

PHASE TRANSITION OF THE LARGEST EIGENVALUE FOR NONNULL COMPLEX SAMPLE COVARIANCE MATRICES

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We compute the limiting distributions of the largest eigenvalue of a complex Gaussian sample covariance matrix when both the number of samples and the number of variables in each sample become large. When all but finitely many, say r , eigenvalues of the covariance matrix are the same, the dependence of the limiting distribution of the largest eigenvalue of the sample covariance matrix on those distinguished r eigenvalues of the covariance matrix is completely characterized in terms of an infinite sequence of new distribution functions that generalize the Tracy–Widom distributions of the random matrix theory. Especially a phase transition phenomenon is observed. Our results also apply to a last passage percolation model and a queueing model.

1. Introduction. Consider M independent, identically distributed samples $\vec{y}_1, \dots, \vec{y}_M$, all of which are $N \times 1$ column vectors. We further assume that the sample vectors \vec{y}_k are Gaussian with mean $\vec{\mu}$ and covariance Σ , where Σ is a fixed $N \times N$ positive matrix; the density of a sample \vec{y} is

$$(1) \quad p(\vec{y}) = \frac{1}{(2\pi)^{N/2}(\det \Sigma)^{1/2}} e^{-(1/2)\langle \vec{y} - \vec{\mu}, \Sigma^{-1}(\vec{y} - \vec{\mu}) \rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors. We denote by ℓ_1, \dots, ℓ_N the eigenvalues of the covariance matrix Σ , called the “population eigenvalues.” The sample mean \bar{Y} is defined by $\bar{Y} := \frac{1}{M}(\vec{y}_1 + \dots + \vec{y}_M)$ and we set $X = [\vec{y}_1 - \bar{Y}, \dots, \vec{y}_M - \bar{Y}]$ to be the (centered) $N \times M$ sample matrix. Let $S = \frac{1}{M}XX'$ be the *sample covariance matrix*. The eigenvalues of S , called the “sample eigenvalues,” are denoted by $\lambda_1 > \lambda_2 > \dots > \lambda_N > 0$. (The eigenvalues are simple with probability 1.) The probability space of λ_j ’s is sometimes called the Wishart ensemble (see, e.g., [29]).

Contrary to the traditional assumptions, it is of current interest to study the case when N is of the same order as M . Indeed when $\Sigma = I$ (null case), several results are known. As $N, M \rightarrow \infty$ such that $M/N \rightarrow \gamma^2 \geq 1$, the following hold.

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(A) *Density of eigenvalues* [27]: For any real x ,

$$(2) \quad \frac{1}{N} \#\{\lambda_j : \lambda_j \leq x\} \rightarrow H(x),$$

where

$$(3) \quad H'(x) = \frac{\gamma^2}{2\pi x} \sqrt{(b-x)(x-a)}, \quad a < x < b,$$

and $a = (\frac{\gamma-1}{\gamma})^2$ and $b = (\frac{1+\gamma}{\gamma})^2$.

(B) *Limit of the largest eigenvalue* [15]:

$$(4) \quad \lambda_1 \rightarrow \left(\frac{1+\gamma}{\gamma}\right)^2 \quad \text{a.s.}$$

(C) *Limiting distribution* [23]: For any real x ,

$$(5) \quad \mathbb{P}\left(\left(\lambda_1 - \left(\frac{\gamma+1}{\gamma}\right)^2\right) \cdot \frac{\gamma M^{2/3}}{(1+\gamma)^{4/3}} \leq x\right) \rightarrow F_{\text{GOE}}(x),$$

where $F_{\text{GOE}}(x)$ is the so-called GOE Tracy–Widom distribution, which is the limiting distribution of the largest eigenvalue of a random real symmetric matrix from the Gaussian orthogonal ensemble (GOE) as the size of the matrix tends to infinity [40].

(D) *Robustness to models* [35]: It turned out that the Gaussian assumption is unnecessary and a result similar to (5) still holds for a quite general class of independent, identically distributed random samples.

From (2) and (4)/(5), we find that the largest sample eigenvalue λ_1 in the null case converges to the rightmost edge of support of the limiting density of eigenvalues. However, in practice (see, e.g., [23]) there often are statistical data for which one or several large sample eigenvalues are separated from the bulk of the eigenvalues. For instance, see Figures 1 and 2 of the paper [23] which plot the sample eigenvalues of the functional data consisting of a speech dataset of 162 instances of a phoneme “dcl” spoken by males calculated at 256 points [9]. Other examples of similar phenomena include mathematical finance [25, 26, 32], wireless communication [37], physics of mixture [34] and data analysis and statistical learning [18]. As suggested in [23], this situation poses a natural question: when $\Sigma \neq I$ (nonnull case), how do a few large sample eigenvalues depend on the population eigenvalues? More concretely, if there are a few large population eigenvalues, do they pull to the sample eigenvalues, and for it to happen, how large should the population eigenvalues be?

