61}$$

Then observe that for any $v \in \mathbb{U}$,

$$\mathbf{P}\big(V_i(x,\varepsilon) = v \mid \widetilde{\mathcal{T}}_\varepsilon^x\big) = \frac{Z_v(\widetilde{\mathcal{T}}_\varepsilon^x)}{\#\widetilde{\tau}_\varepsilon^x}.$$

Thus, (61) and the Cauchy–Schwarz inequality imply

$$\begin{aligned}\mathbf{E}[|\overline{U}_i(x,\varepsilon) - \mathcal{U}_i \#\widetilde{\tau}_\varepsilon^x|] &\le 1 + \mathbf{E}\Bigg[\frac{1}{\#\widetilde{\tau}_\varepsilon^x}\sum_{v\in\mathbb{U}} Z_v(\widetilde{\mathcal{T}}_\varepsilon^x)^2\Bigg] \\ &\le 1 + \mathbf{E}\Bigg[\frac{1}{(\#\widetilde{\tau}_\varepsilon^x)^2}\sum_{v\in\mathbb{U}} Z_v(\widetilde{\mathcal{T}}_\varepsilon^x)^2\Bigg]^{1/2} \mathbf{E}\Bigg[\sum_{v\in\mathbb{U}} Z_v(\widetilde{\mathcal{T}}_\varepsilon^x)^2\Bigg]^{1/2}.\end{aligned}$$

Since $\#\widetilde{\tau}_\varepsilon^x = \sum_{v\in\mathbb{U}} Z_v(\widetilde{\mathcal{T}}_\varepsilon^x)$, we get $\sum_{v\in\mathbb{U}} Z_v(\widetilde{\mathcal{T}}_\varepsilon^x)^2 \le (\#\widetilde{\tau}_\varepsilon^x)^2$ and

$$\mathbf{E}[|\overline{U}_i(x,\varepsilon) - \mathcal{U}_i \#\widetilde{\tau}_\varepsilon^x|] \le 1 + \mathbf{E}\Bigg[\sum_{v\in\mathbb{U}} Z_v(\widetilde{\mathcal{T}}_\varepsilon^x)^2\Bigg]^{1/2}. \tag{62}$$

Remark 3.1 and (43) imply that $\widetilde{\tau}_\varepsilon$ is a GWI-tree with immigration distribution $\nu = \mu$, so that

$$g(x) = f(x) = \frac{u}{1-dx} \quad \text{and} \quad g^{(j)}(1) = j!\left(\frac{d}{u}\right)^j, \qquad j \ge 1.$$

Then, by Proposition 3.4(iii),

$$\mathbf{E}\Bigg[\sum_{v\in\mathbb{U}} Z_v(\widetilde{\mathcal{T}}_\varepsilon^x)^2\Bigg] \le \frac{K_{\mathbf{a}} x_\varepsilon}{1-d/u} = \frac{K_{\mathbf{a}} x}{\varepsilon^2}\big(1+o(1)\big).$$

Thus,

$$\mathbf{E}[\varepsilon^2 |\overline{U}_i(x,\varepsilon) - \mathcal{U}_i \#\widetilde{\tau}_\varepsilon^x|] \le K_{\mathbf{a},x}\varepsilon\big(1+o(1)\big)$$

and (60) follows. $\square$

Then, Lemma 3.7 is a consequence of the convergence in probability:

$$\varepsilon^2\left(\overline{V}_i(x,\varepsilon) - \frac{1}{\gamma}\overline{U}_i(x,\varepsilon)\right) \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{63}$$

