SELF-INTERACTING DIFFUSIONS. III. SYMMETRIC INTERACTIONS[1]

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Dedicated to Morris W. Hirsch 70's birthday

Let M be a compact Riemannian manifold. A *self-interacting diffusion* on M is a stochastic process solution to

$$dX_t = dW_t(X_t) - \frac{1}{t}\left(\int_0^t \nabla V_{X_s}(X_t)\,ds\right)dt,$$

where $\{W_t\}$ is a Brownian vector field on M and $V_x(y) = V(x, y)$ a smooth function. Let $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\,ds$ denote the normalized occupation measure of X_t. We prove that, when V is symmetric, μ_t converges almost surely to the critical set of a certain nonlinear free energy functional J. Furthermore, J has generically finitely many critical points and μ_t converges almost surely toward a local minimum of J. Each local minimum has a positive probability to be selected.

1. Introduction. Let M be a C^∞ d-dimensional, compact connected Riemannian manifold without boundary and $V : M \times M \to \mathbb{R}$ be a smooth function called a *potential*. For every Borel probability measure μ on M, let $V\mu : M \to \mathbb{R}$ denote the smooth function defined by

$$(1) \qquad V\mu(x) = \int_M V(x, u)\mu(du),$$

and let $\nabla(V\mu)$ denote its gradient (computed with respect to the Riemannian metric on M).

A *self-interacting diffusion process* associated to V is a continuous time stochastic process living on M solution to the stochastic differential equation (SDE)

$$(2) \qquad dX_t = \sum_{i=1}^N F_i(X_t) \circ dB_t^i - \tfrac{1}{2}\nabla(V\mu_t)(X_t)\,dt, \qquad X_0 = x \in M,$$

where $(B^1, \ldots, B^N)$ is a standard Brownian motion on $\mathbb{R}^N$, $\{F_i\}$ is a family of smooth vector fields on M such that

$$(3) \qquad \sum_{i=1}^N F_i(F_i f) = \Delta f$$

[for $f \in C^\infty(M)$], where Δ denotes the Laplacian on M; and

$$(4) \qquad \mu_t = \frac{1}{t} \int_0^t \delta_{X_s}\, ds$$

is the *empirical occupation measure* of $\{X_t\}$.

In absence of drift [i.e., $V(x, y) = 0$], $\{X_t\}$ is just a Brownian motion on M. If $V(x, y) = V(x)$, then it is a diffusion process on M. However, for a general function V, such a process is characterized by the fact that the drift term in equation (2) depends both on the position of the process and its empirical occupation measure up to time t.

Self-interacting diffusions (as defined here) were introduced in [3], and we refer the reader to this paper for a more detailed definition and basic properties.

It is worth pointing out that equation (2) presents some strong similarities with the following class of SDE:

$$(5) \qquad dY_t = dB_t - \left(\int_0^t v'(Y_s - Y_t)\, ds \right) dt,$$

whose behavior has been the focus of much attention in the recent years (see, e.g., [9, 10, 12, 14, 21, 24] or [22] for a recent overview and further references about reinforced random processes). The main differences being the following:

(i) The SDE (2) lives on an arbitrary but compact manifold, while (5) lives on $\mathbb{R}$ or $\mathbb{R}^d$.

(ii) The drift term in (5) depends on the *nonnormalized* occupation measure

$$t\mu_t = \int_0^t \delta_{X_s}\, ds.$$

A major goal in understanding (2) is

(a) to provide tools allowing to analyze the long term behavior of $\{\mu_t\}$; and, using these tools,

(b) to identify (at least partially) general classes of potential leading to certain types of behaviors.

A first step in this direction has been achieved in [3], where it is shown that the asymptotic behavior of $\{\mu_t\}$ can be precisely described in terms of a certain deterministic semi-flow $\Psi = \{\Psi_t\}_{t \geq 0}$ defined on the space of Borel probability measures on M. For instance, there are situations (depending on the shape of V) in which $\{\mu_t\}$ converges almost surely to an equilibrium point μ_∞ of Ψ (μ_∞ is random) and other situations where the limit set of $\{\mu_t\}$ coincides almost surely with a periodic orbit for Ψ (see the examples in Section 4 of [3]).

The present paper adresses the second part of this program. The main result here is that

Symmetric interactions (i.e., symmetric potentials) force $\{\mu_t\}$ to converge almost surely toward the critical set of a certain nonlinear free-energy functional.

This result encompasses most of the examples considered in [3] and enlightens the results of [3] and [4]. It also allows to give a sensible definition of *self-attracting* or *repelling* diffusions.

The organization of the paper is as follows. Section 2 defines the class of potentials considered here, gives some examples and states the main results. Section 3 reviews some material from [3] on which rely the analysis. Sections 4, 5, 6 and the Appendix are devoted to the proofs.

2. Hypotheses and main results. We assume throughout that V is a C^3 map (this regularity condition can be slightly weakened (see Hypothesis 1.4 in [3])) and that

HYPOTHESIS 2.1 (*Standing assumption*). V is symmetric:
$$V(x, y) = V(y, x).$$

Recall that λ denotes the Riemannian probability on M. We will sometime use the following additional hypothesis:

HYPOTHESIS 2.2 (*Occasional assumption* 1).   The mapping

(6)
$$V\lambda : x \mapsto V\lambda(x) = \int_M V(x, y)\lambda(dy)$$

is constant.

This later condition has the interpretation that if the empirical occupation measure of X_t is (close to) λ, then the drift term $\nabla(V\mu_t)(X_t)$ is (close to) zero. In other words, if the process has visited M "uniformly" between times 0 and t, then it has no preferred directions and behaves like a Brownian motion.

Notation.   Throughout we let $C^0(M)$ denote the Banach space of real-valued continuous functions $f : M \to \mathbb{R}$, equipped with the supremum norm

$$\|f\|_\infty = \sup_{x \in M} |f(x)|.$$

Given a positive function $g \in C^0(M)$, we let $\langle \cdot, \cdot \rangle_g$ denote the inner product on $C^0(M)$ defined by

$$\langle u, v \rangle_g = \int_M u(x)v(x)g(x)\lambda(dx).$$

When $g = 1$, we usually write $\langle \cdot, \cdot \rangle_\lambda$ (instead of $\langle \cdot, \cdot \rangle_1$) and $\|f\|_\lambda$ for $\sqrt{\langle f, f \rangle_\lambda}$.

The completion of $C^0(M)$ for the norm $\|f\|_\lambda$ is the Hilbert space $L^2(\lambda)$. We sometimes use the notation $\mathbf{1}$ to denote the function on M taking value one everywhere; and

$$L_0^2(\lambda) = \mathbf{1}^\perp = \{h \in L^2(\lambda) : \langle h, \mathbf{1} \rangle_\lambda = 0\}.$$

We let $\mathcal{M}(M)$ denote the space of Borel bounded measures on M and $\mathcal{P}(M)$ the subset of Borel probabilities. For $\mu \in \mathcal{M}(M)$ and $f \in C^0(M)$, we set

$$\mu f = \int_M f(x)\mu(dx) \tag{7}$$

and

$$|\mu| = \sup\{|\mu f| : f \in C^0(M), \ \|f\|_\infty = 1\}. \tag{8}$$

We let $\mathcal{M}_s(M)$ denote the Banach space $(\mathcal{M}(M), |\cdot|)$ [i.e., the dual of $C^0(M)$] and $\mathcal{M}_w(M)$ [resp. $\mathcal{P}_w(M)$] the metric space obtained by equipping $\mathcal{M}(M)$ [resp. $\mathcal{P}(M)$] with the narrow (or weak*) topology. In particular, $\mathcal{P}_w(M)$ is a compact subspace of $\mathcal{M}_w(M)$. Recall that the narrow topology is the topology induced by the family of semi-norms $\{\mu \mapsto |\mu f| : f \in C^0(M)\}$. Hence, $\mu_n \to \mu$ in $\mathcal{M}_w(M)$ if and only if $\mu_n f \to \mu f$ for all $f \in C^0(M)$.

Everywhere in the paper a subset of a topological space inherits the induced topology.

The operator V. The function V induces an operator

$$V : \mathcal{M}_s(M) \to C^0(M),$$

defined by

$$V\mu(x) = \int_M V(x, y)\mu(dy). \tag{9}$$

If $g \in L^2(\lambda)$, we write Vg for $V(g\lambda)$, where $g\lambda$ stands for the measure whose Radon–Nikodym derivative with respect to λ is g.

The following basic lemma will be used in several places:

LEMMA 2.3. (i) *The operator $V : \mathcal{M}_s(M) \to C^0(M)$ and its restriction to $L^2(\lambda)$ [defined by $g \mapsto V(g\lambda)$] are compact operators.*

(ii) *V maps continuously $\mathcal{P}_w(M)$ into $C^0(M)$.*

PROOF. (i) Let $\mu \in \mathcal{M}_s(M)$. Then $\|V\mu\|_\infty \leq \|V\|_\infty |\mu|$ and $|V\mu(u) - V\mu(v)| \leq (\sup_{z \in M} |V(u, z) - V(v, z)|)|\mu|$. Therefore, the set $\{V\mu : |\mu| \leq 1\}$ is bounded and equicontinuous, hence, relatively compact in $C^0(M)$ by Ascoli's theorem. This proves that V is compact.

By definition, $V|L^2(\lambda)$ is the composition of V with the bounded operator $g \in L^2(\lambda) \to g\lambda \in \mathcal{M}_s(M)$. It is then compact.

(ii) Let $\{\mu_n\}$ be a converging sequence in $\mathcal{P}_w(M)$ and $\mu = \lim_{n\to\infty} \mu_n$. Narrow convergence implies that $V\mu_n(u) \to V\mu(u)$ for all $u \in M$. Since, by (i), $\{V\mu_n\}$ is relatively compact in $C^0(M)$, it follows that $V\mu_n \to V\mu$ in $C^0(M)$. $\square$

2.1. *The global convergence theorem.* Let $\Pi = \Pi_V : \mathcal{P}_w(M) \to \mathcal{P}_w(M)$ be the map (we use the notation Π_V for Π when we want to emphasize the dependency on V) defined by

$$(10) \qquad\qquad \Pi(\mu)(dx) = \xi(V\mu)(x)\lambda(dx),$$

where $\xi : C^0(M) \to C^0(M)$ is the function defined by

$$(11) \qquad\qquad \xi(f)(x) = \frac{e^{-f(x)}}{\int_M e^{-f(y)}\lambda(dy)}.$$

The *limit set* of $\{\mu_t\}$ denoted $L(\{\mu_t\})$ is the set of limits [in $\mathcal{P}_w(M)$] of convergent sequences $\{\mu_{t_k}\}$, $t_k \to \infty$.

The following theorem describes $L(\{\mu_t\})$ in terms of Π. It is proved in Section 4.

THEOREM 2.4.   *With probability* 1, $L(\{\mu_t\})$ *is a compact connected subset of*

$$(12) \qquad\qquad \mathsf{Fix}(\Pi) = \{\mu \in \mathcal{P}_w(M) : \mu = \Pi(\mu)\}.$$

This clearly implies the following:

COROLLARY 2.5.   *Assume Π has isolated fixed points. Then $\{\mu_t\}$ converges almost surely to a fixed point of Π.*

REMARK 2.6.   By Theorem 2.10 below, Π has *generically* isolated fixed points. Hence, the generic behavior of $\{\mu_t\}$ is convergence toward one of those fixed points.

2.2. *Fixed points of Π.* With Theorem 2.4 in hand, it is clear that our description of self-interacting diffusions (satisfying Hypothesis 2.1) on M relies on our understanding of the fixed points structure of Π.

Let

$$\mathcal{B}_1 = \{f \in C^0(M) : \langle f, \mathbf{1} \rangle_\lambda = 1\}$$

and

$$\mathcal{B}_0 = \{f \in C^0(M) : \langle f, \mathbf{1} \rangle_\lambda = 0\}.$$

Spaces $\mathcal{B}_0$ and $\mathcal{B}_1$ are, respectively, a Banach space and a Banach affine space parallel to $\mathcal{B}_0$.

Let

$$X = X_V : \mathcal{B}_1 \to \mathcal{B}_0$$

be the C^∞ vector field defined by

$$(13) \qquad\qquad X(f) = -f + \xi(Vf).$$

The following lemma relates fixed points of Π to the zeroes of X.

LEMMA 2.7. *Let $\mu \in \mathcal{P}(M)$. Then, μ is a fixed point of Π if and only if μ is absolutely continuous with respect to λ and $\frac{d\mu}{d\lambda}$ is a zero of X. Furthermore, the map*

(14)
$$j : \mathsf{Fix}(\Pi) \to X^{-1}(0),$$
$$\mu \mapsto \frac{d\mu}{d\lambda}$$

is a homeomorphism. In particular, $X^{-1}(0)$ is compact.

PROOF. The first assertion is immediate from the definitions. Continuity of j follows from the continuity of ξ and Lemma 2.3(ii). Continuity of j^{-1} is immediate since uniform convergence of $\{f_n\} \subset C^0(M)$ clearly implies the narrow convergence of $\{f_n \lambda\}$ to $f\lambda$. □

We shall now prove that the zeroes of X are the critical points of a certain functional. Let $\mathcal{B}_1^+$ be the open subset of $\mathcal{B}_1$ defined by

$$\mathcal{B}_1^+ = \left\{ f \in \mathcal{B}_1 : \inf_{x \in M} f(x) > 0 \right\}$$

and let $J = J_V : \mathcal{B}_1^+ \to \mathbb{R}$ be the C^∞ *free energy* function defined by

(15)
$$J(f) = \tfrac{1}{2} \langle Vf, f \rangle_\lambda + \langle f, \log(f) \rangle_\lambda.$$

REMARK 2.8. It has been pointed out to us by Malrieu [20] that the free energy J occurs naturally in the analysis of certain nonlinear diffusions used in the modeling of granular flows (see [6, 20]); and by Hofbauer [16] that a finite-dimensional version of J appears in the analysis of some ordinary differential equations in evolutionary game theory.

The following proposition shows that the zeroes of X are exactly the critical points of J and have the same type (i.e., sinks or saddles).

PROPOSITION 2.9. *Given $f \in \mathcal{B}_1^+$, let $\mathsf{T}(f) : C^0(M) \to \mathcal{B}_0$ be the operator defined by*

(16)
$$\mathsf{T}(f)h = fh - \langle f, h \rangle_\lambda f.$$

One has:

(i) $\forall u, v \in \mathcal{B}_0$,

$$D^2 J(f)(u, v) = \langle u, v \rangle_{1/f} + \langle Vu, v \rangle_\lambda = \langle (Id + \mathsf{T}(f) \circ V)u, v \rangle_{1/f}.$$

(ii) $\mathscr{B}_0$ *admits a direct sum decomposition*

$$\mathscr{B}_0 = \mathscr{B}_0^u(f) \oplus \mathscr{B}_0^c(f) \oplus \mathscr{B}_0^s(f),$$

where:

(a) $\mathscr{B}_0^u(f), \mathscr{B}_0^c(f), \mathscr{B}_0^s(f)$ *are closed subspaces invariant under* $(Id + \mathsf{T}(f) \circ V)$;

(b) $\mathscr{B}_0^c(f) = \{u \in \mathscr{B}_0 : (Id + \mathsf{T}(f) \circ V)u = 0\}$ *and* $Id + \mathsf{T}(f) \circ V$ *restricted to* $\mathscr{B}_0^u(f)$ *or* $\mathscr{B}_0^s(f)$ *is an isomorphism*;

(c) *Both* $\mathscr{B}_0^u(f)$ *and* $\mathscr{B}_0^c(f)$ *have finite dimension*;

(d) *The bilinear form* $D^2 J(f)$ *restricted to* $\mathscr{B}_0^u(f)$ [*resp.* $\mathscr{B}_0^c(f)$, *resp.* $\mathscr{B}_0^s(f)$] *is definite negative* (*resp. null, resp. definite positive*).

