The Annals of Probability
2005, Vol. 33, No. 5, 1782–1803
DOI 10.1214/009117905000000305

NULL FLOWS, POSITIVE FLOWS AND THE STRUCTURE OF STATIONARY SYMMETRIC STABLE PROCESSES[1]

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This paper elucidates the connection between stationary symmetric α-stable processes with $0 < \alpha < 2$ and nonsingular flows on measure spaces by describing a new and unique decomposition of stationary stable processes into those corresponding to positive flows and those corresponding to null flows. We show that necessary and sufficient for a stationary stable process to be ergodic is that its positive component vanishes.

1. Introduction. We consider the class of measurable stationary symmetric α-stable (henceforth, SαS) processes $\mathbf{X} = (X(t),\ t \in T)$, $0 < \alpha < 2$, both in discrete time ($T = \mathbb{Z}$) and in continuous time ($T = \mathbb{R}$). Unlike their Gaussian counterparts ($\alpha = 2$), which can be neatly described by either covariance function or by the spectral measure, the structure of SαS processes with $0 < \alpha < 2$ remained largely a mystery until the paper of Rosiński [17] which showed that integral representations of stationary symmetric SαS processes can be taken to be of a special, and particularly illuminating, form. The general integral representations of α-stable processes, of the type

$$X(t) = \int_E f_t(x) M(dx), \qquad t \in T, \tag{1.1}$$

where M is an SαS random measure on E with a σ-finite control measure m, and $f_t \in L^\alpha(m)$ for each t, have been known since Bretagnolle, Dacunha-Castelle and Krivine [2] and Schreiber [24]; see also [23] and [8] (for basic information on integrals with respect to stable random measures, one can consult Samorodnitsky and Taqqu [22]). What Rosiński [17] showed was that, for measurable stationary SαS processes, one can choose the kernel $f_t, t \in T$ in (1.1) in a special way, described below. Note that we are considering both real-valued and complex-valued SαS processes. In the complex-valued case we are assuming that the stable process is *isotropic*, or *rotationally invariant* (see [22]). In that case the random measure M in the integral representation can be chosen to be isotropic as well.

Received March 2004; revised December 2004.
[1]Supported in part by NSF Grant DMS-03-03493 and NSA Grant MSPF-02G-183 at Cornell University.
AMS 2000 subject classifications. Primary 60G10, 60G52; secondary 37A40.
Key words and phrases. Stable process, stationary process, integral representation, ergodic theory, nonsingular flow, dissipative flow, conservative flow, null flow, positive flow, ergodicity.

According to Rosiński [17], one can choose the kernel in (1.1) in the form

$$f_t(x) = a_t(x)\left(\frac{dm \circ \phi_t}{dm}(x)\right)^{1/\alpha} f \circ \phi_t(x) \tag{1.2}$$

for $t \in T$ and $x \in E$, where f is a *single* function in $L^\alpha(m)$. Here (ϕ_t) is a measurable family of maps from E onto E such that $\phi_{t+s}(x) = \phi_t(\phi_s(x))$ for all $t, s \in \mathbb{R}$ and $x \in E$, $\phi_0(x) = x$ for all $x \in E$, and $m \circ \phi_t^{-1} \sim m$ for all $t \in \mathbb{R}$. The assumptions mean that the family (ϕ_t) forms a measurable nonsingular flow on E. Finally, (a_t) is a measurable family of $\{-1, 1\}$-valued (unit circle-valued in the complex case) functions on E such that, for every $s, t \in \mathbb{R}$, we have $a_{t+s}(x) = a_s(x)a_t(\phi_s(x))$ m-almost everywhere on E. This means that the family (a_t) forms a cocycle for the flow (ϕ_t).

One can, therefore, relate, in principle, the ergodic-theoretical properties of the flow (ϕ_t) to the probabilistic properties of the SαS process $(X(t), t \in T)$, and under the lack of redundancy assumption

$$\operatorname{supp}\{f \circ \phi_t, t \in T\} = E, \tag{1.3}$$

(which can, obviously, be always assumed) certain important ergodic-theoretical properties of the flow (ϕ_t) remain permanent for a given process. That is, they do not change from representation to representation.

One such permanent feature of stationary SαS processes is the *conservative–dissipative* decomposition. Recall the Hopf decomposition of the flow $E = C \cup D$, where C and D are disjoint flow invariant measurable sets, such that the flow is conservative on C and dissipative on D (see, e.g., [7]). Then by writing

$$X(t) = \int_C f_t(x)M(dx) + \int_D f_t(x)M(dx) := X^C(t) + X^D(t), \qquad t \in T, \tag{1.4}$$

one obtains a unique in law decomposition of a stationary SαS process into a sum of two independent stationary SαS processes, one corresponding to a conservative flow and the other to a dissipative flow [17]. [As far as terminology is concerned, we will say that a process corresponds to a particular flow, or is generated by a particular flow, if the process has an integral representation (1.1) with the kernel of the form (1.2) and (1.3) holds.]

Since the processes corresponding to a dissipative flow have an intuitively clear representation as mixed moving averages (introduced by Surgailis, Rosiński, Mandrekar and Cambanis [26]), attention focused on discerning further structures in the class of stationary SαS processes generated by conservative flows. The class of *cyclic* processes that generalize harmonizable processes was introduced and discussed by Pipiras and Taqqu [12]. Other examples were discussed in [18] and [21].

A further challenge has been to find explicit relations between the ergodic-theoretical properties of the flow underlying a stationary SαS process and probabilistic properties of the process. Certain steps in this direction have been taken, for example, in [11, 15, 16]. The most explicit connection of this kind established up to date is the fact that the partial maxima of the processes generated by conservative flows grow strictly slower than those of the processes generated by dissipative flows; see [20, 21].

In this paper we present a decomposition of stationary SαS processes different from the conservative–dissipative decomposition (1.4), the *positive-null* decomposition. While process generated by a dissipative flow is generated by a null flow, processes generated by conservative flows can be generated by either positive or null flow. This will, therefore, clarify further the structure of stationary SαS processes generated by conservative flows. We will see, further, that a stationary SαS process is ergodic if and only if it is generated by a null flow.

We finish this introductory section by mentioning that the assumption of measurability of the processes we are considering is automatic in the discrete time case, and is equivalent to the assumption of continuity in probability in the continuous time case. See Section 1.6 of [1] or [12].

The following section provides the basic decomposition of a stationary SαS process. Section 3 relates this decomposition to ergodicity of the stable process. Finally, Section 4 discusses a number of examples of stationary stable processes in the context of the theory developed in this paper.

2. Decomposition of stationary SαS processes: positive and null parts. A nonsingular map ϕ on a σ-finite measure space $(E, \mathcal{E}, m)$ is a one-to-one map $E \to E$ such that both ϕ and ϕ^{-1} are measurable and map the measure m into equivalent measures. A nonsingular map ϕ is called *positive* if there is a finite measure μ on $(E, \mathcal{E})$ equivalent to m that is preserved under ϕ. A subset B of E is called weakly wandering if there is a sequence $0 = n_0 < n_1 < n_2 < \cdots$ such that the sets $\phi^{-n_k} B$, $k = 0, 1, 2, \ldots$, are disjoint. The positive-null decomposition of the map ϕ is a decomposition of E into a union of two disjoint ϕ-invariant measurable sets $\mathtt{P}$ and $\mathtt{N}$ (the positive and null parts of ϕ) such that the restriction of ϕ to $\mathtt{P}$ is positive, and $\mathtt{N} = \bigcup_{k=0}^{\infty} \phi^{-n_k} B$, with B a measurable weakly wandering set and a sequence $0 = n_0 < n_1 < n_2 < \cdots$ as above. This decomposition is unique modulo null sets. See Section 1.4 in [1] and Section 3.4 in [7]. (In general, the above two references are a source of the basic facts in ergodic theory used in this paper.)