Though this might be a challenging problem for real sample data, it turned out that one could answer some of the above questions in great detail for complex Gaussian samples. A complex sample covariance matrix has an application in multi-antenna Gaussian channels in wireless communication [37]. Also the results

of the complex case lead us to a guess for aspects of the real case (see Conjecture in Section 1.3 below). Another reason for studying the complex sample covariance matrix is its relation to a last passage percolation model and a queueing theory. See Section 6 below for such a connection.

Before we present our work, we first summarize some known results for the complex sample covariance matrices.

1.1. *Some known results for the eigenvalues of complex sample covariance matrices.* We assume that the samples \vec{y} are *complex Gaussian* with mean $\vec{\mu}$ and covariance Σ . Hence the density of \vec{y} is precisely given by (1) with the understanding that $\langle \cdot, \cdot \rangle$ denotes now the complex inner product. The (centered) sample matrix X and the sample covariance matrix $S = \frac{1}{N} X X^*$ are defined as before where X^* is the transpose followed by the complex conjugation. Recall that the eigenvalues of S , sample eigenvalues, are denoted by $\lambda_1 \geq \dots \geq \lambda_N > 0$, and the eigenvalues of Σ , population eigenvalues, are denoted by $\ell_1, \dots, \ell_N > 0$.

(a) *Density of eigenvalues* [3, 27] (see also Theorem 3.4 of [2]): When all but finitely many eigenvalues ℓ_j of Σ are equal to 1, as $M, N \rightarrow \infty$ such that $M/N \rightarrow \gamma^2 \geq 1$, the limiting density of the sample eigenvalues λ_j is given by

$$(6) \quad \frac{1}{N} \#\{\lambda_j : \lambda_j \leq x\} \rightarrow H(x),$$

where $H(x)$ is again defined by (3).

(b) *Null case:* When $\Sigma = I$, as $M, N \rightarrow \infty$ such that $M/N \rightarrow \gamma^2 \geq 1$ [15],

$$(7) \quad \lambda_1 \rightarrow \left(\frac{1 + \gamma}{\gamma} \right)^2 \quad \text{a.s.}$$

and for any real x (see, e.g., [14, 22])

$$(8) \quad \mathbb{P} \left(\left(\lambda_1 - \left(\frac{1 + \gamma}{\gamma} \right)^2 \right) \cdot \frac{\gamma M^{2/3}}{(1 + \gamma)^{4/3}} \leq x \right) \rightarrow F_{\text{GUE}}(x),$$

where $F_{\text{GUE}}(x)$ is the GUE Tracy–Widom distribution, which is the limiting distribution of the largest eigenvalue of a random complex Hermitian matrix from the Gaussian unitary ensemble (GUE) as the size of the matrix tends to infinity [39]. Moreover, the limit (8) holds true for a quite general class of independent, identically distributed random samples, after suitable scaling [35].

REMARK 1.1. The distribution function F_{GUE} is different from F_{GOE} . A formula of $F_{\text{GUE}}(x)$ is given in (18) below and a formula for $(F_{\text{GOE}}(x))^2$ is given in (24) below.

REMARK 1.2. When $\Sigma = I$, the probability space of the eigenvalues λ_j of S is sometimes called the Laguerre unitary ensemble (LUE) since the correlation

functions of λ_j can be represented in terms of Laguerre polynomials. Similarly, for real samples with $\Sigma = I$, the probability space of the eigenvalues of S is called the Laguerre orthogonal ensemble (LOE). See, for example, [12].

Note that the limiting density of the eigenvalues λ_j is known for general $\Sigma \neq I$, but the convergence (7)/(8) of λ_1 to the edge of the support of the limiting distribution of the eigenvalues was obtained only when $\Sigma = I$. The following result of Péché [31] generalizes (8) and shows that when all but finitely many eigenvalues ℓ_k of Σ are 1 and those distinguished eigenvalues are “not too big,” λ_1 is still not separated from the rest of the eigenvalues.

(c) When $\ell_{r+1} = \dots = \ell_N = 1$ for a fixed integer r and $\ell_1 = \dots = \ell_r < 2$ are fixed, as $M = N \rightarrow \infty$ [31],

$$(9) \quad \mathbb{P}((\lambda_1 - 4) \cdot 2^{-4/3} M^{2/3} \leq x) \rightarrow F_{\text{GUE}}(x).$$

A natural question is then whether the upper bound 2 of $\ell_1 = \dots = \ell_r$ is critical. One of our results in this paper is that it is indeed the critical value. Moreover, we find that if some of ℓ_j are precisely equal to the critical value, then the limiting distribution is changed to something new. And if one or more ℓ_j are bigger than the critical value, the fluctuation order $M^{2/3}$ is changed to the Gaussian-type order \sqrt{M} . In order to state our results, we first need some definitions.

