

THE EVOLUTION OF A RANDOM VORTEX FILAMENT

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We study an evolution problem in the space of continuous loops in a three-dimensional Euclidean space modeled upon the dynamics of vortex lines in 3d incompressible and inviscid fluids. We establish existence of a local solution starting from Hölder regular loops with index greater than $1/3$. When the Hölder regularity of the initial condition X is smaller or equal to $1/2$, we require X to be a *rough path* in the sense of Lyons [Rev. Mat. Iberoamericana **14** (1998) 215–310, System Control and Rough Paths (2002). Oxford Univ. Press]. The solution will then live in an appropriate space of rough paths. In particular, we can construct (local) solution starting from almost every Brownian loop.

1. Introduction. The aim of this work is to study the well-posedness of the evolution problem for a model of a random *vortex filament* in three-dimensional incompressible fluid. If u is the velocity field of the fluid, the vorticity $\omega : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a solenoidal field defined as $\omega = \text{curl } u$. A vortex filament is a field of vorticity ω which is strongly concentrated around a three-dimensional closed curve γ described parametrically as a continuous function $\gamma : [0, 1] \rightarrow \mathbb{R}^3$ such that $\gamma_0 = \gamma_1$. Ideally, neglecting the transverse size of the filament, we can describe the vorticity field ω^γ generated by γ as the distribution

$$(1) \quad \omega^\gamma(x) = \Gamma \int_0^1 \delta(x - \gamma_\xi) d\xi \gamma_\xi, \quad x \in \mathbb{R}^3,$$

where $\Gamma > 0$ is the intensity of vorticity. In \mathbb{R}^3 , the velocity field associated to ω can be reconstructed with the aid of the Biot–Savart formula:

$$(2) \quad u^\gamma(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \wedge \omega^\gamma(y) dy,$$

which is the solution of $\text{curl } u^\gamma = \omega^\gamma$ with enough decay at infinity.

Then

$$(3) \quad u^\gamma(x) = -\frac{\Gamma}{4\pi} \int_0^1 \frac{x - \gamma_\xi}{|x - \gamma_\xi|^3} \wedge d\xi \gamma_\xi,$$

where $a \wedge b$ is the vector product of the vectors $a, b \in \mathbb{R}^3$. The evolution in time of the infinitely thin vortex filament is obtained by imposing that the curve γ is

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transported by the velocity field u^γ :

$$(4) \quad \frac{d}{dt} \gamma(t)_\xi = u^{\gamma(t)}(\gamma(t)_\xi), \quad \xi \in [0, 1]$$

and this gives the initial value problem

$$(5) \quad \frac{d}{dt} \gamma(t)_\xi = -\frac{\Gamma}{4\pi} \int_0^1 \frac{\gamma(t)_\xi - \gamma(t)_\eta}{|\gamma(t)_\xi - \gamma(t)_\eta|^3} \wedge d_\eta \gamma(t)_\eta, \quad \xi \in [0, 1].$$

Even if the curve γ is smooth, this expression is not well defined since the integral is divergent if γ has nonzero curvature.

To overcome this divergence, a natural approach is that of Rosenhead [23], who suggested the following approximate equation of motion based on a regularized kernel:

$$(6) \quad \frac{d}{dt} \gamma(t)_\xi = -\frac{\Gamma}{4\pi} \int_0^1 \frac{\gamma(t)_\xi - \gamma(t)_\eta}{[|\gamma(t)_\xi - \gamma(t)_\eta|^2 + \mu^2]^{3/2}} \wedge d_\eta \gamma(t)_\eta, \quad \xi \in [0, 1],$$

for some $\mu > 0$. This model has clear advantages and has been used in some numerical calculation of aircraft trailing vortices by Moore [21].

We will consider a generalization of the Rosenhead model where the function γ is not necessarily smooth. This is natural when we want to study models of *random* vortex filaments. Indeed, for simple models of random vortex lines, the curve γ is rarely smooth or even of bounded variation. Here we imagine to take as initial condition a typical trajectory of a Brownian loop (since the path must be closed) or other simple models like fractional Brownian loops (to be described precisely in Section 5.1). As we will see later, a major problem is then the interpretation of (6).

The study of the dynamics of random vortex lines is suggested by some work of Chorin [2] and Gallavotti [15]. The main justification for the adoption of a probabilistic point of view comes from two different directions. Chorin builds discrete models of random vortex filaments to explain the phenomenology of turbulence by the statistical mechanics of these coherent structures. Gallavotti instead suggested the use of very irregular random functions to provide a natural regularization of (5). Both approaches are on a physical level of rigor.

On the mathematical side in the recent year there has been some interest in the study of the statistical mechanics of continuous models for vortex filaments. Lions and Majda [17] proposed a statistical model of quasi-3d random vortex lines which are constrained to remain parallel to a given direction and thus cannot fold. Flandoli [7] rigorously studied the problem of the definition of the energy for a random vortex filament modeled over a Brownian motion and Flandoli and Gubinelli [8] introduced a probability measure over Brownian paths to study the statistical mechanics of vortex filaments. The study of the energy of filament configurations has been extended to models based on fractional Brownian motion by Flandoli and Minelli [10] and Nualart, Rovira and Tindel [22]. Moreover,

a model of Brownian vortex filaments capable of reproducing the multi-fractal character [12] of turbulent velocity fields has been introduced in [9].

Problem (6) defines a natural flow on three-dimensional closed curves. The study of this kind of flows has been recently emphasized by Lyons and Li in [19], where they study a class of flows of the form

$$(7) \quad \frac{dY}{dt} = F(I_f(Y)), \quad Y_0 = X,$$

where Y takes values in a (Banach) space of functions, F is a smooth function and $I_f(Y)$ is an Itô map, that is, the map $Y \mapsto Z$ where Z is the solution of the differential equation

$$dZ_\sigma = f(Z_\sigma) dY_\sigma$$

driven by the path Y . They prove that, under suitable conditions, I_f is a smooth map and then that (7) has (local) solutions and, thus, as a by-product, that $F \circ I_f$ can be effectively considered a vector-field on a space of paths.

Our evolution equation does not match the structure of the flows considered by Lyons and Li [19]. A very important difference is that (7), under suitable assumptions on the initial condition X (e.g. X a semi-martingale path), can be solved with standard tools of stochastic analysis (essentially Itô stochastic calculus), while (6) has a structure which is not well adapted to a filtered probability space and prevents even to (easily) set-up the problem in a space of semi-martingale paths. In our opinion this peculiarity makes the problem interesting from the point of view of stochastic analysis and was one of our main motivation to start its study. Using Lyons' rough paths, we will show that it is possible to give a meaning to the evolution problem (6) starting from a (fractional or standard) Brownian loop and that this problem has always a local solution (recall that the existence of a global solution, to our knowledge, has not been proven even in the case of a smooth curve).

The paper is organized as follows: in Section 2 we describe precisely the model we are going to analyze and we make some preliminary observations on the structure of covariation of the solution (in the sense of stochastic analysis and assuming the initial condition has finite covariations). Next, we introduce the functional spaces in which we will set-up the problem of existence of solutions. In Section 3 we build a local solution for initial conditions which are Hölder continuous with exponent greater than $1/2$ and for which the line integrals can be understood à la Young [25]. In Section 4 we build a solution in a class of rough paths (introduced in [16]) for initial conditions which are rough paths of Hölder regularity greater than $1/3$ (which essentially are p -rough paths for $p < 3$, in the terminology of [20]). Moreover, we prove that the solution is Lipschitz continuous with respect to the initial data. Finally, in Section 5 we apply these results to obtain the evolution of random initial conditions of Brownian loop type or its fractional variant. Appendix collects the proof of some lemmas.

2. The model.

2.1. *The evolution equation.* Our aim is to start a study of the three-dimensional evolution of random vortex filaments by the analysis of the well-posedness of the regularized dynamical equations. Inspired by the Rosenhead model (6), we will be interested in studying the evolution described by

$$(8) \quad \frac{\partial Y(t)_\xi}{\partial t} = V^{Y(t)}(Y(t)_\xi), \quad Y(0) = X,$$

with initial condition X belonging to the set \mathcal{C} of closed and continuous curve in \mathbb{R}^3 parametrized by $\xi \in [0, 1]$. For any $Z \in \mathcal{C}$, V^Z is the vector-field given by the line integral

$$(9) \quad V^Z(x) = \int_Z A(x - y) dy = \int_0^1 A(x - Z_\xi) dZ_\xi, \quad x \in \mathbb{R}^3,$$

where $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \otimes \mathbb{R}^3$ is a matrix-valued field. In this setting the Rosenhead model is obtained by taking A of the form

$$A(x)^{ij} = -\frac{\Gamma}{4\pi} \sum_{k=1,2,3} \epsilon_{ijk} \frac{x^k}{[|x|^2 + \mu^2]^{3/2}}, \quad i, j = 1, 2, 3, x \in \mathbb{R}^3,$$

where ϵ_{ijk} is the completely antisymmetric tensor in \mathbb{R}^3 normalized such that $\epsilon_{123} = 1$, $\mu > 0$ is a fixed constant and $(x^k)_{k=1,2,3}$ are the components of the vector $x \in \mathbb{R}^3$.

2.2. *A first approach using covariations for random initial conditions.* Even if the kernel of the paper is fully pathwise, before studying the existence and uniqueness problem (in some sense to be precised), we would like to insert a preamble concerning the stability of the quadratic variation (with respect to the parameter) of a large class of solutions.

Let (Ω, \mathcal{F}, P) be a probability space. In order to simplify a bit the proofs, we have chosen to use the notion of covariation introduced, for instance, in [24].

Given two processes $X = (X_\xi)_{\xi \in [0,1]}$ and $Y = (Y_\xi)_{\xi \in [0,1]}$, the covariation $[X, Y]$ is defined (if it exists) as the limit of the sequence of functions

$$\xi \mapsto \int_0^\xi (X_{\rho+\varepsilon} - X_\rho)(Y_{\rho+\varepsilon} - Y_\rho) \frac{d\rho}{\varepsilon}$$

in the uniform convergence in probability sense (u.c.p.). If X and Y are classical continuous semi-martingales, it is well known that previous $[X, Y]$ coincides with the classical covariation.

A vector (X^1, \dots, X^n) of stochastic processes is said to have *all its mutual covariations* if $[X^i, X^j]$ exist for every $i, j = 1, \dots, n$. Generally here we will consider $n = 3$. Moreover, given a matrix or vector v , we denote v^* its transpose.

It is sometimes practical to have a matrix notation. If $M_\xi = \{m_\xi^{ij}\}_{i,j}$, $N_\xi = \{n_\xi^{ij}\}_{i,j}$, are matrices of stochastic processes such that they are compatible for the matrix product, we denote

$$[M, N]_\xi = \left\{ \sum_{k=1}^n [m^{ik}, n^{kj}]_\xi \right\}_{i,j} .$$

REMARK 1. The following result can be easily deduced from [24]. Let Φ_1, Φ_2 be of class $C^1(\mathbb{R}^3; \mathbb{R}^3)$, $X = (X^1, X^2, X^3)^*$, $Z = (Z^1, Z^2, Z^3)^*$ (understood as row vectors in the matrix calculus) such that (X, Z) has all its mutual covariations. Then $(\Phi_1(X), \Phi_2(Z))$ has all its mutual covariations and

$$[\Phi_1(X), \Phi_2(Z)^*]_\xi = \int_0^\xi (\nabla\Phi_1)(Z_\rho) d[X, Z^*]_\rho (\nabla\Phi_2)(X_\rho)^* .$$

REMARK 2. In reality, we could have chosen the modified Föllmer [11] approach appearing in [5], based on discretization procedures for which the common reader would be more accustomed.

In that case the same results stated in Remark 1 and Proposition 1 will be valid also in this discretization framework. We recall briefly that context.