Given a (measurable) nonsingular flow (ϕ_t), each map ϕ_t is a nonsingular map, and has a corresponding positive-null decomposition $E = \mathtt{P}_t \cup \mathtt{N}_t$. One can show [7] that there are invariant measurable sets $\mathtt{P}$ and $\mathtt{N}$ (the positive and null parts of the flow) such that $\mathtt{P}_t = \mathtt{P}$ and $\mathtt{N}_t = \mathtt{N}$ modulo null sets for all $t > 0$. The flow is positive if $\mathtt{P} = E$ and null if $\mathtt{N} = E$ modulo null sets.

Our first result shows the invariant nature of the property of being generated by a positive or by a null flow, and is parallel to the corresponding result for conservative and dissipative flows in [17]. Let $\mathcal{W}$ be class of functions $w : T \to$

$[0, \infty)$ such that w is nondecreasing on $T \cap (-\infty, 0]$, nonincreasing on $T \cap [0, \infty)$ and

$$\int_{T\cap(-\infty,0]} w(t)\lambda(dt) = \int_{T\cap[0,\infty)} w(t)\lambda(dt) = \infty. \tag{2.1}$$

Here λ is the counting measure if $T = \mathbb{Z}$ and the Lebesgue measure if $T = \mathbb{R}$. The (implicit) assumption in the theorem below is that the process is given in the form (1.1) with the kernel of the form (1.2) and (1.3) holds.

THEOREM 2.1. (i) *A stationary $S\alpha S$ process is generated by a positive flow if and only if, for any $w \in \mathcal{W}$, we have*

$$\int_T w(t)|f|^\alpha \circ \phi_t \frac{dm \circ \phi_t}{dm} \lambda(dt) = \infty \qquad m\text{-a.e.} \tag{2.2}$$

(ii) *A stationary $S\alpha S$ process is generated by a null flow if and only if, for some $w \in \mathcal{W}$, we have*

$$\int_T w(t)|f|^\alpha \circ \phi_t \frac{dm \circ \phi_t}{dm} \lambda(dt) < \infty \qquad m\text{-a.e.} \tag{2.3}$$

(iii) *If a stationary $S\alpha S$ process is generated by a positive (resp. null) flow in one representation, then in every other representation it will be generated by a positive (resp. null) flow. In particular, the classes of stationary $S\alpha S$ processes generated by positive and null flows are disjoint.*

PROOF. (i) We start with the discrete time case $T = \mathbb{Z}$. Suppose that (2.2) holds for all $w \in \mathcal{W}$ and assume that, to the contrary, $\mathtt{N}$ is not empty m-a.e. For a nonsingular map ϕ and a nonincreasing nonnegative nonsummable sequence $w = (w_n, n = 0, 1, 2, \ldots)$, there is an (uniquely determined modulo null sets) invariant set $\mathtt{P}_w^\phi$ such that, for every strictly positive $g \in L^1(m)$,

$$\sum_{n=0}^{\infty} w(n)|g \circ \phi_n(x)|^\alpha \frac{dm \circ \phi_n}{dm}(x) = \infty \qquad m\text{-a.e. on } \mathtt{P}_w^\phi \tag{2.4}$$

and

$$\sum_{n=0}^{\infty} w(n)|g \circ \phi_n(x)|^\alpha \frac{dm \circ \phi_n}{dm}(x) < \infty \qquad m\text{-a.e. on } \mathtt{N}_w^\phi = E - \mathtt{P}_w^\phi, \tag{2.5}$$

see Theorem 2 in [6].

It follows by Theorems 2 and 3 of [6] that there is a nonincreasing nonnegative nonsummable sequence $w^{(+)} = (w_n^{(+)}, n = 0, 1, 2, \ldots)$ such that $\mathtt{N}_{w^{(+)}}^\phi = \mathtt{N}$ m-a.e.

Note, furthermore, that the null parts of ϕ and ϕ^{-1} coincide modulo null sets. Therefore, appealing once again to [6], we see that there is a nonincreasing nonnegative nonsummable sequence $w^{(-)} = (w_n^{(-)}, n = 0, 1, 2, \dots)$ such that $\mathtt{N}_{w^{(-)}}^{\phi^{-1}} = \mathtt{N}$ m-a.e.

Defining $w_0 = w_0^{(+)} + w_0^{(-)}$, $w_n = w_n^{(+)}$ for $n \ge 1$ and $w_n = w_n^{(-)}$ for $n \le -1$, we obtain a $w \in \mathcal{W}$, for which (2.2) fails at almost every point of $\mathtt{N}$. Hence, ϕ is positive.

In the opposite direction, suppose that ϕ is positive. By Theorem 2 of Krengel [6], for every $w \in \mathcal{W}$,

$$\sum_{n=0}^{\infty} w(n)|f \circ \phi_n|^\alpha \frac{dm \circ \phi_n}{dm} = \infty \qquad m\text{-a.e. on } \left\{\sum_{n=0}^{\infty} |f \circ \phi_n|^\alpha > 0\right\}$$

and

$$\sum_{n=-\infty}^{0} w(n)|f \circ \phi_n|^\alpha \frac{dm \circ \phi_n}{dm} = \infty \qquad m\text{-a.e. on } \left\{\sum_{n=-\infty}^{0} |f \circ \phi_n|^\alpha > 0\right\}.$$

Since

$$E = \left\{\sum_{n=0}^{\infty} |f \circ \phi_n|^\alpha > 0\right\} \cup \left\{\sum_{n=0}^{\infty} |f \circ \phi_n|^\alpha > 0\right\}$$

m-a.e., we obtain (2.2). This completes the proof of part (i) in the case $T = \mathbb{Z}$.

Consider now the case $T = \mathbb{R}$. Suppose that the flow (ϕ_t) is positive, and let $w \in \mathcal{W}$. We have

$$\begin{aligned} &\int_0^\infty w(t)|f|^\alpha \circ \phi_t \frac{dm \circ \phi_t}{dm} \lambda(dt) \\ &\qquad \ge \sum_{j=1}^{\infty} w(j) \int_0^1 |f|^\alpha \circ \phi_{t+j-1} \frac{dm \circ \phi_{t+j-1}}{dm} \lambda(dt) \\ &\qquad = \sum_{j=1}^{\infty} w(j) g_+ \circ \phi_j \frac{dm \circ \phi_j}{dm}, \end{aligned} \tag{2.6}$$

where

$$g_+(x) = \int_0^1 |f \circ \phi_{t-1}(x)|^\alpha \frac{dm \circ \phi_{t-1}}{dm}(x) \lambda(dt), \qquad x \in E. \tag{2.7}$$

Therefore, by Theorem 2 of [6],

$$\int_0^\infty w(t)|f|^\alpha \circ \phi_t \frac{dm \circ \phi_t}{dm} \lambda(dt) = \infty \qquad m\text{-a.e. on } \left\{\sum_{j=0}^{\infty} g_+ \circ \phi_j > 0\right\}. \tag{2.8}$$

Similarly,

$$\int_{-\infty}^0 w(t)|f|^\alpha \circ \phi_t \frac{dm \circ \phi_t}{dm} \lambda(dt) = \infty \qquad m\text{-a.e. on } \left\{\sum_{j=0}^{\infty} g_- \circ \phi_{-j} > 0\right\}, \tag{2.9}$$

where

$$g_-(x) = \int_0^1 |f \circ \phi_{-t+1}(x)|^\alpha \frac{dm \circ \phi_{-t+1}}{dm}(x)\lambda(dt), \qquad x \in E. \tag{2.10}$$