(iii) *We have*

$$DJ(f) = 0 \quad \Longleftrightarrow \quad X(f) = 0,$$

and in this case, for all $u \in \mathscr{B}_0$,

$$DX(f)u = -(Id + \mathsf{T}(f) \circ V)u.$$

PROOF.   (i) For all $u \in \mathscr{B}_0$,

(17) $DJ(f)u = \langle Vf + \log(f) + 1, u \rangle_\lambda = \langle Vf + \log(f), u \rangle_\lambda.$

Therefore,

$$D^2 J(f)(u, v) = \left\langle Vu + \frac{1}{f}u, v \right\rangle_\lambda = \langle Vu, v \rangle_\lambda + \langle u, v \rangle_{1/f},$$

which gives the first expression for $D^2 J(f)$. Since, for all $u, v \in \mathscr{B}_0$,

(18) $\langle \mathsf{T}(f)Vu, v \rangle_{1/f} = \langle Vu, v \rangle_\lambda - \langle f, Vu \rangle_\lambda \langle \mathbf{1}, v \rangle_\lambda = \langle Vu, v \rangle_\lambda,$

we get the second expression for $D^2 J(f)$.

(ii) Let K denote the operator $\mathsf{T}(f) \circ V$ restricted to $L_0^2(\lambda)$. Then K is compact (by Lemma 2.3) and self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{1/f}$ [by (18)]. It then follows, from the spectral theory of compact self-adjoint operators (see [19], Chapters XVII and XVIII), that:

(a) K has at most countably many real eigenvalues.

(b) The set of nonzero eigenvalues is either finite or can be ordered as $|c_1| > |c_2| > \cdots > 0$ with $\lim_{i \to \infty} c_i = 0$.

(c) The family $\{\mathscr{H}_c\}$ of eigenspaces, where c ranges over all the eigenvalues (including 0), forms an orthogonal decomposition of $L_0^2(\lambda)$.

(d) Each $\mathscr{H}_c$ has finite dimension provided $c \neq 0$.

We now set $\mathcal{B}_0^c(f) = \mathcal{H}_1$, $\mathcal{B}_0^u(f) = \oplus \mathcal{H}_d$, where d ranges over all eigenvalues > 1 and $\mathcal{B}_0^s(f) = (\mathcal{B}_0^c(f) \oplus \mathcal{B}_0^u(f))^{\perp} \cap \mathcal{B}_0$.

(iii) From (17), and by density of $\mathcal{B}_0$ in $L_0^2(\lambda)$, $DJ(f) = 0$ if and only if $Vf + \log(f) \in \mathbb{R}\mathbf{1}$. Since $f \in \mathcal{B}_1$, this is equivalent to $f = \xi(Vf)$. Now,

$$(19) \qquad\qquad DX(f) = -Id - \mathsf{T}(\xi(Vf)) \circ V.$$

Hence, $DX(f) = -Id - \mathsf{T}(f) \circ V$ when $X(f) = 0$. $\quad\square$

Let $f \in X^{-1}(0)$ or, equivalently, $\mu = f\lambda \in \mathsf{Fix}(\Pi)$. We say that f (resp. μ) is a *nondegenerate* zero or *equilibrium* of X (resp. a *nongenerate fixed point* of Π) if the space $\mathcal{B}_0^c(f)$ in the above decomposition reduces to zero. The *index* of f (resp. μ) is defined to be the dimension of $\mathcal{B}_0^u(f)$.

A nondegenerate zero of X (fixed point of Π) is called a *sink* if it has zero index and a *saddle* otherwise.

Let $C_{\mathrm{sym}}^k(M \times M)$, $k \geq 0$, denote the Banach space of C^k symmetric functions $V : M \times M \to \mathbb{R}$, endowed with the topology of C^k convergence. The following theorem gives some sense to the hypothesis (made in Theorems 2.12, 2.24 and 2.27 below) that fixed points of Π are nondegenerate. However, we will not make any other use of this theorem. The proof is given in the Appendix.

THEOREM 2.10. *Let $\mathcal{G}$ denote the set of $V \in C_{\mathrm{sym}}^k(M \times M)$ such that Π_V has nondegenerate fixed points. Then $\mathcal{G}$ is open and dense.*

REMARK 2.11. The key argument that will be used in the proof of the genericity Theorem 2.10 is Smale's infinite-dimensional version of Sard's theorem for Fredholm maps. This result by Smale is also at the origin of the Brouwer degree theory for Fredholm maps initially developed by Elworthy and Tromba [13]. A consequence of this degree theory (applied to X) is the following result:

THEOREM 2.12. *Suppose that every $\mu^* \in \mathsf{Fix}(\Pi)$ is nondegenerate. Let C_k, $k \geq 0$, denote the number of fixed point for Π having index k. Then*

$$\sum_{k \geq 0} (-1)^k C_k = 1.$$

2.3. *Self-repelling diffusions.* A function $K : M \times M \to \mathbb{R}$ is called a *Mercer kernel*, if K is continuous, symmetric and defines a positive operator in the sense that

$$\langle Kf, f \rangle_{\lambda} \geq 0$$

for all $f \in L^2(\lambda)$.

If, up to an additive constant [the dynamics (2) are unchanged if one replaces $V(x, y)$ by $V(x, y) + \beta$], V (resp. $-V$) is a Mercer kernel, we call $\{X_t\}$ [given

by (2)] a *self-repelling* (resp. *self-attracting* process). The following result and the examples below give some sense to this terminology (see, e.g., Examples 2.15, 2.16 and 2.19).

THEOREM 2.13.   *Suppose that, up to an additive constant, V is a Mercer kernel. Then*:

  (i) *$J = J_V$ is strictly convex.*
  (ii) Fix(Π) *reduces to a singleton $\{\mu^*\}$ and $\lim_{t \to \infty} \mu_t = \mu^*$ almost surely. If we, furthermore, assume that Hypothesis* 2.2 *holds, then $\mu^* = \lambda$.*

PROOF.   It follows from the definition of J, Proposition 2.9 and Theorem 2.4. $\qquad\square$

EXAMPLE 2.14.   Let C be a metric space, ν a probability over C and $G : M \times C \to \mathbb{R}$ a continuous bounded function. Then

$$K(x, y) = \int_C G(x, u)G(y, u)\nu(du)$$

is a Mercer kernel. Indeed, K is clearly continuous, symmetric and

$$\langle Kf, f \rangle_\lambda = \int_C \left(\int_M G(x, u)f(x)\lambda(dx) \right)^2 \nu(du) \geq 0.$$

Note that when $C = M$ and $\nu = \lambda$, then $K = G^2$ as an operator on $L^2(\lambda)$.

EXAMPLE 2.15.   (i) Let $M = S^d \subset \mathbb{R}^{d+1}$ be the unit sphere of $\mathbb{R}^{d+1}$ and let $K(x, y) = \langle x, y \rangle = \sum_{i=1}^{d+1} x_i y_i$. Then K is a Mercer kernel [take $C = \{1, \ldots, d + 1\}$, ν the uniform measure on C, and $G(i, x) = \sqrt{d+1} \times x_i$].

EXAMPLE 2.16.   Let Δ denote the Laplacian on M and $\{K_t(x, y)\}$ the Heat kernel of $e^{\Delta t}$. Fix $\tau > 0$ and let $K = K_\tau$. The function $G(x, y) = K_{\tau/2}(x, y)$ is a symmetric C^∞ Markov kernel so that K is a Mercer kernel in view of the Example 2.14 (take $C = M$ and $\nu = \lambda$).

EXAMPLE 2.17.   The example above can be generalized as follows. Let $\{P_t\}_{t \geq 0}$ be a continuous time Markov semigroup reversible with respect to some probability measure ν on M. Assume that $P_t(x, dy)$ is absolutely continuous with respect to ν with smooth density $K_t(x, y)$. Then for all positive τ, K_τ is a Mercer kernel.

EXAMPLE 2.18.   (i) Let $M = T^d = \mathbb{R}^d / (2\pi \mathbb{Z})^d$ be the flat d-dimensional torus, and let $\kappa : T^d \to \mathbb{R}$ be an even [i.e., $\kappa(x) = \kappa(-x)$] continuous function. Set

(20) $$K(x, y) = \kappa(x - y).$$

Given $k \in \mathbb{Z}^d$, let

$$(21) \qquad \kappa_k = \int_{T^d} \kappa(x) e^{-ik \cdot x} \lambda(dx)$$

be the kth Fourier coefficient of κ. Here $k \cdot x = \sum_{i=1}^{d} k_i x_i$ and λ is the normalized Lebesgue measure on $T^d \sim [0, 2\pi[^d$. Since v is real and even, $\kappa_{-k} = \kappa_k = \bar{\kappa}_k$. If we furthermore assume that

$$\forall k \in \mathbb{Z}^d \qquad \kappa_k \geq 0,$$

then K is a Mercer kernel, since

$$\langle Kf, f \rangle_\lambda = \sum_k \kappa_k |f_k|^2$$

for all $f \in L^2(\lambda)$ and f_k the kth Fourier coefficient of f.

EXAMPLE 2.19. A function $f : [0, \infty[\to \mathbb{R}$ is said to be *completely monotonic* if it is C^∞ and, for all $t > 0$ and $k \geq 0$,

$$(-1)^k \frac{d^k f}{dx^k}(t) \geq 0.$$

Examples of such functions are $f(t) = \beta e^{-t/\sigma^2}$ and $f(t) = \beta(\sigma^2 + t)^{-\alpha}$ for $\sigma \neq 0, \alpha, \beta > 0$.

Suppose $M \subset \mathbb{R}^n$, and $K(x, y) = f(\|x - y\|^2)$, where f is completely monotonic and $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^n$. Then it was proved by Schoenberg [25] that K is a Mercer kernel.

Weakly self-repelling diffusions. When V is not a Mercer kernel but can be written as the difference of two Mercer kernels, it is still possible to give a condition ensuring strict convexity of J.

We will need the following consequence of Mercer's theorem:

LEMMA 2.20. *Let K be a Mercer kernel. Then there exists continuous symmetric functions $G^n : M \times M \to \mathbb{R}, n \geq 1$, such that*

$$K(x, y) = \lim_{n \to \infty} \langle G_x^n, G_y^n \rangle_\lambda$$

uniformly on $M \times M$. Here G_x^n stands for the function $u \mapsto G^n(x, u)$.

PROOF. The kernel K defines a compact positive and self-adjoint operator on $L^2(\lambda)$. Hence, by the spectral theorem, K has countably (or finitely) many nonnegative eigenvalues $(c_k^2)_{k \geq 1}$ and the corresponding eigenfunctions (e_k) can be chosen to form an orthonormal system. Furthermore, by Mercer's theorem (see Chapter XI-6 in [11]), $K(x, y) = \sum_i c_i^2 e_i(x) e_i(y)$, where the convergence is absolute and uniform. Now set $G_x^n(y) = G^n(x, y) = \sum_{i=1}^{n} c_i e_i(x) e_i(y)$. $\square$

To a Mercer kernel K, we associate the function $D_K : M \times M \to \mathbb{R}^+$ given by

(22)
$$D_K^2(x, y) = \left[\frac{K(x, x) + K(y, y)}{2} - K(x, y) \right]$$
$$= \lim_{n \to \infty} \frac{1}{2} \| G_x^n - G_y^n \|_\lambda^2,$$

where the (G^n) are like in Lemma 2.20.

Note that D_K is a semi-distance on M (i.e., D_K is nonnegative, symmetric, verifies the triangle inequality and vanishes on the diagonal). We let

$$\mathrm{diam}_K(M) = \sup_{x, y \in M} D_K(x, y)$$

denote the diameter of M for D_K.

Another useful quantity is

$$K(x, x) = \lim_{n \to \infty} \| G_x^n \|_\lambda^2.$$

We let

$$\mathrm{diag}_K(M) = \sup_{x \in M} K(x, x).$$

REMARK 2.21.   Note that $\mathrm{diam}_K(M) \leq 2\,\mathrm{diag}_K(M)$. But there is no obvious way to compare $\mathrm{diam}_K(M)$ and $\mathrm{diag}_K(M)$. For instance, if K is the kernel given in Example 2.19, then

$$\mathrm{diam}_K(M) = f(0) - f\left(\sup_{x,y} \|x - y\|^2 \right) \leq \mathrm{diag}_K(M) = f(0),$$

while

$$\mathrm{diam}_K(M) = 2 > \mathrm{diag}_K(M) = 1,$$

with K the kernel given in Example 2.15.

THEOREM 2.22.   *Suppose that, up to an additive constant,*

(23)
$$V = V^+ - V^-,$$

where V^+ and V^- are Mercer kernels.

If $\mathrm{diam}_{V^-}(M) < 1$, or $\mathrm{diag}_{V^-}(M) < 1$, then the conclusions of Theorem 2.13 hold.

PROOF.   First note that $J_V(f) = \frac{1}{2}\langle V^+ f, f \rangle + J_{-V^-}(f)$, and since $f \mapsto \langle V^+ f, f \rangle_\lambda$ is convex, it suffices to prove that J_{-V^-} is strictly convex. We can therefore assume, without loss of generality, that $V^+ = 0$. Or, in other words, that $-V$ is a Mercer kernel. We proceed in two steps.

Step 1. We suppose here that $V(x, y) = -\langle G_x, G_y \rangle_\lambda$ for some continuous symmetric function $G : (x, u) \mapsto G_x(u)$. By Proposition 2.9, proving that $D^2 J_V(f)$ is definite positive reduces to show that $Id + \mathsf{T}(f)V = Id - \mathsf{T}(f)G^2$ has eigenvalues > 0, or, equivalently, that $\mathsf{T}(f)G^2$ has eigenvalues < 1.

Let λ be an eigenvalue for $\mathsf{T}(f)G^2$ and $u \in \mathcal{B}_0$ a corresponding eigenvector. Set $v = Gu$. Then

$$\mathsf{T}(f)Gv = \lambda u.$$

This implies that $v \neq 0$ (because $u \neq 0$) and that

(24) $$G\mathsf{T}(f)Gv = \lambda v.$$

Thus, using the fact that G is symmetric,

$$\langle \mathsf{T}(f)Gv, Gv \rangle_\lambda = \lambda \|v\|_\lambda^2.$$

That is,

(25) $$\mathsf{Var}_f(Gv) = \lambda \|v\|_\lambda^2,$$

where

(26) $$\mathsf{Var}_f(u) = \langle \mathsf{T}(f)u, u \rangle_\lambda$$
$$= \int_M u^2(x) f(x) \lambda(dx) - \left(\int_M u(x) f(x) \lambda(dx) \right)^2.$$

Now

(27) $$\mathsf{Var}_f(Gv) = \tfrac{1}{2} \int_{M \times M} (Gv(x) - Gv(y))^2 f(x) f(y) \lambda(dx) \lambda(dy).$$

On the other hand,

$$(Gv(x) - Gv(y))^2 = \langle G_x - G_y, v \rangle_\lambda^2$$
$$\leq \|G_x - G_y\|^2 \|v\|^2$$
$$= 2(D_{-V}(x, y))^2 \|v\|_\lambda^2.$$

Thus,

(28) $$\mathsf{Var}_f(Gv) \leq (\mathsf{diam}_{-V})^2 \|v\|_\lambda^2.$$

Combining (25) and (28) leads to $\lambda \leq (\mathsf{diam}_{-V})^2 < 1$.

To obtain the second estimate, observe that [by (26)]

$$\mathsf{Var}_f(Gv) \leq \int_M (\langle G_x, v \rangle)^2 f(x) \lambda(dx)$$
$$\leq \|v\|_\lambda^2 \int \|G_x\|^2 f(x) \lambda(dx) \leq \mathsf{diag}_{-V}(M) \|v\|_\lambda^2.$$

Step 2. In the general case, by Lemma 2.20, we have $V(x, y) = \lim_{n\to\infty} V^n(x, y)$ uniformly on $M \times M$, where $V^n(x, y) = -\langle G_x^n, G_y^n \rangle_\lambda$.