Consider a family of subdivisions $0 = \xi_0^n < \dots < \xi_n^n = 1$ of the interval $[0, 1]$. We will say that the mesh of the subdivision converges to zero if $|\xi_{i+1}^n - \xi_i^n|$ go to zero as $n \rightarrow \infty$.

In this framework, the covariation $[X, Y]$ is defined (if it exists) as the limit of

$$\xi \mapsto \sum_{i=0}^{n-1} (X_{\xi_{i+1}^n \wedge \xi}^n - X_{\xi_i^n \wedge \xi}^n)(Y_{\xi_{i+1}^n \wedge \xi}^n - Y_{\xi_i^n \wedge \xi}^n)$$

in the u.c.p. sense with respect to ξ and the limit does not depend on the chosen family of subdivisions.

Suppose there exists a sub Banach space B of \mathcal{C} and let $V : (\gamma, y) \rightarrow V^\gamma(y)$ be a Borel map from $B \times \mathbb{R}^3$ to \mathbb{R}^3 such that:

- (V1) for fixed $\gamma \in B$, $y \mapsto V^\gamma(y)$ is $C_b^1(\mathbb{R}^3; \mathbb{R}^3)$;
- (V2) the application $\gamma \mapsto \|\nabla V^\gamma\|_\infty$ is locally bounded from B to \mathbb{R} .

The main motivation for this abstract framework comes from the setting described in the following section. Indeed, as we will see, there exists natural Banach sub-spaces B of \mathcal{C} such that the map V defined as

$$V^\gamma(x) = \int_0^1 A(x - \gamma_\xi) d^* \gamma_\xi ,$$

where d^* denotes some kind of path integration defined for every $\gamma \in B$, satisfy the above hypotheses (V1) and (V2).

PROPOSITION 1. *Suppose that a random field $(Y(t)_\xi)_{\xi \in [0,1], t \in [0,T]}$ with values in B is a continuous solution of*

$$(10) \quad Y(t)_\xi = X_\xi + \int_0^t V^{Y(s)}(Y(s)_\xi) ds; \quad \xi \in [0, 1], t \in [0, T],$$

with an initial condition X having all its mutual covariations. Then at each time $t \in [0, T]$, the path $Y(t)$ has all its mutual covariations. Moreover,

$$(11) \quad [Y(t), Y^*(t)]_\xi = \int_0^\xi M(t)_\rho d[X, X^*]_\rho (M(t)_\rho)^*,$$

where

$$(12) \quad M(t)_\xi := \exp \left[\int_0^t (\nabla V^{Y(s)})(Y(s)_\xi) ds \right].$$

REMARK 3. Note the following:

1. It is possible to adapt this proof to the situation where the solution exists up to a random time.
2. Since we are in the multidimensional case, we recall that $E_\xi(t) = \exp[\int_0^t Q_\xi(s) ds]$ is defined as the matrix-valued function satisfying the differential equation

$$E_\xi(t) \in \mathbb{R}^3 \otimes \mathbb{R}^3, \quad \frac{dE_\xi(t)}{dt} = Q_\xi(t)E_\xi(t), \quad E_\xi(0) = \text{Id}.$$

3. A typical case of initial condition of process having all its mutual covariation is a three-dimensional Brownian loop.

PROOF OF PROPOSITION 1. For simplicity, we prolongate the processes X parametrized by $[0, 1]$, setting $X_\xi = X_1$, $\xi \geq 1$. Let $\varepsilon > 0$. For $t \in [0, T]$, $\xi \in [0, 1]$, write

$$Z^\varepsilon(t)_\xi = Y(t)_{\xi+\varepsilon} - Y(t)_\xi, \quad X_\xi^\varepsilon = X_{\xi+\varepsilon} - X_\xi,$$

then

$$\begin{aligned} Z^\varepsilon(t)_\xi &= X_\xi^\varepsilon + \int_0^t [V^{Y(s)}(Y(s)_{\xi+\varepsilon}) - V^{Y(s)}(Y(s)_\xi)] ds \\ &= X_\xi^\varepsilon + \int_0^t (\nabla V^{Y(s)})(Y(s)_\xi) Z^\varepsilon(s)_\xi ds \\ &\quad + \int_0^t R^\varepsilon(s)_\xi Z(s)_\xi ds, \end{aligned}$$

where

$$R^\varepsilon(s)_\xi = \int_0^t ds \int_0^1 da [(\nabla V^{Y(s)})(Y(s)_\xi + aZ^\varepsilon(s)_\xi) - (\nabla V^{Y(s)})(Y(s)_\xi)]$$

so that

$$\sup_{s \leq T} |R^\varepsilon(s)_\xi| \rightarrow 0 \quad \text{a.s.}$$

Therefore,

$$Z^\varepsilon(t)_\xi = \exp \left[\int_0^t ds (\nabla V^{Y(s)})(Y(s)_\xi) + \int_0^t ds R^\varepsilon(s)_\xi \right] X_\xi^\varepsilon =: M(t)_\xi^\varepsilon X_\xi^\varepsilon.$$

Multiplying both sides by their transposed, dividing by ε and integrating from 0 to y , we get

$$\int_0^\xi \frac{Z^\varepsilon(t)_\rho (Z^\varepsilon)^*(t)_\rho}{\varepsilon} d\rho = \int_0^\xi M(t)_\rho^\varepsilon \frac{X_\rho^\varepsilon (X^\varepsilon)_\rho^*}{\varepsilon} (M(t)_\rho^\varepsilon)^* d\rho.$$

Then since, as $\varepsilon \rightarrow 0$,

$$M(t)_\xi^\varepsilon \rightarrow \exp \left[\int_0^t (\nabla V^{Y(s)})(Y(s)_\xi) ds \right] = M(t)_\xi$$

uniformly in t and ξ almost surely, using Lebesgue dominated convergence theorem, and similar arguments to Proposition 2.1 of [24], it is enough to show that

$$\xi \mapsto \int_0^\xi \exp \left[\int_0^t ds (\nabla V^{Y(s)})(Y(s)_\rho) \right] \frac{X_\rho^\varepsilon (X^\varepsilon)_\rho^*}{\varepsilon} \exp \left[\int_0^t ds (\nabla V^{Y(s)})(Y(s)_\rho) \right]^* d\rho$$

converges u.c.p. to the right member of (11).

This is obvious since

$$\int_0^\xi \frac{X_\rho^\varepsilon (X^\varepsilon)_\rho^*}{\varepsilon} d\xi \rightarrow [X, X^*]_\xi$$

u.c.p. with respect to $\xi \in [0, 1]$ and so, modulo extraction of a subsequence, we can make use of the weak \star -topology. \square

REMARK 4. Suppose that the initial condition has all its n -mutual covariations $n \geq 3$, see for this [4]. Proceeding in a similar way as above, it is possible to show that $Y(t)$ has all its mutual n -covariations.

A typical example of process having a strong finite n -variation is fractional Brownian motion with Hurst index $H = 1/n$.

2.3. *The functional space framework.* The filament evolution problem when X has finite-length has been previously studied in [1] where it is proved that, under some regularity conditions on A , there exists a unique local solution living in the space H_c^1 of closed curves with L^1 derivative.

We would like to be able to solve the Cauchy problem (8) starting from a random curve X like a 3d-Brownian loop (since it must be closed) or a fractional Brownian

loop. In these cases X is a.s. not in H_c^1 and, as a consequence, we need a sensible definition to the path-integral appearing in (9).

Even if X is a Brownian loop do not exists a simple strategy to give a well-defined meaning to the evolution problem (8) through the techniques of stochastic calculus. Indeed, we could try to define the integral in V^Y as an Itô or Stratonovich integral which requires Y to be a semi-martingale with respect to some filtration \mathcal{F} (e.g., the filtration generated by X). However, we readily note that the problem has no relationship with any natural filtration \mathcal{F} since, for example, to compute the velocity field $V^Y(x)$ in some point x , we need information about the whole trajectory of Y .

A viable (and relatively straightforward) strategy is then to give a well-defined meaning to the problem using a path-wise approach.

We require that the initial data has γ -Hölder regularity. When $\gamma > 1/2$, the line integral appearing in the definition (9) of the instantaneous velocity field $V^{Y(t)}$ can be understood à la Young [25]. The corresponding results will be presented in Section 3.

When $1/2 \geq \gamma > 1/3$, an appropriate notion of line integral has been formulated by Lyons in [18, 20]. In Section 4 we will show that, given an initial γ -Hölder path X (and an associated *area process* \mathbb{X}^2), there exists a unique local solution of the problem (13) in the class \mathcal{D}_X of paths *weakly-controlled* by X . The class \mathcal{D}_X has been introduced in [16] to provide an alternative formulation of Lyons' theory of integration and corresponds to paths $Z \in \mathcal{C}$ which locally behaves as X in the sense that

$$Z_\xi - Z_\eta = F_\eta(X_\xi - X_\eta) + O(|\xi - \eta|^{2\gamma}),$$

where $F \in C([0, 1], \mathbb{R}^3 \otimes \mathbb{R}^3)$ is a path taking values in the bounded endomorphisms of \mathbb{R}^3 .

In particular, these results provide solutions of the problem when X is a fractional Brownian loop of Hurst-index $H > 1/3$ (see Section 5).

3. Evolution for γ -Hölder curves with $\gamma > 1/2$.

3.1. *Setting and notation.* For any $X \in \mathcal{C}$, let

$$\|X\|_\gamma := \sup_{\xi, \eta \in [0, 1]} \frac{|X_\xi - X_\eta|}{|\xi - \eta|^\gamma}, \quad \|X\|_\infty := \sup_{\xi \in [0, 1]} |X_\xi|$$

and

$$\|X\|_\gamma^* := \|X\|_\infty + \|X\|_\gamma.$$

Denote \mathcal{C}^γ the set of paths $X \in \mathcal{C}$ such that $\|X\|_\gamma^* < \infty$.

All along this section we will assume that γ is a fixed number greater than $1/2$. In this case the following result states that there exists a unique extension to the Riemann–Stieltjes integral $\int f dg$ defined for smooth functions f, g to all $f, g \in \mathcal{C}^\gamma$.

PROPOSITION 2 (Young’s integral). *Let $X, Y \in \mathcal{C}^\gamma$, then $\int_\eta^\xi X_\rho dY_\rho$ is well defined, coincides with the Riemann–Stieltjes integral when the latter exists and satisfies the following bound:*

$$\left| \int_\eta^\xi (X_\rho - X_\eta) dY_\rho \right| \leq C_\gamma \|X\|_\gamma \|Y\|_\gamma |\xi - \eta|^{2\gamma}$$

for all $\xi, \eta \in [0, 1]$, where $C_\gamma \geq 1$ is a constant depending only on γ .

PROOF. See, for example, [18, 25]. \square

It will be convenient to introduce the integrated form of (8) as

$$(13) \quad Y(t)_\xi = X_\xi + \int_0^t V^{Y(s)}(Y(s)_\xi) ds.$$

Consider the Banach space $\mathbf{X}_T := C([0, T], \mathcal{C}^\gamma)$, with norm

$$\|Y\|_{\mathbf{X}_T} := \sup_{t \in [0, T]} \|Y(t)\|_\gamma^*, \quad Y \in \mathbf{X}_T.$$

Solutions of (13) will then be found as fixed points of the nonlinear map $F : \mathbf{X}_T \rightarrow \mathbf{X}_T$ defined as

$$(14) \quad F(Y)(t)_\xi := X_\xi + \int_0^t V^{Y(s)}(Y(s)_\xi) ds, \quad t \in [0, T], \xi \in [0, 1],$$

where the application $Z \mapsto V^Z$ is defined for any $Z \in \mathcal{C}^\gamma$ as in (9) with the line integral understood according to Proposition 2 and with the matrix field A satisfying regularity conditions which will be shortly specified.