Therefore, to prove (2.2), we need to show that

$$\left\{\sum_{j=0}^{\infty} g_+ \circ \phi_j > 0\right\} \cup \left\{\sum_{j=0}^{\infty} g_- \circ \phi_{-j} > 0\right\} = E \qquad m\text{-a.e.} \tag{2.11}$$

To this end, note that

$$\begin{aligned}
&\left\{\sum_{j=0}^{\infty} g_+ \circ \phi_j > 0\right\} \cup \left\{\sum_{j=0}^{\infty} g_- \circ \phi_{-j} > 0\right\} \\
&\qquad = \left\{\sum_{j=0}^{\infty} g_+ \circ \phi_j \frac{dm \circ \phi_j}{dm} > 0\right\} \cup \left\{\sum_{j=0}^{\infty} g_- \circ \phi_{-j} \frac{dm \circ \phi_{-j}}{dm} > 0\right\} \qquad m\text{-a.e.}
\end{aligned}$$

and that

$$\sum_{j=0}^{\infty} g_+ \circ \phi_j \frac{dm \circ \phi_j}{dm} = \int_{-1}^{\infty} |f|^\alpha \circ \phi_t \frac{dm \circ \phi_t}{dm} \lambda(dt)$$

and

$$\sum_{j=0}^{\infty} g_- \circ \phi_{-j} \frac{dm \circ \phi_{-j}}{dm} = \int_{-\infty}^{1} |f|^\alpha \circ \phi_t \frac{dm \circ \phi_t}{dm} \lambda(dt).$$

If (2.11) fails, then, for some set A with $m(A) > 0$, we have, for all $x \in A$, $f_t(x) = 0$ for almost every $t \in \mathbb{R}$. By Fubini's theorem, we conclude that, for almost every $t \in \mathbb{R}$, we have $f_t(x) = 0$ m-a.e. on A. Let now $t \in \mathbb{R}$, and choose a sequence $t_n \to t$ such that $f_{t_n}(x) = 0$ m-a.e. on A for each n. By the continuity in probability,

$$0 = \lim_{n\to\infty} \int_E \big|f_t(x) - f_{t_n}(x)\big|^\alpha m(dx) \geq \int_E |f_t(x)|^\alpha m(dx),$$

and for each $t \in \mathbb{R}$, we have $f_t(x) = 0$ m-a.e. on A, contradicting the assumption of full support (1.3). Therefore, (2.2) follows.

In the opposite direction, suppose that (2.2) holds for all $w \in \mathcal{W}$ and assume that, to the contrary, $\mathtt{N}$ is not empty m-a.e. Applying, once again, to Theorems 2 and 3 of [6], we conclude that there is a nonincreasing nonnegative nonsummable sequence $w^{(+)} = (w_n^{(+)}, n = 0, 1, 2, \ldots)$ such that

$$\sum_{n=0}^{\infty} w(n)|g_+ \circ \phi_n(x)|^\alpha \frac{dm \circ \phi_n}{dm}(x) < \infty \qquad \text{a.e. on } \mathtt{N},$$

with the function g_+ given in (2.7). Letting $w(t) = w_n^{(+)}$ for $n-1 \le t < n$, $n = 2, 3, \dots$, and $w(t) = w_0^{(+)}$ for $0 < t < 1$, we see that

$$\int_0^\infty w(t)|f|^\alpha \circ \phi_t \frac{dm \circ \phi_t}{dm} \lambda(dt) < \infty \qquad \text{a.e. on } \mathtt{N}. \tag{2.12}$$

Similarly, we can find a nonincreasing nonnegative nonsummable sequence $w^{(-)} = (w_n^{(-)}, n = 0, 1, 2, \dots)$ such that, with $w(-t) = w_n^{(-)}$ for $n-1 \le t < n$, $n = 2, 3, \dots$, and $w(-t) = w_0^{(-)}$ for $0 < t < 1$, we have

$$\int_{-\infty}^0 w(t)|f|^\alpha \circ \phi_t \frac{dm \circ \phi_t}{dm} \lambda(dt) < \infty \qquad \text{a.e. on } \mathtt{N}. \tag{2.13}$$

If we select $w(0)$ large enough, we obtain $w \in \mathcal{W}$ for which (2.2) fails at a.e. point of $\mathtt{N}$. This contradiction finishes the proof of part (i) in all cases.

(ii) Consider first the case $T = \mathbb{Z}$. If (2.3) holds for some $w \in \mathcal{W}$, then the assumption of full support (1.3) and Theorem 2 in [6] show that $\mathtt{P} = \varnothing$ and so ϕ is null. If, on the other hand, ϕ is null, then appealing, as above, to Theorem 3 in [6], we can construct $w \in \mathcal{W}$ for which (2.3) holds. In the case $T = \mathbb{R}$ we reduce the argument to the discrete time case as in the proof of (i).

(iii) Suppose that $\mathbf{X}$ is generated by a positive flow in the representation (1.1) with the kernel of the form (1.2), and consider another representation (in law) of $\mathbf{X}$,

$$X(t) = \int_S g_t(y) M_1(dy), \qquad t \in T,$$

where M_1 has a control measure m_1 and

$$g_t(y) = a_t^{(1)}(y) \left(\frac{dm_1 \circ \psi_t}{dm_1}(y) \right)^{1/\alpha} g \circ \psi_t(y),$$

with (ψ_t) a measurable nonsingular flow and $(a^{(1)})$ a cocycle for this flow, and the functions (g_t) have full support. We need to prove that the flow (ψ_t) is positive, and by part (i), we need to show that, for every $w \in \mathcal{W}$,

$$\int_T w(t)|g_t|^\alpha \lambda(dt) = \infty \qquad m_1\text{-a.e.} \tag{2.14}$$

Let

$$\mathtt{P}_w^f = \left\{ x \in E : \int_T w(t+k)|f|^\alpha \circ \phi_t(x) \frac{dm \circ \phi_t}{dm}(x) \lambda(dt) = \infty \text{ for all } k \in \mathbb{Z} \right\}.$$

Then $\mathtt{P}_w^f$ is a flow invariant set (both in the cases $T = \mathbb{Z}$ and $T = \mathbb{R}$) and, since, for every k, $w(\cdot + k)$ differs from a function in $\mathcal{W}$ only on a compact interval, it has a full measure by part (i). Therefore, one can restrict the integral in (1.1) and the flow to $\mathtt{P}_w^f$. By Theorem 1.1 in [17], there are measurable maps $\Phi : S \to \mathtt{P}_w^f$ and

$h : S \to \mathbb{R} - \{0\}$ in the real-valued case and $h : S \to \mathbb{C} - \{0\}$ in the complex-valued case such that

$$g_t(y) = h(y) f_t(\Phi(y)) \qquad \text{for } m_1 \times \lambda\text{-a.e. } (y, t).$$

Choose a measurable set $S_1 \subset S$ of full measure such that, for every $y \in S_1$, this relation holds for λ-a.e. t. We have, for any $y \in S_1$,

$$\int_T w(t)|g_t(y)|^\alpha \lambda(dt) = |h(y)| \int_T w(t)|f_t(\Phi(y))|^\alpha \lambda(dt) = \infty,$$

since $\Phi(y) \in \mathrm{P}_w^f$. Therefore, (ψ_t) is positive.

The argument for null flows is similar. $\square$

Similarly to the conservative–dissipative decomposition, one can check conditions of the type given in parts (i) and (ii) of Theorem 2.1 without having to have a specific form of integral representation of a stationary process. Here we are allowing any *measurable* representation (1.1). This is a representation in which the function $f_t(x)$ is jointly measurable in the variables $x \in E$ and $t \in T$. Every measurable SαS process has such representation; see Section 11.1 in [22]. In addition, we will assume the full support condition (1.3).