PROOF OF (63). We need several preliminary estimates (Lemmas 3.8 and 3.9) whose proofs rely on Propositions 3.3 and 3.4. We first consider a random marked GW-forest with l elements $\mathcal{F}_\varepsilon = (\varphi_\varepsilon; (\mu_u, u \in \varphi_\varepsilon))$ as defined in Proposition 3.3: Recall that $\varphi_\varepsilon = (\tau_1, \dots, \tau_l)$ is a forest of l i.i.d. GW(μ)-trees and that the marks $(\mu_u, u \in \varphi)$ are i.i.d. conditional on φ_ε, their conditional distribution being given by $\mathbf{a}$. Set $\mathcal{T}_{1,\varepsilon} = (\tau_1; (\mu_u, u \in \tau_1))$ and define

$$1/\gamma_\varepsilon = \frac{\mathbf{E}[\sum_{v\in\mathbb{U}} \mathbf{1}_{\{Z_v(\mathcal{T}_{1,\varepsilon})>0\}}]}{\mathbf{E}[\sum_{v\in\mathbb{U}} Z_v(\mathcal{T}_{1,\varepsilon})]}.$$

We also set

$$\beta(\mathcal{F}_\varepsilon) = \sum_{v\in\mathbb{U}} Z_v(\mathcal{F}_\varepsilon) - \gamma_\varepsilon \mathbf{1}_{\{Z_v(\mathcal{F}_\varepsilon)>0\}}.$$ □

LEMMA 3.8. (i) $\lim_{\varepsilon\to 0}\gamma_\varepsilon = \gamma$,
(ii) *for any* $l\geq 1$, $0\leq \mathbf{E}[\beta(\mathcal{F}_\varepsilon)]\leq K_{\mathbf{a}} l(l-1)$,
(iii) $\mathbf{E}[\beta(\mathcal{F}_\varepsilon)^2]\leq K_{\mathbf{a}}\frac{l^4}{1-d/u}$ *and thus* $\mathbf{E}[|\beta(\mathcal{F}_\varepsilon)|]\leq K_{\mathbf{a}} l^2(1-d/u)^{-1/2}$.

PROOF. Let us prove (i): First observe that

$$1/\gamma_\varepsilon = \frac{\sum_{v\in\mathbb{U}} 1 - f_v(0)}{\sum_{v\in\mathbb{U}} f'_v(1)}.$$

Then, Proposition 3.3 implies that for any $v = m_1\dots m_n\in\mathbb{U}$

$$\sum_{v\in\mathbb{U}} f'_v(1) = \sum_{v\in\mathbb{U}} a_v(d/u)^{|v|} = \frac{1}{1-d/u}$$

and also

$$1 - f_v(0) = a_v\left(\frac{d}{u}\right)^{|v|}\left[\left(\frac{d}{u}\right)^n a_{m_n}\cdots a_{m_1} + \left(\frac{d}{u}\right)^{n-1} a_{m_n}\cdots a_{m_2} + \cdots + 1\right]^{-1}.$$

Thus, we get

$$1/\gamma_\varepsilon = \mathbf{E}\left[\left(1 + X_1\frac{d}{u} + \cdots + X_1X_2\cdots X_G\left(\frac{d}{u}\right)^G\right)^{-1}\right],$$

where we recall that the sequence of random variables $(X_n; n\geq 0)$ is distributed as specified after formula (1), and where G stands for an independent random variable whose distribution is given by $\mathbf{P}(G=n) = (1-d/u)(d/u)^n$, $n\geq 0$. Since $\lim_{\varepsilon\to 0} d/u = 1$, an elementary argument implies

$$\lim_{\varepsilon\to 0} 1/\gamma_\varepsilon = \mathbf{E}[(1 + X_1 + X_1X_2 + X_1X_2X_3 + \cdots)^{-1}] = 1/\gamma.$$

Let us prove (ii): Deduce from Proposition 3.3 that

$$\mathbf{E}[\beta(\mathcal{F}_\varepsilon)] = \sum_{v\in\mathbb{U}} l f'_v(1) - \gamma_\varepsilon\big(1 - f_v(0)^l\big). \tag{64}$$

The definition of γ_ε implies

$$l\sum_{v\in\mathbb{U}} f'_v(1) - \gamma_\varepsilon\big(1 - f_v(0)\big) = \sum_{v\in\mathbb{U}} l f'_v(1) - l\gamma_\varepsilon\big(1 - f_v(0)\big) = 0.$$

We then subtract this expression from (64) and we get

$$\mathbf{E}[\beta(\mathcal{F}_\varepsilon)] = \gamma_\varepsilon\sum_{v\in\mathbb{U}} f_v(0)^l - 1 + l\big(1 - f_v(0)\big).$$