On m -tensor field $\varphi : \mathbb{R}^3 \rightarrow (\mathbb{R}^3)^{\otimes m}$ and for any integer $n \geq 0$, we define the following norm:

$$\|\varphi\|_n := \sum_{k=0}^n \|\nabla^k \varphi\| \quad \text{where } \|\varphi\| := \sup_{x \in \mathbb{R}^3} |\varphi(x)|,$$

where the norm $|M|$ of a matrix $M \in \mathbb{R}^3 \otimes \mathbb{R}^3$ or, more generally, of an n -tensor $M \in (\mathbb{R}^3)^{\otimes n}$ is given by

$$|M| = \sum_{i_1=1}^3 \dots \sum_{i_n=1}^3 |M^{i_1 \dots i_n}|,$$

with $(M^{i_1 \dots i_n})_{i_1, \dots, i_n}$ the components of the tensor in the canonical basis of \mathbb{R}^3 .

Then we can state the following:

THEOREM 1. *Assume $\|\nabla A\|_2 < \infty$ and $X \in \mathcal{C}^\gamma$. Then there exists a time T_0 depending only on $\|\nabla A\|_1, \|X\|_\gamma^*, \gamma$ such that (13) has a unique solution bounded in \mathcal{C}^γ .*

PROOF. Consider the initial condition X fixed. According to Lemma 3 below, there exists $T_0 > 0$ and $B_{T_0} > 0$ depending on $\|\nabla A\|_1, \|X\|_\gamma^*, \gamma$ such that the set $\{Y \in \mathbf{X}_{T_0} : Y(0) = X, \|Y\|_{\mathbf{X}_{T_0}} \leq B_{T_0}\}$ is invariant under F . Lemma 4 then asserts that, provided $\|\nabla A\|_2 < \infty$, the map F is a strict contraction over a smaller time interval $[0, \bar{T}]$ with $\bar{T} \leq T_0$. Proceeding by induction on the intervals $[0, \bar{T}], [\bar{T}, 2\bar{T}]$, and so on \dots , it is possible to construct the unique solution of the evolution problem up to the time T_0 . \square

Before giving the lemmas used in the proof, we will state two useful results. The first is just a straightforward computation on Hölder functions, the second will allow to control the velocity field V^Y in terms of the regularity of Y and of A .

LEMMA 1. *Let $Y, \tilde{Y} \in \mathcal{C}^\gamma$ and $\varphi \in C^2$, then*

$$(15) \quad \|\varphi(Y)\|_\gamma \leq \|\nabla\varphi\| \|Y\|_\gamma$$

and

$$(16) \quad \|\varphi(Y) - \varphi(\tilde{Y})\|_\gamma \leq \|\nabla\varphi\|_1 (1 + \|Y\|_\gamma) \|Y - \tilde{Y}\|_\gamma.$$

PROOF. See Section A.1. \square

LEMMA 2. *Let $Y, \tilde{Y} \in \mathcal{C}^\gamma$. For any integer $n \geq 0$, the following estimates holds:*

$$(17) \quad \|\nabla^n V^Y\| \leq C_\gamma \|\nabla^{n+1} A\| \|Y\|_\gamma^2$$

$$(18) \quad \begin{aligned} &\|\nabla^n V^Y - \nabla^n V^{\tilde{Y}}\| \\ &\leq C_\gamma \|\nabla^{n+1} A\|_1 (\|Y\|_\gamma + \|\tilde{Y}\|_\gamma + \|\tilde{Y}\|_\gamma \|Y\|_\gamma) \|Y - \tilde{Y}\|_\gamma^*. \end{aligned}$$

PROOF. See Section A.2. \square

3.2. Local existence and uniqueness.

LEMMA 3. *Assume $\|\nabla A\|_1 < \infty$. For any initial datum $X \in \mathcal{C}^\gamma$, there exists a time $T_0 > 0$, such that for any time $T < T_0$, the set*

$$Q_T := \{Y \in \mathbf{X}_T : Y(0) = X, \|Y\|_{\mathbf{X}_T} \leq B_T\},$$

where B_T is a suitable constant, is invariant under F .

PROOF. Let us compute

$$|F(Y)(t)_\xi| \leq |X_\xi| + \int_0^t |V^{Y(s)}(Y(s)_\xi)| ds \leq |X_\xi| + \int_0^t \|V^{Y(s)}\|_\infty ds$$

so

$$\begin{aligned} \|F(Y)(t)\|_\infty &\leq \|X\|_\infty + \int_0^t \|V^{Y(s)}\| ds \\ &\leq \|X\|_\infty + C_\gamma \|\nabla A\| \int_0^T \|Y(s)\|_\gamma^2 ds \\ &\leq \|X\|_\infty + TC_\gamma \|\nabla A\| \|Y\|_{\mathbf{X}_T}^2. \end{aligned}$$

The γ -Hölder norm of the path $F(Y)(t)$ can be estimated in a similar fashion:

$$\begin{aligned} \|F(Y)(t)\|_\gamma &\leq \|X\|_\gamma + \int_0^t \|V^{Y(s)}(Y(s), \cdot)\|_\gamma ds \\ &\leq \|X\|_\gamma + \int_0^t \|\nabla V^{Y(s)}\| \|Y(s)\|_\gamma ds \\ &\leq \|X\|_\gamma + C_\gamma \|\nabla^2 A\| \int_0^T \|Y(s)\|_\gamma^3 ds \\ &\leq \|X\|_\gamma + TC_\gamma \|\nabla^2 A\| \|Y\|_{\mathbf{X}_T}^3, \end{aligned}$$

where we used (15) in the second line and (17) in the third.

Then

$$\|F(Y)\|_{\mathbf{X}_T} \leq \|X\|_\gamma^* + C_\gamma T \|\nabla A\|_1 \|Y\|_{\mathbf{X}_T}^2 (1 + \|Y\|_{\mathbf{X}_T}).$$

Let B_T be a solution of

$$B_T \leq \|X\|_\gamma^* + C_\gamma T \|\nabla A\|_1 B_T^2 (1 + B_T)$$

which exists for any $T \leq T_0$, where T_0 is a constant depending only on $\|X\|_\gamma^*$, $\|\nabla A\|_1$ and γ . Then if $\|Y\|_{\mathbf{X}_T} \leq B_T$, we have $\|F(Y)\|_{\mathbf{X}_T} \leq B_T$ and Q_T is invariant under F . \square

Given another initial condition $\tilde{X} \in \mathcal{C}^\gamma$, consider the associated map

$$(19) \quad \tilde{F}(Y)(t)_\xi = \tilde{X}_\xi + \int_0^t V^{Y(s)}(Y(s)_\xi) ds, \quad t \in [0, T], \xi \in [0, 1].$$

LEMMA 4. Assume $\|\nabla A\|_2 < \infty$. We have

$$(20) \quad \begin{aligned} &\|F(Y) - \tilde{F}(\tilde{Y})\|_{\mathbf{X}_T} \\ &\leq \|X - \tilde{X}\|_\gamma^* + C_\gamma T \|\nabla A\|_2 (1 + \|Y\|_{\mathbf{X}_T} + \|\tilde{Y}\|_{\mathbf{X}_T})^3 \|Y - \tilde{Y}\|_{\mathbf{X}_T}. \end{aligned}$$

In particular, there exists a time $\bar{T} \leq T_0$ such that the map F is a strict contraction on $Q_{\bar{T}}$.

PROOF. We proceed as follows: take $Y, \tilde{Y} \in \mathcal{C}^\gamma$, then

$$\begin{aligned} & \|F(Y)(t) - \tilde{F}(\tilde{Y})(t)\|_\infty \\ & \leq \|X - \tilde{X}\|_\infty + \int_0^T [\|V^{Y(s)}(Y(s).\!) - V^{Y(s)}(\tilde{Y}(s).\!)\|_\infty \\ & \qquad \qquad \qquad + \|V^{Y(s)}(\tilde{Y}(s).\!) - V^{\tilde{Y}(s)}(\tilde{Y}(s).\!)\|_\infty] ds \\ & \leq \|X - \tilde{X}\|_\infty + \int_0^T [\|\nabla V^{Y(s)}\| \|Y(s) - \tilde{Y}(s)\|_\infty + \|V^{Y(s)} - V^{\tilde{Y}(s)}\|] ds \\ & \leq \|X - \tilde{X}\|_\infty + C_\gamma \int_0^T [\|\nabla^2 A\| \|Y(s)\|_\gamma^2 \|Y(s) - \tilde{Y}(s)\|_\infty \\ & \qquad \qquad \qquad + \|\nabla A\|_1 (\|Y(s)\|_\gamma + \|\tilde{Y}(s)\|_\gamma \\ & \qquad \qquad \qquad \qquad \qquad \qquad + \|\tilde{Y}(s)\|_\gamma \|Y(s)\|_\gamma) \|Y(s) - \tilde{Y}(s)\|_\gamma^*] ds \\ & \leq \|X - \tilde{X}\|_\infty + C_\gamma T \|\nabla A\|_1 \|Y - \tilde{Y}\|_{\mathbf{x}_T} (1 + \|Y\|_{\mathbf{x}_T} + \|\tilde{Y}\|_{\mathbf{x}_T})^2 \end{aligned}$$

and

$$\begin{aligned} & \|F(Y)(t) - \tilde{F}(\tilde{Y})(t)\|_\gamma \\ & \leq \|X - \tilde{X}\|_\gamma + \int_0^T [\|V^{Y(s)}(Y(s).\!) - V^{Y(s)}(\tilde{Y}(s).\!)\|_\gamma \\ & \qquad \qquad \qquad + \|V^{Y(s)}(\tilde{Y}(s).\!) - V^{\tilde{Y}(s)}(\tilde{Y}(s).\!)\|_\gamma] ds \\ & \leq \|X - \tilde{X}\|_\gamma + \int_0^T [\|\nabla V^{Y(s)}\|_1 \|Y(s) - \tilde{Y}(s)\|_\gamma^* (1 + \|Y(s)\|_\gamma) \\ & \qquad \qquad \qquad \qquad \qquad \qquad + \|\nabla V^{Y(s)} - \nabla V^{\tilde{Y}(s)}\| \|\tilde{Y}(s)\|_\gamma] ds \\ & \leq \|X - \tilde{X}\|_\gamma \\ & \qquad \qquad \qquad + C_\gamma \int_0^T [\|\nabla^2 A\|_1 \|Y(s)\|_\gamma^2 (1 + \|Y(s)\|_\gamma) \|Y(s) - \tilde{Y}(s)\|_\gamma^* \\ & \qquad \qquad \qquad \qquad \qquad \qquad + \|\nabla^2 A\|_1 (\|Y(s)\|_\gamma + \|\tilde{Y}(s)\|_\gamma + \|\tilde{Y}(s)\|_\gamma \|Y(s)\|_\gamma) \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \times \|\tilde{Y}(s)\|_\gamma \|Y(s) - \tilde{Y}(s)\|_\gamma^*] ds \\ & \leq \|X - \tilde{X}\|_\gamma + C_\gamma T \|\nabla^2 A\|_1 \|Y - \tilde{Y}\|_{\mathbf{x}_T} (1 + \|Y\|_{\mathbf{x}_T} + \|\tilde{Y}\|_{\mathbf{x}_T})^3. \end{aligned}$$

Putting together the two estimates, we get

$$\|F(Y) - F(\tilde{Y})\|_{\mathbf{x}_T} \leq \|X - \tilde{X}\|_\gamma^* + C_\gamma T \|\nabla A\|_2 (1 + \|Y\|_{\mathbf{x}_T} + \|\tilde{Y}\|_{\mathbf{x}_T})^3 \|Y - \tilde{Y}\|_{\mathbf{x}_T}.$$

There exists $\bar{T} \leq T_0$ such that

$$C_\gamma \bar{T} \|\nabla A\|_2 (1 + 2B_{\bar{T}})^3 =: \alpha < 1,$$

then if $Y, \tilde{Y} \in Q_{\bar{T}}$, we have

$$\|F(Y) - F(\tilde{Y})\|_{\mathbf{X}_T} \leq C_\gamma T \|\nabla A\|_2 (1 + 2B_{\bar{T}})^3 \|Y - \tilde{Y}\|_{\mathbf{X}_T} = \alpha \|Y - \tilde{Y}\|_{\mathbf{X}_T}$$

so that F is a strict contraction on $Q_{\bar{T}}$ with a unique fixed-point. \square

REMARK 5. Here and in the proofs for the case $\gamma > 1/3$, some conditions on A can be slightly relaxed using better estimates. For example, in the proof of Lemma 3 the condition $\|\nabla A\|_1 < \infty$ can be relaxed to require ∇A to be a Hölder continuous function of index $(1 - \gamma + \epsilon)/\gamma$ for some $\epsilon > 0$. However, these refinements are not able to improve qualitatively the results.