COROLLARY 2.2. *A stationary SαS with a measurable and full support representation* (1.1) *is generated by a positive flow if and only, for any $w \in \mathcal{W}$, we have*

$$\int_T w(t)|f_t|^\alpha \lambda(dt) = \infty \qquad m\text{-a.e.} \tag{2.15}$$

A process is generated by a null flow if and only, for some $w \in \mathcal{W}$, we have

$$\int_T w(t)|f_t|^\alpha \lambda(dt) < \infty \qquad m\text{-a.e.} \tag{2.16}$$

PROOF. The same argument as in the proof of Corollary 4.2 in [17] reduces the situation to that in Theorem 2.1. $\square$

REMARK 2.3. Note that the direct parts of Corollary 2.2 hold without the assumption of full support. That is, if (2.15) [resp. (2.16)] holds, then the process is generated by a positive (resp. null) flow. Indeed, the above statements will still be valid if one reduces the integration to the support of the kernel (f_t).

Given a stationary SαS process $\mathbf{X}$ and any integral representation (1.1) of the form (1.2) and of full support, use the positive-null decomposition $E = \mathrm{P} \cup \mathrm{N}$ to write the following analog of (1.4):

$$X(t) = \int_{\mathrm{P}} f_t(x) M(dx) + \int_{\mathrm{N}} f_t(x) M(dx) := X^P(t) + X^N(t), \qquad t \in T, \tag{2.17}$$

a sum of two independent stationary SαS processes, one of which is generated by a positive flow, and the other one by a null flow. The next result shows that this decomposition is unique.

THEOREM 2.4. *A decomposition of a stationary SαS process into a sum of two independent stationary SαS processes, one of which is generated by a positive flow, and the other one by a null flow, is unique in law.*

PROOF. Consider any fixed *minimal* representation of the process $\mathbf{X}$,

$$X(t) = \int_S g_t(y) M_1(dy), \qquad t \in T,$$

where M_1 has a control measure m_1 and

$$g_t(y) = a_t^{(1)}(y)\left(\frac{dm_1 \circ \psi_t}{dm_1}(y)\right)^{1/\alpha} g \circ \psi_t(y),$$

with (ψ_t) a measurable nonsingular flow and $(a^{(1)})$ a cocycle for this flow, and the functions (g_t) have full support. We refer the reader to [17] about basic facts and properties on minimal representations. Let $\mathbf{X} = \mathbf{X}_\psi^P + \mathbf{X}_\psi^N$ be the decomposition in (2.17) with respect to that representation.

Consider now any full support representation (1.1) of the form (1.2) of the process $\mathbf{X}$, and let $\mathbf{X} = \mathbf{X}^P + \mathbf{X}^N$ be the corresponding decomposition. We will prove that this decomposition coincides with the above decomposition with respect to the fixed minimal representation. By Remark 2.5 in [17], there is a measurable map $\Phi : E \to S$ and a nonvanishing (real or complex-valued) measurable function h such that, for each $t \in T$, $f_t(x) = h(x) g_t(\Phi(x))$ m-a.e., and $m \sim m_1 \circ \Phi^{-1}$. Let $E = \mathtt{P} \cup \mathtt{N}$ be the positive-null decomposition of the flow (ϕ_t) and $S = \mathtt{P}^\psi \cup \mathtt{N}^\psi$ be the positive-null decomposition of the flow (ψ_t). Let us prove that

$$\mathtt{P} = \Phi^{-1}(\mathtt{P}^\psi), \qquad \mathtt{N} = \Phi^{-1}(\mathtt{N}^\psi) \tag{2.18}$$

modulo sets of m-measure zero.

If $T = \mathbb{Z}$, select (using Theorem 3 in [6]) a $w \in \mathcal{W}$ such that

$$\mathtt{P}^\psi = \left\{ y \in S : \sum_{n=-\infty}^{\infty} w(n) |g \circ \psi_n(y)|^\alpha \frac{dm_1 \circ \psi_n}{dm_1}(y) = \infty \right\}$$

up to a set of m_1-measure zero. Notice (once again, by Theorem 3 in [6]) that, for m-a.e. $x \in \mathtt{P}$,

$$\sum_{n=-\infty}^{\infty} w(n) |f \circ \phi_n(x)|^\alpha \frac{dm \circ \phi_n}{dm}(x) = \infty.$$

Therefore, also

$$\sum_{n=-\infty}^{\infty} w(n) |g \circ \psi_n(\Phi(x))|^\alpha \frac{dm_1 \circ \psi_n}{dm_1}(\Phi(x)) = \infty,$$

and so $\Phi(x) \in \mathtt{P}^\psi$ for m-a.e. $x \in \mathtt{P}$. Hence,

$$\Phi^{-1}(\mathtt{N}^\psi) \subset \mathtt{N} \qquad \text{modulo a set of } m\text{-measure zero.} \tag{2.19}$$

Select now a (perhaps different) $w \in \mathcal{W}$ such that

$$\mathtt{P} = \left\{ x \in E : \sum_{n=-\infty}^{\infty} w(n) |f \circ \phi_n(x)|^\alpha \frac{dm \circ \phi_n}{dm}(x) = \infty \right\}$$

up to a set of m-measure zero. Then for m_1-a.e. $y \in \mathtt{P}^\psi$,

$$\sum_{n=-\infty}^{\infty} w(n) |g \circ \psi_n(y)|^\alpha \frac{dm_1 \circ \psi_n}{dm_1}(y) = \infty.$$

Since, for m-a.e. $x \in \mathtt{N}$,

$$\sum_{n=-\infty}^{\infty} w(n) |g \circ \psi_n(\Phi(x))|^\alpha \frac{dm_1 \circ \psi_n}{dm_1}(\Phi(x)) < \infty,$$

we see that $\Phi(x) \in \mathtt{N}^\psi$ for m-a.e. $x \in \mathtt{N}$. Hence,

$$\Phi^{-1}(\mathtt{P}^\psi) \subset \mathtt{P} \qquad \text{modulo a set of } m\text{-measure zero,} \tag{2.20}$$

and (2.18) follows from (2.19) and (2.20).

In the case $T = \mathbb{R}$, construct the functions g_+ and g_- as in (2.7) and (2.10) for the flow (ϕ_t) and the corresponding functions g_+^ψ and g_-^ψ for the flow (ψ_t). Notice that $g_+(x) = |h(x)|^\alpha g_+^\psi(\Phi(x))$ and $g_-(x) = |h(x)|^\alpha g_-^\psi(\Phi(x))$ m-a.e. This reduced everything to the discrete time case. For example, there is a $w \in \mathcal{W}$ such that

$$\begin{aligned} \mathtt{P}^\psi = \Bigg\{ y \in S : &\sum_{n=0}^{\infty} w(n) |g_+^\psi \circ \psi_n(y)| \frac{dm_1 \circ \psi_n}{dm_1}(y) \\ &+ \sum_{n=-\infty}^{0} w(n) |g_-^\psi \circ \psi_n(y)| \frac{dm_1 \circ \psi_n}{dm_1}(y) = \infty \Bigg\} \end{aligned}$$

up to a set of m_1-measure zero, and so on, and so we obtain (2.18) as in the discrete time case.