Hence, assuming $\mathsf{diam}_{-V}(M) < 1$, we get that $\mathsf{diam}_{-V^n}(M) < 1$ for $n \geq n_0$ large enough. Then, by step 1, there exists $\alpha > 0$ such that, for all $n \geq n_0$,

$$D^2 J_{V^n}(u, u) = \langle u + \mathsf{T}(f)V^n u, u \rangle_{1/f} \geq \alpha \|u\|_{1/f}^2$$

for all $u \in \mathcal{B}_0$. Passing to the limit when $n \to \infty$ leads to

$$D^2 J_V(u, u) \geq \alpha \|u\|_{1/f}^2.$$

The proof of the second estimate is similar. $\quad\square$

EXAMPLE 2.15 (ii), (continued). Suppose $M = S^d \subset \mathbb{R}^{d+1}$ and

$$V(x, y) = a \times \langle x, y \rangle = a \times \sum_{i=1}^{d+1} x_i y_i$$

for some $a \in \mathbb{R}$. The kernel $K = \mathsf{sign}(a)V$ is a Mercer kernel, and $\mathsf{diag}_K(M) = |a|$. Hence, by Theorem 2.22, $\mu_t \to \lambda$ a.s. for $a > -1$.

This condition is far from being sharp since it actually follows from Theorem 4.5 in [3] that

$$a \geq -(d+1) \quad \Longleftrightarrow \quad \mu_t \to \lambda \quad \text{a.s.}$$

EXAMPLE 2.18 (ii), (continued). Let v be an even C^3 real-valued function defined on the flat d-dimensional torus (see Example 2.18) and

$$V(x, y) = v(x - y).$$

As a consequence of Theorem 2.22, we get the following result which generalizes largely Theorem 4.14 of [3]. It also corrects a mistake in the proof of this theorem.

PROPOSITION 2.23. *Let $(v_k)_{k \in \mathbb{Z}^d}$ denote the Fourier coefficients of v as defined by (21). Assume that*

$$\sum_{k \in \mathbb{Z}^d \setminus \{0\}} \inf(v_k, 0) > -1.$$

Then $\mu_t \to \lambda$ almost surely.

PROOF. Integrating by part 3 times, and using the fact that $v \in C^3$, proves that, for all $k \in \mathbb{Z}^d$, $|v_k| \leq \frac{C}{\|k\|^3}$, where $\|k\| = \sup_i |k_i|$ and C is some positive constant. Hence, the Fourier series

$$v_n(x) = \sum_{\{k \in \mathbb{Z}^d \,:\, \|k\| \leq n\}} v_k e^{ik \cdot x}$$

converges uniformly to v. Set

$$v^-(x) = - \sum_{\{k \in \mathbb{Z}^d \setminus \{0\} : v_k < 0\}} v_k e^{ik \cdot x}.$$

Then $v(x) = v^+(x) - v^-(x) + v_0$, $V = V^+ - V^- + v_0$, where $V^+(x, y) = v^+(x - y)$ and $V^-(x, y) = v^-(x - y)$ are Mercer kernels. Clearly,

$$\mathsf{diag}_{V^-}(T^d) = v^-(0) = - \sum_{\{k \neq 0 : v_k < 0\}} v_k$$

and the result follows from Theorem 2.22. $\quad\square$

2.4. *Self-attracting diffusions.* The results of this section are motivated by the analysis of self-attracting diffusions (i.e., $-V$ is a Mercer kernel), but apply to a more general setting.

Recall that $\mu^* \in \mathsf{Fix}(\Pi)$ is a sink if μ^* is nondegenerate and has zero index (thus, it corresponds to a nondegenerate local minimum of J). We denote by $\mathsf{Sink}(\Pi)$ the set of sinks.

The following result is proved in Section 5.

THEOREM 2.24. *Let $\mu^* \in \mathsf{Sink}(\Pi)$. Then*

$$\mathsf{P}\left[\lim_{t \to \infty} \mu_t = \mu^* \right] > 0.$$

The next theorem is a converse to Theorem 2.24 under a supplementary condition on V that we now explain.

From the spectral theory of compact self-adjoint operators (see, e.g., [19], Chapters XVII and XVIII), $L^2(\lambda)$ admits an orthogonal decomposition invariant under V,

$$L^2(\lambda) = E_V^0 \oplus E_V^+ \oplus E_V^-,$$

where E_V^0 stands for the kernel of V and the restriction of V to E_V^+ (resp. the restriction of $-V$ to E_V^-) is a positive operator.

Let π_+ and π_- be, respectively, the orthogonal projections from $L^2(\lambda)$ onto E_V^+ and E_V^-. Set

(29) $$V_+ = V \circ \pi_+ \quad \text{and} \quad V_- = -V \circ \pi_-$$

so that $V = V_+ - V_-$.

HYPOTHESIS 2.25 (*Occasional assumption* 2). V_+ and V_- are Mercer kernels.

Recall that $\mu^* \in \mathsf{Fix}(\Pi)$ is a saddle if μ^* is nondegenerate and has positive index. The following theorem is proved in Section 6.

THEOREM 2.26.   *Assume that Hypothesis 2.25 holds. Let $\mu^* \in \mathsf{Fix}(\Pi)$ be a saddle. Then*

$$\mathsf{P}\left[\lim_{t \to \infty} \mu_t = \mu^*\right] = 0.$$

COROLLARY 2.27.   *Suppose that Hypothesis 2.25 holds and that every $\mu^* \in \mathsf{Fix}(\Pi)$ is nondegenerate. Then there exists a random variable μ_∞ such that*:

(i) $\lim_{t \to \infty} \mu_t = \mu_\infty$ *a.s.*,
(ii) $\mathsf{P}[\mu_\infty \in \mathsf{Sink}(\Pi)] = 1$ *and*
(iii) *for all $\mu^* \in \mathsf{Sink}(\Pi)$,*

$$\mathsf{P}[\mu_\infty = \mu^*] > 0.$$

PROOF.   It follows from Theorems 2.4, 2.24 and 2.26.   □

2.5. *Localization.*   In this section we assume that Hypothesis 2.2 holds. In this case, λ is always a fixed point for Π, hence, a possible limit point for $\{\mu_t\}$. We will say that the self-interacting diffusion "localizes" provided $\mathsf{P}[\mu_t \to \lambda] = 0$. We have already seen (see Theorems 2.13 and 2.22) that self-repelling diffusions and weakly self-attracting diffusions never localize.

THEOREM 2.28.   *Suppose that Hypothesis 2.2 holds. Let*

(30) $$\rho(V) = \inf\{\langle Vu, u\rangle_\lambda : u \in L_0^2(\lambda), \|u\|_\lambda = 1\}.$$

Assume that $\rho(V) > -1$, then

(31) $$\mathsf{P}\left[\lim_{t \to \infty} \mu_t = \lambda\right] > 0.$$

Assume that $\rho(V) < -1$ and that Hypothesis 2.25 holds, then

(32) $$\mathsf{P}\left[\lim_{t \to \infty} \mu_t = \lambda\right] = 0.$$

PROOF.   Under Hypothesis 2.2, $\xi(V\lambda) = 1$. Then, by Proposition 2.9,

$$D^2 J(1)(u, v) = -\langle DX(1)u, v\rangle_\lambda = \langle u + Vu, v\rangle_\lambda.$$

The result then follows from Theorems 2.24 and 2.26.   □

EXAMPLE 2.18 (iii), (continued).   With V as in Example 2.18(ii),

$$\rho(V) = \inf_{k \in \mathbb{Z}^d \setminus \{0\}} v_k.$$

EXAMPLE 2.16 (ii), (continued).   Suppose $V(x, y) = aK_\tau(x, y)$ for some $a \leq 0$ and $\tau > 0$, where $\{K_t\}_{t>0}$ is the Heat kernel of $e^{\Delta t}$. Then $\rho(V) = ae^{-\lambda \tau}$, where λ is the smallest nonzero eigenvalue of Δ. Note that there exist numerous estimates of λ in terms of the geometry of M.

3. Review of former results. We recall here some notation and results from [3] on which rely our analysis. There is no assumption in this section that V satisfies one of the Hypotheses 2.1 or 2.2. The only required assumption is that V is smooth enough, say C^3 (see [3] for a more precise assumption).

The map Π defined by (10) extends to a map $\Pi : \mathcal{M}(M) \to \mathcal{P}(M)$ given by the same formulae. Let $F : \mathcal{M}_s(M) \to \mathcal{M}_s(M)$ be the vector field defined by

$$(33) \qquad F(\mu) = -\mu + \Pi(\mu).$$

Then (see [3], Lemma 3.2) F induces a C^∞ flow $\{\Phi_t\}_{t\in\mathbb{R}}$ on $\mathcal{M}_s(M)$.

The *limiting dynamical system associated to V* is the mapping

$$(34) \qquad \begin{aligned} \Psi : \mathbb{R} \times \mathcal{P}_w(M) &\to \mathcal{M}_w(M), \\ (t, \mu) &\mapsto \Psi_t(\mu) = \Phi_t(\mu). \end{aligned}$$

Because Φ is a flow, Ψ satisfies the flow property

$$(35) \qquad \Psi_{t+s}(\mu) = \Psi_t \circ \Psi_s(\mu)$$

for all $t, s \in \mathbb{R}$ and $\mu \in \mathcal{P}(M) \cap \Phi_{-s}(\mathcal{P}(M))$. Furthermore (see Lemmas 3.2 and 3.3 of [3]), Ψ is continuous and leaves $\mathcal{P}(M)$ positively invariant:

$$(36) \qquad \Psi_t(\mathcal{P}(M)) \subset \mathcal{P}(M) \qquad \text{for all } t \geq 0.$$

The key tool for analyzing self-interacting diffusion is Theorem 3.2 below (Theorem 3.8 of [3]), according to which the long term behavior of the sequence $\{\mu_t\}$ can be described in terms of certain invariant sets for Ψ. Before stating this theorem, we first recall some definitions from dynamical systems theory.

Attractor free sets and the limit set theorem. A subset $A \subset \mathcal{P}_w(M)$ is said to be *invariant* for Ψ if $\Psi_t(A) \subset A$ for all $t \in \mathbb{R}$. Let A be an invariant set for Ψ. Then Ψ induces a flow on A, $\Psi|A$ defined by taking the restriction of Ψ to A. That is, $(\Psi|A)_t = \Psi_t|A$.

Given an invariant set A, a set $K \subset A$ is called an *attractor* (in the sense of [7]) for $\Psi|A$, if it is compact, invariant and has a neighborhood W in A such that

$$(37) \qquad \lim_{t\to\infty} \mathrm{dist}_w(\Psi_t(\mu), K) = 0$$

uniformly in $\mu \in W$. Here dist_w is any metric on $\mathcal{P}_w(M)$ compatible with the narrow convergence.

An attractor $K \subset A$ for $\Psi|A$ which is different from $\varnothing$ and A is called *proper*. An *attractor free set* for Ψ is a nonempty compact invariant set $A \subset \mathcal{P}_w(M)$ with the property that $\Psi|A$ has no proper attractor. Equivalently, A is a nonempty compact connected invariant set such that $\Psi|A$ is a *chain-recurrent* flow [7].

REMARK 3.1. The definitions (invariant sets, attractors, attractor free sets) given here for Ψ extend obviously to any (local) flow on a metric space. This will be used below.

The *limit set* of $\{\mu_t\}$ denoted $L(\{\mu_t\})$ is the set of limits of convergent sequences $\{\mu_{t_k}\}$, $t_k \to \infty$. That is,

$$(38) \qquad L(\{\mu_t\}) = \bigcap_{t \geq 0} \overline{\{\mu_s : s \geq t\}},$$

where $\bar{A}$ stands for the closure of A in $\mathcal{P}_w(M)$.

THEOREM 3.2 ([3], Theorem 3.8).   *With probability 1, $L(\{\mu_t\})$ is an attractor free set of Ψ.*

This result allows, in various situations, to characterize exactly the asymptotic of $\{\mu_t\}$ in term of the potential V and the geometry of M. We refer the reader to [3] for several examples and further results. Among the general consequences of Theorem 3.2, the two following corollaries will be useful here.

COROLLARY 3.3.   *Let $A \subset \mathcal{P}_w(M)$ be an attractor and*

$$(39) \qquad B(A) = \left\{ \mu \in \mathcal{P}_w(M) : \lim_{t \to \infty} \mathrm{dist}_w(\Psi_t(\mu), A) = 0 \right\}$$

its basin of attraction. Then the events

$$(40) \qquad \{L(\{\mu_t\}) \cap B(A) \neq \varnothing\} \quad \text{and} \quad \{L(\{\mu_t\}) \subset A\}$$

coincide almost surely.

For a proof, see [3], Proposition 3.9.

COROLLARY 3.4.   *With probability 1, every point $\mu^* \in L(\{\mu_t\})$ can be written as*

$$(41) \qquad \mu^* = \int_{\mathcal{P}_w(M)} \Pi(\mu)\rho(d\mu),$$

where ρ is a Borel probability measure over $\mathcal{P}_w(M)$. In particular, if V is C^k, then μ^* has a C^k density with respect to λ.

This last result follows from Corollary 3.3 as follows: Let

$$(42) \qquad C_\Pi(\mathcal{P}_w(M)) = \left\{ \int_{\mathcal{P}(M)} \Pi(\mu)\rho(d\mu) : \rho \in \mathcal{P}(\mathcal{P}_w(M)) \right\},$$

where $\mathcal{P}(\mathcal{P}_w(M))$ is the set of Borel probability measures over $\mathcal{P}_w(M)$. It is not hard to prove that $C_\Pi(\mathcal{P}_w(M))$ contains a global attractor for Ψ; that is, an attractor whose basin is $\mathcal{P}_w(M)$. Hence, $L(\{\mu_t\}) \subset C_\Pi(\mathcal{P}_w(M))$ by Corollary 3.3. For details, see [3], Theorem 4.1.

4. Convergence of $\{\mu_t\}$ toward Fix(Π). This section is devoted to the proof of Theorem 2.4. Hypothesis 2.1 is implicitly assumed.

4.1. *The flow induced by X.* Recall that $\mathcal{B}_1^+ = \{f \in \mathcal{B}_1 : f > 0\}$, where $\mathcal{B}_1 = \{f \in C^0(M) : \int f\, d\lambda = 1\}$.

PROPOSITION 4.1. *The vector field X given by* (13) *induces a global smooth flow $\Phi^X = \{\Phi_t^X\}$ on $\mathcal{B}_1$. Furthermore:*

(i) $\Phi_t^X(f) \in \mathcal{B}_1^+$ *for all $t \geq 0$ and $f \in \mathcal{B}_1^+$.*
(ii) *For all $f \in \mathcal{B}_1^+$ and $t > 0$, $J(\Phi_t^X(f)) < J(f)$ if f is not an equilibrium.*

PROOF. The vector field X being smooth, it induces a smooth local flow Φ^X on $\mathcal{B}_1$. To check that this flow is global observe that

$$\|-f + \xi(Vf)\|_{L^1(\lambda)} \leq \|f\|_{L^1(\lambda)} + 1.$$

Hence, by standard results, the differential equation

$$\frac{df}{dt} = -f + \xi(Vf)$$

generates a smooth global flow on $L^1(\lambda)$ whose restriction to $\mathcal{B}_1$ is exactly Φ.

(i) For $f \in \mathcal{B}_1^+$, $\|Vf\|_\infty \leq \|V\|_\infty$. Thus, $X(f)(x) \geq -f(x) + \delta$ for all $x \in M$, where $\delta = e^{-2\|V\|_\infty}$. It follows that $\Phi_t^X(f)(x) \geq e^{-t}(f(x) - \delta) + \delta \geq \delta(1 - e^{-t}) > 0$ for all $t > 0$.