Then, use the elementary inequality $(1-x)^l - 1 + lx \le l(l-1)x^2/2$, $x \in [0,1]$, to get

$$\mathbf{E}[\beta(\mathcal{F}_\varepsilon)] \le \frac{\gamma_\varepsilon l(l-1)}{2} \sum_{v\in\mathbb{U}} (1 - f_v(0))^2. \tag{65}$$

Deduce from the explicit computation of $1 - f_v(0)$ recalled above that

$$(1 - f_v(0))^2 \le \left(\frac{d}{u}\right)^{2|v|} a_v^2 \le a_+^{|v|} a_v.$$

Thus,

$$\sum_{v\in\mathbb{U}} (1 - f_v(0))^2 \le \sum_{n\ge 0} a_+^n \sum_{m_1,\dots,m_n\in\mathbb{N}^*} a_{m_1}\cdots a_{m_n} \le (1-a_+)^{-1}$$

and (ii) follows from (i).

It remains to prove (iii): For convenience of notation, we simply write β and Z_v instead of $\beta(\mathcal{F}_\varepsilon)$ and $Z_v(\mathcal{F}_\varepsilon)$. Check that

$$\mathbf{E}[\beta^2] = \mathbf{E}[E_1] + \mathbf{E}[E_2], \tag{66}$$

where

$$E_1 = \sum_{\substack{v,v'\in\mathbb{U}\\ v\wedge v'\notin\{v,v'\}}} (Z_v - \gamma_\varepsilon \mathbf{1}_{\{Z_v>0\}})(Z_{v'} - \gamma_\varepsilon \mathbf{1}_{\{Z_{v'}>0\}})$$

and

$$E_2 = \sum_{\substack{v,v'\in\mathbb{U}\\ v\wedge v'\in\{v,v'\}}} (Z_v - \gamma_\varepsilon \mathbf{1}_{\{Z_v>0\}})(Z_{v'} - \gamma_\varepsilon \mathbf{1}_{\{Z_{v'}>0\}})$$

(note that in the two sums all but a finite number of terms vanish). Define for any $w \in \mathbb{U}$

$$\beta_w = \sum_{v\in\mathbb{U}} Z_{wv} - \gamma_\varepsilon \mathbf{1}_{\{Z_{wv}>0\}}.$$

E_1 can be rewritten as follows:

$$E_1 = \sum_{w\in\mathbb{U}} \sum_{i\ne j\in\mathbb{N}^*} \beta_{wi}\beta_{wj}.$$

Deduce from Proposition 3.3 that conditional on (Z_{wi}, Z_{wj}) (with $i \ne j$) the random variables β_{w_i} and β_{w_j} are independent and distributed as β with, respectively, $l = Z_{wi}$ and $l = Z_{wi}$. Use (ii) to get

$$\begin{aligned}\mathbf{E}[\beta_{w_i}\beta_{w_j} | (Z_{wi}, Z_{wj})] &= \mathbf{E}[\beta_{w_i} | Z_{wi}]\mathbf{E}[\beta_{w_j} | Z_{wj}] \\ &\le K_{\mathbf{a}} Z_{wi}(Z_{wi}-1) Z_{wj}(Z_{wj}-1).\end{aligned}$$

By Proposition 3.3 again, we get

$$\mathbf{E}[x^{Z_{wi}}y^{Z_{wj}}|Z_w] = f(1-a_i-a_j+a_ix+a_jy)^{Z_w}.$$

Recall that

$$\frac{d^k f^n}{dx^k}(x) = \left(\frac{d}{u}\right)^k \frac{(n+k-1)!}{(n-1)!} f(x)^{n+k}. \tag{67}$$

Then,

$$\mathbf{E}[Z_{wi}(Z_{wi}-1)Z_{wj}(Z_{wj}-1)|Z_w] = a_i^2a_j^2\left(\frac{d}{u}\right)^4 (Z_w+3)_4 \le 12Z_w^4.$$