1.2. *Definitions of some distribution functions.*

1.2.1. *Airy-type distributions.* Let $\text{Ai}(u)$ be the Airy function which has the integral representation

$$(10) \quad \text{Ai}(u) = \frac{1}{2\pi} \int e^{iua+i(1/3)a^3} da,$$

where the contour is from $\infty e^{5i\pi/6}$ to $\infty e^{i\pi/6}$. Define the *Airy kernel* (see, e.g., [39]) by

$$(11) \quad \mathbf{A}(u, v) = \frac{\text{Ai}(u) \text{Ai}'(v) - \text{Ai}'(u) \text{Ai}(v)}{u - v}$$

and let \mathbf{A}_x be the operator acting on $L^2((x, \infty))$ with kernel $\mathbf{A}(u, v)$. An alternative formula of the Airy kernel is

$$(12) \quad \mathbf{A}(u, v) = \int_0^\infty \text{Ai}(u+z) \text{Ai}(z+v) dz,$$

which can be checked directly by using the relation $\text{Ai}''(u) = u \text{Ai}(u)$ and integrating by parts. For $m = 1, 2, 3, \dots$, set

$$(13) \quad s^{(m)}(u) = \frac{1}{2\pi} \int e^{iua+i(1/3)a^3} \frac{1}{(ia)^m} da,$$

where the contour is from $\infty e^{5i\pi/6}$ to $\infty e^{i\pi/6}$ such that the point $a = 0$ lies above the contour. Also set

$$(14) \quad t^{(m)}(v) = \frac{1}{2\pi} \int e^{iva+i(1/3)a^3} (-ia)^{m-1} da,$$

where the contour is from $\infty e^{5i\pi/6}$ to $\infty e^{i\pi/6}$. Alternatively,

$$(15) \quad s^{(m)}(u) = \sum_{\substack{\ell+3n=m-1 \\ \ell,n=0,1,2,\dots}} \left\{ \frac{(-1)^n}{3^n \ell! n!} u^\ell + \frac{1}{(m-1)!} \int_\infty^u (u-y)^{m-1} \text{Ai}(y) dy \right\}$$

and

$$(16) \quad t^{(m)}(v) = \left(-\frac{d}{dv} \right)^{m-1} \text{Ai}(v).$$

See Lemma 3.3 below for the proof that the two formulas of $s^{(m)}(u)$ are the same.

DEFINITION 1.1. For $k = 1, 2, \dots$, define for real x ,

$$(17) \quad F_k(x) = \det(1 - \mathbf{A}_x) \cdot \det \left(\delta_{mn} - \left\langle \frac{1}{1 - \mathbf{A}_x} s^{(m)}, t^{(n)} \right\rangle \right)_{1 \leq m, n \leq k},$$

where $\langle \cdot, \cdot \rangle$ denotes the (real) inner product of functions in $L^2((x, \infty))$. Let $F_0(x) = \det(1 - \mathbf{A}_x)$.

The fact that the inner product in (17) makes sense and hence $F_k(x)$ is well defined is proved in Lemma 3.3 below.

It is well known that (see, e.g., [14, 39])

$$(18) \quad F_0(x) = \det(1 - \mathbf{A}_x) = F_{\text{GUE}}(x)$$

and hence F_0 is the GUE Tracy–Widom distribution function. There is an alternative expression of F_0 . Let $u(x)$ be the solution to the Painlevé II equation

$$(19) \quad u'' = 2u^3 + xu$$

satisfying the condition

$$(20) \quad u(x) \sim -\text{Ai}(x), \quad x \rightarrow +\infty.$$

There is a unique, global solution [17], and satisfies (see, e.g., [10, 17])

$$(21) \quad u(x) = -\frac{e^{-(2/3)x^{3/2}}}{2\sqrt{\pi}x^{1/4}} + O\left(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\right) \quad \text{as } x \rightarrow +\infty,$$

$$(22) \quad u(x) = -\sqrt{\frac{-x}{2}}(1 + O(x^{-2})) \quad \text{as } x \rightarrow -\infty.$$

Then [39]

$$(23) \quad F_0(x) = \det(1 - \mathbf{A}_x^{(0)}) = \exp\left(-\int_x^\infty (y-x)u^2(y) dy\right).$$

In addition to being a beautiful identity, the right-hand side of (23) provides a practical formula to plot the graph of F_0 .