(ii) For $f \in \mathcal{B}_1^+$, let $K_f : \mathcal{B}_1^+ \to \mathbb{R}$ be the "free energy" function associated to the potential Vf

$$K_f(g) = \langle Vf, g\rangle_\lambda + \langle g, \log(g)\rangle_\lambda.$$

The function K_f is a C^∞, strictly convex function and reaches its global minimum at the "Gibbs" measure $\xi(Vf)$. Indeed, a direct computation shows that, for $h \in \mathcal{B}_0$,

$$DK_f(g) \cdot h = \langle \log(g) + Vf, h\rangle_\lambda$$

and for h and k in $\mathcal{B}_0$,

$$D^2 K_f(g)(h, k) = \langle h, k\rangle_{1/g}.$$

Thus, $DK_f(g) = 0$ if and only if $g = \xi(Vf)$ and $D^2 K_f(g)$ is positive definite for all g. Then, since

$$(43) \quad DK_f(g) \cdot [g - \xi(Vf)] = [DK_f(g) - DK_f(\xi(Vf))] \cdot [g - \xi(Vf)],$$

by strict convexity, we then deduce that

$$(44) \qquad\qquad DK_f(g) \cdot [g - \xi(Vf)] \geq 0,$$

with equality if and only if $g = \xi(Vf)$.

Now observe that $DJ(f) = DK_f(f)$. Hence, by (44),

$$DJ(f) \cdot X(f) \leq 0$$

with equality if and only if $X(f) = 0$. This proves (ii). $\quad\square$

4.2. *Proof of Theorem 2.4.*

LEMMA 4.2. *The map $i : C_\Pi(\mathcal{P}_w(M)) \to \mathcal{B}_1^+ \subset C^0(M)$ defined by $i(\mu) = \frac{d\mu}{d\lambda}$ is continuous.*

PROOF. Let $\mu_n = \int_{\mathcal{P}(M)} \Pi(\nu)\rho_n(d\nu) \in C_\Pi(\mathcal{P}_w(M))$ be such that $\mu_n \to \mu$ (for the narrow topology). By the Lipschitz continuity of V, the family $\{\xi(V\nu) : \nu \in \mathcal{P}(M)\}$ is uniformly bounded and equicontinuous. Hence, the sequence of densities $f_n = \int_{\mathcal{P}(M)} \xi(V\nu)\rho_n(d\nu)$, $n \geq 0$, is uniformly bounded and equicontinuous. By the Ascoli theorem, it is relatively compact in $C^0(M)$. It easily follows that $f_n \to f = \frac{d\mu}{d\lambda}$ in $C^0(M)$. $\quad\square$

LEMMA 4.3. *Let $K \subset \mathcal{P}_w(M)$ be a compact invariant set for Ψ. Then for all $\mu \in K$ and $t \in \mathbb{R}$,*

$$\Phi_t^X \circ i(\mu) = i \circ \Psi_t(\mu).$$

PROOF. Note that for all $\mu \in C_\Pi(\mathcal{P}(M))$, $X \circ i(\mu) = i \circ F(\mu)$ from which the result follows since $K \subset C_\Pi(\mathcal{P}(M))$ is invariant. $\quad\square$

To shorten notation, we set here $L = L(\{\mu_t\})$. Recall that $L \subset C_\Pi(\mathcal{P}(M))$ (Corollary 3.4) and that L is attractor free for Ψ (Theorem 3.2).

LEMMA 4.4. *$i(L)$ is an attractor free set for Φ.*

PROOF. This easily follows from the continuity of i (Lemma 4.2), compactness of L and the conjugacy property (Lemma 4.3) (cf. to Corollary 3.10 in [3]). $\quad\square$

COROLLARY 4.5. *$i(L)$ is a connected subset of $X^{-1}(0)$.*

Before proving this corollary, remark that it implies Theorem 2.4 since $i^{-1}(X^{-1}(0)) = \mathsf{Fix}(\Pi)$.

PROOF OF COROLLARY 4.5. The proof of this corollary relies on the following result ([2], Proposition 6.4).

PROPOSITION 4.6.   *Let Λ be a compact invariant set for a flow $\Theta = \{\Theta_t\}_{t \in \mathbb{R}}$ on a metric space E. Assume there exists a continuous function $\mathcal{V} : E \to \mathbb{R}$ such that*:

(a) $\mathcal{V}(\Theta_t(x)) < \mathcal{V}(x)$ *for $x \in E \setminus \Lambda$ and $t > 0$.*
(b) $\mathcal{V}(\Theta_t(x)) = \mathcal{V}(x)$ *for $x \in \Lambda$ and $t \in \mathbb{R}$.*

Such a $\mathcal{V}$ is called a Lyapounov function *for (Λ, Θ). If $\mathcal{V}(\Lambda)$ has empty interior, then every attractor free set K for Θ is contained in Λ. Furthermore, $\mathcal{V}|K$ ($\mathcal{V}$ restricted to K) is constant.*

Set $E = i(L)$, $\Theta = \Phi^X|i(L)$, $\Lambda = X^{-1}(0) \cap i(L)$ and $\mathcal{V} = J|i(L)$. Then Λ is a compact set (Lemma 2.7), and $\mathcal{V}$ is a Lyapounov function for (Λ, Θ) by Proposition 4.1. By Lemma 4.4, $i(L)$ is an attractor free set. Therefore, to apply Proposition 4.6, it suffices to check that $J(X^{-1}(0))$ has an empty interior. This is a consequence of the infinite-dimensional version of Sard's theorem for C^∞ functionals proved by Tromba (see Theorem 1 and Remark 7 of [29]). Thus, Proposition 4.6 proves that $i(L) \subset X^{-1}(0)$.

THEOREM 4.7 (Tromba [29]).   *Let $\mathcal{B}$ be a C^∞ Banach manifold, X a C^∞ vector field on $\mathcal{B}$ and $J : \mathcal{B} \to \mathbb{R}$ a C^∞ function. Assume the following*:

(a) $DJ(f) = 0$ *if and only if $X(f) = 0$.*
(b) $X^{-1}(0)$ *is compact.*
(c) *For each $f \in X^{-1}(0)$, $DX(f) : T_f \mathcal{B} \to T_f \mathcal{B}$ is a Fredholm operator.*

Then $J(X^{-1}(0))$ has an empty interior.

The verification that Tromba's theorem applies to the present setting is immediate. Indeed, assertion (a) follows from Proposition 2.9 and assertion (b) from Lemma 2.7. Recall that a bounded operaror T from one Banach space E_1 to a Banach space E_2 is Fredholm if its kernel $\mathsf{Ker}(T)$ has finite dimension and its range $\mathsf{Im}(T)$ has finite codimension. Hence, assertion (c) follows from Proposition 2.9. This concludes the proof of Corollary 4.5. $\square$

5. Convergence toward sinks. The purpose of this section is to prove Theorem 2.24.

5.1. *The vector field $Y = Y_V$.*   In order to prove Theorem 2.24, it is convenient to introduce a new vector field

(45)
$$Y = Y_V : C^0(M) \to C^0(M),$$
$$f \mapsto -f + V\xi(f),$$

as well as the stochastic process $\{V_t\}_{t \geq 0}$ defined by

(46)
$$V_t = V\mu_{e^t}.$$

The reason for this is, roughly speaking, the following. The measure μ_t is singular with respect to λ, while Φ^X is defined on a space of continuous densities. This is not a problem if we are dealing with *qualitative* properties of $L(\{\mu_t\})$ (like in Theorem 2.4) since we know (by Corollary 3.4) that $L(\{\mu_t\})$ consists of measures having smooth densities.

Proving Theorem 2.24 requires *quantitative* estimates on the way $\{\mu_t\}$ approaches its limit set. We shall do this by showing that $\{V_{t+s}\}_{s \geq 0}$ "shadows" at a certain rate the deterministic solution to the Cauchy problem

$$\dot{f} = Y(f)$$

with initial condition $f_0 = V_t$.

LEMMA 5.1.   *The vector field Y induces a global smooth flow $\Phi^Y = \{\Phi_t^Y\}$ on $C^0(M)$. Furthermore*:

(i) $V\Phi_t^X(f) = \Phi_t^Y(Vf)$ *for all $f \in \mathcal{B}_1$ and $t \in \mathbb{R}$.*
(ii) V *maps homeomorphically $X^{-1}(0)$ to $Y^{-1}(0)$, sinks to sinks and saddles to saddles.*

PROOF.   The vector field Y is C^∞ and sublinear because $\|Y(f)\|_\infty \leq \|f\|_\infty + \|V\|_\infty$. It then induces a global smooth flow.

(i) Follows from the conjugacy $V \circ X = Y \circ V$.

(ii) It is easy to verify that V induces a homeomorphism from $X^{-1}(0)$ to $Y^{-1}(0)$ whose inverse is ξ. Let $f \in X^{-1}(0)$ and $g = Vf$. Then with the notation of Proposition 2.9, $DX(f) = -(Id + \mathsf{T}(f) \circ V)$ and $DY(g) = -(Id + V \circ \mathsf{T}(\xi(g)) = -(Id + V \circ \mathsf{T}(f))$.

For all $\alpha \in \mathbb{R}$, let

$$E^\alpha = \{u \in L^2(\lambda), \mathsf{T}(f)Vu = \alpha u\},$$

$$H^\alpha = \{u \in L^2(\lambda), V\mathsf{T}(f)u = \alpha u\}.$$

The operators $\mathsf{T}(f)V$ and $V\mathsf{T}(f)$ are compact operators acting on $L^2(\lambda)$. The adjoint of $\mathsf{T}(f)V$ is $V\mathsf{T}(f)$. This implies that, for $\alpha \neq 0$, E^α and H^α are isomorphic, with $V : E^\alpha \to H^\alpha$ having for inverse function $\frac{1}{\alpha}\mathsf{T}(f)$. Therefore, if f is nondegenerate (resp. a sink, resp. a saddle) for X, then Vf is nondegenerate (resp. a sink, resp. a saddle) for Y. $\square$

5.2. *Proof of Theorem* 2.24.   We now follow the line of the proof of Theorem 4.12(b) in [3]. We let $\mathcal{F}_t$ denote the sigma field generated by the random variables $(B_s^i : s \leq e^t, i = 1, \ldots, N)$.

LEMMA 5.2.   *There exists a constant K (depending on V) such that, for all $T > 0$ and $\delta > 0$,*

(47) $\mathsf{P}\left[\sup_{0 \leq s \leq T} \|V_{t+s} - \Phi_s^Y(V_t)\|_\infty \geq \delta | \mathcal{F}_t \right] \leq \dfrac{K}{\delta^{d+2}} e^{-t}.$

PROOF. Given $t \geq 0$ and $s \geq 0$, let $\varepsilon_t(s) \in \mathcal{M}(M)$ be the measure defined by

$$(48) \qquad \varepsilon_t(s) = \int_t^{t+s} \left(\delta_{X_{e^r}} - \Pi(\mu_{e^r}) \right) dr.$$

Let us first show the following:

LEMMA 5.3. *There exists a constant K (depending on V) such that, for all $T > 0$ and $\delta > 0$,*

$$(49) \qquad \mathsf{P}\left[\sup_{0 \leq s \leq T} \|V\varepsilon_t(s)\|_\infty \geq \delta | \mathcal{F}_t \right] \leq \frac{K}{\delta^{d+2}} e^{-t}.$$

PROOF. According to Theorem 3.6(i)(a) in [3], there exists a constant K such that, for all $\delta > 0$ and $f \in C^\infty(M)$,

$$(50) \qquad \mathsf{P}\left[\sup_{0 \leq s \leq T} |\varepsilon_t(s)f| \geq \delta | \mathcal{F}_t \right] \leq \frac{K}{\delta^2} \|f\|_\infty^2 e^{-t}.$$

Note that this also holds for all $f \in C^0(M)$ (for a larger constant K) since f can be uniformly approximated by smooth functions. By compactness of M and Lipschitz continuity of V, there exists a finite set $\{x_1, \ldots, x_m\} \in M$ such that, for all $x \in M$,

$$|V(x, y) - V(x_i, y)| \leq \frac{\delta}{4T}$$

for some $i \in \{1, \ldots, m\}$. Therefore,

$$\sup_{0 \leq s \leq T} \|V\varepsilon_t(s)\|_\infty \leq \sup_{i=1,\ldots,m} \sup_{0 \leq s \leq T} \|V\varepsilon_t(s)(x_i)\| + \delta/2.$$

Hence,

$$\mathsf{P}\left[\sup_{0 \leq s \leq T} \|V\varepsilon_t(s)\|_\infty \geq \delta | \mathcal{F}_t \right]$$

$$\leq \mathsf{P}\left[\sup_{i=1,\ldots,m} \sup_{0 \leq s \leq T} |\varepsilon_t(s)V_{x_i}| \geq \frac{\delta}{2} \Big| \mathcal{F}_t \right]$$

$$\leq \frac{4mK\|V\|_\infty^2}{\delta^2} \times e^{-t}.$$

Since M has dimension d, m can be chosen to be $m = O(\delta^{-d})$ and the result follows. $\square$

Note that for all $u \in M$,

$$\frac{dV_t(u)}{dt} = -V_t(u) + V(u, X_{e^t})$$

$$= [VF(\mu_{e^t}) + V(\delta_{X_{e^t}} - \Pi(\mu_{e^t}))](u).$$

Thus, using the fact that $VF(\mu) = Y(V\mu)$, we obtain

$$V_{t+s}(u) - V_t(u) = \int_t^{t+s} VF(\mu_{e^r})(u)\,dr + V\varepsilon_t(s)(u)$$

$$= \int_t^{t+s} Y(V_r)(u)\,dr + V\varepsilon_t(s)(u)$$

$$= \int_0^s Y(V_{t+r})(u)\,dr + V\varepsilon_t(s)(u)$$

for all $u \in M$. In short,

$$(51) \qquad V_{t+s} - V_t = \int_0^s Y(V_{t+r})\,dr + V\varepsilon_t(s).$$

Let $v(s) = \|V_{t+s} - \Phi_s^Y(V_t)\|_\infty$. Then for $0 \le s \le T$,

$$(52) \qquad v(s) \le \int_0^s \|Y(V_{t+r}) - Y(\Phi_r^Y(V_t))\|_\infty\,dr + \sup_{0 \le s \le T} \|V\varepsilon_t(s)\|_\infty.$$

Now, for t, $r \ge 0$, both V_{t+r} and $\Phi_r^Y(V_t)$ lie in $V\mathcal{P}_w(M)$, which is a compact subset of $C^0(M)$ (by Lemma 2.3). Therefore, by Gronwall's lemma,

$$(53) \qquad \sup_{0 \le s \le T} v(s) \le e^{LT} \sup_{0 \le s \le T} \|V\varepsilon_t(s)\|_\infty,$$

where L is the Lipschitz constant of Y restricted to $V\mathcal{P}_w(M)$.

Then, with the estimate (53), Lemma 5.2 follows from Lemma 5.3. □

The following lemma is Theorem 7.3 of [2] (see also Proposition 4.13 of [3]) restated in the present context.

LEMMA 5.4. *Let $A \subset C^0(M)$ be an attractor for Φ^Y with basin of attraction $B(A)$. Let $U \subset B(A)$ be an open set with closure $\bar{U} \subset B(A)$. Then there exist positive numbers δ and T (depending on U and $\{\Phi^Y\}$) such that*

$$(54) \qquad \mathsf{P}\left[\lim_{t \to \infty} \mathrm{dist}(V_t, A) = 0\right] \ge \left(1 - \frac{K}{\delta^{d+2}} e^{-t}\right) \times \mathsf{P}[\exists\, s \ge t : V_s \in U],$$

where K is given by Lemma 5.2 and $\mathrm{dist}(\cdot, \cdot)$ is the distance associated to $\|\cdot\|_\infty$.

LEMMA 5.5. *Let $\mu \in \mathcal{P}(M)$, $f = V\mu$ and U a neighborhood of f in $C^0(M)$. Then for all $t > 0$,*

$$(55) \qquad \mathsf{P}[V_t \in U] > 0.$$

PROOF. Let Ω_M (resp. $\Omega_{\mathbb{R}^N}$) denote the space of continous paths from $\mathbb{R}^+$ to M (resp. $\mathbb{R}^N$), equipped with the topology of uniform convergence on compact intervals and the associated Borel σ-field.

Let $B_t = (B_t^1, \ldots, B_t^N)$ be a standard Brownian motion on $\mathbb{R}^N$. We let P denote the law of $(B_t : t \geq 0) \in \Omega_{\mathbb{R}^N}$ and E the associated expectation.