Thus, by Proposition 3.3,

$$\mathbf{E}[E_1] \le K_{\mathbf{a}}\mathbf{E}\left[\sum_{w\in\mathbb{U}} Z_w^4\right] \le K_{\mathbf{a}}\frac{l^4}{1-d/u}. \tag{68}$$

We get a similar upper bound for $\mathbf{E}[E_2]$ by first noting that

$$E_2 \le 2\sum_{w\in\mathbb{U}} (Z_w - \gamma_\varepsilon \mathbf{1}_{\{Z_w>0\}})\beta_w.$$

Apply Proposition 3.3(i) and Lemma 3.8(ii) to get

$$\mathbf{E}\big[(Z_w - \gamma_\varepsilon \mathbf{1}_{\{Z_w>0\}})\beta_w|Z_w\big] \le K_{\mathbf{a}}Z_w^3.$$

By Proposition 3.3(iii) again

$$\mathbf{E}[E_2] \le K_{\mathbf{a}}\mathbf{E}\left[\sum_{w\in\mathbb{U}} Z_w^3\right] \le K_{\mathbf{a}}\frac{l^3}{1-d/u}. \tag{69}$$

Then (iii) follows from (66), (68) and (69). □

We need similar estimates for a marked GWI(μ, r)-forest $\mathcal{F}_{0,\varepsilon}$ whose distribution is the same as in Proposition 3.4: Recall that r is some fixed repartition probability measure on $\{(k,l) \in \mathbb{N}^* \times \mathbb{N}^* : l \le k\}$. We denote by ν the corresponding immigration probability measure given by $\nu(k-1) = \sum_{1\le l\le k} r(k,l)$, $k \ge 1$, and we set $g(r) = \sum_{k\ge 0}\nu(k)r^k$. We define $\mathcal{F}_{0,\varepsilon}$ as $(\varphi_{0,\varepsilon}; (\mu_u, u \in \varphi_{0,\varepsilon}))$ where $\varphi_{0,\varepsilon} = (\tau_0, \tau_1, \dots, \tau_l)$, the τ_i's are mutually independent, $\tau_1, \dots, \tau_l$ are i.i.d. GW(μ)-trees, τ_0 is a GWI(μ, r)-tree and conditional on $\varphi_{0,\varepsilon}$ the marks μ_u are i.i.d. random variables distributed in accordance with $\mathbf{a}$. Recall notation

$$u_n^* = u_n^*(\varphi_{0,\varepsilon}), \qquad v_n^* = \mathbf{Tr}_{\mathcal{F}_{0,\varepsilon}}(u_n^*), \qquad \mathbf{Sp} = \{v_n^* i, i \in \mathbb{N}^* \setminus \{\mu_{u_n^*}\}, n \ge 0\},$$

and recall that $\mathcal{S}$ is the σ-field generated by the random variables $(\mu_{u_n^*}; n \ge 0)$ and $(Z_w(\mathcal{F}_{0,\varepsilon}), w \in \mathbf{Sp})$. For any $n \ge 1$ we also set

$$\mathbf{Sp}(n) = \{w \in \mathbf{Sp} : |w| \le n\} \cup \{v_n^*\}.$$

We set

$$\beta_w([\mathcal{F}_{0,\varepsilon}]_{u_n^*}) = \sum_{v\in\mathbb{U}} Z_{wv}([\mathcal{F}_{0,\varepsilon}]_{u_n^*}) - \gamma_\varepsilon \mathbf{1}_{\{Z_{wv}([\mathcal{F}_{0,\varepsilon}]_{u_n^*})>0\}}.$$

LEMMA 3.9. *For any $n \geq 1$,*

$$\mathbf{E}\left[\sup_{A\subset \mathbf{Sp}(n)}\left|\sum_{w\in A}\beta_w\big([\mathcal{F}_{0,\varepsilon}]_{u_n^*}\big)\right|\right] \leq K_{\mathbf{a}} n(l+1)^2(1-d/u)^{-1/2}\max\big(1, g'(1)^2, g''(1)^2\big).$$