For $k = 1$, it is known that (see (3.34) of [8, 11])

$$(24) \quad F_1(x) = \det(1 - \mathbf{A}_x) \cdot \left(1 - \left\langle \frac{1}{1 - \mathbf{A}_x} s^{(1)}, t^{(1)} \right\rangle\right) = (F_{\text{GOE}}(x))^2.$$

The function F_{GOE} also has a Painlevé formula [40] and

$$(25) \quad F_1(x) = F_0(x) \exp\left(\int_x^\infty u(y) dy\right).$$

The functions $F_k, k \geq 2$, seem to be new. The Painlevé formula of F_k for general $k \geq 2$ will be presented in [4]. For each $k \geq 2$, $F_k(x)$ is clearly a continuous function in x . Being a limit of nondecreasing functions as Theorem 1.1 below shows, $F_k(x)$ is a nondecreasing function. It is also not difficult to check by using a steepest-descent analysis that $F_k(x) \rightarrow 1$ as $x \rightarrow +\infty$ (cf. proof of Lemma 3.3). However, the proof that $F_k(x) \rightarrow 0$ as $x \rightarrow -\infty$ is not trivial. The fact that $F_k(x) \rightarrow 0$ as $x \rightarrow -\infty$ is obtained in [4] using the Painlevé formula. Therefore $F_k(x), k \geq 2$, are distribution functions, which generalize the Tracy–Widom distribution functions. (The functions F_0, F_1 are known to be distribution functions.)

1.2.2. *Finite GUE distributions.* Consider the density of k particles ξ_1, \dots, ξ_k on the real line defined by

$$(26) \quad p(\xi_1, \dots, \xi_k) = \frac{1}{Z_k} \prod_{1 \leq i < j \leq k} |\xi_i - \xi_j|^2 \cdot \prod_{i=1}^k e^{-(1/2)\xi_i^2},$$

where Z_k is the normalization constant,

$$(27) \quad \begin{aligned} Z_k &:= \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \prod_{1 \leq i < j \leq k} |\xi_i - \xi_j|^2 \cdot \prod_{i=1}^k e^{-(1/2)\xi_i^2} d\xi_1 \cdots d\xi_k \\ &= (2\pi)^{k/2} \prod_{j=1}^k j!, \end{aligned}$$

which is called Selberg’s integral (see, e.g., [28]). This is the density of the eigenvalues of the Gaussian unitary ensemble (GUE), the probability space of $k \times k$ Hermitian matrices H whose entries are independent Gaussian random variables with mean 0 and standard deviation 1 for the diagonal entries, and mean 0 and standard deviation 1/2 for each of the real and complex parts of the off-diagonal entries (see, e.g., [28]).

DEFINITION 1.2. For $k = 1, 2, 3, \dots$, define the distribution $G_k(x)$ by

$$(28) \quad G_k(x) = \frac{1}{Z_k} \int_{-\infty}^x \cdots \int_{-\infty}^x \prod_{1 \leq i < j \leq k} |\xi_i - \xi_j|^2 \cdot \prod_{i=1}^k e^{-(1/2)\xi_i^2} d\xi_1 \cdots d\xi_k.$$

In other words, G_k is the distribution of the *largest eigenvalue* of $k \times k$ GUE. When $k = 1$, this is the Gaussian distribution,

$$(29) \quad G_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-(1/2)\xi_1^2} d\xi_1 = \text{erf}(x).$$

There is an alternative expression of G_k in terms of a Fredholm determinant similar to the formula (17) of F_k . Let $p_n(x) = c_n x^n + \dots$ be the polynomial of degree n ($c_n > 0$ is the leading coefficient) determined by the orthogonality condition

$$(30) \quad \int_{-\infty}^{\infty} p_m(x) p_n(x) e^{-(1/2)x^2} dx = \delta_{mn}.$$

The orthonormal polynomial p_n is given by

$$(31) \quad p_n(\xi) := \frac{1}{(2\pi)^{1/4} 2^{n/2} \sqrt{n!}} H_n\left(\frac{\xi}{\sqrt{2}}\right),$$

where H_n is the Hermite polynomial. The leading coefficient c_n of p_n is (see, e.g., [24])

$$(32) \quad c_n = \frac{1}{(2\pi)^{1/4} \sqrt{n!}}.$$

Then the so-called orthogonal polynomial method in the random matrix theory establishes that:

LEMMA 1.1. For any $k = 1, 2, \dots$ and $x \in \mathbb{R}$,

$$(33) \quad G_k(x) = \det(1 - \mathbf{H}_x^{(k)}),$$

where $\mathbf{H}_x^{(k)}$ is the operator acting on $L^2((x, \infty))$ defined by the kernel

$$(34) \quad \mathbf{H}^{(k)}(u, v) = \frac{c_{k-1}}{c_k} \frac{p_k(u)p_{k-1}(v) - p_{k-1}(u)p_k(v)}{u - v} e^{-(u^2+v^2)/4}.$$

This is a standard result in the theory of random matrices. The proof can be found, for example, in [28, 41]. There is also an identity of the form (23) for $\det(1 - \mathbf{H}_x^{(k)})$, now in terms of the Painlevé IV equation. See [38].