3.3. *Dependence on the initial condition.* Denote with $T_0(\|X\|_\gamma^*)$ the existence time of the solution build in Theorem 1, where we have considered explicitly its dependence on the norm of the initial condition. For any $r > 0$, let $\mathcal{B}(\mathcal{C}^\gamma; r)$, the open ball of \mathcal{C}^γ with radius r and centered in zero. Now, fix $r > 0$ and let $\Gamma : \mathcal{B}(\mathcal{C}^\gamma; r) \rightarrow \mathbf{X}_{T_0}$, where $T_0 = T_0(r)$, denote the solution of the evolution problem starting from the initial condition $X \in \mathcal{B}(\mathcal{C}^\gamma; r)$ and living up to time $T_0(r)$.

THEOREM 2. *Under the conditions of Theorem 1, the map $X \mapsto \Gamma(X)$ is Lipschitz.*

PROOF. Consider two initial conditions $X, \tilde{X} \in \mathcal{B}(\mathcal{C}^\gamma; r)$. Note that $F(\Gamma(X)) = \Gamma(X)$ in \mathbf{X}_{T_0} . By Lemma 4, we have that, for $T < T_0$,

$$\begin{aligned} & \|F(\Gamma(X)) - \tilde{F}(\Gamma(\tilde{X}))\|_{\mathbf{X}_T} \\ &= \|F(\Gamma(X)) - \tilde{F}(\Gamma(\tilde{X}))\|_{\mathbf{X}_T} \\ &\leq \|X - \tilde{X}\|_\gamma^* \\ &\quad + C_\gamma T \|\nabla A\|_2 (1 + \|\Gamma(X)\|_{\mathbf{X}_T} + \|\Gamma(\tilde{X})\|_{\mathbf{X}_T})^3 \|\Gamma(X) - \Gamma(\tilde{X})\|_{\mathbf{X}_T}. \end{aligned}$$

Since the norm of the initial condition is bounded by r in \mathcal{C}^γ , by Lemma 3, there exists a constant $B_{T_0}(r)$ such that $\|\Gamma(X)\|_{\mathbf{X}_{T_0}} \leq B_{T_0}(r)$ for any $X \in \mathcal{B}(\mathcal{C}^\gamma; r)$. Then for $T < T_0$ sufficiently small, we have

$$\|\Gamma(X) - \Gamma(\tilde{X})\|_{\mathbf{X}_T} \leq (1 - \alpha)^{-1} \|X - \tilde{X}\|_\gamma^*,$$

where

$$\alpha := C_\gamma T \|\nabla A\|_2 (1 + \|\Gamma(X)\|_{\mathbf{X}_T} + \|\Gamma(\tilde{X})\|_{\mathbf{X}_T})^3 < 1;$$

which gives the Lipschitz continuity of the map on \mathbf{X}_T . By an easy induction argument, it follows the Lipschitz continuity on all \mathbf{X}_{T_0} (see, e.g., [16]). \square

3.4. *Blow-up estimate.* From the previous results, it is clear that if the norm $\|Y(t)\|_\gamma$ of a solution Y with initial condition $X \in \mathcal{C}^\gamma$ is bounded by some number M in some interval $[0, \bar{T}]$, then the solution can be extended on a strictly larger interval $[0, \bar{T} + \delta_M]$ with δ_M depending only on M (and on the data of the problem). This implies that the only case in which we cannot find a global solution (for any positive time) is when there is some time $\hat{t}_\gamma(X)$ such that $\lim_{t \rightarrow \hat{t}_\gamma(X)^-} \|Y(t)\|_\gamma = +\infty$. This time is an *epoch of irregularity* for the evolution in the class \mathcal{C}^γ . Near this epoch we can establish a lower bound for the norm $\|Y(t)\|_\gamma$.

PROPOSITION 3. Assume $\hat{t}_\gamma(X) > 0$ is the smallest epoch of irregularity for a solution Y in the class \mathcal{C}^γ . Then we have

$$(21) \quad \|Y(t)\|_\gamma \geq \frac{C}{(\hat{t}_\gamma(X) - t)^{1/2}}$$

for any $t \in [0, \hat{t}_\gamma(X))$.

PROOF.

$$(22) \quad \begin{aligned} \|Y(t)\|_\gamma - \|Y(s)\|_\gamma &\leq \int_s^t \|V^{Y(u)}(Y(u))\|_\gamma \, du \\ &\leq \int_s^t \|\nabla V^{Y(u)}\|_\infty \|Y(u)\|_\gamma \, du, \end{aligned}$$

and using Lemma 2, we have

$$\|Y(t)\|_\gamma - \|Y(s)\|_\gamma \leq C \int_s^t \|Y(u)\|_\gamma^3 \, du$$

for some constant C depending only on A and γ , so that

$$\frac{d}{dt} \|Y(t)\|_\gamma \leq C \|Y(t)\|_\gamma^3,$$

letting $y(t) = \|Y(t)\|_\gamma$ and integrating the differential inequality between times $t > s$, we obtain

$$\frac{1}{y(s)^2} - \frac{1}{y(t)^2} \leq 2C(t - s).$$

Now, assume that there exists a time $\hat{t}_\gamma(X)$ such that $\lim_{t \rightarrow \hat{t}_\gamma(X)^-} y(t) = +\infty$, then, for any $s < \hat{t}_\gamma(X)$, we have the following lower bound for the explosion of the \mathcal{C}^γ norm of Y :

$$\|Y(s)\|_\gamma = y(s)^{1/2} \geq \frac{1}{(2C)^{1/2}(\hat{t}_\gamma(X) - s)^{1/2}}. \quad \square$$

Estimate (22) used in the previous proof implies also that

$$(23) \quad z(t) \leq z(0) + \int_0^t \|\nabla V^{Y(s)}\|_\infty z(s) ds,$$

where $z(t) = \sup_{s \in [0,t]} \|Y(s)\|_\gamma$. By the Gronwall lemma,

$$z(t) \leq z(0) \exp\left(\int_0^t \|\nabla V^{Y(s)}\|_\infty ds\right).$$

This bound allows the continuation of any solution on the interval $[0, t]$ if the integral $\int_0^t \|\nabla V^{Y(s)}\|_\infty ds$ is finite. Then if $\hat{t}_\gamma(X)$ is the first irregularity epoch in the class \mathcal{C}^γ , we must have that $\hat{t}_\gamma(X) = \hat{t}(X) = \sup_{1/2 < \gamma \leq 1} \hat{t}_\gamma(X)$ for any $1/2 < \gamma \leq 1$. Indeed, is easy to see that, for any $t < \hat{t}(X)$, there exists a finite constant M_t such that $\sup_{s \in [0,t]} \|\nabla V^{Y(s)}\|_\infty \leq M_t$.

COROLLARY 1. *Let $X \in \mathcal{C}^{\gamma^*}$ with $\gamma^* > 1/2$, then, for any $1/2 < \gamma \leq \gamma^*$, there exists a unique solution $Y^\gamma \in C([0, \hat{t}_\gamma(X)), \mathcal{C}^\gamma)$ with initial condition X . Moreover, the first irregularity epoch $\hat{t}_\gamma(X)$ for the solution in \mathcal{C}^γ does not depend on $\gamma \geq \gamma^*$.*

4. Evolution for $\gamma > 1/3$.

4.1. *Rough path-integrals.* When $\gamma \leq 1/2$, there are difficulties in defining the path-integral appearing in (9) for the velocity field V . A successful approach to such irregular integrals has been found by Lyons to consist in enriching the notion of *path* (see, e.g., [18, 20] and for some recent contributions [6, 13, 16]).

For any $\gamma > 1/3$, a γ -*rough path* (of degree two) is a couple $\mathbb{X} = (X, \mathbb{X}^2)$, where $X \in \mathcal{C}^\gamma$ and $\mathbb{X}^2 \in C([0, 1]^2, \mathbb{R}^3 \otimes \mathbb{R}^3)$ is a matrix-valued function (called the *area process*) on the square $[0, 1]^2$ verifying the following compatibility condition with X :

$$(24) \quad \mathbb{X}_{\xi\rho}^{2,ij} - \mathbb{X}_{\xi\eta}^{2,ij} - \mathbb{X}_{\eta\rho}^{2,ij} = (X_\xi^i - X_\eta^i)(X_\eta^j - X_\rho^j), \quad \xi, \eta, \rho \in [0, 1]^2,$$

($i, j = 1, 2, 3$ are vector indexes) and such that

$$(25) \quad \|\mathbb{X}^2\|_{2\gamma} := \sup_{\xi, \eta \in [0,1]} \frac{|\mathbb{X}_{\xi\eta}^2|}{|\xi - \eta|^{2\gamma}} < \infty.$$

REMARK 6. If $\gamma > 1/2$, then a natural choice for the area process \mathbb{X}^2 is the *geometric* one given by

$$(26) \quad (\mathbb{X}_{\xi\eta}^2)_{\text{geom}}^{ij} = \int_\xi^\eta (X_\rho - X_\xi)^i dX_\rho^j,$$

which naturally satisfy (24) (as can be directly checked) and (25) (using Lemma 2).

As shown by Lyons [18], when $\gamma > 1/3$, any integral of the form

$$\int \varphi(X) dX$$

can be defined to depend in a continuous way on the γ -rough path (X, \mathbb{X}^2) for sufficiently regular φ .

In [16] it is pointed out that any γ -rough path \mathbb{X} define a natural class of paths for which path-integrals are meaningful. Define the Banach space \mathcal{D}_X of paths weakly-controlled by X as the set of paths Y that can be decomposed as

$$(27) \quad Y_\xi - Y_\eta = Y'_\eta(X_\xi - X_\eta) + R^Y_{\eta\xi},$$

with $Y' \in C^\gamma([0, 1], \mathbb{R}^3 \otimes \mathbb{R}^3)$ and $\|R^Y\|_{2\gamma} < \infty$. Define the norm for $Y \in \mathcal{D}_X$ as

$$\|Y\|_D := \|Y'\|_\gamma + \|R^Y\|_{2\gamma} + \|Y'\|_\infty;$$

moreover, let

$$\|Y\|_D^* := \|Y\|_D + \|Y\|_\infty.$$

Since we will need to consider only closed paths, we will require for $Y \in \mathcal{D}_X$ that $Y_0 = Y_1$. Then it is easy to show that

$$\|Y\|_\gamma \leq \|Y\|_D(1 + \|X\|_\gamma)$$

and that $\mathcal{D}_X \subseteq \mathcal{C}^\gamma$.