Once (2.18) has been established, the claim $\mathbf{X}^P \stackrel{\mathrm{d}}{=} \mathbf{X}_\psi^P$ and $\mathbf{X}^N \stackrel{\mathrm{d}}{=} \mathbf{X}_\psi^N$ follows as in the proof of Theorem 4.3 in [17]. $\square$

REMARK 2.5. Notice the relationship between the conservative–dissipative decomposition (1.4) and the positive-null decomposition (2.4) of a stationary SαS process. Since every dissipative flow is null, we have a unique in law decomposition

$$X(t) = X^D(t) + X^{CN}(t) + X^P(t), \qquad t \in T, \tag{2.21}$$

into a sum of three independent stationary SαS processes: one generated by a dissipative flow (the dissipative component of $\mathbf{X}$), one generated by a conservative null flow (the conservative null part of $\mathbf{X}$) and one generated by a positive flow (the positive part of $\mathbf{X}$). In particular, we have the relations

$$X^N(t) = X^D(t) + X^{CN}(t), \qquad t \in T, \tag{2.22}$$

and

$$X^C(t) = X^P(t) + X^{CN}(t), \qquad t \in T. \tag{2.23}$$

REMARK 2.6. More explicit descriptions of different components in the decompositions described in Remark 2.5 are desirable. For the dissipative component $\mathbf{X}^D$, we have the mixed moving average representation of Rosiński [17]:

$$X^D(t) = \int_W \int_{\mathbb{R}} f(v, x - t)\, M(dv, dx), \qquad t \in \mathbb{R}, \tag{2.24}$$

with M an SαS random measure on a product measurable space $(W \times \mathbb{R}, \mathcal{W} \times \mathcal{B})$ with control measure $m = \nu \times \text{Leb}$, where ν is a σ-finite measure on $(W, \mathcal{W})$, and $f \in L^\alpha(m, \mathcal{W} \times \mathcal{B})$. This representation has proven to be very useful in investigations of properties of processes generated by dissipative flows; see, for example, [11, 14, 20, 21].

Consider now the positive component $\mathbf{X}^P$. Assuming this component is not degenerate (identically zero), take any representation (1.1) of the form (1.2), and recall that there is a probability measure μ on $\mathtt{P}$ invariant under the flow (ϕ_t) and equivalent to the restriction of the control measure m to $\mathtt{P}$. Therefore, we can write, in law,

$$X^P(t) = \int_{\mathtt{P}} a_t(x) g \circ \phi_t(x) M_\mu(dx), \qquad t \in T, \tag{2.25}$$

where M_μ is an SαS random measure on E with control measure μ, (a_t) is a length 1 cocycle for the flow (ϕ_t), and

$$g(x) = f(x)\left(\frac{dm}{d\mu}(x)\right)^{1/\alpha}, \qquad x \in E.$$

This leads us to the following representation of stationary SαS processes generated by positive flows.

Let $(\Omega', \mathcal{F}', P')$ be a probability space, and $(\theta_t)_{t\in T}$ a measurable group of shift operators (maps of Ω' into itself). Recall that a stochastic process $\mathbf{A} = (A(t), t \in T)$ is called a (raw) multiplicative functional if, for every $s, t \in \mathbb{R}$,

$$A(t) = A(s)A(t) \circ \theta_s \qquad P'\text{-a.s.},$$

see, for example, [25]. Let $\mathbf{A}$ be a multiplicative process with $|A(t)| = 1$ for all $t \in T$. Let $\mathbf{B} = (B(t), t \in T) = (B(0) \circ \theta_t, t \in T)$ be a compatible with the shift (θ_t) stationary process on $(\Omega', \mathcal{F}', P')$, such that $E'|B(0)|^\alpha < \infty$.

Let, finally, M be an SαS random measure on $(\Omega', \mathcal{F}')$ with control measure P' [defined on a generic probability space $(\Omega, \mathcal{F}, P)$]. Then

$$X(t) = \int_{\Omega'} A(t)B(t)\, dM', \qquad t \in T \tag{2.26}$$

is, clearly, a symmetric SαS process [on $(\Omega, \mathcal{F}, P)$] corresponding to a positive flow. Furthermore, the expression (2.25) shows that every process corresponding to a positive flow has a representation of the form (2.26).

The representation in (2.26) describes the positive component of a stationary SαS process in terms of a stationary process with a finite moment of order α and a multiplicative functional of absolute value 1. It is an attractive representation from purely probabilistic point of view because both stationary processes and multiplicative functionals are important and widely studied objects of their own.

Several particular cases have been long considered in literature. If the multiplicative functional $\mathbf{A}$ is identically equal to 1, then the SαS process has the form

$$X(t) = \int_{\Omega'} B(t)\, dM', \qquad t \in T, \tag{2.27}$$

an integral of an L^α stationary process. Processes of the form (2.27) have been known as *doubly stationary processes* (the term introduced, apparently, by Cambanis, Hardin and Weron [3]) with a finite control measure. The further particular cases where the stationary process $\mathbf{B}$ is a stationary zero mean Gaussian process or a symmetric β-stable process with $\alpha < \beta < 2$ are known as, correspondingly, sub-Gaussian or sub-stable processes and have an alternative representation as

$$X(t) = c_{\alpha,\beta} W^{1/\beta} B(t), \qquad t \in T, \tag{2.28}$$

where now $\mathbf{X}$ and $\mathbf{B}$ live on the same probability space, $c_{\alpha,\beta}$ is a positive constant for $\alpha < \beta \leq 2$, and W is an independent of $\mathbf{B}$ positive strictly α/β-stable random variable. See [22].

Another important example of processes generated by positive flows that has been long studied is that of *harmonizable processes*. It corresponds to taking the stationary process $\mathbf{B}$ to be constant and the multiplicative functional $\mathbf{A}$ of a special form. See [17].

No "canonical" representation of the component $\mathbf{X}^{CN}$ seems to be known at this time!

3. Ergodicity. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $(\theta_t)_{t\in T}$ a measurable group of shift operators preserving P. Let $X(t) = X(0) \circ \theta_t$, $t \in \mathbb{R}$ be a stationary process. Recall that a process $\mathbf{X}$ is *ergodic* if, for every events $A, B \in \sigma(X(t), t \in \mathbb{R})$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P(A \cap B \circ \theta_t)\lambda(dt) = P(A)P(B).$$

A process $\mathbf{X}$ is *weakly mixing* if, for all A and B as above,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |P(A\cap B\circ\theta_t)-P(A)P(B)|\,\lambda(dt)=0.$$

In general, weak mixing is a stronger condition than ergodicity, but the two notions coincide for both stationary Gaussian processes (see [9]) and for stationary infinitely divisible processes, see [19] (for stationary SαS processes, this statement was established by Podgórski [13]). Furthermore, a stationary Gaussian processes is ergodic (weakly mixing) if and only if its spectral measure is atomless (again, [9]).

The following result shows that, for stationary SαS processes with $0<\alpha<2$, ergodicity is related to absence of the component corresponding to a positive flow.

THEOREM 3.1. *A stationary SαS process is ergodic* (*equivalently*, *weakly mixing*) *if and only if the component* $\mathbf{X}^P$ *in* (2.21) *corresponding to a positive flow vanishes*.