1.3. *Main results.* We are now ready to state our main results.

THEOREM 1.1. *Let λ_1 be the largest eigenvalue of the sample covariance matrix constructed from M independent, identically distributed complex Gaussian sample vectors of N variables. Let ℓ_1, \dots, ℓ_N denote the eigenvalues of the covariance matrix of the samples. Suppose that for a fixed integer $r \geq 0$,*

$$(35) \quad \ell_{r+1} = \ell_{r+2} = \dots = \ell_N = 1.$$

As $M, N \rightarrow \infty$ while $M/N = \gamma^2$ is in a compact subset of $[1, \infty)$, the following hold for any real x in a compact set.

(a) *When for some $0 \leq k \leq r$,*

$$(36) \quad \ell_1 = \dots = \ell_k = 1 + \gamma^{-1}$$

and $\ell_{k+1}, \dots, \ell_r$ are in a compact subset of $(0, 1 + \gamma^{-1})$,

$$(37) \quad \mathbb{P}\left((\lambda_1 - (1 + \gamma^{-1})^2) \cdot \frac{\gamma}{(1 + \gamma)^{4/3}} M^{2/3} \leq x\right) \rightarrow F_k(x),$$

where $F_k(x)$ is defined in (17).

(b) *When for some $1 \leq k \leq r$,*

$$(38) \quad \ell_1 = \dots = \ell_k \text{ is in a compact set of } (1 + \gamma^{-1}, \infty)$$

and $\ell_{k+1}, \dots, \ell_r$ are in a compact subset of $(0, \ell_1)$,

$$(39) \quad \mathbb{P}\left(\left(\lambda_1 - \left(\ell_1 + \frac{\ell_1 \gamma^{-2}}{\ell_1 - 1}\right)\right) \cdot \frac{\sqrt{M}}{\sqrt{\ell_1^2 - \ell_1^2 \gamma^{-2} / (\ell_1 - 1)^2}} \leq x\right) \rightarrow G_k(x),$$

where $G_k(x)$ is defined in (28).

Hence, for instance, when $r = 2$,

$$(40) \quad \ell_3 = \dots = \ell_N = 1$$

and there are two distinguished eigenvalues ℓ_1 and ℓ_2 of the covariance matrix. Assume without loss of generality that $\ell_1 \geq \ell_2$. Then

$$(41) \quad \begin{aligned} & \mathbb{P}\left((\lambda_1 - (1 + \gamma^{-1})^2) \cdot \frac{\gamma}{(1 + \gamma)^{4/3}} M^{2/3} \leq x\right) \\ & \rightarrow \begin{cases} F_0(x), & 0 < \ell_1, \ell_2 < 1 + \gamma^{-1}, \\ F_1(x), & 0 < \ell_2 < 1 + \gamma^{-1} = \ell_1, \\ F_2(x), & \ell_1 = \ell_2 = 1 + \gamma^{-1}, \end{cases} \end{aligned}$$

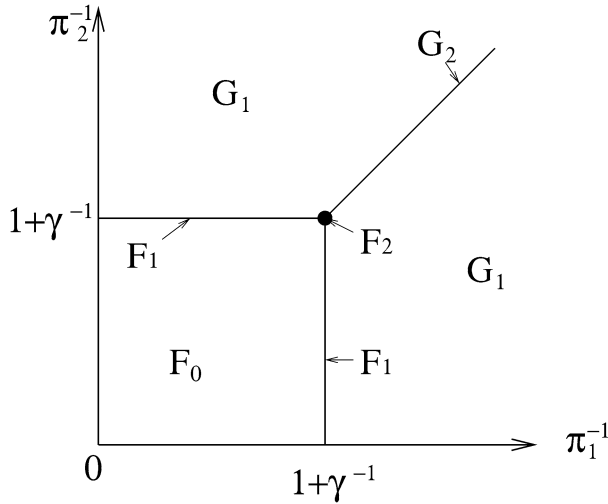


FIG. 1. Diagram of the limiting distributions for various choices of $\ell_1 = \pi_1^{-1}$ and $\ell_2 = \pi_2^{-1}$ while $\ell_3 = \dots = \ell_N = 1$.