PROOF. To simplify notation we write β_w and Z_w instead of $\beta_w([\mathcal{F}_{0,\varepsilon}]_{u_n^*})$ and $Z_w([\mathcal{F}_{0,\varepsilon}]_{u_n^*})$. We also denote by $\mathbf{E}^{\mathcal{S}}$ the $\mathcal{S}$-conditional expectation. Let $A \subset \mathbf{Sp}(n)$. From Proposition 3.4(i) we deduce that conditional on $\mathcal{S}$ the $(\beta_w; w \in \mathbf{Sp}(n))$ are independent random variables and that for each $w \in \mathbf{Sp}(n)$, conditional on $Z_w = l$, β_w is distributed as the random variable $\beta(\mathcal{F}_\varepsilon)$ defined at Lemma 3.8. Apply Lemma 3.8 to get

$$\mathbf{E}^{\mathcal{S}}\left[\left|\sum_{w\in A}\beta_w\right|\right] \leq \sum_{w\in A}\mathbf{E}^{\mathcal{S}}[|\beta_w|] \leq K_{\mathbf{a}}(1-d/u)^{-1/2}\sum_{w\in \mathbf{Sp}(n)} Z_w^2.$$

Next, use Proposition 3.4(ii) to get

$$\mathbf{E}\left[\sum_{w\in \mathbf{Sp}(n)} Z_w^2\right] \leq \sum_{k=0}^{n-1}\sum_{i\in\mathbb{N}^*}\mathbf{E}[Z_{v_k^* i}^2] \leq K_{\mathbf{a}} n(l+1)^2\max\big(1, g'(1)^2, g''(1)^2\big),$$

which achieves the proof of the lemma. □

We now come back to the proof of (63) and we apply the previous results to the marked sin-tree $\widetilde{\mathcal{T}}_\varepsilon = (\widetilde{\tau}_\varepsilon; (\widetilde{\mu}_u, u \in \widetilde{\tau}_\varepsilon))$. For convenience of notation, we fix i and we set

$$U = U_i(x,\varepsilon), \qquad V = V_i(x,\varepsilon), \qquad \overline{U} = \overline{U}_i(x,\varepsilon), \qquad \overline{V} = \overline{V}_i(x,\varepsilon).$$

We keep the notation $u_n^* = u_n^*(\widetilde{\tau}_\varepsilon)$, $v_n^* = \mathbf{Tr}_{\widetilde{\mathcal{T}}_\varepsilon}(u_n^*)$, $\mathbf{Sp}$, $\mathbf{Sp}(n)$ and $\mathcal{S}$. Recall that $\widetilde{\mathcal{T}}_\varepsilon^x = [\widetilde{\mathcal{T}}_\varepsilon]_{u_{x_\varepsilon}^*}$ and that for any $v \in \mathbb{U}$ that is not a descendant of $v_{x_\varepsilon}^*$

$$Z_w(\widetilde{\mathcal{T}}_\varepsilon^x) = Z_w(\widetilde{\mathcal{T}}_\varepsilon). \tag{70}$$

For convenience of notation, we set for any $w \in \mathbb{U}$

$$Z_w = Z_w(\widetilde{\mathcal{T}}_\varepsilon^x) \quad \text{and} \quad \beta_w = \sum_{v\in\mathbb{U}} Z_{wv} - \gamma_\varepsilon \mathbf{1}\{Z_{wv} > 0\}.$$

Since $\lim_{\varepsilon\to 0}\gamma_\varepsilon = \gamma$, we only have to show

$$\varepsilon^2(\overline{U} - \gamma_\varepsilon\overline{V}) = \varepsilon^2\sum_{v\leq V} Z_v - \gamma_\varepsilon\mathbf{1}_{\{Z_v>0\}} \underset{\varepsilon\to 0}{\longrightarrow} 0 \tag{71}$$

in probability. To that end, we first introduce the random word

$$W = \max\{w \in \mathbf{Sp}(x_\varepsilon) : w \le V\},$$

where the maximum is taken with respect to the lexicographical order on $\mathbb{U}$. There are two cases:

(1) If $V \notin [\![\varnothing, v^*_{x_\varepsilon - 1}]\!]$, then we can find $V' \in \mathbb{U}$ such that $V = WV'$ and we set in that case $A = \{w \in \mathbf{Sp}(x_\varepsilon) : w < W\}$.