The next lemma states that \mathcal{D}_X behaves nicely under maps by regular functions:

LEMMA 5. *If φ is a C^2 function and $Y \in \mathcal{D}_X$, then $\varphi(Y) \in \mathcal{D}_X$ with $\varphi(Y)' = \nabla\varphi(Y)Y'$ and there exists a constant $K \geq 1$ such that*

$$(28) \quad \|\varphi(Y)\|_D \leq K \|\nabla\varphi\|_1 \|Y\|_D(1 + \|Y\|_D)(1 + \|X\|_\gamma)^2.$$

Moreover, if $Y, \tilde{Y} \in \mathcal{D}_X$, we have

$$(29) \quad \begin{aligned} &\|\varphi(Y) - \varphi(\tilde{Y})\|_D \\ &\leq K \|\nabla\varphi\|_2 \|Y\|_D(1 + \|Y\|_D + \|\tilde{Y}\|_D)^2(1 + \|X\|_\gamma)^4 \|Y - \tilde{Y}\|_D. \end{aligned}$$

PROOF. See [16], Proposition 4. \square

The main result about weakly-controlled paths is that they can be integrated one against the other with a good control of the resulting object:

LEMMA 6 (Integration of weakly-controlled paths). *If $Y, Z \in \mathcal{D}_X$, then the integral*

$$\int_\xi^\eta Y dZ := Y_\xi(Z_\eta - Z_\xi) + Y'_\xi Z'_\xi \mathbb{X}^2_{\eta\xi} + Q_{\xi\eta}, \quad \eta, \xi \in [0, 1]$$

is well defined with

$$\|Q\|_{3\gamma} \leq C'_\gamma C_X \|Y\|_D \|Z\|_D,$$

where $C'_\gamma > 1$ and

$$C_X = (1 + \|X\|_\gamma + \|\mathbb{X}^2\|_{2\gamma}).$$

The integral $\int_\xi^\eta Y dZ$ is the limit of the following “renormalized” finite sums:

$$\sum_{i=0}^{n-1} [Y_{\xi_i} (Z_{\xi_{i+1}} - Z_{\xi_i}) + Y'_{\xi_i} Z'_{\xi_i} \mathbb{X}^2_{\xi_{i+1}\xi_i}],$$

(where $\xi_0 = \xi < \xi_1 < \dots < \xi_n = \eta$ is a finite partition of $[\xi, \eta]$) as the size of the partition goes to zero.

Moreover, if $\tilde{Y}, \tilde{Z} \in \mathcal{D}_{\tilde{X}}$, then

$$\begin{aligned} & \int_\xi^\eta Y dZ - \int_\xi^\eta \tilde{Y} d\tilde{Z} \\ &= Y_\xi (Z_\eta - Z_\xi) - \tilde{Y}_\xi (\tilde{Z}_\eta - \tilde{Z}_\xi) + (Y'_\xi Z'_\xi - \tilde{Y}'_\xi \tilde{Z}'_\xi) \mathbb{X}^2_{\eta\xi} + Q_{\xi\eta} - \tilde{Q}_{\xi\eta} \end{aligned}$$

and

$$\|Q - \tilde{Q}\|_{3\gamma} \leq C'_\gamma C_X (\|Y\|_{D\epsilon_Y} + \|Z\|_{D\epsilon_Z})$$

with

$$\begin{aligned} \epsilon_Y &= \|Y' - \tilde{Y}'\|_\infty + \|Y' - \tilde{Y}'\|_\gamma + \|R^Y - R^{\tilde{Y}}\|_{2\gamma}, \\ \epsilon_Z &= \|Z' - \tilde{Z}'\|_\infty + \|Z' - \tilde{Z}'\|_\gamma + \|R^Z - R^{\tilde{Z}}\|_{2\gamma}. \end{aligned}$$

PROOF. See [16], Theorem 1. \square

A weakly-controlled path $(Y, Y') \in \mathcal{D}_X$ can be naturally lifted to a γ -rough path by setting

$$(30) \quad \mathbb{Y}^{2,ij}_{\rho\xi} := \int_\rho^\xi (Y_\eta^i - Y_\rho^i) dY_\eta^j,$$

where the integral is the rough integral in Lemma 6.

4.2. *Local existence and uniqueness.* Given $T > 0$, consider the Banach space $\mathbf{D}_{X,T} = C([0, T], \mathcal{D}_X)$ endowed with the norm

$$\|Y\|_{\mathbf{D}_{X,T}} := \sup_{t \in [0, T]} \|Y(t)\|_D^*$$

and, as above, the application $F : \mathbf{D}_{X,T} \rightarrow \mathbf{D}_{X,T}$ defined as

$$F(Y)(t)_\xi := X_\xi + \int_0^t V^{Y(s)}(Y(s)_\xi) ds$$

with

$$(31) \quad V^Y(x) := \int_0^1 A(x - Y_\eta) dY_\eta$$

understood as a rough integral. Lemma 5 guarantees, under suitable smoothness of the function $x \mapsto V^{Y(s)}(x)$, that $F(Y) \in \mathbf{D}_{X,T}$ if $Y \in \mathbf{D}_{X,T}$ with $F(Y)'$ given by

$$(32) \quad [F(Y)(t)']_\xi^{ij} := \delta_{ij} + \sum_{k=1}^3 \int_0^t \nabla_k [V^{Y(s)}(Y(s)_\xi)]^i [Y(s)'_\xi]^{kj} ds,$$

$i, j = 1, 2, 3,$

where δ_{ij} is the Kronecker symbol.

A result proven in [16], page 103, implies that the rough integral in (33) can be indifferently understood as an integral with respect to the weakly-controlled path $Y \in \mathcal{D}_X$ or as an integral over the lifted rough path (Y, \mathbb{Y}^2) , where \mathbb{Y}^2 is defined as in (30).

We will state now the main result of this section, namely, the existence and uniqueness of solutions to the vortex line equation in the space \mathcal{D}_X .

THEOREM 3. *Assume $\|\nabla A\|_4 < \infty$ and \mathbb{X} is a γ -rough path. Then there exists a time $T_0 > 0$ depending only on $\|\nabla A\|_3, X, \mathbb{X}, \gamma$ such that (13) has a unique solution bounded in \mathcal{D}_X for any $T \leq T_0$.*

PROOF. Lemmas 8 and 9 prove that, on a small enough time interval $[0, T]$, the map F is a strict contraction on some ball of $\mathbf{D}_{X,T}$ having a unique fixed point. The arguments are similar to those used in the case $\gamma > 1/2$. \square

Since V is defined through rough integrals, we can obtain the following bounds on its regularity:

LEMMA 7. *Let $Y, \tilde{Y} \in \mathcal{D}_X$, for any integer $n \geq 0$,*

$$(33) \quad \|\nabla^n V^Y\| \leq 4C'_\gamma \|\nabla^{n+1} A\|_1 C_X^3 \|Y\|_D^2 (1 + \|Y\|_D)$$

and

$$(34) \quad \|\nabla^n V^Y - \nabla^n V^{\tilde{Y}}\| \leq 16C'_\gamma C_X^3 \|\nabla^{n+1} A\|_2 \|Y - \tilde{Y}\|_D^* (1 + \|Y\|_D)^2 \|Y\|_D.$$

PROOF. See Section A.3. \square

LEMMA 8. *Assume $\|\nabla A\|_3 < \infty$. For any initial γ -rough path \mathbb{X} with $\gamma > 1/3$, there exists a time T_0 such that, for any time $T \leq T_0$, the set*

$$Q_T := \{Y \in \mathbf{D}_{X,T} : Y(0) = X, \|Y\|_{\mathbf{D}_{X,T}} \leq B_T\},$$

where B_T is a suitable constant, is invariant under F .

PROOF. Fix a time $T > 0$. First of all we have, for any $t \in [0, T]$,

$$\begin{aligned} |F(Y)(t)_\xi| &\leq |X_\xi| + \int_0^t \|V^{Y(s)}\|_\infty ds \\ &\leq \|X\|_\infty + 4C'_\gamma \|\nabla A\|_1 C_X^3 \int_0^t ds \|Y(s)\|_D^2 (1 + \|Y(s)\|_D) \\ &\leq \|X\|_\infty + 4C'_\gamma T C_X^3 \|\nabla A\|_1 \|Y\|_{\mathbf{D}_{X,T}}^2 (1 + \|Y\|_{\mathbf{D}_{X,T}}) \end{aligned}$$

so that

$$\sup_{t \in [0, T]} \|F(Y)(t)\| \leq \|X\|_\infty + 4TC'_\gamma C_X^3 \|\nabla A\|_1 \|Y\|_{\mathbf{D}_{X,T}}^2 (1 + \|Y\|_{\mathbf{D}_{X,T}}).$$

Next,

$$\begin{aligned} \|F(Y)(t)\|_D &\leq \|X\|_D + \int_0^t \|V^{Y(s)}(Y(s))\|_D ds \\ &\leq \|X\|_D + KC_X^2 \int_0^t \|\nabla V^{Y(s)}\|_1 \|Y(s)\|_D (1 + \|Y(s)\|_D) ds, \end{aligned}$$

where we used Lemma 5.

Lemma 7 gives then

$$\|F(Y)(t)\|_D \leq \|X\|_D + 16KC'_\gamma C_X^5 \int_0^t \|Y(s)\|_D^3 (1 + \|Y(s)\|_D)^2 ds,$$

from which we easily obtain

$$\begin{aligned} \|F(Y)\|_{\mathbf{D}_{X,T}} &\leq \|X\|_D^* + C'_\gamma T [16KC_X^5 \|Y\|_{\mathbf{D}_{X,T}}^3 (1 + \|Y\|_{\mathbf{D}_{X,T}})^2 \\ &\quad + 4C_X^3 \|Y\|_{\mathbf{D}_{X,T}}^2 (1 + \|Y\|_{\mathbf{D}_{X,T}})] \\ &\leq \|X\|_\infty + 1 + 20KC'_\gamma C_X^5 T \|\nabla A\|_3 \|Y\|_{\mathbf{D}_{X,T}}^2 (1 + \|Y\|_{\mathbf{D}_{X,T}})^3 \end{aligned}$$

and for T small enough ($T \leq T_0$ with T_0 depending only on \mathbb{X} and $\|\nabla A\|_3$), we have that there exists a constant B_T such that if $\|Y\|_{\mathbf{D}_{X,T}} \leq B_T$, then $\|F(Y)\|_{\mathbf{D}_{X,T}} \leq B_T$. \square

LEMMA 9. Assume $\|\nabla A\|_4 < \infty$. There exists a time $\bar{T} \leq T_0$ such that the map F is a strict contraction on $Q_{\bar{T}}$.

PROOF. Let $Z(t) := F(Y)(t)$ and $\tilde{Z}(t) := F(\tilde{Y})(t)$, with $Y, \tilde{Y} \in Q_T$. Let $H := Z - \tilde{Z}$. We start with the estimation of the sup norm of $H(t)$:

$$\begin{aligned} H(t)_\xi &= Z(t)_\xi - \tilde{Z}(t)_\xi = \int_0^t ds [V^{Y(s)}(Y(s)_\xi) - V^{\tilde{Y}(s)}(\tilde{Y}(s)_\xi)] \\ &= \int_0^t ds [V^{Y(s)}(Y(s)_\xi) - V^{Y(s)}(\tilde{Y}(s)_\xi) + V^{Y(s)}(\tilde{Y}(s)_\xi) - V^{\tilde{Y}(s)}(\tilde{Y}(s)_\xi)], \end{aligned}$$

which gives

$$\begin{aligned}
 \|H(t)\|_\infty &\leq \int_0^t ds [\|\nabla V^{Y(s)}\|_\infty \|Y(s) - \tilde{Y}(s)\|_\infty + \|V^{Y(s)} - V^{\tilde{Y}(s)}\|_\infty] \\
 &\leq C'_\gamma T \|Y - \tilde{Y}\|_{\mathbf{D}_{X,T}} [4C_X^3 \|\nabla^2 A\|_1 \|Y\|_{\mathbf{D}_{X,T}}^2 (1 + \|Y\|_{\mathbf{D}_{X,T}}) \\
 &\qquad\qquad\qquad + 16C_X^3 \|\nabla A\|_2 \|Y\|_{\mathbf{D}_{X,T}} (1 + \|Y\|_{\mathbf{D}_{X,T}})^2] \\
 &\leq 20C'_\gamma T C_X^3 \|\nabla A\|_2 (1 + \|Y\|_{\mathbf{D}_{X,T}})^3 \|Y - \tilde{Y}\|_{\mathbf{D}_{X,T}}.
 \end{aligned}
 \tag{35}$$