Let $\{W_t^x\}$ be the solution to the SDE

$$(56) \qquad dW_t^x = \sum_{i=1}^N F_i(W_t^x) \circ dB_t^i : W_0^x = X_0 = x \in M.$$

Then $W^x \in \Omega$ is a Brownian motion on M starting at x. Let

$$
(57) \qquad
\begin{aligned}
M(t) = \exp\Bigg(& \int_0^t \sum_i \langle \nabla V_{\mu_s(W)}(W_s), F_i(W_s) \rangle \, dB_s^i \\
& - \tfrac{1}{2} \int_0^t \| \nabla V_{\mu_s(W)}(W_s) \|^2 \, ds \Bigg),
\end{aligned}
$$

where, for all path $\omega \in \Omega$,

$$(58) \qquad \mu_t(\omega) = \frac{1}{t} \int_0^t \delta_{\omega_s} \, ds.$$

Then, $\{M_t\}$ is a martingale with respect to $(\Omega_{\mathbb{R}^N}, \{\sigma(B_s, s \leq t)\}_{t \geq 0}, \mathsf{P})$, and by the transformation of drift formula (Girsanov's theorem) (see Section IV 4.1 and Theorem IV 4.2 of [17]),

$$(59) \qquad \mathsf{P}[V_t \in U] = \mathsf{P}[V\mu_{e^t} \in U] = \mathsf{E}\big[M(e^t)\mathbb{1}_{\{V\mu_{e^t}(W) \in U\}}\big].$$

By continuity of the maps $V : \mathcal{P}_w(M) \to C^0(M)$ (Lemma 2.3) and $\omega \in \Omega_M \mapsto \mu_t(\omega) \in \mathcal{P}_w(M)$, the set $\mathcal{U} = \{\omega \in \Omega : V\mu_{e^t}(\omega) \in U\}$ is an open subset of Ω_M. Its Wiener measure $\mathsf{P}[W \in \mathcal{U}] = \mathsf{P}[V\mu_{e^t}(W) \in U]$ is then positive. This implies that $\mathsf{E}[M(e^t)\mathbb{1}_{\{V\mu_{e^t}(W) \in U\}}] > 0$. $\quad \square$

The proof of Theorem 2.24 is now clear. Let μ^* be a sink for Π. Then $V^* = V\mu^*$ is a sink for Y according to Lemma 5.1, and Lemmas 5.4 and 5.5 imply that

$$\mathsf{P}[V_t \to V^*] > 0.$$

On the event $\{V_t \to V^*\}$,

$$L(\{\mu_t\}) \subset \{\mu \in \mathsf{Fix}(\Pi) : V\mu = V^*\}.$$

Note that $\mu \in \mathsf{Fix}(\Pi)$ with $V\mu = V^*$ implying that $\mu = \mu^*$. Therefore, on the event $\{V_t \to V^*\}$, we have $\lim_{t \to \infty} \mu_t = \mu^*$. This proves Theorem 2.24.

6. Nonconvergence toward unstable equilibria. The purpose of this section is to prove Theorem 2.26. That is,

$$P[\mu_t \to \mu^*] = 0, \tag{60}$$

provided $\mu^* \in \text{Fix}(\Pi)$ is a nondegenerate unstable equilibrium and Hypothesis 2.25 holds.

The proof of this result is somewhat long and technical. For the reader's convenenience, we first briefly explain our strategy.

- Set $h_t = V\mu_t$. To prove that $\mu_t \not\to \mu^*$, we will prove that $h_t \not\to h^*$. We see h_t as a random perturbation of a deterministic dynamical system induced by a vector field $\tilde{Y}$. The vector field $\tilde{Y}$ is introduced in Section 6.2. It is defined like the vector field Y (see Section 5) but on a subset $\mathcal{H}^K$ of $C^0(M)$ equipped with a convenient Hilbert space structure (Section 6.1).
- The fact that μ^* is a saddle makes h^* a saddle for $\tilde{Y}$. According to the stable manifold theorem, the set of points whose forward trajectory (under $\tilde{Y}$) remains close to h^* is a smooth submanifold $W^s_{\text{loc}}(h^*)$ of nonzero finite codimension. We construct in Section 6.3 a "Lyapounov function" η which increases strictly along forward trajectory of $\tilde{Y}$ off $W^s_{\text{loc}}(h^*)$ and vanishes on $W^s_{\text{loc}}(h^*)$.
- The strategy of the proof now consists to show that $\eta(h_t) \not\to 0$ [since $\mu_t \to \mu^*$ implies $\eta(h_t) \to 0$]. Using stochastic calculus (in $\mathcal{H}^K$), we derive the stochastic evolution of $\eta(h_t)$ (Section 6.5) and then prove the theorem in Sections 6.6 and 6.7.

In the different (but related) context of urn processes and stochastic approximations, the idea of using the stable manifold theorem to prove the nonconvergence toward unstable equilibria is due to Pemantle [23]. Pemantle's probabilistic estimates have been revisited and improved by Tarrès in his Ph.D. thesis [27, 28].

The present section is clearly inspired by the work of these authors.

6.1. *Mercer kernels.* Recall that a Mercer kernel is a continuous symmetric function $K : M \times M \to \mathbb{R}$ inducing a positive operator on $L^2(\lambda)$ (i.e., $\langle Kf, f\rangle_\lambda \geq 0$). The following theorem is a fairly standard result in the theory of reproducing kernel Hilbert spaces (see, e.g., [1] or [8], Chapter III, 3).

THEOREM 6.1. *Let K be a Mercer kernel. Then there exists a unique Hilbert space $\mathcal{H}^K \subset C^0(M)$, the self-reproducing space, such that*:

  (i) *For all $\mu \in \mathcal{M}(M)$, $K\mu \in \mathcal{H}^K$.*
 (ii) *For all μ and ν in $\mathcal{M}(M)$,*

$$\langle K\mu, K\nu\rangle_K = \int\int K(x, y)\mu(dx)\nu(dy). \tag{61}$$

 (iii) *$K(L^2(\lambda))$, $\{K_x, x \in M\}$ and $K(\mathcal{M}(M))$ are dense in $\mathcal{H}^K$.*

(iv) *For all $h \in \mathcal{H}^K$ and $\mu \in \mathcal{M}(M)$,*

$$\mu h = \langle K\mu, h \rangle_K. \tag{62}$$

Moreover, the mappings $K : \mathcal{M}_s(M) \to \mathcal{H}^K$ and $K : C^0(M) \to \mathcal{H}^K$ are linear continuous and for all $h \in \mathcal{H}^K$,

$$\|h\|_\infty \le \|K\|_\infty^{1/2} \|h\|_K. \tag{63}$$

Hence, the mapping $i_K : \mathcal{H}^K \to C^0(M)$ defined by $i_K(h) = h$ is continuous.

From now on and throughout the remainder of the section, we assume that Hypothesis 2.25 holds and we set

$$K = V_+ + V_-, \tag{64}$$

where V_+ and V_- have been defined by (29). According to Hypothesis 2.25, V_+ and V_-, hence, K are Mercer kernels.

PROPOSITION 6.2. (i) *One has the orthogonal decomposition (in $\mathcal{H}^K$)*

$$\mathcal{H}^K = \mathcal{H}^{V_+} \oplus \mathcal{H}^{V_-}.$$

(ii) *Let π^+ and π^- be the orthogonal projections onto $\mathcal{H}^{V_+}$ and onto $\mathcal{H}^{V_-}$ (note that $\pi^\pm = \pi_\pm$ restricted to $\mathcal{H}^K$). Then for all $h \in \mathcal{H}^K$,*

$$\|h\|_K^2 = \|\pi^+ h\|_{V_+}^2 + \|\pi^- h\|_{V_-}^2. \tag{65}$$

(iii) *$V(\mathcal{M}(M)) = K(\mathcal{M}(M))$ and for all $\mu \in \mathcal{M}(M)$ and $h \in \mathcal{H}^K$,*

$$\langle V\mu, h \rangle_K = \mu\pi^+ h - \mu\pi^- h. \tag{66}$$

PROOF. We have the orthogonal decomposition (in $\mathcal{H}^K$) $K(L^2(\lambda)) = V_+(L^2(\lambda)) \oplus V_-(L^2(\lambda))$ (since $\langle V_+ f, V_- g \rangle_K = \langle K\pi_+ f, K\pi_- g \rangle_K = \langle K\pi_+ f, \pi_- g \rangle_\lambda = 0$). This implies the orthogonal decomposition $\mathcal{H}^K = \mathcal{H}^{V_+} \oplus \mathcal{H}^{V_-}$, because $\mathcal{H}^{V_+}$ and $\mathcal{H}^{V_-}$ are, respectively, the closures of $V_+(L^2(\lambda))$ and of $V_-(L^2(\lambda))$ in $\mathcal{H}^K$ (since $\langle V_+ f, V_+ g \rangle_{V_+} = \langle V_+ f, g \rangle_\lambda = \langle K\pi_+ f, \pi_+ g \rangle_\lambda = \langle V_+ f, V_+ g \rangle_K$). Assertions (ii) and (iii) easily follow. $\quad\square$

REMARK 6.3. Let $(e_i)_i$ be an orthonormal basis of $\mathcal{H}^K$ such that, for all i, e_i belongs to $\mathcal{H}^{V_+}$ or to $\mathcal{H}^{V_-}$ and we set $\epsilon_i = \pm 1$ when $e_i \in H^{V_\pm}$. Then we have

$$V_\pm(x, y) = \sum_i \mathbb{1}_{\epsilon_i = \pm 1} e_i(x) e_i(y),$$

$$K(x, y) = \sum_i e_i(x) e_i(y),$$

$$V(x, y) = \sum_i \epsilon_i e_i(x) e_i(y),$$

the convergence being uniform by Mercer theorem (see, e.g., Chapter XI-6 in [11] or [8]).

LEMMA 6.4. *The mappings* $V : \mathcal{M}_s(M) \to \mathcal{H}^K$ *and* $V : C^0(M) \to \mathcal{H}^K$ *are bounded operators.*

PROOF. This follows from the fact that, for every $\mu \in \mathcal{M}(M)$ and every $f \in C^0(M)$,

$$\|V\mu\|_K^2 = \mu^{\otimes 2} K \leq \|K\|_\infty \times |\mu|^2;$$
$$\|Vf\|_K^2 \leq \|K\|_\infty \times \|f\|_\infty^2. \qquad \square$$

6.2. *The vector field* $\tilde{Y} = \tilde{Y}_V$. We denote by $\mathcal{H}_0^K$ the closure in $\mathcal{H}^K$ of $V(\mathcal{M}_0(M)) = K(\mathcal{M}_0(M))$ and we set $\mathcal{H}_1^K = V1 + \mathcal{H}_0^K$, the closure of $V(\mathcal{M}_1(M)) = K(\mathcal{M}_1(M))$. Equipped with the scalar product $\langle \cdot, \cdot \rangle_K$, $\mathcal{H}_0^K$ and $\mathcal{H}_1^K$ are, respectively, a Hilbert space and an affine Hilbert space.

We let $\tilde{Y} = \tilde{Y}_V : \mathcal{H}_1^K \to \mathcal{H}_0^K$ be the vector field defined by

(67) $$\tilde{Y}(h) = -h + V\xi(h).$$

Observe that $\tilde{Y}$ is exactly defined like the vector field Y (introduced in the Section 5.1) but for the fact that $\tilde{Y}$ is a vector field on $\mathcal{H}_1^K$ [rather than on $C^0(M)$].

Recall that we let Φ denote the smooth flow on $\mathcal{M}_s(M)$ induced by the vector field F defined in Section 3 [equation (33)]. The proof of the following lemma is similar to the proof of Lemma 5.1.

LEMMA 6.5. *The vector field* $\tilde{Y}$ *induces a global smooth flow* $\tilde{\Phi}$ *on* $\mathcal{H}_1^K(M)$. *Furthermore*:

  (i) $V\Phi_t(\mu) = \tilde{\Phi}_t(V\mu)$ *for all* $\mu \in \mathcal{M}_s(M)$ *and* $t \in \mathbb{R}$.
 (ii) V *maps homeomorphically* $\mathsf{Fix}(\Pi)$ *to* $\tilde{Y}^{-1}(0)$, *sinks to sinks and saddles to saddles.*

6.3. *The stable manifold theorem and the function* η. Let μ^* be a nondegenerate unstable fixed point of Π and let

(68) $$h^* = V\mu^*.$$

By Lemma 6.5, h^* is a saddle for $\tilde{Y}$. Therefore, there exists constants $C, \lambda > 0$ and a splitting

(69) $$\mathcal{H}_0^K = H^s \oplus H^u,$$

with $H^u \neq \{0\}$, invariant under $D\tilde{\Phi}$ such that, for all $t \geq 0$ and $v \in H^u$,

(70) $$\|D\tilde{\Phi}_t(h^*)v\|_K \geq Ce^{\lambda t}\|v\|_K$$

and

$$(71) \qquad \|D\tilde{\Phi}_{-t}(h^*)v\|_K \geq Ce^{\lambda t}\|v\|_K.$$

REMARK 6.6. Let, for $\alpha \in \mathbb{R}$, $H^\alpha = \{u \in L^2(\lambda), V\mathsf{T}(h^*)u = \alpha u\}$, where $\mathsf{T}(f)$ is the operator defined in Proposition 2.9. From the proof of Lemma 5.1, it is easy to see that

$$H^u = \bigoplus_{\alpha < -1} H^\alpha$$

and

$$H^s = \bigoplus_{\alpha > -1} H^\alpha.$$

In particular, H^u has finite dimension.

The stable manifold theorem. Let $(h_s^*, h_u^*) \in H^s \times H^u$ be such that $h^* = h_s^* + h_u^*$. By the *stable manifold theorem* (see, e.g., [15] or [18]), there exists a neighborhood $\mathcal{N}_0 = \mathcal{N}_0^s \oplus \mathcal{N}_0^u$ of h^*, with $\mathcal{N}_0^s$ (resp. $\mathcal{N}_0^u$) a ball around h_s^* in H^s, (resp. h_u^* in H^u) and a smooth function $\Gamma : \mathcal{N}_0^s \to \mathcal{N}_0^u$ such that:

(a) $D\Gamma(h_s^*) = 0$.
(b) The graph of Γ:

$$\mathrm{Graph}(\Gamma) = \{v + \Gamma(v) : v \in \mathcal{N}_0^s\},$$

equals the local stable manifold of h^*:

$$W_{\mathrm{loc}}^s(h^*) = \left\{ h \in \mathcal{H}_1^K : \forall t \geq 0, \ \tilde{\Phi}_t(h) \in \mathcal{N}_0 \right.$$

$$\left. \text{and } \lim_{t \to \infty} \tilde{\Phi}_t(h) = h^* \right\}$$

$$= \{h \in \mathcal{H}_1^K : \forall t \geq 0, \ \tilde{\Phi}_t(h) \in \mathcal{N}_0\}.$$

(c) $W_{\mathrm{loc}}^s(h^*)$ is an invariant manifold. That is, for all $t \in \mathbb{R}$,

$$\tilde{\Phi}_t(W_{\mathrm{loc}}^s(h^*)) \cap \mathcal{N}_0 \subset W_{\mathrm{loc}}^s(h^*).$$

The function η. Let $r : \mathcal{N}_0 = \mathcal{N}_0^s \oplus \mathcal{N}_0^u \to W_{\mathrm{loc}}^s(h^*)$ and $R : \mathcal{N}_0 \to \mathbb{R}$ be the functions defined by

$$r(h_s + h_u) = h_s + \Gamma(h_s)$$

and

$$R(h) = \|h - r(h)\|_K^2.$$

Then r and R are smooth and R vanishes on $W_{\mathrm{loc}}^s(h^*)$.