and

$$\begin{aligned}
 & \mathbb{P}\left(\left(\lambda_1 - \left(\ell_1 + \frac{\ell_1 \gamma^{-2}}{\ell_1 - 1}\right)\right) \cdot \frac{\sqrt{M}}{\sqrt{\ell_1^2 - \ell_1^2 \gamma^{-2} / (\ell_1 - 1)^2}} \leq x\right) \\
 (42) \quad & \rightarrow \begin{cases} G_1(x), & \ell_1 > 1 + \gamma^{-1}, \ell_1 > \ell_2, \\ G_2(x), & \ell_1 = \ell_2 > 1 + \gamma^{-1}, \end{cases}
 \end{aligned}$$

assuming that ℓ_1, ℓ_2 are in compact sets in each case. See Figure 1 for a diagram.

Note the different fluctuation orders $M^{2/3}$ and \sqrt{M} depending on the values of ℓ_1, ℓ_2 . This type of “phase transition” was also observed in [5, 7, 33] for different models in combinatorics and last passage percolation, in which a few limiting distribution functions were also computed depending on parameters. But the functions $F_k, k \geq 2$, in Theorem 1.1 seem to be new in this paper. The last passage percolation model considered in [5, 33] has some relevance to our problem; see Section 6 below.

Theorem 1.1 and the fact that F_k and G_k are distribution functions yield the following consequence.

COROLLARY 1.1. *Under the same assumption of Theorem 1.1, the following hold.*

(a) *When for some $0 \leq k \leq r$,*

$$(43) \quad \ell_1 = \dots = \ell_k = 1 + \gamma^{-1},$$

and $\ell_{k+1}, \dots, \ell_r$ are in a compact subset of $(0, 1 + \gamma^{-1})$,

$$(44) \quad \lambda_1 \rightarrow (1 + \gamma^{-1})^2 \quad \text{in probability.}$$

(b) When for some $1 \leq k \leq r$,

$$(45) \quad \ell_1 = \dots = \ell_k > 1 + \gamma^{-1}$$

and $\ell_{k+1}, \dots, \ell_r$ are in a compact subset of $(0, \ell_1)$,

$$(46) \quad \lambda_1 \rightarrow \ell_1 \left(1 + \frac{\gamma^{-2}}{\ell_1 - 1} \right) \quad \text{in probability.}$$

PROOF. Suppose we are in the case of (a). For any fixed $\varepsilon > 0$ and $x \in \mathbb{R}$,

$$(47) \quad \begin{aligned} & \limsup_{M \rightarrow \infty} \mathbb{P}(\lambda_1 \leq (1 - \varepsilon)(1 + \gamma^{-1})^2) \\ & \leq \limsup_{M \rightarrow \infty} \mathbb{P}\left(\lambda_1 \leq (1 + \gamma^{-1})^2 + \frac{xM^{1/3}\gamma}{(1 + \gamma)^{4/3}}\right) = F_k(x). \end{aligned}$$

By taking $x \rightarrow -\infty$, we find that

$$(48) \quad \lim_{M \rightarrow \infty} \mathbb{P}(\lambda_1 \leq (1 - \varepsilon)(1 + \gamma^{-1})^2) = 0.$$

Similar arguments imply that $\mathbb{P}(\lambda_1 \geq (1 + \varepsilon)(1 + \gamma^{-1})^2) \rightarrow 0$. The case of (b) follows from the same argument. \square

Together with (6), Theorem 1.1/Corollary 1.1 imply that under the Gaussian assumption, when all but finitely many eigenvalues of Σ are 1, λ_1 is separated from the rest of the eigenvalues if and only if at least one eigenvalue of Σ is greater than $1 + \gamma^{-1}$. Theorem 1.1 also claims that when λ_1 is separated from the rest, the fluctuation of λ_1 is of order $M^{1/2}$ rather than $M^{2/3}$. Here the critical value $1 + \gamma^{-1}$ comes from a detail of computations and we do not have an intuitive reason yet. However, see Section 6 below for a heuristic argument from a last passage percolation model.

Compare the case (b) of Theorem 1.1/Corollary 1.1 with the following result for samples of finite number of variables.

PROPOSITION 1.1. *Suppose that there are M samples of $N = k$ variables. Assume that all the eigenvalues of the covariance matrix are the same;*

$$(49) \quad \ell_1 = \dots = \ell_k.$$

Then for fixed $N = k$, as $M \rightarrow \infty$,

$$(50) \quad \lim_{M \rightarrow \infty} \mathbb{P}\left((\lambda_1 - \ell_1) \frac{1}{\ell_1} \sqrt{M} x\right) = G_k(x)$$

and

$$(51) \quad \lambda_1 \rightarrow \ell_1 \quad \text{in probability.}$$

This result shows that the model in the case (b) of Theorem 1.1/Corollary 1.1 is not entirely dominated by the distinguished eigenvalues $\ell_1 = \dots = \ell_k$ of the covariance matrix. Instead the contribution to λ_1 comes from both $\ell_1 = \dots = \ell_k$ and infinitely many unit eigenvalues. The proof of Proposition 1.1 is given in Section 5.