(2) If $V \in [\![\varnothing, v^*_{x_\varepsilon - 1}]\!]$, then we set $A = \{w \in \mathbf{Sp}(x_\varepsilon) : w \le W\}$.

Then, check that $\overline{U} - \gamma_\varepsilon \overline{V} = e_1(\varepsilon) + e_2(\varepsilon) + e_3(\varepsilon)$ with

$$e_1(\varepsilon) = \sum_{\substack{v \in [\![\varnothing, v^*_{x_\varepsilon - 1}]\!] \\ v \le V}} Z_v - \gamma_\varepsilon \mathbf{1}_{\{Z_v > 0\}},$$

$$e_2(\varepsilon) = \sum_{w \in A} \beta_w,$$

$$e_3(\varepsilon) = \mathbf{1}_{\{V \notin [\![\varnothing, v^*_{x_\varepsilon - 1}]\!]\}} \sum_{v \le V'} Z_{Wv} - \gamma_\varepsilon \mathbf{1}_{\{Z_{Wv} > 0\}}.$$

The limit (71) is then implied by the following convergences:

$$\varepsilon^2 \mathbf{E}[|e_1(\varepsilon)|] \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{72}$$

$$\varepsilon^2 \mathbf{E}[|e_2(\varepsilon)|] \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{73}$$

$$\varepsilon^4 \mathbf{E}[e_3(\varepsilon)^2] \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{74}$$

PROOF OF (72). Use Proposition 3.4(ii) with $p = 1$, $l = 0$, $n = x_\varepsilon - 1$ and $g(x) = f(x) = u/(1 - dx)$ to get

$$\begin{aligned} \mathbf{E}[|e_1(\varepsilon)|] &\le \sum_{i=0}^{x_\varepsilon - 1} \mathbf{E}[Z_{v_i^*}] + (x_\varepsilon - 1)\gamma_\varepsilon \\ &\le K_{\mathbf{a}} d(x_\varepsilon - 1)/u + (x_\varepsilon - 1)\gamma_\varepsilon \le K_{\mathbf{a},x} \varepsilon^{-1} \end{aligned}$$

which obviously implies (72). □

PROOF OF (73). We use Lemma 3.9 with $n = x_\varepsilon$, $l = 0$ and $g(x) = f(x) = u/(1 - dx)$ and thus $g^{(j)}(1) = j!(d/u)^j$, to get

$$\mathbf{E}[|e_2(\varepsilon)|] \le K_{\mathbf{a}} x_\varepsilon (1 - d/u)^{-1/2} \le K_{\mathbf{a},x} \varepsilon^{-3/2}$$

which implies (73). □

PROOF OF (74). It requires more complicated arguments. Let $w_0 \in \mathbb{U}$ and let l be a positive integer. We define $E(w_0, l)$ as the event $\{W = w_0; Z_{w_0} = l\}$. We first get an upper bound for

$$\xi(w_0, l) = \mathbf{E}[e_3(\varepsilon)^2 | E(w_0, l)].$$

Let $\mathcal{F} = (\varphi; (\mu_u, u \in \varphi))$ be a marked GW-forest with l elements as defined in Proposition 3.3. Pick uniformly at random a vertex $\mathcal{U}(\mathcal{F})$ in φ and define $\mathcal{V}(\mathcal{F}) \in \mathbb{U}$ by $\mathcal{V}(\mathcal{F}) = \mathbf{Tr}_{\mathcal{F}}(\mathcal{U}(\mathcal{F}))$. As a consequence of Propositions 3.3(i) and 3.4(i), we get the following identity:

$$(75)\qquad (Z_{w_0 v}, v \in \mathbb{U}; V') \quad \text{under} \quad \mathbf{P}(\cdot \mid E(w_0, l)) \overset{\text{(law)}}{=} (Z_v(\mathcal{F}), v \in \mathbb{U}; \mathcal{V}(\mathcal{F})).$$