Next we need to estimate the \mathcal{D}_X norm of $H(t)$:

$$\begin{aligned}
 \|H(t)\|_D &\leq \int_0^T \|V^{Y(s)}(Y(s,\cdot)) - V^{\tilde{Y}(s)}(\tilde{Y}(s,\cdot))\|_D ds \\
 &\leq \int_0^T [\|V^{Y(s)}(Y(s,\cdot)) - V^{\tilde{Y}(s)}(Y(s,\cdot))\|_D \\
 &\qquad\qquad\qquad + \|V^{\tilde{Y}(s)}(Y(s,\cdot)) - V^{\tilde{Y}(s)}(\tilde{Y}(s,\cdot))\|_D] ds.
 \end{aligned}$$

Next we estimate the first contribution in the integral by using Lemmas 5 and 7 as

$$\begin{aligned}
 &\|V^{Y(s)}(Y(s,\cdot)) - V^{\tilde{Y}(s)}(Y(s,\cdot))\|_D \\
 &\leq KC_X^2 \|\nabla V^{Y(s)} - \nabla V^{\tilde{Y}(s)}\|_1 \|Y(s)\|_D (1 + \|Y(s)\|_D) \\
 &\leq 32KC'_\gamma C_X^5 \|\nabla^2 A\|_3 \|Y(s) - \tilde{Y}(s)\|_D^* \|Y(s)\|_D^2 (1 + \|Y(s)\|_D)^3 \\
 &\leq 32KC'_\gamma C_X^5 \|\nabla^2 A\|_3 \|Y - \tilde{Y}\|_{\mathbf{D}_{X,T}} \|Y\|_{\mathbf{D}_{X,T}}^2 (1 + \|Y\|_{\mathbf{D}_{X,T}})^3
 \end{aligned}$$

and the second as

$$\begin{aligned}
 &\|V^{\tilde{Y}(s)}(Y(s,\cdot)) - V^{\tilde{Y}(s)}(\tilde{Y}(s,\cdot))\|_D \\
 &\leq KC_X^4 \|\nabla V^{\tilde{Y}(s)}\|_2 (1 + \|\tilde{Y}(s)\|_D + \|Y(s)\|_D)^2 \|Y - \tilde{Y}\|_D \\
 &\leq 8KC'_\gamma \|\nabla^2 A\|_3 C_X^7 (1 + \|\tilde{Y}(s)\|_D + \|Y(s)\|_D)^3 \|\tilde{Y}(s)\|_D^2 \|Y(s) - \tilde{Y}(s)\|_D \\
 &\leq 8KC'_\gamma \|\nabla^2 A\|_3 C_X^7 (1 + \|\tilde{Y}\|_{\mathbf{D}_{X,T}} + \|Y\|_{\mathbf{D}_{X,T}})^3 \|\tilde{Y}\|_{\mathbf{D}_{X,T}}^2 \|Y - \tilde{Y}\|_{\mathbf{D}_{X,T}},
 \end{aligned}$$

giving

$$\|H(t)\|_D \leq 40TKC'_\gamma \|\nabla^2 A\|_3 C_X^7 (1 + \|Y\|_{\mathbf{D}_{X,T}} + \|Y\|_{\mathbf{D}_{X,T}})^5 \|Y - \tilde{Y}\|_{\mathbf{D}_{X,T}}.$$

Finally, collecting together the bounds,

$$\begin{aligned}
 &\|F(Y) - F(\tilde{Y})\|_{\mathbf{D}_{X,T}} \\
 &\leq 60TC'_\gamma KC_X^7 \|\nabla A\|_4 (1 + \|Y\|_{\mathbf{D}_{X,T}} + \|\tilde{Y}\|_{\mathbf{D}_{X,T}})^5 \|Y - \tilde{Y}\|_{\mathbf{D}_{X,T}}.
 \end{aligned}$$

When $Y, \tilde{Y} \in Q_T$, we have

$$\|F(Y) - F(\tilde{Y})\|_{\mathbf{D}_{X,T}} \leq 60TC'_\gamma KC_X^7 \|\nabla A\|_4 (1 + 2B_T)^5 \|Y - \tilde{Y}\|_{\mathbf{D}_{X,T}}.$$

Proving that for $\bar{T} < T_0$ small enough so that

$$60\bar{T}C'_\gamma KC_X^7 \|\nabla A\|_4 (1 + 2B_{\bar{T}})^5 < 1,$$

F is a contraction in the ball $Q_{\bar{T}} \subset \mathbf{D}_{X,\bar{T}}$. \square

REMARK 7. By imposing enough regularity on A and requiring that X can be completed to a geometric rough path with bounded p -variation (in the sense of Lyons), it is likely that the above proof of existence and uniqueness can be extended to cover the case of rougher initial conditions (e.g., paths living in some \mathcal{C}^γ with $\gamma < 1/3$ suitably lifted to rough paths).

REMARK 8. The solution (Y, Y') in $\mathbf{D}_{X,\bar{T}}$, satisfies [cf. with (32)]

$$(36) \quad Y(t)'_\eta = \text{Id} + \int_0^t \nabla V^{Y(s)}(Y(s)_\eta) Y'(s)_\eta ds$$

and, as can be easily verified,

$$(37) \quad \begin{aligned} & [R^Y(t)]^i_{\xi\eta} \\ &= \int_0^t ds \left[\nabla^k V^{Y(s)i}(Y(s)_\eta) [R^Y(s)]^k_{\xi\eta} \right. \\ & \quad \left. + \int_0^1 dr \int_0^r dw \right. \\ & \quad \left. \times \sum_{k,l=1,2,3} \nabla^k \nabla^l [V^{Y(s)}(Y(s)_\xi + wY(s)_{\xi\eta})]^i Y(s)^k_{\xi\eta} Y(s)^l_{\xi\eta} \right], \end{aligned}$$

where $Y(s)_{\xi\eta} := Y(s)_\eta - Y(s)_\xi$.

4.3. *Dependence on the initial condition.* Let Σ^γ be the set of γ -rough paths (X, \mathbb{X}^2) (with $\gamma > 1/3$). On Σ^γ consider the distance

$$d(\mathbb{X}, \tilde{\mathbb{X}}) = \|X - \tilde{X}\|_\gamma + \|\mathbb{X}^2 - \tilde{\mathbb{X}}^2\|_{2\gamma},$$

where $\mathbb{X} = (X, \mathbb{X}^2)$ and $\tilde{\mathbb{X}} = (\tilde{X}, \tilde{\mathbb{X}}^2)$ are two points in Σ^γ .

Fix $r > 0$ and let $\mathcal{B}(\Sigma^\gamma; r)$ be the open ball of Σ^γ with radius r centered at the trivial rough path $(0, 0)$. Let $T_0(r) > 0$ be the smaller existence time of solution to the evolution problem, as given by Theorem 8, for initial data living in $\mathcal{B}(\Sigma^\gamma; r)$.

Define the map $\Gamma : \mathcal{B}(\Sigma^\gamma; r) \rightarrow C([0, T_0], \Sigma^\gamma)$ as follows: for any $t \in [0, T_0]$ and any $\mathbb{X} = (X, \mathbb{X}^2) \in \mathcal{B}(\Sigma^\gamma; r)$, let $\Gamma(\mathbb{X}) := \mathbb{Y}$, where $\mathbb{Y} = (Y, \mathbb{Y}^2)$ with

$Y \in \mathbf{D}_{X, T_0}$ the solution starting at \mathbb{X} and \mathbb{Y}^2 the corresponding area process defined as

$$(38) \quad [\mathbb{Y}^2(t)]_{\rho\xi}^{ij} := \int_{\rho}^{\xi} (Y(t)_\eta^i - Y(t)_\rho^i) dY(t)_\eta^j$$

for any $t \in [0, T_0]$ where the integral is understood as an integral over the weakly-controlled path $Y(t) \in \mathcal{D}_X$.

Then

THEOREM 4. *The map Γ is Lipschitz from $\mathcal{B}(\Sigma^\gamma; r)$ to $C([0, T_0], \Sigma^\gamma)$ endowed with the uniform distance.*

PROOF. Take two initial conditions in $\mathcal{B}(\Sigma^\gamma; r)$: $\mathbb{X} = (X, \mathbb{X}^2)$ and $\tilde{\mathbb{X}} = (\tilde{X}, \tilde{\mathbb{X}}^2)$. Let $Y \in \mathbf{D}_{X, T_0}$ (resp. $\tilde{Y} \in \mathbf{D}_{\tilde{X}, T_0}$) be the solution starting from \mathbb{X} ($\tilde{\mathbb{X}}$).

Let

$$\Psi(t) := \|Y'(t) - \tilde{Y}'(t)\|_\gamma^* + \|R^Y(t) - R^{\tilde{Y}}(t)\|_{2\gamma}.$$

Using results from [16], it is not too difficult to prove that (cf. Lemmas 6 and 7)

$$(39) \quad \|\mathbb{Y}^2(t) - \tilde{\mathbb{Y}}^2(t)\|_{2\gamma} \leq D_1[d(X, \tilde{X}) + \Psi(t)]$$

and

$$(40) \quad \|\nabla^n V^{Y(t)} - \nabla^n V^{\tilde{Y}(t)}\| \leq D_2 \|\nabla^{n+1} A\|_2 [d(X, \tilde{X}) + \Psi(t)], \quad n \geq 0,$$

uniformly for $t \in [0, T_0]$, where here and in the following $D_k > 0$ are constants depending only on r and on γ .

At this point $\Psi(t)$ can be estimated using the expression given in (36) and (37) for $Y'(t)$, $\tilde{Y}'(t)$, $R^Y(t)$ and $R^{\tilde{Y}}(t)$ to obtain

$$\Psi(t) \leq D_3 \|\nabla A\|_4 \int_0^t [d(\mathbb{X}, \tilde{\mathbb{X}}) + \Psi(s)] ds.$$

For \bar{T} small enough so that $D_3 \|\nabla A\|_4 \bar{T} \leq 1/2$, we have

$$\sup_{t \in [0, \bar{T}]} \Psi(t) \leq d(\mathbb{X}, \tilde{\mathbb{X}}),$$

which implies that

$$\sup_{t \in [0, \bar{T}]} d(\mathbb{Y}(t), \tilde{\mathbb{Y}}(t)) \leq D_4 d(\mathbb{X}, \tilde{\mathbb{X}})$$

[using (39)]. Then again, a simple induction argument allows to extend this result to the whole interval $[0, T_0]$ proving the claim. \square

The continuity of Γ implies, in particular, that if we have a sequence of smooth loops $(X^{(n)})_{n \geq 1}$ which can be naturally lifted to a sequence of γ -rough paths

$(\mathbb{X}^{(n)})_{n \geq 1}$ [using (26) for the area process $\mathbb{X}^{(n),2}$] and such that it converges to the rough path \mathbb{X} (in the topology of Σ^γ), then the sequence of solutions $\Gamma(\mathbb{X}^{(n)})$ converge to the solution $\Gamma(\mathbb{X})$.

Since integrals over a smooth path $X^{(n)}$ coincide with integrals over the (geometrically lifted) rough path $\mathbb{X}^{(n)}$, we have that, for smooth initial conditions, classical solutions to the vortex line equation coincide with the projection of the solution $\Gamma(\mathbb{X}^{(n)})$ built in the space of rough paths.

Then the sequence of classical solutions (once lifted to a γ -rough path) converges in the sense of γ -rough paths (which is stronger than the γ -Hölder topology) to the solution $\Gamma(\mathbb{X})$.

4.4. *Dynamics of the covariations.* Recall the framework described in Section 2.2 on the covariation structure of the solution. If we assume that the initial condition (X, \mathbb{X}^2) is a random variable a.s. with values in the space of γ -rough paths (with $\gamma > 1/3$) and that X is a process with all its mutual covariations, then the solution $Y(t)$ at any instant of time t less than a random time T_0 (depending on the initial condition) is still a process with all its mutual covariations (due to Proposition 1).