PROOF. We start with proving that a process generated by a null flow is weakly mixing. Consider first the case $T=\mathbb{Z}$. Let $\mathbf{X}$ be given in the form (1.1) with the kernel of the form (1.2). By Gross [4] (see also the discussion of weak mixing in [6]), weak mixing is equivalent to the following statement: for every compact set K bounded away from zero and $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} m\bigl(x:|f_0(x)|^\alpha\in K,\ |f_j(x)|^\alpha>\varepsilon\bigr)=0. \tag{3.1}$$

Fix K as above and let

$$A=\{x:|f_0(x)|^\alpha\in K\}.$$

Then $m(A)<\infty$. For $\theta>0$, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{j=0}^{n-1} m\bigl(x\in A:|f_j(x)|^\alpha>\varepsilon\bigr)\\
&\qquad=\int_A\frac{1}{n}\sum_{j=0}^{n-1}\mathbf{1}\bigl(x\in A:|f_j(x)|^\alpha>\varepsilon\bigr)\,m(dx)\\
&\qquad=\int_A\Biggl(\frac{1}{n}\sum_{j=0}^{n-1}\mathbf{1}\bigl(x\in A:|f_j(x)|^\alpha>\varepsilon\bigr)\Biggr)\mathbf{1}\Biggl(\frac{1}{n}\sum_{j=0}^{n-1}|f_j(x)|^\alpha>\theta\Biggr)m(dx)\\
&\qquad\quad+\int_A\Biggl(\frac{1}{n}\sum_{j=0}^{n-1}\mathbf{1}\bigl(x\in A:|f_j(x)|^\alpha>\varepsilon\bigr)\Biggr)\mathbf{1}\Biggl(\frac{1}{n}\sum_{j=0}^{n-1}|f_j(x)|^\alpha\le\theta\Biggr)m(dx)
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&\le m\left(x \in A : \frac{1}{n}\sum_{j=0}^{n-1} |f_j(x)|^\alpha > \theta\right) \\
&\quad + \varepsilon^{-1} \int_A \frac{1}{n}\sum_{j=0}^{n-1} |f_j(x)|^\alpha \mathbf{1}\left(\frac{1}{n}\sum_{j=0}^{n-1} |f_j(x)|^\alpha \le \theta\right) m(dx) \\
&\le m\left(x \in A : \frac{1}{n}\sum_{j=0}^{n-1} |f_j(x)|^\alpha > \theta\right) + \frac{\theta}{\varepsilon} m(A).
\end{aligned}
$$

Let $T : L^1(m) \to L^1(m)$ be defined by

$$
Tg = \frac{dm \circ \phi}{dm} g \circ \phi_1.
$$

Note that T is a positive contraction (actually, an isometry) on $L^1(m)$. Furthermore, $|f_j|^\alpha = T^j |f|^\alpha$ for all $j \ge 0$.

By the stochastic ergodic theorem (see Theorem 4.9, page 143 in [7]), the average

$$
\frac{1}{n}\sum_{j=0}^{n-1} |f_j|^\alpha = \frac{1}{n}\sum_{j=0}^{n-1} T^j |f|^\alpha
$$

converges in measure, on any set of a finite measure, to a nonnegative function $h \in L^1(m)$ which is invariant under T. This means that the finite measure $d\mu = h\,dm$ is invariant under the map ϕ. Since the flow is null, the measure μ must be the null measure and, hence, the limit function h must be equal to zero. The convergence in measure then gives us that

$$
\lim_{n\to\infty} m\left(x \in A : \frac{1}{n}\sum_{j=0}^{n-1} |f_j(x)|^\alpha > \theta\right) = 0
$$

for every $\theta > 0$. Letting in (3.2), first $n \to \infty$, and then letting $\theta \to 0$ verifies (3.1) and, hence, shows that a process generated by a null flow is weakly mixing.

Suppose now that $T = \mathbb{R}$. By Theorem 2 of [19], ergodicity of $\mathbf{X}$ is equivalent to

$$
\lim_{M\to\infty} \frac{1}{M} \int_0^M \exp\{2\|X(0)\|_\alpha^\alpha - \|X(t) - X(0)\|_\alpha^\alpha\}\, dt = 1. \tag{3.3}
$$

Here $\|X(t)\|_\alpha$ is simply the scaling parameter of the SαS random variable $X(t)$; see [22].

Recall that our processes are continuous in probability. Therefore, for every $\varepsilon > 0$, we can choose $\delta > 0$ such that $\|X(t) - X(s)\|_\alpha \le \varepsilon$ if $|t - s| \le \delta$. For such a pair of ε and δ, we have, for $N = 1, 2, \dots,$

$$
\frac{1}{N\delta} \int_0^{N\delta} \exp\{2\|X(0)\|_\alpha^\alpha - \|X(t) - X(0)\|_\alpha^\alpha\}\, dt = \frac{1}{N\delta} \sum_{j=1}^{N} I_j(\delta), \tag{3.4}
$$

where

$$I_j(\delta) = \int_{(j-1)\delta}^{j\delta} \exp\{2\|X(0)\|_\alpha^\alpha - \|X(t) - X(0)\|_\alpha^\alpha\}\, dt,$$

$j = 1, \ldots, N$.

Suppose that $1 < \alpha < 2$. Then $\|\cdot\|_\alpha$ satisfies the triangle inequality, and by the inequality

$$(a+b)^\alpha \le a^\alpha + \alpha a b^{\alpha-1} + b^\alpha,$$

for $a, b \ge 0$, we have, for every $j = 1, \ldots, N$ and $t \in [(j-1)\delta, j\delta]$,

$$\begin{aligned}\|X(t) - X(0)\|_\alpha^\alpha &\le \bigl(\varepsilon + \|X((j-1)\delta) - X(0)\|_\alpha\bigr)^\alpha \\ &\le \|X((j-1)\delta) - X(0)\|_\alpha^\alpha + \alpha\varepsilon \|X((j-1)\delta) - X(0)\|_\alpha^{\alpha-1} + \varepsilon^\alpha.\end{aligned}$$

Therefore,

$$\begin{aligned}I_j(\delta) &\ge \delta \exp\bigl\{2\|X(0)\|_\alpha^\alpha - \|X((j-1)\delta) - X(0)\|_\alpha^\alpha\bigr\} \\ &\quad \times \exp\bigl\{-\varepsilon^\alpha - \alpha\varepsilon \|X((j-1)\delta) - X(0)\|_\alpha^{\alpha-1}\bigr\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\liminf_{M\to\infty} \frac{1}{M} \int_0^M \exp\{2\|X(0)\|_\alpha^\alpha - \|X(t) - X(0)\|_\alpha^\alpha\}\, dt \\ &\quad \ge \exp\{-\varepsilon^\alpha - 2\alpha\varepsilon \|X(0)\|_\alpha^{\alpha-1}\} \\ &\qquad \times \liminf_{N\to\infty} \frac{1}{N} \sum_{j=1}^N \exp\bigl\{2\|X(0)\|_\alpha^\alpha - \|X((j-1)\delta) - X(0)\|_\alpha^\alpha\bigr\}.\end{aligned}$$

Since the flow (ϕ_t) is null, so is the discrete time flow (ϕ_n) and, hence, the discrete time process $(X(n), n \in \mathbb{Z})$ is ergodic. Therefore,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^N \exp\bigl\{2\|X(0)\|_\alpha^\alpha - \|X((j-1)\delta) - X(0)\|_\alpha^\alpha\bigr\} = 1.$$

Letting $\varepsilon \to 0$, we conclude that

$$\liminf_{M\to\infty} \frac{1}{M} \int_0^M \exp\{2\|X(0)\|_\alpha^\alpha - \|X(t) - X(0)\|_\alpha^\alpha\}\, dt \ge 1.$$

Similarly, using the inequality

$$(a-b)_+^\alpha \ge a^\alpha - \alpha a b^{\alpha-1},$$

for $1 < \alpha < 2$ and $a, b \ge 0$, we have, for every $j = 1, \ldots, N$ and $t \in [(j-1)\delta, j\delta]$,

$$\begin{aligned}\|X(t) - X(0)\|_\alpha^\alpha &\ge \bigl(\|X((j-1)\delta) - X(0)\|_\alpha - \varepsilon\bigr)_+^\alpha \\ &\ge \|X((j-1)\delta) - X(0)\|_\alpha^\alpha - \alpha\varepsilon^{\alpha-1} \|X((j-1)\delta) - X(0)\|_\alpha,\end{aligned}$$

and the same argument as above gives us

$$\limsup_{M\to\infty} \frac{1}{M} \int_0^M \exp\{2\|X(0)\|_\alpha^\alpha - \|X(t) - X(0)\|_\alpha^\alpha\} \, dt \le 1.$$

Therefore, (3.3) holds. The argument for (3.3) in the case $0 < \alpha \le 1$ is similar (and even easier). Hence, a process generated by a null flow is ergodic (weakly mixing) in the case $T = \mathbb{R}$ as well.