Further detailed analysis along the line of this paper would yield the convergence of the moments of λ_1 under the scaling of Theorem 1.1. This will be presented somewhere else.

The real question is the real sample covariance. In the null cases, by comparing (5) and (8), we note that even though the limiting distributions are different, the scalings are identical. In view of this, we conjecture the following:

CONJECTURE. For real sample covariance, Theorem 1.1 still holds true for different limiting distributions but *with the same scaling*. In particular, the critical value of distinguished eigenvalues ℓ_j of the covariance matrix is again expected to be $1 + \gamma^{-1}$.

1.4. *Around the transition point; interpolating distributions.* We also investigate the nature of the transition at $\ell_j = 1 + \gamma^{-1}$. The following result shows that if ℓ_j themselves scale properly in M , there are interpolating limiting distributions.

We first need more definitions. For $m = 1, 2, 3, \dots$, and for $w_1, \dots, w_m \in \mathbb{C}$, set

$$(52) \quad s^{(m)}(u; w_1, \dots, w_m) = \frac{1}{2\pi} \int e^{iua+i(1/3)a^3} \prod_{j=1}^m \frac{1}{w_j + ia} da,$$

where the contour is from $\infty e^{5i\pi/6}$ to $\infty e^{i\pi/6}$ such that the points $a = iw_1, \dots, iw_m$ lie above the contour. Also set

$$(53) \quad t^{(m)}(v; w_1, \dots, w_{m-1}) = \frac{1}{2\pi} \int e^{ivb+i(1/3)b^3} \prod_{j=1}^{m-1} (w_j - ib) db,$$

where the contour is from $\infty e^{5i\pi/6}$ to $\infty e^{i\pi/6}$.

DEFINITION 1.3. For $k = 1, 2, \dots$, define for real x and w_1, \dots, w_k ,

$$(54) \quad \begin{aligned} &F_k(x; w_1, \dots, w_k) \\ &= \det(1 - \mathbf{A}_x) \cdot \det \left(1 - \left\langle \frac{1}{1 - \mathbf{A}_x} s^{(m)}(w_1, \dots, w_m), \right. \right. \\ &\quad \left. \left. t^{(n)}(w_1, \dots, w_{n-1}) \right\rangle \right)_{1 \leq m, n \leq k}. \end{aligned}$$

The function $F_1(x; w)$ previously appeared in [13] in a disguised form. See (4.18) and (4.12) of [13].

Formula (54) may seem to depend on the ordering of the parameters w_1, \dots, w_k . But as the following result (56) shows, it is independent of the ordering of the parameters. This can also be seen from a formula of [4].

Like F_k , it is not difficult to check that the function $F_k(x; w_1, \dots, w_k)$ is continuous, nondecreasing and becomes 1 as $x \rightarrow +\infty$. The proof that $F_k(x; w_1, \dots, w_k) \rightarrow 0$ as $x \rightarrow -\infty$ is in [4]. Therefore, $F_k(x; w_1, \dots, w_k)$ is a distribution function. It is direct to check that $F_k(x; w_1, \dots, w_k)$ interpolates $F_0(x), \dots, F_k(x)$. For example, $\lim_{w_2 \rightarrow +\infty} F_2(x, 0, w_2) = F_1(x)$, $\lim_{w_1 \rightarrow +\infty} \lim_{w_2 \rightarrow +\infty} F_2(x; w_1, w_2) = F_0(x)$, and so on.

THEOREM 1.2. *Suppose that for a fixed r , $\ell_{r+1} = \ell_{r+2} = \dots = \ell_N = 1$. Set for some $1 \leq k \leq r$,*

$$(55) \quad \ell_j = 1 + \gamma^{-1} - \frac{(1 + \gamma)^{2/3} w_j}{\gamma M^{1/3}}, \quad j = 1, 2, \dots, k.$$

When $w_j, 1 \leq j \leq k$, is in a compact subset of \mathbb{R} , and $\ell_j, k + 1 \leq j \leq r$, is in a compact subset of $(0, 1 + \gamma^{-1})$, as $M, N \rightarrow \infty$ such that $M/N = \gamma^2$ is in a compact subset of $[1, \infty)$,

$$(56) \quad \mathbb{P}\left((\lambda_1 - (1 + \gamma^{-1})^2) \cdot \frac{\gamma}{(1 + \gamma)^{4/3}} M^{2/3} \leq x\right) \rightarrow F_k(x; w_1, \dots, w_k)$$

for any x in a compact subset of \mathbb{R} .