LEMMA 6.7.   *There exists $T > 0$ and a neighborhood $\mathcal{N}_1 \subset \mathcal{N}_0$ of h^* in $\mathcal{H}_1^K$ such that, for all $h \in \mathcal{N}_1$, $\tilde{\Phi}_T(h) \in \mathcal{N}_0$ and*

$$(72) \qquad\qquad R(\tilde{\Phi}_T(h)) \geq R(h).$$

PROOF.   Using (70), we choose T large enough so that, for all $v \in H^u$,

$$(73) \qquad\qquad \|D\tilde{\Phi}_T(h^*)v\|_K \geq 4\|v\|_K.$$

Hence, there exists a neighborhood $\mathcal{N}_0' \subset \mathcal{N}_0$ of h^* such that, for all $h \in \mathcal{N}_0'$, $\tilde{\Phi}_T(h) \in \mathcal{N}_0$, and for all $v \in H^u$,

$$(74) \qquad\qquad \|D\tilde{\Phi}_T(h)v\|_K \geq 3\|v\|_K.$$

One may furthermore assume that, for all $h \in \mathcal{N}_0'$ (taking $\mathcal{N}_0'$ small enough),

$$(75) \qquad\qquad \|D(r \circ \tilde{\Phi}_T)(h) - D(r \circ \tilde{\Phi}_T)(h^*)\|_K \leq 1.$$

Now, one has

$$(76) \qquad \tilde{\Phi}_T(h) - \tilde{\Phi}_T(r(h)) - D\tilde{\Phi}_T(r(h))(h - r(h)) = o(\|h - r(h)\|_K).$$

Using first the invariance of $W_{\text{loc}}^s(h^*)$, then (76) with the fact that $D(r \circ \tilde{\Phi}_T)(h^*)v = Dr(h^*)D\tilde{\Phi}_T(h^*)v = 0$ for all $v \in H^u$, we get

$$
\begin{aligned}
r(\tilde{\Phi}_T(h)) - \tilde{\Phi}_T(r(h)) &= r(\tilde{\Phi}_T(h)) - r(\tilde{\Phi}_T(r(h))) \\
&= D(r \circ \tilde{\Phi}_T)(r(h))(h - r(h)) + o(\|h - r(h)\|_K) \\
&= [D(r \circ \tilde{\Phi}_T)(r(h)) - D(r \circ \tilde{\Phi}_T)(h^*)](h - r(h)) \\
&\quad + o(\|h - r(h)\|_K).
\end{aligned}
$$

Thus, using (75), (76) and the previous equation, we obtain the upper-estimate

$$
\begin{aligned}
\big\|\tilde{\Phi}_T(h) - r(\tilde{\Phi}_T(h)) &- D\tilde{\Phi}_T(r(h))(h - r(h))\big\|_K \\
&\leq \|h - r(h)\|_K + o(\|h - r(h)\|_K).
\end{aligned}
$$

This yields, using (74),

$$\|\tilde{\Phi}_T(h) - r(\tilde{\Phi}_T(h))\|_K \geq 2\|h - r(h)\|_K + o(\|h - r(h)\|_K).$$

We finish the proof of this lemma by taking $\mathcal{N}_1 \subset \mathcal{N}_0$, a neighborhood of h^*, such that for every $h \in \mathcal{N}_1$, $o(\|h - r(h)\|_K) \geq -\|h - r(h)\|_K$. $\square$

Let $\mathcal{N}_2 \subset \mathcal{N}_1$ be a neighborhood of h^* such that, for every $h \in \mathcal{N}_2$ and every $t \in [0, T]$, $\tilde{\Phi}_{-t}(h) \in \mathcal{N}_1$ (T being the constant given in the previous lemma). For every $h \in \mathcal{N}_2$, set

$$(77) \qquad\qquad \eta(h) = \int_0^T R(\tilde{\Phi}_{-s}(h))\, ds.$$

Then η satisfies the following:

LEMMA 6.8. (i) $\eta(h) = 0$ for every $h \in \mathcal{N}_2 \cap W_{\text{loc}}^s(h^*)$.

 (ii) η is C^2 on $\mathcal{N}_2$.

 (iii) For every $h \in \mathcal{N}_2$,

$$D\eta(h)\tilde{Y}(h) \geq 0.$$

 (iv) For every positive ϵ, there exists $\mathcal{N}_2^\epsilon \subset \mathcal{N}_2$ and $D > 0$ such that, for all $h \in \mathcal{N}_2^\epsilon$, u and v in $\mathcal{H}_0^K$,

$$|D_{u,v}^2 \eta(h) - D_{u,v}^2 \eta(h^*)| \leq \epsilon \times \|u\|_K \times \|v\|_K,$$

$$|D_{u,v}^2 \eta(h^*)| \leq D \times \|u\|_K \times \|v\|_K.$$

 (v) $D_{u,u}^2 \eta(h^*) = 0$ implies that $u \in H^s$.

 (vi) There exists a constant C_η such that, for all $u \in \mathcal{H}_0^K$ and $h \in \mathcal{N}_2$,

$$|D\eta(h)u| \leq C_\eta \times \|u\|_K \times \sqrt{\eta(h)},$$

$$2\eta(h)D_{u,u}^2 \eta(h) - (D_u \eta(h))^2 \geq -C_\eta \times \|u\|_K^2 \times \eta(h)^{3/2}.$$

PROOF. (i) and (ii) are clear. We have, for $h \in \mathcal{N}_2$,

$$D\eta(h)\tilde{Y}(h) = \lim_{s \to 0} \frac{1}{s}\big(\eta(\tilde{\Phi}_s(h)) - \eta(h)\big)$$

$$= \lim_{s \to 0} \frac{1}{s}\left(\int_0^s R(\tilde{\Phi}_t(h))\,dt - \int_{T-s}^T R(\tilde{\Phi}_{-t}(h))\,dt\right)$$

$$= R(h) - R(\tilde{\Phi}_{-T}(h)) \geq 0 \qquad \text{(by Lemma 6.7)}.$$

This shows (iii). Assertion (iv) follows from the fact that η is C^2.

For $h \in \mathcal{N}_2$, $u \in \mathcal{H}_0^K$ and $s \in [0,t]$, we set $h_s = \tilde{\Phi}_{-s}(h)$, $u_s = D\tilde{\Phi}_{-s}(h)u$ and $v_s = D_{u,u}^2 \tilde{\Phi}_{-s}(h)$. Then $h_s \in \mathcal{N}_1 \subset \mathcal{N}_0$ and

(78)
$$D\eta(h)u = 2\int_0^T \langle h_s - r(h_s), (Id - Dr(h_s))u_s \rangle_K \, ds$$

$$D_{u,u}^2 \eta(h) = 2\int_0^T \|(Id - Dr(h_s))u_s\|_K^2 \, ds$$

(79)
$$- 2\int_0^T \langle h_s - r(h_s), D_{u_s,u_s}^2 r(h_s) \rangle_K \, ds$$

$$+ 2\int_0^T \langle h_s - r(h_s), (Id - Dr(h_s))v_s \rangle_K \, ds.$$

Using the Cauchy–Schwarz inequality, (78) implies

(80)
$$|D\eta(h)u|^2 \leq 4\eta(h) \times \int_0^T \|(Id - Dr(h_s))u_s\|_K^2 \, ds,$$

which implies the first estimate of assertion (vi).

Since $r(h^*) = h^*$ and $h_s = h^*$ for all s, (79) implies

$$(81) \qquad D_{u,u}^2 \eta(h^*) = 2 \int_0^T \left\| (Id - Dr(h^*)) D\tilde{\Phi}_{-s}(h^*) u \right\|_K^2 ds.$$

Since $Dr(h^*)$ is the projection onto H^s parallel to H^u one sees that $D_{u,u}^2 \eta(h^*) = 0$ if and only if $D\tilde{\Phi}_{-s}(h^*) u \in H^u$ for all s. This proves (v) after remarking that, for $s = 0$, $D\tilde{\Phi}_{-s}(h^*) u = u$.

We now prove the last estimate of (vi). Equations (78), (79) and (80) imply the relation

$$2\eta(h) D_{u,u}^2 \eta(h) - (D_u \eta(h))^2$$

$$\geq -4\eta(h) \int_0^T \langle h_s - r(h_s), D_{u_s,u_s}^2 r(h_s) \rangle_K \, ds$$

$$+ 4\eta(h) \int_0^T \langle h_s - r(h_s), (Id - Dr(h_s)) v_s \rangle_K \, ds.$$

The last estimate of (vi) follows after using the Cauchy–Schwarz inequality. $\quad\square$

6.4. *Semigroups estimates.* In the following, $\mathcal{D}_2$ denotes the L^2-domain of the Laplacian on M. For $h \in C^1(M)$, set $A_h : \mathcal{D}_2 \to L^2(\lambda)$ defined by

$$(82) \qquad A_h f = -\Delta f + \langle \nabla h, \nabla f \rangle,$$

and $Q_h : L^2(\lambda) \to \mathcal{D}_2$ such that

$$(83) \qquad -Q_h A_h f = f - \langle \xi(h), f \rangle_\lambda.$$

Let P_t^h be the Markovian semigroup symmetric with respect to $\mu_h = \xi(h)\lambda$ and with generator A_h. Note that Q_h can be defined by

$$(84) \qquad Q_h f = \int_0^\infty (\mathsf{P}_t^h f - \mu_h f) \, dt.$$

LEMMA 6.9. *There exists a constant K_1 such that, for all $f \in C^0(M)$ and $h \in \mathcal{H}_1^K$ satisfying $\|h\|_\infty \leq \|V\|_\infty$, $Q_h f \in C^1(M) \cap \mathcal{D}_2$ and*

$$(85) \qquad \|\nabla Q_h f\|_\infty \leq K_1 \|f\|_\infty.$$

PROOF. The proof of Lemma 5.1 in [3] can be easily adapted to prove this lemma. $\quad\square$

We denote by $C^{1,1}(M^2)$ the class of functions $f \in C^0(M^2)$ such that, for all $1 \leq k, l \leq n$, $\frac{\partial}{\partial x^k} \frac{\partial}{\partial y^l} f(x, y)$ exists and belongs to $C^0(M^2)$, where $(x^k)_k$ is a system of local coordinates. For $f \in C^{1,1}(M^2)$, we define $\nabla^{\otimes 2} f \in C^0(TM \times TM)$ by

$$\nabla^{\otimes 2} f((x, u), (y, v)) = (\nabla_u \otimes \nabla_v) f(x, y)$$

$$= \sum_{k,l} u^k v^l \frac{\partial}{\partial x^k} \frac{\partial}{\partial y^l} f(x, y),$$

in a system of local coordinates. We also define $\mathrm{Tr}(\nabla^{\otimes 2} f) \in C^0(M)$, the trace of $\nabla^{\otimes 2} f$, by (d denotes the dimension of M)

$$\mathrm{Tr}(\nabla^{\otimes 2} f)(x) = \sum_{k=1}^{d} \frac{\partial}{\partial x^k} \frac{\partial}{\partial y^k} f(x, x).$$

This definition is, of course, independent of the chosen system of local coordinates.

REMARK 6.10. Lemma 6.9 implies that, for all $f \in C^0(M^2)$ and $h \in \mathcal{H}_1^K$ satisfying $\|h\|_\infty \leq \|V\|_\infty$, we have $Q_h^{\otimes 2} f \in C^{1,1}(M^2)$ and

(86) $$\|\nabla^{\otimes 2} Q_h^{\otimes 2} f\|_\infty \leq K_1^2 \|f\|_\infty.$$

This estimate implies that

(87) $$\|\mathrm{Tr}(\nabla^{\otimes 2} Q_h^{\otimes 2} f)\|_\infty \leq d K_1^2 \|f\|_\infty.$$

LEMMA 6.11. *There exists a constant $K_2 (= K_1^2)$ such that, for all $f \in C^0(M)$, h_1 and h_2 in $\mathcal{H}_1^K$ satisfying $\|h_1\|_\infty \vee \|h_2\|_\infty \leq \|V\|_\infty$, we have*

(88) $$\|\nabla Q_{h_2} f - \nabla Q_{h_1} f\|_\infty \leq K_2 \|f\|_\infty \|\nabla h_2 - \nabla h_1\|_\infty.$$

PROOF. Set $u = Q_{h_1} f$. Then

$$-A_{h_1} u = f - \langle \xi(h_1), f \rangle_\lambda$$

and since $A_{h_2} u - A_{h_1} u = \langle \nabla(h_2 - h_1), \nabla u \rangle$,

$$Q_{h_2} f = -Q_{h_2}\big(A_{h_1} u - \langle \xi(h_1), f \rangle_\lambda\big)$$
$$= -Q_{h_2} A_{h_1} u$$
$$= -Q_{h_2} A_{h_2} u + Q_{h_2} f_h,$$

where $h = h_2 - h_1$ and $f_h = \langle \nabla h, \nabla u \rangle$. Thus,

$$Q_{h_2} f = Q_{h_1} f - \langle \xi(h_2), Q_{h_1} f \rangle_\lambda + Q_{h_2} f_h$$

and

$$\nabla Q_{h_2} f - \nabla Q_{h_1} f = \nabla Q_{h_2} f_h.$$

Lemma 6.9 implies that

$$\|\nabla Q_{h_2} f_h\|_\infty \leq K_1 \|f_h\|_\infty$$

and

$$\|\nabla Q_{h_1} f\|_\infty \leq K_1 \|f\|_\infty.$$

We conclude since $\|f_h\|_\infty \leq \|\nabla h\|_\infty \|\nabla Q_{h_1} f\|_\infty$. □

REMARK 6.12.   Lemma 6.11 implies that, for all $f \in C^0(M^2)$, h_1 and h_2 in $\mathcal{H}_1^K$ satisfying $\|h_1\|_\infty \vee \|h_2\|_\infty \le \|V\|_\infty$, we have

$$(89) \qquad \|\nabla^{\otimes 2}(Q_{h_2} - Q_{h_1})^{\otimes 2} f\|_\infty \le K_2^2 \|f\|_\infty \|\nabla h_2 - \nabla h_1\|_\infty^2.$$

This implies that

$$(90) \qquad \|\mathrm{Tr}(\nabla^{\otimes 2}(Q_{h_2} - Q_{h_1})^{\otimes 2} f)\|_\infty \le d K_2^2 \|f\|_\infty \|\nabla h_2 - \nabla h_1\|_\infty^2.$$

6.5. *Itô calculus.*   Set $h_t = V\mu_t$. Given a smooth (at least C^2) function

$$\mathbb{R} \times M \to \mathbb{R},$$

$$(t, x) \mapsto F_t(x),$$

Itô's formula reads

$$(91) \qquad dF_t(X_t) = \partial_t F_t(X_t)\, dt + A_{h_t} F_t(X_t)\, dt + dM_t,$$

where M is a martingale with ($\langle \cdot, \cdot \rangle_t$ denotes the martingale bracket)

$$\frac{d}{dt}\langle M^f \rangle_t = \frac{1}{t^2}\|\nabla F_t(X_t)\|^2.$$

Set $Q_t = Q_{h_t}$ and $F_t(x) = \frac{1}{t} Q_t f(x)$ for some $f \in C^0(M)$. Then (91) [note that Itô's formula also holds if $(t, x) \mapsto F_t(x)$ is C^1 in t and for all t, $F_t \in \mathcal{D}_2$, which holds here] combined with (83) gives

$$(92) \qquad d\left(\frac{1}{t}Q_t f(X_t)\right) = \frac{H_t f}{t^2}\, dt + \frac{\langle \xi(h_t), f\rangle_\lambda - f(X_t)}{t} + dM_t^f,$$

where H_t is the measure defined by

$$(93) \qquad H_t f = -Q_t f(X_t) + t\left(\frac{d}{dt}Q_t\right)f(X_t),$$

M^f is a martingale with

$$(94) \qquad \frac{d}{dt}\langle M^f \rangle_t = \frac{1}{t^2}\|\nabla Q_t f(X_t)\|^2.$$

Using the fact that

$$\frac{d}{dt}\mu_t f = \frac{f(X_t) - \mu_t f}{t},$$

together with the definition of the vector field F, (92) can be rewritten as [recall that $F(\mu) = -\mu + \Pi(\mu)$ and that $\Pi(\mu) = \xi(V\mu)\lambda$]

$$(95) \qquad d\mu_t f = \frac{F(\mu_t)f}{t}\, dt - d\left(\frac{1}{t}Q_t f(X_t)\right) + \frac{H_t f}{t^2}\, dt + dM_t^f.$$

Note that there exists a constant H such that, for all $t \geq 0$ and $f \in C^0(M)$, $|H_t f| \leq H\|f\|_\infty$ (see Lemmas 5.1 and 5.6 in [3]).