In the opposite direction, suppose that the component $\mathbf{X}^P$ in (2.21) corresponding to a positive flow does not vanish. The fact that $\mathbf{X}^P$ is not ergodic follows from Corollary 4.2 in [4]. Therefore, there are two stationary processes with different finite-dimensional distributions, $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ and $0 < p < 1$, such that

$$\mathbf{X}^P = \begin{cases} \mathbf{Y}^{(1)} & \text{with probability } p, \\ \mathbf{Y}^{(2)} & \text{with probability } 1 - p \end{cases}$$

(to see this, view the process $\mathbf{X}^P$ as defined on the canonical path space, take an invariant event A with probability $0 < p < 1$, and let $\mathbf{Y}^{(1)}$ have the law of $\mathbf{X}^P$ conditioned on belonging to A, and let $\mathbf{Y}^{(2)}$ have the law of $\mathbf{X}^P$ conditioned on belonging to A^c).

Therefore, by (2.17),

$$\mathbf{X} = \begin{cases} \mathbf{X}^N + \mathbf{Y}^{(1)} & \text{with probability } p, \\ \mathbf{X}^N + \mathbf{Y}^{(2)} & \text{with probability } 1 - p. \end{cases} \tag{3.5}$$

Since the characteristic functions of infinitely divisible random vectors (and, in particular, of SαS random vectors) do not vanish, we see that the processes $\mathbf{X}^N + \mathbf{Y}^{(1)}$ and $\mathbf{X}^N + \mathbf{Y}^{(2)}$ have different finite dimensional distributions. Therefore, (3.5) implies that $\mathbf{X}$ is not ergodic. $\square$

REMARK 3.2. It is known (see [26]) that all stationary SαS processes generated by dissipative flows are mixing, and we have seen that all processes generated by positive flows are not ergodic. On the other hand, there are processes generated by conservative null flows that are ergodic but not mixing, simply because ergodicity and mixing are not equivalent for stationary SαS processes; see [5].

Combining the results of Theorem 3.1 above and of Theorem 4.1 in [20], we obtain the following interesting characterization of processes corresponding to conservative null flows.

COROLLARY 3.3. *A stationary SαS process $(X_n, n \in \mathbb{Z})$ is generated by a conservative null flow if and only if it is ergodic and $n^{-1/\alpha} \max_{j=1,\dots,n} |X_j| \to 0$ in probability as $n \to \infty$.*

A similar characterization holds for locally bounded continuous time stationary SαS processes (see [21]).

4. Examples. In this section we consider several examples of both discrete and continuous time stationary SαS processes and see how the notions developed in the previous sections apply here.

EXAMPLE 4.1. Consider a bilateral real-valued Markov chain with law $P_x(\cdot)$, $x \in \mathbb{R}$, admitting an infinite σ-finite invariant measure π. Define a σ-finite measure m on $E = \mathbb{R}^{\mathbb{Z}}$ by

$$m(\cdot) = \int_{\mathbb{R}} P_x(\cdot)\pi(dx).$$

Suppose that the Markov chain is m-irreducible and Harris recurrent (see, e.g., [10]).

Let $\mathbf{X}$ be a stationary discrete-time SαS process defined by the integral representation (1.1), with M being an SαS random measure with control measure m, and

$$f_n(x) = a_n(x) f \circ \phi_n(x), \qquad x \in E,\ n = 0, 1, 2, \dots,$$

where ϕ is the left shift operator on E, $f \in L^\alpha(m)$, and a a ± 1-valued cocycle for the flow (ϕ_n).

This process was considered in [18] who showed that this process is generated by a conservative flow and is mixing. By Theorem 3.1, we conclude that this process corresponds to a conservative null flow.

EXAMPLE 4.2. Here we consider the class of continuous time stationary SαS processes corresponding to the so-called *cyclic flows* introduced by Pipiras and Taqqu [12]. These processes have a representation of the form

$$X(t) = \int_S \int_{[0,q(z))} b(z)^{[v+s(z)t]_{q(z)}} g\big(z, \{v + s(z)t\}_{q(z)}\big) M(dz, dv), \qquad t \in \mathbb{R}, \tag{4.1}$$

where $(S, \mathcal{S})$ is a measurable space, b, q and s are measurable (possibly, complex-valued) functions on S, such that $|b(z)| = 1$, $q(z) > 0$ and $s(z) \neq 0$ for all $z \in S$. Furthermore, M is an SαS random measure on $E = \{(z, v) \in S \times \mathbb{R}_+ : 0 \le v < q(z)\}$ with control measure $m = \sigma \times \lambda$, where σ is a σ-finite measure on S. Finally, $g \in L^\alpha(m)$ is assumed to be such that

$$\sigma\{z \in S : g(z, \cdot) = 0 \text{ a.e. on } [0, q(z))\} = 0 \tag{4.2}$$

(the full support assumption). In (4.1), $[\cdot]_a$ and $\{\cdot\}_a$ are, respectively, the integer part and the fractional part with respect to a positive number a.

We will see that such a process is generated by a positive flow. We use Corollary 2.2. Assume, for example, that $s(z) > 0$. For any $w \in \mathcal{W}$,

$$
\begin{aligned}
\int_{-\infty}^{\infty} & w(t)|f_t(z,v)|^\alpha \lambda(dt) \\
&= \int_{-\infty}^{\infty} w(t)\big|g\big(z, \{v + s(z)t\}_{q(z)}\big)\big|^\alpha \lambda(dt) \\
&= \frac{1}{s(z)} \int_{-\infty}^{\infty} w\left(\frac{t-v}{s(z)}\right)\big|g\big(z, \{t\}_{q(z)}\big)\big|^\alpha \lambda(dt) \qquad (4.3)\\
&= \frac{1}{s(z)} \sum_{n=-\infty}^{\infty} \int_0^{q(z)} w\left(\frac{t-v+nq(z)}{s(z)}\right)|g(z,t)|^\alpha \lambda(dt) \\
&= \frac{1}{s(z)} \int_0^{q(z)} |g(z,t)|^\alpha \left(\sum_{n=-\infty}^{\infty} w\left(\frac{t-v+nq(z)}{s(z)}\right)\right)\lambda(dt).
\end{aligned}
$$

Notice that, for any v, z and t,

$$\sum_{n=-\infty}^{\infty} w\left(\frac{t-v+nq(z)}{s(z)}\right) \geq \int_{(v-t)_+/q(z)+1}^{\infty} w\left(\frac{t-v+xq(z)}{s(z)}\right) dx = \infty.$$

Therefore, by (4.3) and the full support assumption (4.2), we conclude that

$$\int_{-\infty}^{\infty} w(t)|f_t(z,v)|^\alpha \lambda(dt) = \infty$$

for m-a.e. $(z, v) \in E$ such that $s(z) > 0$. Since the argument in the case $s(z) < 0$ is the same, by Corollary 2.2, the cyclic process is generated by a positive flow.

EXAMPLE 4.3. Let $E = \Omega_1 \times \mathbb{R}$, where $(\Omega_1, \mathcal{F}_1, P_1)$ is the canonical probability space $\Omega_1 = \mathbb{R}^{\mathbb{R}}$ with the cylindrical σ-field, and P_1 such that $(Y(t,\omega_1) = \omega_1(t),\ t \in \mathbb{R},\ \omega_1 \in \Omega_1)$ is a measurable process with stationary increments.