Let G be the function on $\mathbb{F}$ defined by

$$G([\mathcal{F}]_{\mathcal{U}(\mathcal{F})}) = \sum_{v \le \mathcal{V}(\mathcal{F})} Z_v(\mathcal{F}) - \gamma_\varepsilon \mathbf{1}_{\{Z_v(\mathcal{F}) > 0\}}.$$

Then, (75) implies

$$(76)\qquad \xi(w_0, l) = \mathbf{E}[G([\mathcal{F}]_{\mathcal{U}(\mathcal{F})})^2] = \mathbf{E}\left[\frac{1}{\#\varphi} \sum_{u \in \varphi} G([\mathcal{F}]_u)^2\right]$$

$$(77)\qquad \le (1 + \gamma_\varepsilon)\mathbf{E}\left[\sum_{u \in \varphi} |G([\mathcal{F}]_u)|\right],$$

since for any $u \in \varphi$,

$$\frac{1}{\#\varphi} |G([\mathcal{F}]_u)| \le \frac{1 + \gamma_\varepsilon}{\#\varphi} \sum_{v \le \mathbf{Tr}_{\mathcal{F}}(u)} Z_v(\mathcal{F}) \le 1 + \gamma_\varepsilon.$$

We now estimate the right member of (77) thanks to (4): Recall the notation $\varphi_\flat$ for a size-biased forest with l elements, that is, a GWI(μ, r)-forest with l elements where r is given by $r(k, j) = ud^k/\bar{\mu}$, $1 \le j \le k$, with $\bar{\mu} = \sum_{k \ge 0} k\mu(k) = d/u$. Thus the corresponding immigration distribution is $\nu(k) = (k+1)u^2 d^k$, $k \ge 0$, and its generating function is $g(r) = u^2/(1 - dr)^2$. Let us define the random marked GWI-forest $\mathcal{F}_\flat$ as $(\varphi_\flat; (\overset{\flat}{\mu}_u \in \varphi_\flat))$ where conditional on $\varphi_\flat$ the $\overset{\flat}{\mu}_u$'s are i.i.d. with distribution $\mathbf{a}$. Deduce from (4) that

$$(78)\qquad \mathbf{E}\left[\sum_{u \in \varphi} |G([\mathcal{F}]_u)|\right] = \sum_{n \ge 0} l \left(\frac{d}{u}\right)^n \mathbf{E}[|G([\mathcal{F}_\flat]_{u_n^*(\mathcal{F}_\flat)})|].$$

Set as usual $v_n^*(\mathcal{F}_\flat) = \mathbf{Tr}_{\mathcal{F}_\flat}(u_n^*(\mathcal{F}_\flat))$ and observe for any $n \ge 0$

$$G([\mathcal{F}_\flat]_{u_n^*(\mathcal{F}_\flat)}) = \sum_{v < v_n^*(\mathcal{F}_\flat)} Z_v(\mathcal{F}_\flat) - \gamma_\varepsilon \mathbf{1}_{\{Z_v(\mathcal{F}_\flat) > 0\}}$$

$$= \sum_{w \in A_\flat} \beta_w(\mathcal{F}_\flat) + \sum_{i=0}^{n-1} Z_{v_i^*}(\mathcal{F}_\flat) - \gamma_\varepsilon$$

where we have set

$$\beta_w(\mathcal{F}_\flat) = \sum_{v \in \mathbb{U}} Z_{wv}(\mathcal{F}_\flat) - \gamma_\varepsilon \mathbf{1}_{\{Z_{wv}(\mathcal{F}_\flat) > 0\}}$$

and $A_\flat = \{w \in \mathbf{Sp}_\flat : w < v_n^*(\mathcal{F}_\flat)\}$ with

$$\mathbf{Sp}_\flat = \{v_{k-1}^*(\mathcal{F}_\flat)i; i \in \mathbb{N}^* \setminus \{\mu_{u_k^*(\mathcal{F}_\flat)}\}, k \geq 1\}.$$