Let ν_t be the measure defined by

$$(96) \qquad \nu_t f = \mu_t f + \frac{1}{t} Q_t f(X_t), \qquad f \in C^0(M).$$

Then $|\mu_t - \nu_t| \to 0$ and

$$(97) \qquad d\nu_t f = \frac{F(\nu_t)f}{t} dt + \frac{N_t f}{t^2} dt + dM_t^f,$$

with N_t the measure defined by $N_t f = H_t f + t(F(\mu_t) - F(\nu_t))f$. Since F is Lipschitz, there exists a constant N such that, for all $t \geq 0$ and $f \in C^0(M)$,

$$(98) \qquad |N_t f| \leq N\|f\|_\infty.$$

For every $t \geq 1$, set $g_t = V\nu_t$. Then using the fact that $VF(\mu) = \tilde{Y}(V\mu)$,

$$(99) \qquad dg_t(x) = \frac{\tilde{Y}(g_t)(x)}{t} dt + \frac{N_t V_x}{t^2} dt + dM_t^{V_x},$$

where $V_x(y) = V(x, y)$.

Note that $(g_t)_{t \geq 1}$ is a $\mathcal{H}_0^K$-valued continuous semimartingale. We denote its martingale part M_t, with $M_t(x) = M_t^{V_x} - M_1^{V_x}$. In the following, (e_i) denotes an orthonormal basis of $\mathcal{H}^K$ like in Remark 6.3. Then $M_t = \sum_i M_t^i e_i$, with $M_t^i = \langle M_t, e_i \rangle_K$. Using the fact that, for all $\mu \in \mathcal{M}_0(M)$,

$$\langle M_t, K\mu \rangle_K = \int M_t(x)\mu(dx),$$

we have

$$\frac{d}{ds}\langle \langle M_\cdot, K\mu \rangle_K \rangle_s = \iint \frac{d}{ds}\langle M^{V_x}, M^{V_y} \rangle_s \mu(dx)\mu(dy)$$

$$= \iint \frac{1}{s^2} \times \langle \nabla Q_s V_x(X_s), \nabla Q_s V_y(X_s) \rangle \mu(dx)\mu(dy)$$

$$= \frac{1}{s^2} \times \|\nabla Q_s(V\mu)(X_s)\|^2.$$

This implies that, for h in $\mathcal{H}^{V_+}$ or in $\mathcal{H}^{V_-}$,

$$(100) \qquad \frac{d}{ds}\langle \langle M_\cdot, h \rangle_K \rangle_s = \frac{1}{s^2} \times \|\nabla Q_s h(X_s)\|^2$$

and

$$(101) \qquad \frac{d}{ds}\langle M^i, M^j \rangle_s = \frac{\epsilon_i \epsilon_j}{s^2} \times \langle \nabla Q_s e_i(X_s), \nabla Q_s e_j(X_s) \rangle.$$

LEMMA 6.13.   *There exists a constant C_1 such that, for every $s \geq 1$,*

(102)
$$\mathsf{E}[\|M_s\|_K^2] \leq C_1.$$

PROOF.   We have

$$\frac{d}{ds}\mathsf{E}[\|M_s\|_K^2] = \sum_i \frac{d}{ds}\mathsf{E}[\langle M^i, M^i \rangle_s]$$

$$= \frac{1}{s^2} \times \mathsf{E}\left[\sum_i \|\nabla Q_s e_i(X_s)\|^2\right]$$

$$= \frac{1}{s^2} \times \mathsf{E}[\mathrm{Tr}(\nabla^{\otimes 2} Q_s^{\otimes 2} K)(X_s, X_s)]$$

since $K = \sum_i e_i \otimes e_i$. We conclude using Remark 6.10 and taking $C_1 = dK_1^2\|K\|_\infty$.   □

6.6. *A first lemma.*   Let L be a positive constant we will fix later on. Set $\eta_t = \eta(g_t)\mathbb{1}_{g_t \in \mathcal{N}_2}$, where $\mathcal{N}_2$ is like in Lemma 6.8. Let $\mathcal{N}$ be a neighborhood of μ^* (for the narrow topology). For every $t \geq 1$, set $S_t = \inf\{s \geq t, \ \eta_s \geq L^2/s\}$ and $U_t^{\mathcal{N}} = \inf\{s \geq t, \mu_s \notin \mathcal{N}\}$ (note that for t large enough, $\{S_t < U_t^{\mathcal{N}}\} = \{\mu_t \in \mathcal{N}\} \cap \{S_t < \infty\}$). The purpose of this section is to prove the following:

LEMMA 6.14.   *There exist a neighborhood $\mathcal{N}$ of μ^*, $p \in\,]0, 1]$ and $T_1 > 0$ such that, for all $t > T_1$,*

(103)
$$\mathsf{P}[S_t \wedge U_t^{\mathcal{N}} < \infty | \mathcal{B}_t] \geq p,$$

where $\mathcal{B}_t$ is the sigma field generated by $\{B_s^i : i = 1, \ldots, N, s \leq t\}$.

PROOF.   We fix $\epsilon > 0$. Since $V : \mathcal{P}_w(M) \to \mathcal{H}^K$ is continuous and $|\nu_t - \mu_t| \to 0$, there exist τ_1 large enough and $\mathcal{N}_\epsilon$ a neighborhood of μ^* such that, for all $t \geq \tau_1$, $\mu_t \in \mathcal{N}_\epsilon$ implies that $\nu_t \in V^{-1}(\mathcal{N}_2^\epsilon)$, where $\mathcal{N}_2^\epsilon$ is the neighborhood defined in Lemma 6.8. In particular, $\mu_t \in \mathcal{N}_\epsilon$ implies that $g_t = V\nu_t \in \mathcal{N}_2^\epsilon$.

For every neighborhood $\mathcal{N} \subset \mathcal{N}_\epsilon$ of μ^* and every $s \in [t, U_t^{\mathcal{N}}]$, $\eta_s = \eta(g_s)$. Then Itô's formula with formulas (99) and (101) gives

(104)
$$d\eta(g_s) = \frac{D\eta(g_s)\tilde{Y}(g_s)}{s} ds + \frac{D\eta(g_s)(VN_s)}{s^2} ds + dM_s^\eta$$

$$+ \frac{1}{2}\sum_{i,j} D_{i,j}^2 \eta(g_s) \times \langle \epsilon_i \nabla Q_s e_i(X_s), \epsilon_j \nabla Q_s e_j(X_s) \rangle \times \frac{ds}{s^2},$$

where $VN_s(x) = N_s V_x$ and M^η is the martingale defined by

(105)
$$dM_s^\eta = D\eta(g_s)\, dM_s.$$

We now intend to prove that

$$(106) \quad \mathsf{E}\big[\eta\big(g_{S_t \wedge U_t^N}\big)|\mathcal{B}_t\big] - \eta(g_t) \geq -C\epsilon/t + (K^*/t)\mathsf{P}[S_t \wedge U_t^N = \infty|\mathcal{B}_t],$$

where C and K^* are positive constants. In order to do this, we bound from below the four terms in the right-hand side of (104).

Lemma 6.8(iii) implies that $D\eta(g_s)\tilde{Y}(g_s) \geq 0$. Using Lemma 6.8(vi) and inequality (98), it can be easily seen that there exists a constant N_η such that, for $s \in [t, U_t^N]$,

$$|D\eta(g_s)VN_s| \leq N_\eta\sqrt{\eta(g_s)}.$$

Then

$$\int_t^{S_t \wedge U_t^N} \frac{D\eta(g_s)VN_s}{s^2}\,ds \geq -LN_\eta \int_t^\infty \frac{ds}{s^{5/2}}.$$

We choose $\tau_2 \geq \tau_1$ large enough such that, for all $t \geq \tau_2$,

$$(107) \qquad\qquad LN_\eta \int_t^\infty \frac{ds}{s^{5/2}} \leq \frac{\epsilon}{t}.$$

This gives an estimate of the second term. Since the third term is a martingale increment, after taking the expectation, this term will vanish.

We now estimate the last term. For $s > 0$, set

$$(108) \qquad \Gamma_s = \sum_{i,j} D_{i,j}^2\eta(g_s) \times \langle \epsilon_i\nabla Q_s e_i(X_s), \epsilon_j\nabla Q_s e_j(X_s)\rangle$$

and, for $\mu \in \mathcal{P}(M)$ and $x \in M$, set

$$(109) \qquad \Gamma(\mu, x) = \sum_{i,j} D_{i,j}^2\eta(h^*) \times \langle \epsilon_i\nabla Q_{V\mu} e_i(x), \epsilon_j\nabla Q_{V\mu} e_j(x)\rangle.$$

Lemma 6.8(iv) implies that, for $s \in [t, U_t^N]$ (to prove this upper-estimate, one can use a system of local coordinates and use the fact that $K = \sum_i e_i \otimes e_i$),

$$|\Gamma_s - \Gamma(\mu_s, X_s)| \leq \epsilon \times \sum_i \|\nabla Q_s e_i(X_s)\|^2$$

$$\leq \epsilon \times \mathrm{Tr}(\nabla^{\otimes 2}Q_s^{\otimes 2}K)(X_s).$$

Thus, $|\Gamma_s - \Gamma(\mu_s, X_s)| \leq C_1 \times \epsilon$, where C_1 is the same constant as the one given in Lemma 6.13.

LEMMA 6.15. $\Gamma: \mathcal{P}_w(M) \times M \to \mathbb{R}^+$ is continuous.

PROOF. We only prove the continuity in μ. For μ and ν in $\mathcal{P}(M)$ and $x \in M$,

$$\Gamma(\mu, x) - \Gamma(\nu, x) = \sum_{i,j} D_{i,j}^2\eta(h^*)\langle u_i(\mu, x) - u_i(\nu, x), u_j(\mu, x) + u_j(\nu, x)\rangle,$$

where $u_i(\mu, x) = \epsilon_i \nabla Q_{V\mu} e_i(x)$. Using Lemma 6.8(iv),

$$
|\Gamma(\mu, x) - \Gamma(\nu, x)| \leq D \times \left(\text{Tr}(\nabla^{\otimes 2}(Q_{V\mu} - Q_{V_\nu})^{\otimes 2} K)(x) \right)^{1/2}
$$
$$
\times \left(\text{Tr}(\nabla^{\otimes 2}(Q_{V\mu} + Q_{V_\nu})^{\otimes 2} K)(x) \right)^{1/2}.
$$

Remarks 6.10 and 6.12 imply that

$$
|\Gamma(\mu, x) - \Gamma(\nu, x)| \leq D \times 2dK_2 K_1 \|K\|_\infty \times \|\nabla V\mu - \nabla V\nu\|_\infty,
$$

which converges toward 0 as $\text{dist}_w(\mu, \nu) \to 0$. The proof of the continuity in x is similar. $\square$

Lemma 6.15 implies that we can choose the neighborhood $\mathcal{N} \subset \mathcal{N}_\epsilon$ of μ^* such that, for all $s \in [t, U_t^{\mathcal{N}}]$,

$$
\tag{110} |\Gamma(\mu_s, X_s) - \Gamma(\mu^*, X_s)| \leq \epsilon.
$$

We now set $\Gamma^*(x) = \Gamma(\mu^*, x)$. Thus, we now have

$$
\tag{111}
\begin{aligned}
\Gamma_s &= \left(\Gamma_s - \Gamma(\mu_s, X_s) \right) + \left(\Gamma(\mu_s, X_s) - \Gamma^*(X_s) \right) + \Gamma^*(X_s) \\
&\geq -(C_1 + 1) \times \epsilon + \Gamma^*(X_s).
\end{aligned}
$$

Finally, using (107) and (111) (with the convention $\eta_{S_t \wedge U_t^{\mathcal{N}}} = 0$ when $S_t \wedge U_t^{\mathcal{N}} = \infty$),

$$
\begin{aligned}
\mathsf{E}\big[\eta_{S_t \wedge U_t^{\mathcal{N}}} | \mathcal{B}_t \big] - \eta_t &\geq -\frac{(2 + C_1)\epsilon}{t} \\
&\quad + \frac{1}{2} \mathsf{E}\left[\int_t^\infty \Gamma^*(X_s) \frac{ds}{s^2} \mathbb{1}_{\{S_t \wedge U_t^{\mathcal{N}} = \infty\}} | \mathcal{B}_t \right].
\end{aligned}
$$

For all s, set $K(s) = \mu_s \Gamma^*$. Since $\Gamma^*(X_s) = K(s) + sK'(s)$ (recall that $\mu_s = \frac{1}{s} \int_0^s \delta_{X_u} du$), integrating by parts, we get

$$
\int_t^\infty \Gamma^*(X_s) \frac{ds}{s^2} = -\frac{K(t)}{t} + 2 \int_t^\infty \frac{K(s)}{s^2} ds.
$$

Since $\mu \mapsto \mu \Gamma^*$ is continuous, we can choose the neighborhood $\mathcal{N}$ of μ^* such that, for all $\mu \in \mathcal{N}$,

$$
|\mu \Gamma^* - K^*| < \epsilon/3,
$$

where $K^* = \mu^* \Gamma^*$. Then, on the event $\{S_t \wedge U_t^{\mathcal{N}} = \infty\}$, for all $s \geq t$,

$$
|K(s) - K^*| < \epsilon/3
$$

and

$$
\int_t^\infty \Gamma^*(X_s) \frac{ds}{s^2} \geq \frac{K^* - \epsilon}{t}.
$$

Thus,

$$
\tag{112} \mathsf{E}\big[\eta_{S_t \wedge U_t^{\mathcal{N}}} | \mathcal{B}_t \big] - \eta_t \geq -(3 + C_1)\epsilon/t + (K^*/t) \mathsf{P}[S_t \wedge U_t^{\mathcal{N}} = \infty | \mathcal{B}_t].
$$

LEMMA 6.16.   *The constant $K^* = \int \Gamma^*(x)\mu^*(dx)$ is positive.*

PROOF.   We first remark that, for all f and g in $C^0(M)$,

$$\langle \nabla Q_{h^*}f, \nabla Q_{h^*}g \rangle_{\mu^*} = \langle f - \mu^*f, Q_{h^*}g \rangle_{\mu^*}$$

$$= \int_0^\infty \langle f - \mu^*f, \mathsf{P}_t^{h^*}(g - \mu^*g) \rangle_{\mu^*}\,dt$$

$$= \int_0^\infty \langle \mathsf{P}_{t/2}^{h^*}(f - \mu^*f), \mathsf{P}_{t/2}^{h^*}(g - \mu^*g) \rangle_{\mu^*}\,dt.$$

Using this relation, we get that

$$K^* = \sum_{i,j} D_{i,j}^2 \eta(h^*) \times \langle \epsilon_i \nabla Q_{h^*}e_i, \epsilon_j \nabla Q_{h^*}e_j \rangle_{\mu^*}$$

$$= \int_0^\infty \sum_{i,j} D_{i,j}^2 \eta(h^*) \times \langle \epsilon_i (\mathsf{P}_{t/2}^{h^*}e_i - \mu^*e_i), \epsilon_j (\mathsf{P}_{t/2}^{h^*}e_j - \mu^*e_j) \rangle_{\mu^*}\,dt$$

$$= \int_0^\infty \int D^2\eta(h^*)(u_t^x, u_t^x)\,\mu^*(dx) \times dt,$$

where

$$u_t^x = \sum_i \epsilon_i (\mathsf{P}_{t/2}^{h^*}e_i(x) - \mu^*e_i)e_i$$

$$= V(\mathsf{P}_{t/2}^{h^*}(x)) - V\mu^*$$

[$\mathsf{P}_{t/2}^{h^*}(x)$ denotes the measure defined by $\mathsf{P}_{t/2}^{h^*}(x)f = \mathsf{P}_{t/2}^{h^*}f(x)$].