Let M be an SαS random measure on E with control measure $P_1 \times \lambda$ (where λ is, as usual, the Lebesgue measure on $\mathbb{R}$). Let $\varphi \in L^\alpha(\lambda)$ and

$$X(t) = \int_{\Omega_1} \int_{-\infty}^{\infty} \varphi\big(Y(t,\omega_1) + z\big) M(d\omega_1, dz), \qquad t \in \mathbb{R}. \tag{4.4}$$

Notice that

$$
\begin{aligned}
\int_{\Omega_1} \int_{-\infty}^{\infty} & \big|\varphi\big(Y(t,\omega_1) + z\big)\big|^\alpha m(d\omega_1, dz) \\
&= \int_{-\infty}^{\infty} E_1\big|\varphi\big(Y(t) + z\big)\big|^\alpha \lambda(dz) \\
&= \int_{-\infty}^{\infty} \varphi(z)\lambda(dz) < \infty
\end{aligned}
$$

for every $t \in \mathbb{R}$, and so the process $\mathbf{X}$ in (4.4) is well defined. Furthermore, let $(\theta_t)_{t\in\mathbb{R}}$ be the measurable group of left shift operators on Ω_1, and let $\phi_t(\omega_1, z) = (\theta_t(\omega_1), z)$ be the induced flow on E. Notice that the measure m is preserved under this flow. To see this notice that, for every $t_1 < t_2 < \cdots < t_k$ and $t > 0$,

$$\begin{aligned}
&m\{(\omega_1, z) \in E : Y(t_1 + t, \omega_1) + z \in A, Y(t_2 + t, \omega_1) - Y(t_1 + t, \omega_1) \in B_1, \dots, \\
&\qquad\qquad\qquad\qquad Y(t_k + t, \omega_1) - Y(t_1 + t, \omega_1) \in B_{k-1}\} \\
&\quad = E_1\biggl[\mathbf{1}\bigl(Y(t_2 + t, \omega_1) - Y(t_1 + t, \omega_1) \in B_1, \dots, \\
&\qquad\qquad Y(t_k + t, \omega_1) - Y(t_1 + t, \omega_1) \in B_{k-1}\bigr) \\
&\qquad\qquad \times \int_{-\infty}^{\infty} \mathbf{1}\bigl(Y(t_1 + t, \omega_1) + z \in A\bigr)\lambda(dz)\biggr] \\
&\quad = \lambda(A) P_1\bigl(Y(t_2 + t, \omega_1) - Y(t_1 + t, \omega_1) \in B_1, \dots, \\
&\qquad\qquad Y(t_k + t, \omega_1) - Y(t_1 + t, \omega_1) \in B_{k-1}\bigr) \\
&\quad = \lambda(A) P_1\bigl(Y(t_2, \omega_1) - Y(t_1, \omega_1) \in B_1, \dots, Y(t_k, \omega_1) - Y(t_1, \omega_1) \in B_{k-1}\bigr) \\
&\quad = m\{(\omega_1, z) \in E : Y(t_1, \omega_1) + z \in A, Y(t_2, \omega_1) - Y(t_1, \omega_1) \in B_1, \dots, \\
&\qquad\qquad\qquad\qquad Y(t_k, \omega_1) - Y(t_1, \omega_1) \in B_{k-1}\}
\end{aligned}$$

by the stationarity of the increments of $\mathbf{Y}$, for all Borel sets $A, B_1, \dots, B_{k-1}$ with $\lambda(A) < \infty$. Hence, the flow (ϕ_t) preserves the measure of every measurable finite dimensional rectangle, hence, it preserves the measure m. Therefore, the process in (4.4) is a stationary SαS process.

We will prove that, under the assumption

$$|Y(t)| \to \infty \qquad \text{in } P_1\text{-probability as } |t| \to \infty, \tag{4.5}$$

the process in (4.4) corresponds to a null flow.

To this end, we use Corollary 2.2. It is enough to show that, for some $w \in \mathcal{W}$,

$$\int_{\Omega_1} \int_{-\infty}^{\infty} \biggl(\int_{-\infty}^{\infty} w(t) \bigl|\varphi\bigl(Y(t, \omega_1) + z\bigr)\bigr|^{\alpha} \, dt\biggr) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz \, P_1(d\omega_1) < \infty. \tag{4.6}$$

Indeed, the integral in (4.6) can be written in the form $\int_{-\infty}^{\infty} w(t)h(t)\,dt$, where

$$h(t) = E_1\bigl|\varphi\bigl(Y(t) + G\bigr)\bigr|^{\alpha}, \qquad t \in \mathbb{R},$$

and G is an independent (under P_1) of $Y(t)$ standard normal random variable. Therefore, if we show that

$$\lim_{|t|\to\infty} h(t) = 0, \tag{4.7}$$

then it is clear how to choose a $w \in \mathcal{W}$ such that (4.6) holds. In order to prove (4.7), note that, for every $M > 0$,

$$\begin{aligned} h(t) &= E_1\bigl[\bigl|\varphi\bigl(Y(t)+G\bigr)\bigr|^\alpha \mathbf{1}\bigl(|Y(t)| \le M\bigr)\bigr] + E_1\bigl[\bigl|\varphi\bigl(Y(t)+G\bigr)\bigr|^\alpha \mathbf{1}\bigl(|Y(t)| > M\bigr)\bigr] \\ &:= h_1(t; M) + h_2(t; M). \end{aligned}$$

By (4.5), we immediately see that, for every M,

$$h_1(t; M) \le \frac{1}{\sqrt{2\pi}} \|\varphi\|_\alpha^\alpha P_1\bigl(|Y(t)| \le M\bigr) \to 0$$

as $|t| \to \infty$. On the other hand, by the dominated convergence theorem,

$$\lim_{|y|\to\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-y)^2/2} |\varphi(z)|^\alpha \, dz = 0.$$

Therefore,

$$\lim_{M\to\infty} \limsup_{|t|\to\infty} h_2(t; M) = 0,$$

and so (4.7) follows.

Notice that if (4.5) fails, then the process $\mathbf{X}$ will not, in general, be generated by a null flow. For example, if the process $\mathbf{Y}$ is, actually, stationary under P_1, then the process $\mathbf{X}$ is generated by a positive flow. To see this, notice that, in this case, the process $\mathbf{X}$ can be represented in the form (2.26) with $\Omega' = \Omega_1 \times \mathbb{R}$, $P' = P_1 \times P_G$, $\mathbf{A} \equiv 1$ and $B(t) = f(z)^{-1/\alpha} \varphi(Y(t, \omega_1) + z)$, $t \in \mathbb{R}$, where P_G is the standard Gaussian law on $\mathbb{R}$ and f is the density of this law.

Under the assumption (4.5), both conservative null and dissipative flows are possible. For example, if we strengthen the assumption (4.5) to

$$|Y(t)| \to \infty \qquad P_1\text{-a.s. as } |t| \to \infty, \tag{4.8}$$

then the process $\mathbf{X}$ is generated by a dissipative flow. To see this, note that, under assumption (4.8), there is a bounded strictly positive and monotone on both $(-\infty, 0]$ and $[0, \infty)$ deterministic function $\tilde{\varphi} \in L^\alpha(\lambda)$ such that

$$\int_{-\infty}^{\infty} \tilde{\varphi}\bigl(\tfrac{1}{2} Y(t)\bigr) \, dt < \infty$$

P_1-a.s. If we replace the function φ in the definition of the process $\mathbf{X}$ by $\tilde{\varphi}$, we immediately conclude by Corollary 4.2 of [17] that the new SαS process is generated by a dissipative flow. Since the kernel of the new process has support at least as large as that of the kernel of the process $\mathbf{X}$, the latter process is also generated by a dissipative flow.

On the other hand, if, for example, the process $\mathbf{Y}$ is, under P_1, a fractional Brownian motion with any $0 < H < 1$, then the process $\mathbf{X}$ is generated by a conservative flow (see Section 3 in [21]) and, hence, it corresponds to a conservative null flow.

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