The covariations of Y satisfy the equation

$$(41) \quad [Y(t)^i, Y(t)^j]_\eta = \sum_{k,l=1,2,3} \int_0^\eta (Y(t)')_\rho^{ik} (Y(t)')_\rho^{jl} d_\rho [X^k, X^l]_\rho.$$

Indeed, comparing (36) with (12), we can identify the function $M(t)_\xi$ in Proposition 1 with $Y(t)'_\xi$.

REMARK 9. The same result can be obtained noting that, for our solution,

$$\sum_i |Y(t)_{\xi_{i+1}\xi_i}|^2 = \sum_i Y(t)'_{\xi_i} X_{\xi_{i+1}\xi_i} Y'(t)_{\xi_i} X_{\xi_{i+1}\xi_i} + \sum_i O(|\xi_{i+1} - \xi_i|^{3\gamma}).$$

Equation (41) has a differential counterpart in the following equation:

$$(42) \quad \frac{d}{dt} W(t)_\xi = \int_0^\xi (H(t)_\rho d_\rho W(t)_\rho + d_\rho W(t)_\rho H(t)_\rho^*),$$

where we let $W(t)_\xi := [Y(t), Y(t)^*]_\xi$ as a matrix-valued process, and $H(t)_\xi := \nabla V^{Y(t)}(Y(t)_\xi)$. To understand better this evolution equation, let us split the matrix $H(t)_\xi$ into its symmetric S and anti-symmetric T components:

$$H(t)_\xi = S(t)_\xi + T(t)_\xi, \quad S(t)_\xi = S(t)_\xi^*, \quad T(t)_\xi = -T(t)_\xi^*.$$

Moreover, define $Q(t)_\xi$ as the solution of the Cauchy problem

$$\frac{d}{dt} Q(t)_\xi = -Q(t)_\xi T(t)_\xi, \quad Q(0)_\xi = \text{Id},$$

that is,

$$Q(t)_\xi = \exp\left[-\int_0^t T(s)_\xi ds\right].$$

Since T is antisymmetric, the matrix Q is orthogonal, that is, $Q(t)_\xi^{-1} = Q(t)_\xi^*$. This matrix describes the rotation of the local frame of reference at the point $Y(t)_\xi$ caused by the motion of the curve.

Then define

$$\tilde{W}(t)_\xi := \int_0^\xi Q(t)_\rho d_\rho W(t)_\rho Q(t)_\rho^{-1}$$

and, analogously, $\tilde{T}(t)_\xi := Q(t)_\xi T(t)_\xi Q(t)_\xi^{-1}$ and $\tilde{S}(t)_\xi = Q(t)_\xi S(t)_\xi Q(t)_\xi^{-1}$, and compute the following time-derivative:

$$\begin{aligned} \frac{d}{dt} d_\xi \tilde{W}(t)_\xi &= \frac{dQ(t)_\xi}{dt} Q(t)_\xi^{-1} d_\xi \tilde{W}(t)_\xi \\ &\quad + d_\xi \tilde{W}(t)_\xi Q(t)_\xi \frac{dQ(t)_\xi^{-1}}{dt} + Q(t)_\xi \left(\frac{d}{dt} d_\xi W(t)_\xi \right) Q(t)_\xi^{-1} \\ &= -\tilde{T}(t)_\xi d_\xi \tilde{W}(t)_\xi + d_\xi \tilde{W}(t)_\xi \tilde{T}(t)_\xi \\ &\quad + Q(t)_\xi [H(t)_\xi d_\xi W(t)_\xi + d_\xi W(t)_\xi H(t)_\xi^*] Q(t)_\xi^{-1} \\ &= \tilde{S}(t)_\xi d_\xi \tilde{W}(t)_\xi + d_\xi \tilde{W}(t)_\xi \tilde{S}(t)_\xi. \end{aligned}$$

This result implies that $d_\xi W(t)_\xi$ can be decomposed as

$$(43) \quad d_\xi W(t)_\xi = Q(t)_\xi^{-1} \exp\left[\int_0^t \tilde{S}(s)_\xi ds\right] d_\xi W(0)_\xi \exp\left[\int_0^t \tilde{S}(s)_\xi ds\right]^* Q(t)_\xi.$$

The relevance of this decomposition is the following. Modulo rotations, the matrix $\tilde{S}(t)_\xi$, corresponds to the symmetric part of the tensor field $\nabla V^{Y(t)}(x)$ in the point $x = Y(t)_\xi$. This symmetric component describes the stretching of the volume element around x due to the flow generated by the (time-dependent) vector field $V^{Y(t)}$. The magnitude of the covariation then varies with time, due to this stretching contribution, according to (43).

5. Random vortex filaments.

5.1. *Fractional Brownian loops with $H > 1/2$.* Consider the following probabilistic model of Gaussian vortex filament. Let $(\tilde{X}_\xi)_{\xi \in [0,1]}$ be a 3d-fractional Brownian motion (FBM) of Hurst index H , that is, a centered Gaussian process on \mathbb{R}^3 defined on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$ such that

$$\mathbb{E} \tilde{X}_\xi^i \tilde{X}_\eta^j = \frac{\delta_{ij}}{2} (|\xi|^{2H} + |\eta|^{2H} - |\xi - \eta|^{2H}), \quad i, j = 1, 2, 3, \xi, \eta \in [0, 1],$$

with $H > 1/2$ and $X_0 = 0$. Define the Gaussian process $(X_\xi)_{\xi \in [0,1]}$ as

$$(44) \quad X_\xi := \tilde{X}_\xi - \frac{C(\xi, 1)}{C(1, 1)} \tilde{X}_1,$$

where $C(\xi, \eta) := (|\xi|^{2H} + |\eta|^{2H} - |\xi - \eta|^{2H})$. Then $X_0 = X_1 = 0$ a.s., moreover, the process $(X_\xi)_\xi$ is independent of the r.v. \tilde{X}_1 . We call X a fractional Brownian loop (FBL). Using the standard Kolmogorov criterion, it is easy to show that X in a.s. Hölder continuous for any index $\gamma < H$. Since $H > 1/2$, then we can choose $\gamma \in (1/2, H)$ and apply the results of Section 3 to obtain the evolution of a random vortex filament modeled on an FBL.

5.2. *Evolution of Brownian loops.* As an example of application of Theorem 3, we can consider the evolution of an initial random curve whose law is that of a Brownian bridge on $[0, 1]$ starting at an arbitrary point x_0 . A standard three-dimensional Brownian bridge $\{B_\xi\}_{\xi \in [0,1]}$ such that $B_0 = B_1 = x_0 \in \mathbb{R}^3$ is a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose law is the law of a Brownian motion starting at x_0 and conditioned to reach x_0 at “time” 1. As in the previous section, it can be obtained starting from a standard Brownian motion $\{\tilde{B}\}_{\xi \in [0,1]}$ as

$$B_\xi = \tilde{B}_\xi - \xi \tilde{B}_1, \quad \xi \in [0, 1].$$

The Brownian bridge is a semi-martingale with respect to its own filtration $\{\mathcal{F}_\xi^B : 0 \leq \xi \leq 1\}$, with decomposition

$$dB_\xi = \frac{B_\xi - x_0}{1 - \xi} d\xi + d\beta_\xi,$$

where $\{\beta_\xi\}_{\xi \in [0,1]}$ is a standard 3d-Brownian motion. Using the results in [16], it is easy to see that B is a γ -Hölder rough path if we consider it together with the area process defined as

$$(45) \quad \mathbb{B}_{\xi\eta}^{2,ij} = \int_\xi^\eta (B_\rho^i - B_\eta^i) \circ dB_\rho^j,$$

where the integral is understood in Stratonovich sense. Indeed, there exists a version of the process $(\xi, \eta) \mapsto \mathbb{B}_{\xi\eta}^2$ which is continuous in both parameters and such that $\|\mathbb{B}\|_{2,\gamma}$ is almost surely finite (also all moments are finite). Then, outside an event of \mathbb{P} -measure zero, the couple (B, \mathbb{B}) is a γ -Hölder rough path and, by Theorem 3, there exists a solution of the problem (8) starting at B . Of course, in this case, the solution depends a priori on the choice (45) we made for the area process. Indeed, if in (45) we consider, for example, the Itô integral (for which the regularity result on \mathbb{B} still holds), we would have obtained a different solution, even if the path B is unchanged.

REMARK 10. Consider the discussion in Section 4.4 and note that, for our Brownian loop B , the covariation is $[B, B]_\xi = \text{Id} \cdot \xi$, we can say that the covariation of the solution Y starting at B will be

$$d_\xi[Y(t), Y(t)^*]_\xi = Q(t)_\xi^{-1} \exp\left[\int_0^t \tilde{S}(s)_\xi ds\right] \exp\left[\int_0^t \tilde{S}(s)_\xi ds\right]^* Q(t)_\xi$$

(the notation is the same as in Section 4.4).

5.3. *Evolution of fractional Brownian loops with $1/3 < H \leq 1/2$.* The above results on the Brownian loop are a particular case of the more general case of a fractional Brownian loop X of Hurst index $H \in (1/3, 1/2]$. In general, lifting X to a γ -rough path (with $1/3 < \gamma < H$) require to build an area process \mathbb{X}^2 with appropriate regularity (when $H \neq 1/2$ cannot be obtained by semi-martingale stochastic calculus as in Section 5.2 above).

In [3] the authors give a construction of the *area process* $\tilde{\mathbb{X}}^2$ in the case where \tilde{X} is an FBM of Hurst index $H > 1/4$. Moreover, they prove that the sequence $(\tilde{X}^{(n)}, \tilde{\mathbb{X}}^{(n),2})_{n \in \mathbb{N}}$ of piece-wise linear approximations of \tilde{X} , together with the associated geometric area process $\tilde{\mathbb{X}}^{(n),2}$, converges to $(\tilde{X}, \tilde{\mathbb{X}}^2)$ in the generalized p -variation sense for any $1/H < p < 4$ (for the definition of this kind of convergence, see [3]). It is not difficult to prove that we have convergence also as γ -rough paths for any $1/3 < \gamma < H$, that is, that

$$\|\tilde{X}^{(n)} - \tilde{X}\|_\gamma + \|\tilde{\mathbb{X}}^{(n),2} - \tilde{\mathbb{X}}^2\|_{2\gamma} \rightarrow 0$$

as $n \rightarrow \infty$.

To identify an appropriate area process for the fractional Brownian loop X , we can consider the following definition:

$$\begin{aligned} \mathbb{X}_{\xi\rho}^2 &:= \tilde{\mathbb{X}}_{\xi\rho}^2 + \int_\xi^\rho (h_\eta - h_\xi) \otimes d\tilde{X}_\eta \\ &+ \int_\xi^\rho (\tilde{X}_\eta - \tilde{X}_\xi) \otimes dh_\eta + \int_\xi^\rho (h_\eta - h_\xi) \otimes dh_\eta, \end{aligned} \tag{46}$$

where

$$h_\xi := -\frac{C(\xi, 1)}{C(1, 1)} \tilde{X}_1$$

[cf. (44)]. The function h is $2H$ -Hölder continuous so the integrals in (45) can be understood as Young integrals when $2H + \gamma > 3\gamma > 1$.

It is then straightforward to check that \mathbb{X}^2 satisfies (24) and that (X, \mathbb{X}^2) is a γ -rough path for any $1/3 < \gamma < H$.

Moreover, by exploiting the continuity of the Young integral and the results in [3] mentioned above, we have that piece-wise linear approximations $(X^{(n)}, \mathbb{X}^{(n),2})$ of (X, \mathbb{X}^2) converge to (X, \mathbb{X}^2) as γ -rough paths.

REMARK 11. This construction of the area process of a fractional Brownian loop with $H > 1/3$ is a particular case of a more general result about translations on the space of rough paths [6, 14, 20].