If $K^* = 0$, then for all $x \in M$ and $t \geq 0$, $u_t^x \in H^s$ since $D_{u,u}^2\eta(h^*) = 0$ implies $u \in H^s$. Thus, for all $x \in M$, $V_x - V\mu^* \in H^s$, and for all x and y in M, $V_x - V_y \in H^s$. Therefore, for every $\mu \in \mathcal{M}_0(M)$, $V\mu \in H^s$. This proves that $\mathcal{H}_0^K \subset H^s$ and $H^u = \{0\}$. This gives a contradiction since the dimension of H^u is larger than 1. $\square$

On the other hand,

$$\mathsf{E}[\eta_{S_t \wedge U_t^{\mathcal{N}}}|\mathcal{B}_t] - \eta_t \leq \mathsf{E}[L^2/S_t \wedge U_t^{\mathcal{N}}|\mathcal{B}_t].$$

Therefore,

(113) $$L^2\mathsf{E}[t/S_t \wedge U_t^{\mathcal{N}}|\mathcal{B}_t] \geq -(3 + C_1)\epsilon + K^*\mathsf{P}[S_t \wedge U_t^{\mathcal{N}} = \infty|\mathcal{B}_t],$$

and, since

$$\mathsf{P}[S_t \wedge U_t^{\mathcal{N}} < \infty|\mathcal{B}_t] \geq \mathsf{E}[t/S_t \wedge U_t^{\mathcal{N}}|\mathcal{B}_t],$$

we have

(114) $$\mathsf{P}[S_t \wedge U_t^{\mathcal{N}} < \infty|\mathcal{B}_t] \geq \frac{K^* - (3 + C_1)\epsilon}{L^2 + K^*}.$$

Choosing $\epsilon < K^*/(3 + C_1)$, this proves the lemma. $\square$

6.7. *A second lemma.* We choose $\mathcal{N}$, p and T_1 like in Lemma 6.14. Set

(115) $$H = \{\liminf \eta_t > 0\}.$$

LEMMA 6.17. *There exists $T_2 > 0$ such that, for all $t > T_2$, on the event* $\{S_t < U_t^{\mathcal{N}}\}$,

(116) $$\mathsf{P}[H|\mathcal{B}_{S_t}] \geq \tfrac{1}{2}.$$

PROOF. Fix $t > 0$. Set

(117) $$I_t = \inf_{s \in [S_t, U_t^{\mathcal{N}}]} \left(\frac{1}{2} \int_{S_t}^s \frac{dM_s^\eta}{\sqrt{\eta_s}} \right)$$

and

(118) $$T_t = \inf\{s > S_t, \eta_s = 0\}.$$

On the event $\{S_t < U_t^{\mathcal{N}}\} \cap \{I_t \geq -\frac{L}{2\sqrt{S_t}}\}$, for $s \in [S_t, T_t \wedge U_t^{\mathcal{N}}]$,

$$\sqrt{\eta_s} = \sqrt{\eta_{S_t}} + \int_{S_t}^s \frac{D\eta(g_u)\tilde{Y}(g_u)}{2u\sqrt{\eta(g_u)}} \, du + \int_{S_t}^s \frac{D\eta(g_u)(V N_u)}{2u^2 \sqrt{\eta(g_u)}} \, du$$
$$+ \frac{1}{2} \int_{S_t}^s \frac{dM_u^\eta}{\sqrt{\eta_u}} + \frac{1}{2} \int_{S_t}^s \sum_{i,j} D_{i,j}^2 \sqrt{\eta}(g_u) \, d\langle M^i, M^j \rangle_u.$$

Using (vi), we have

$$\sum_{i,j} D_{i,j}^2 \sqrt{\eta}(g_u) \frac{d}{du} \langle M^i, M^j \rangle_u \geq -\frac{C_\eta}{4u^2} \times \mathrm{Tr}(\nabla^{\otimes 2} Q_u^{\otimes 2} K) \geq -\frac{C_\eta'}{u^2}$$

for some constant C_η'. This implies that there exists a constant k such that

$$\sqrt{\eta_s} \geq \frac{L}{\sqrt{S_t}} - \frac{k}{S_t} - \frac{L}{2\sqrt{S_t}}.$$

Therefore, for $t \geq T_2$ large enough, $\sqrt{\eta_s} \geq -\frac{L}{4\sqrt{S_t}}$. Thus, for $t \geq T_2$,

$$\liminf_{s \to \infty} \sqrt{\eta_s} \geq \frac{L}{4\sqrt{S_t}}$$

and

$$\{S_t < U_t^{\mathcal{N}}\} \cap \left\{ I_t \geq -\frac{L}{2\sqrt{S_t}} \right\} \subset H.$$

Now, on the event $\{S_t < U_t^{\mathcal{N}}\}$,

$$\mathsf{P}\left[I_t < -\frac{L}{2\sqrt{S_t}}\Big|\mathcal{B}_{S_t}\right] = \mathsf{P}\left[\sup_{s\in[S_t,U_t^{\mathcal{N}}]} -\left(\frac{1}{2}\int_{S_t}^{s}\frac{dM_u^{\eta}}{\sqrt{\eta_u}}\right) > \frac{L}{2\sqrt{S_t}}\Big|\mathcal{B}_{S_t}\right]$$

$$\leq \frac{4S_t}{L^2} \times \mathsf{E}\left[\int_{S_t}^{s}\frac{d\langle M^{\eta}\rangle_u}{4\eta_u}\Big|\mathcal{B}_{S_t}\right],$$

by the Doob inequality. For $s \in [S_t, U_t^{\mathcal{N}}]$,

$$d\langle M^{\eta}\rangle_s = \sum_{i,j} D_i\eta(g_s)D_j\eta(g_s)\,d\langle M^i, M^j\rangle_s$$

$$= \frac{ds}{s^2}\sum_{i,j} D_i\eta(g_s)D_j\eta(g_s)\langle\epsilon_i\nabla Q_s e_i(X_s), \epsilon_j\nabla Q_s e_j(X_s)\rangle_s.$$

Lemma 6.8(vi) implies that (recall that $K = \sum_i e_i \otimes e_i$)

$$\frac{d}{ds}\langle M^{\eta}\rangle_s \leq \frac{1}{s^2}C_\eta^2 \times \eta_s \times \mathrm{Tr}(\nabla^{\otimes 2}Q_s^{\otimes 2}K)(X_s) \leq \frac{C\eta_s}{s^2},$$

with $C = C_1 C_\eta^2$. Thus, $\int_{S_t}^{s}\frac{d\langle M^{\eta}\rangle_u}{4\eta_u} \leq \frac{C}{4S_t}$ and on the event $\{S_t < U_t^{\mathcal{N}}\}$, we have

$$\mathsf{P}\left[I_t < -\frac{L}{2\sqrt{S_t}}\Big|\mathcal{B}_{S_t}\right] \leq \frac{C}{L^2}.$$

We choose L such that $C/L^2 < 1/2$. Then for $t \geq T_2$, on the event $\{S_t < U_t^{\mathcal{N}}\}$,

$$\mathsf{P}[H|\mathcal{B}_{S_t}] \geq \mathsf{P}\left[I_t \geq -\frac{L}{2\sqrt{S_t}}\Big|\mathcal{B}_{S_t}\right] \geq \frac{1}{2}.$$

This proves the lemma. $\square$

6.8. *Proof of Theorem* 2.26. We fix $\mathcal{N}, p, T_1$ and T_2 like in Lemmas 6.14 and 6.17. Let $A = \{\exists t, U_t^{\mathcal{N}} = \infty\}$. Then for $t \geq T = T_1 \vee T_2$, using Lemmas 6.14 and 6.17,

$$\mathsf{P}[H|\mathcal{B}_t] \geq \mathsf{E}\big[\mathbb{1}_H\mathbb{1}_{S_t<U_t^{\mathcal{N}}}|\mathcal{B}_t\big]$$

$$\geq \mathsf{E}\big[\mathsf{P}[H|\mathcal{B}_{S_t}]\mathbb{1}_{S_t<U_t^{\mathcal{N}}}|\mathcal{B}_t\big]$$

$$\geq \frac{1}{2} \times \mathsf{P}[S_t < U_t^{\mathcal{N}}|\mathcal{B}_t]$$

$$\geq \frac{1}{2}(p - \mathsf{P}[U_t^{\mathcal{N}} < \infty|\mathcal{B}_t]).$$

On one hand,

$$\lim_{t\to\infty}\mathsf{P}[H|\mathcal{B}_t] = \mathbb{1}_H \qquad \text{a.s.}$$

On the other hand,

$$\lim_{t \to \infty} \mathbb{1}_{\{U_t^{\mathcal{N}} = \infty\}} = \mathbb{1}_A \qquad \text{a.s.}$$

and

$$\begin{aligned}
\mathsf{E}\big[\big|\mathbb{1}_A - \mathsf{P}[U_t^{\mathcal{N}} = \infty | \mathcal{B}_t]\big|\big] &\leq \mathsf{E}\big[\big|\mathbb{1}_A - \mathsf{P}[A | \mathcal{B}_t]\big|\big] \\
&\quad + \mathsf{E}\big[\big|\mathsf{P}[A | \mathcal{B}_t] - \mathsf{P}[U_t^{\mathcal{N}} = \infty | \mathcal{B}_t]\big|\big] \\
&\leq \mathsf{E}\big[\big|\mathbb{1}_A - \mathsf{P}[A | \mathcal{B}_t]\big|\big] + \mathsf{E}\big[\big|\mathbb{1}_A - \mathbb{1}_{\{U_t^{\mathcal{N}} = \infty\}}\big|\big],
\end{aligned}$$

which converges toward 0 as $t \to \infty$. Thus, $\lim_{t \to \infty} \mathsf{P}[U_t^{\mathcal{N}} < \infty | \mathcal{B}_t] = \mathbb{1}_{A^c}$ in L^1 and

(119) $$\mathbb{1}_H \geq \tfrac{1}{2}(p - \mathbb{1}_{A^c}) \qquad \text{a.s.}$$

This implies that a.s., $A \subset H$. But since $H \subset \{\mu_t \nrightarrow \mu^*\}$ and $\{\mu_t \to \mu^*\} \subset A$, we have $\{\mu_t \to \mu^*\} \subset \{\mu_t \nrightarrow \mu^*\}$ a.s. This implies that $\mathsf{P}[\mu_t \to \mu^*] = 0$.

APPENDIX

Recall that we let $\mathcal{G}$ denote the set of $V \in C^k_{\text{sym}}(M \times M)$ such that Π_V has nondegenerate fixed points. Our purpose here is to prove Theorem 2.10. That is, that $\mathcal{G}$ is open and dense.

Openess. We first prove that $\mathcal{G}$ is open. Let $V^* \in \mathcal{G}$. Then the zeros of X_{V^*} are isolated (by the inverse function theorem) and since $(X_{V^*})^{-1}(0)$ is compact (Lemma 2.7), $X_{V^*}{}^{-1}(0)$ is a finite set. Say, $X_{V^*}{}^{-1}(0) = \{f_1, \ldots, f_d\}$.

By the implicit function theorem applied to the map $(V, f) \mapsto X_V(f)$, there exist open neighborhoods U_i of f_i, W_i of V^* and smooth maps $R_i : W_i \to U_i$ such that:

(a) $X_V(f) = 0 \Leftrightarrow f = R_i(V)$, for all $V \in W_i$, $f \in U_i$,
(b) $R_i(V^*) = f_i$,
(c) $DX_V(f)$ is invertible at $f = R_i(V)$.

It remains to show that there exists an open neigborhood of V^* $W \subset \bigcap_i W_i$ such that, for all $V \in W$, equilibria of X_V lie in $\bigcup U_i$. In view of (a) and (c) above, this will imply that $W \subset \mathcal{G}$, concluding the proof of openess. Assume, to the contrary, that there is no such neighborhood. Then there exists $V_n \to V^*$ and $f_n \in \mathcal{B}_1 \setminus \bigcup_i U_i$ such that $X_{V_n}(f_n) = 0$. That is,

(120) $$f_n = \xi(V_n f_n).$$

Then by Lemma 2.3, we can extract from $\{V^* f_n\}$ a subsequence $\{V^* f_{n_k}\}$ converging to some $g \in C^0(M)$. Now, $\|V_n f_n - V f_n\|_\infty \leq \|V_n - V^*\|_\infty$. Thus, $V_{n_k} f_{n_k} \to g$. Equation (120) then implies that $f_{n_k} \to f = \xi(g)$ and $f = \xi(V^* f)$. Hence, $f \in \bigcup_i U_i$. A contradiction.

Density. We now pass to the proof of the density. Recall that if Z is a smooth map from one Banach manifold to another, a point $h \in \mathcal{B}_2$ is called a *regular value* of Z, provided $DZ(f)$ is subjective for all $f \in Z^{-1}(h)$. Here, saying that 0 is a regular value for X_V is equivalent to saying that X_V has nondegenerate equilibria.

Let $\mathcal{B}_1^k = \mathcal{B}_1 \cap C^k(M)$, $\mathcal{B}_0^k = \mathcal{B}_0 \cap C^k(M)$ and $\mathcal{B}_1^{+,k} = \mathcal{B}_1^+ \cap C^k(M)$. For all $V \in C_{\text{sym}}^k(M \times M)$, let $Z_V : \mathcal{B}_1^{+,k} \to \mathcal{B}_0^k$ denote the C^∞ vector field defined by

$$Z_V(f) = Vf + \log(f) - \langle Vf + \log(f), \mathbf{1} \rangle.$$

Remark that, for all $h \in \mathcal{B}_0^k$,

$$DJ_V(f)h = \langle Z_V(f), h \rangle.$$

Hence, by Proposition 2.9, X_V and Z_V have the same set of equilibria and 0 is a regular value for X_V if and only if it is a regular value for Z_V.

Given $h \in \mathcal{B}_0^k$, let $V[h]$ be the symmetric function defined by

$$V[h](x, y) = V(x, y) - h(x) - h(y).$$

One has

$$Z_{V[h]}(f) = Z_V(f) - h.$$

Therefore, h is a regular value of Z_V if and only if 0 is a regular value of $Z_{V[h]}$ or, equivalently, a regular value of $X_{V[h]}$.

We claim that Z_V is a *Fredholm map*. That is, a map whose derivative $DZ_V(f)$ is a Fredholm operator for each $f \in \mathcal{B}_1^{+,k}$ (see Section 4 for the definition of a Fredholm operator). Hence, by a theorem of Smale [26], generalizing Sard's theorem to Fredholm maps) R_{Z_V} is a residual (i.e., a countable intersection of open dense sets) set. Being residual, it is dense. Therefore, for any $\epsilon > 0$, we can find $h \in \mathsf{R}_{Z_V}$ with $\|h\|_{C^k} \leq \epsilon$. With this choice of h,

$$\|V - V[h]\|_{C^k} \leq \epsilon$$

and $X_{V[h]}$ has nondegenerate equilibria. This concludes the proof of the density.

To see that $DZ_V(f)$ is Fredholm, write $DZ_V(f) = A \circ B \circ C$, where $C : \mathcal{B}_0^k \to C^k(M)$, $B : C^k(M) \to C^k(M)$ and $A : C^k(M) \to \mathcal{B}_0^k$ are, respectively, defined by $Ch = f \cdot (Vh) + h$, $Bh = \frac{1}{f}h$ and $Ah = h - \langle h, \mathbf{1} \rangle$.

The operator C is the sum of a compact operator and identity. Hence, by a classical result, (see, e.g., [19], Theorem 2.1, Chapter XVII) it is Fredholm. Operators B and A are clearly Fredholm since $\text{Ker}(B) = \{0\}$, $\text{Im}(B) = C^k(M)$, $\text{Ker}(A) = \mathbb{R}\mathbf{1}$ and $\text{Im}(A) = \mathcal{B}_0^k$. Since the composition of Fredholm operators is Fredholm ([19], Corollary 2.6, Chapter XVII), $DZ_V(f)$ is Fredholm.

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