Then,

$$\mathbf{E}[|G([\mathcal{F}_\flat]_{u_n^*(\mathcal{F}_\flat)})|] \leq \mathbf{E}\left[\left|\sum_{w \in A_\flat} \beta_w(\mathcal{F}_\flat)\right|\right] + \mathbf{E}\left[\left|\sum_{i=0}^{n-1} Z_{v_i^*}(\mathcal{F}_\flat) - \gamma_\varepsilon\right|\right].$$

Use Lemma 3.9 with $g(x) = u^2/(1 - dx)^2$ to get

$$\mathbf{E}\left[\left|\sum_{w \in A_\flat} \beta_w(\mathcal{F}_\flat)\right|\right] \leq K_{\mathbf{a}} n l^2 (1 - d/u)^{-1/2}$$

and use Proposition 3.4(ii) with $p = 1$ and $g(x) = u^2/(1 - dx)^2$ to get

$$\mathbf{E}\left[\left|\sum_{i=0}^{n-1} Z_{v_i^*}(\mathcal{F}_\flat) - \gamma_\varepsilon\right|\right] \leq K_{\mathbf{a}} n l.$$

These inequalities imply

$$\xi(w_0, l) \leq K_{\mathbf{a}} l^3 (1 - d/u)^{-1/2} \sum_{n \geq 0} n\left(\frac{d}{u}\right)^n \leq K_{\mathbf{a}} \frac{l^3}{(1 - d/u)^{5/2}}. \tag{79}$$

We now come back to the proof of (74): by (79), we get

$$\begin{aligned}\mathbf{E}[e_3(\varepsilon)^2] &= \sum_{\substack{w_0 \in \mathbb{U}, \\ l \geq 1}} \xi(w_0, l) \mathbf{P}(W = w_0; Z_W = l) \\ &\leq \frac{K_{\mathbf{a}}}{(1 - d/u)^{5/2}} \mathbf{E}\left[\sum_{w_0 \in \mathbf{Sp}(x_\varepsilon)} Z_{w_0}^3 \mathbf{1}_{\{W = w_0\}}\right].\end{aligned}$$

Then, set for any $w_0 \in \mathbf{Sp}(x_\varepsilon)$, $\zeta_{w_0} = \sum_{v \in \mathbb{U}} Z_{w_0 v}$ and observe that $\mathbf{P}(W = w_0 | \mathcal{S}) = \zeta_{w_0} / \#\widetilde{\tau}_\varepsilon^x$. Thus the previous inequality implies

$$\begin{aligned}\mathbf{E}[e_3(\varepsilon)^2] &\leq \frac{K_{\mathbf{a}}}{(1 - d/u)^{5/2}} \mathbf{E}\left[\sum_{w_0 \in \mathbf{Sp}(x_\varepsilon)} Z_{w_0}^3 \frac{\zeta_{w_0}}{\#\widetilde{\tau}_\varepsilon^x}\right] \\ &\leq \frac{K_{\mathbf{a}}}{(1 - d/u)^{5/2}} \mathbf{E}\left[\sum_{w_0 \in \mathbf{Sp}(x_\varepsilon)} Z_{w_0}^6\right]^{1/2} \mathbf{E}\left[\sum_{w_0 \in \mathbf{Sp}(x_\varepsilon)} \frac{\zeta_{w_0}^2}{(\#\widetilde{\tau}_\varepsilon^x)^2}\right]^{1/2}.\end{aligned}$$

But $\sum_{w_0 \in \mathbf{Sp}(x_\varepsilon)} \zeta_{w_0}^2 \leq (\#\widetilde{\tau}_\varepsilon^x)^2$ since $1 + x_\varepsilon + \sum_{w_0 \in \mathbf{Sp}(x_\varepsilon)} \zeta_{w_0} = \#\widetilde{\tau}_\varepsilon^x$. Then, use Proposition 3.4(ii) with $p = 6$, $l = 0$ and $g(x) = f(x)$ to get

$$\mathbf{E}[e_3(\varepsilon)^2] \leq K_{\mathbf{a}} \frac{x_\varepsilon^{1/2}}{(1 - d/u)^{5/2}} \leq K_{\mathbf{a},x} \varepsilon^{-3},$$

which implies (74). □

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