APPENDIX: PROOFS OF SOME LEMMAS

In the proofs we will often need to use Taylor expansions with integral remainders, so, for convenience, we introduce a special notation: given $X \in \mathcal{C}$, let $X_{\eta\xi} := X_\eta - X_\xi$ and $X_{\eta\xi}^r := J_r(X_\eta, X_\xi)$, where $J_r(x, y)$ is the linear interpolation

$$J_r(x, y) = (x - y)r + y$$

for $r \in [0, 1]$.

A.1. Proof of Lemma 1. The bound in (15) is easy. Let us prove the second by considering the following decomposition:

$$\begin{aligned} & [\varphi(Y_\xi) - \varphi(\tilde{Y}_\xi)] - [\varphi(Y_\eta) - \varphi(\tilde{Y}_\eta)] \\ &= Y_{\xi\eta} \int_0^1 \nabla\varphi(Y_{\xi\eta}^r) dr - \tilde{Y}_{\xi\eta} \int_0^1 \nabla\varphi(\tilde{Y}_{\xi\eta}^r) dr \\ &= (Y_{\xi\eta} - \tilde{Y}_{\xi\eta}) \int_0^1 \nabla\varphi(\tilde{Y}_{\xi\eta}^r) dr + Y_{\xi\eta} \left[\int_0^1 \nabla\varphi(Y_{\xi\eta}^r) dr - \int_0^1 \nabla\varphi(\tilde{Y}_{\xi\eta}^r) dr \right] \\ &= (Y_{\xi\eta} - \tilde{Y}_{\xi\eta}) \int_0^1 \nabla\varphi(\tilde{Y}_{\xi\eta}^r) dr \\ &\quad - Y_{\xi\eta} \int_0^1 dr (\tilde{Y}_{\xi\eta}^r - Y_{\xi\eta}^r) \int_0^1 dw \nabla^2\varphi(J_w(\tilde{Y}_{\xi\eta}^r, Y_{\xi\eta}^r)), \end{aligned}$$

where $Y_{\xi\eta}^r := J_r(Y_\xi, Y_\eta)$, we obtain

$$\|\varphi(Y) - \varphi(\tilde{Y})\|_\gamma \leq \|Y - \tilde{Y}\|_\gamma \|\nabla\varphi\| + \|Y\|_\gamma \|Y - \tilde{Y}\|_\infty \|\nabla^2 A\|,$$

which implies

$$(47) \quad \|\varphi(Y) - \varphi(\tilde{Y})\|_\gamma \leq \|\nabla\varphi\|_1 (1 + \|Y\|_\gamma) \|Y - \tilde{Y}\|_\gamma.$$

A.2. Proof of Lemma 2. By (15) and (16), we have

$$(48) \quad \|\nabla^n A(Y)\|_\gamma \leq \|\nabla^{n+1} A\| \|Y\|_\gamma$$

and

$$(49) \quad \|\nabla^n A(Y) - \nabla^n A(\tilde{Y})\|_\gamma \leq \|\nabla^{n+1} A\|_1 (1 + \|Y\|_\gamma) \|Y - \tilde{Y}\|_\gamma.$$

Now, it is enough to consider $n = 0$, the proof for general n being similar. Using Lemma 2 and bounds (48) and (49), estimates (17) and (18) on the velocity vector-field follow as

$$\begin{aligned} |V^Y(x)| &= \left| \int_0^1 A(x - Y_\eta) dY_\eta \right| \leq C_\gamma \|A(x - Y)\|_\gamma \|Y\|_\gamma \\ &\leq C_\gamma \|\nabla A\| \|Y\|_\gamma^2. \end{aligned}$$

Moreover, for the difference $V^Y - V^{\tilde{Y}}$, we have the decomposition

$$\begin{aligned} V^Y(x) - V^{\tilde{Y}}(x) &= \int_0^1 [A(x - Y_\eta) dY_\eta - A(x - \tilde{Y}_\eta) d\tilde{Y}_\eta] \\ &= \int_0^1 A(x - Y_\eta) d(Y - \tilde{Y})_\eta + \int_0^1 [A(x - Y_\eta) - A(x - \tilde{Y}_\eta)] d\tilde{Y}_\eta, \end{aligned}$$

which, in turn, can be estimated as

$$\begin{aligned} |V^Y(x) - V^{\tilde{Y}}(x)| &\leq C_\gamma \|A(x - Y)\|_\gamma \|Y - \tilde{Y}\|_\gamma \\ &\quad + C_\gamma \|A(x - Y) - A(x - \tilde{Y})\|_\gamma \|\tilde{Y}\|_\gamma \\ &\leq C_\gamma \|\nabla A\| \|Y\|_\gamma \|Y - \tilde{Y}\|_\gamma \\ &\quad + C_\gamma \|\tilde{Y}\|_\gamma \|\nabla A\|_1 \|Y - \tilde{Y}\|_\gamma^* (1 + \|Y\|_\gamma) \\ &\leq C_\gamma \|\nabla A\|_1 (\|Y\|_\gamma + \|\tilde{Y}\|_\gamma + \|\tilde{Y}\|_\gamma \|Y\|_\gamma) \|Y - \tilde{Y}\|_\gamma^*, \end{aligned}$$

giving (18).

A.3. Proof of Lemma 7. Consider the case $n = 0$, the general case being similar. The path $Z_\xi = A(x - Y_\xi)$ belongs to \mathcal{D}_X and has the following decomposition:

$$\begin{aligned} Z_{\xi\eta} &= \nabla A(x - Y_\eta) Y_{\xi\eta} + Y_{\xi\eta} Y_{\xi\eta} \int_0^1 dr \int_0^r dw \nabla^2 A(x - Y_{\xi\eta}^w) \\ &= \nabla A(x - Y_\eta) Y'_\eta X_{\xi\eta} + \nabla A(x - Y_\eta) R^Y_{\xi\eta} \\ &\quad + Y_{\xi\eta} Y_{\xi\eta} \int_0^1 dr \int_0^r dw \nabla^2 A(x - Y_{\xi\eta}^w) \\ &= Z'_\eta X_{\xi\eta} + R^Z_{\xi\eta}, \end{aligned}$$

with

$$Z'_\eta = \nabla A(x - Y_\eta) Y'_\eta$$

and

$$R^Z_{\xi\eta} = \nabla A(x - Y_\eta) R^Y_{\xi\eta} + Y_{\xi\eta} Y_{\xi\eta} \int_0^1 dr \int_0^r dw \nabla^2 A(x - Y_{\xi\eta}^w).$$

The \mathcal{D}_X norm of Z can be estimated as follows:

$$\begin{aligned}
 \|Z\|_D &= \|Z'\|_\infty + \|Z'\|_\gamma + \|R^Z\|_{2\gamma} \\
 &\leq \|\nabla A\|_\infty \|Y'\|_\infty + \|\nabla^2 A\|_\infty \|Y\|_\gamma + \|\nabla A\|_\infty \|Y'\|_\gamma \\
 &\quad + \|\nabla A\|_\infty \|R^Y\|_{2\gamma} + \|Y\|_\gamma^2 \|\nabla^2 A\|_\infty \\
 (50) \quad &\leq \|\nabla A\|_1 (\|Y\|_D + \|Y\|_\gamma + \|Y\|_\gamma^2) \\
 &\leq \|\nabla A\|_1 [(1 + C_X)\|Y\|_D + C_X^2 \|Y\|_D^2] \\
 &\leq C_X^2 \|\nabla A\|_1 [2\|Y\|_D + \|Y\|_D^2],
 \end{aligned}$$

where we used the fact that

$$\begin{aligned}
 \|Y\|_\gamma &\leq \|Y'\|_\infty \|X\|_\gamma + \|R^Y\|_\gamma \\
 (51) \quad &\leq \|Y'\|_\infty \|X\|_\gamma + \|R^Y\|_{2\gamma} \\
 &\leq (1 + \|X\|_\gamma)\|Y\|_D \leq C_X \|Y\|_D.
 \end{aligned}$$

Then

$$V^Y(x) = \int_0^1 A(x - Y_\eta) dY_\eta = \int_0^1 Z_\eta dY_\eta Z_0(Y_1 - Y_0) + Z'_0 Y'_0 \mathbb{X}_{01}^2 + Q_{01},$$

with

$$\|Q\|_{3\gamma} \leq C'_\gamma C_X \|Z\|_D \|Y\|_D$$

and

$$\begin{aligned}
 |V^Y(x)| &\leq \|Z'\|_\infty \|Y'\|_\infty \|\mathbb{X}^2\|_{2\gamma} + \|Q\|_{3\gamma} \\
 &\leq 2C'_\gamma C_X \|Z\|_D \|Y\|_D \\
 &\leq 4C'_\gamma \|\nabla A\|_1 C_X^3 \|Y\|_D^2 (1 + \|Y\|_D),
 \end{aligned}$$

where we used the fact that $C'_\gamma \geq 1$.

To bound $V^Y(x) - V^{\tilde{Y}}(x)$, we need the \mathcal{D}_X norm of the difference $A(x - Y) - A(x - \tilde{Y})$. Let $\phi(y) = A(x - y)$ and consider the expansion

$$\begin{aligned}
 &\phi(Y_\eta) - \phi(Y_\xi) - (\phi(\tilde{Y}_\eta) - \phi(\tilde{Y}_\xi)) \\
 &= [\nabla\phi(Y_\xi)Y_{\eta\xi} - \nabla\phi(\tilde{Y}_\xi)\tilde{Y}_{\eta\xi}] \\
 &\quad + \int_0^1 dr \int_0^r dw [\nabla^2\phi(Y_{\eta\xi}^w)Y_{\eta\xi}Y_{\eta\xi} - \nabla^2\phi(\tilde{Y}_{\eta\xi}^w)\tilde{Y}_{\eta\xi}\tilde{Y}_{\eta\xi}],
 \end{aligned}$$

which, by arguments similar to those leading to (50), gives a related estimate:

$$\begin{aligned}
 \|\phi(Y) - \phi(\tilde{Y})\|_D &\leq \|\nabla\phi\|_\infty \|Y - \tilde{Y}\|_D + \|\nabla^2\phi\|_\infty \|Y - \tilde{Y}\|_\infty \|Y\|_D \\
 &\quad + 3\|\nabla^3\phi\| \|Y - \tilde{Y}\|_\infty \|Y\|_\gamma^2 + 2\|\nabla^2\phi\|_\infty \|Y - \tilde{Y}\|_\gamma \|Y\|_\gamma \\
 &\leq 6\|\nabla\phi\|_2 C_X^2 (1 + \|Y\|_D)^2 \|Y - \tilde{Y}\|_D^*
 \end{aligned}$$

so that

$$\|A(x - Y) - A(x - \tilde{Y})\|_D \leq 6\|\nabla A\|_2 C_X^2 (1 + \|Y\|_D)^2 \|Y - \tilde{Y}\|_D^*.$$

Now,

$$\begin{aligned} V^Y(x) - V^{\tilde{Y}}(x) &= \int_0^1 A(x - Y_\eta) dY_\eta - \int_0^1 A(x - \tilde{Y}_\eta) d\tilde{Y}_\eta \\ &= \int_0^1 [A(x - Y_\eta) - A(x - \tilde{Y}_\eta)] dY_\eta + \int_0^1 A(x - \tilde{Y}_\eta) d(Y - \tilde{Y})_\eta \end{aligned}$$

and we can conclude by observing that

$$\begin{aligned} |V^Y(x) - V^{\tilde{Y}}(x)| &\leq 2C'_\gamma C_X (\|A(x - Y) - A(x - \tilde{Y})\|_D \|Y\|_D \\ &\quad + \|A(x - Y)\|_D \|Y - \tilde{Y}\|_D) \\ &\leq 16C'_\gamma C_X^3 \|\nabla A\|_2 \|Y - \tilde{Y}\|_D^* (1 + \|Y\|_D)^2 \|Y\|_D. \end{aligned}$$

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