The Painlevé II-type expression for $F_k(x; w_1, \dots, w_k)$ will be presented in [4].

This paper is organized as follows. The basic algebraic formula of the distribution of λ_1 in terms of a Fredholm determinant is given in Section 2, where an outline of the asymptotic analysis of the Fredholm determinant is also presented. The proofs of Theorem 1.1(a) and Theorem 1.2 are given in Section 3. The proof of Theorem 1.1(b) is in Section 4 and the proof of Proposition 1.1 is presented in Section 5. In Section 6 we indicate a connection between the sample covariance matrices, and a last passage percolation model and also a queueing theory.

NOTATIONAL REMARK. Throughout the paper, we set

$$(57) \quad \pi_j = \ell_j^{-1}.$$

This is only because the formulas below involving ℓ_j^{-1} become simpler with π_j .

2. Basic formulas.

NOTATIONAL REMARK. The notation $V(x)$ denotes the Vandermonde determinant

$$(58) \quad V(x) = \prod_{i < j} (x_i - x_j)$$

of a (finite) sequence $x = (x_1, x_2, \dots)$.

2.1. *Eigenvalue density; algebraic formula.* For complex Gaussian samples, the density of the sample covariance matrix was already known to Wishart around 1928 (see, e.g., [29]):

$$(59) \quad p(S) = \frac{1}{C} e^{-M \cdot \text{tr}(\Sigma^{-1}S)} (\det S)^{M-N}$$

for some normalization constant $C > 0$. As Σ and S are Hermitian, we can set $\Sigma = UDU^{-1}$ and $S = HLH^{-1}$ where U and H are unitary matrices, $D = \text{diag}(\ell_1, \dots, \ell_N) = \text{diag}(\pi_1^{-1}, \dots, \pi_N^{-1})$ and $L = \text{diag}(\lambda_1, \dots, \lambda_N)$. By taking the change of variables $S \mapsto (L, H)$ using the Jacobian formula $dS = cV(L)^2 dL dH$ for some constant $c > 0$, and then integrating over H , the density of the eigenvalues is (see, e.g., [19])

$$(60) \quad p(\lambda) = \frac{1}{C} V(\lambda)^2 \prod_{j=1}^N \lambda_j^{M-N} \cdot \int_{Q \in U(N)} e^{-M \cdot \text{tr}(D^{-1}QLQ^{-1})} dQ$$

for some (new) constant $C > 0$ where $U(N)$ is the set of $N \times N$ unitary matrices and $\lambda = (\lambda_1, \dots, \lambda_N)$. The last integral is known as the Harish–Chandra–Itzykson–Zuber integral (see, e.g., [28]) and we find

$$(61) \quad p(\lambda) = \frac{1}{C} \frac{\det(e^{-M\pi_j\lambda_k})_{1 \leq j, k \leq N}}{V(\pi)} V(\lambda) \prod_{j=1}^N \lambda_j^{M-N}.$$

Here when some of the π_j 's coincide, we interpret the formula using l'Hôpital's rule. We note that for a real sample covariance matrix, it is not known if the corresponding integral over the orthogonal group $O(N)$ is computable as above. Instead one usually defines hypergeometric functions of matrix argument and studies their algebraic properties (see, e.g., [29]). Consequently, the techniques below that we will use for the density of the form (61) are not applicable to real sample matrices.

For the density (61), the distribution function of the largest eigenvalue λ_1 can be expressed in terms of a Fredholm determinant, which will be the starting point of our asymptotic analysis. The following result can be obtained by suitably reinterpreting and taking a limit of a result of [30]. A different proof is given in [31]. For the convenience of the reader we include yet another proof by Johansson [20] which uses an idea from random matrix theory (see, e.g., [41]).

PROPOSITION 2.1. *For any fixed q satisfying $0 < q < \min\{\pi_j\}_{j=1}^N$, let $\mathbf{K}_{M,N}|_{(\xi, \infty)}$ be the operator acting on $L^2((\xi, \infty))$ with kernel*

$$(62) \quad \mathbf{K}_{M,N}(\eta, \zeta) = \frac{M}{(2\pi i)^2} \int_{\Gamma} dz \int_{\Sigma} dw e^{-\eta M(z-q) + \zeta M(w-q)} \frac{1}{w-z} \left(\frac{z}{w}\right)^M \prod_{k=1}^N \frac{\pi_k - w}{\pi_k - z}$$

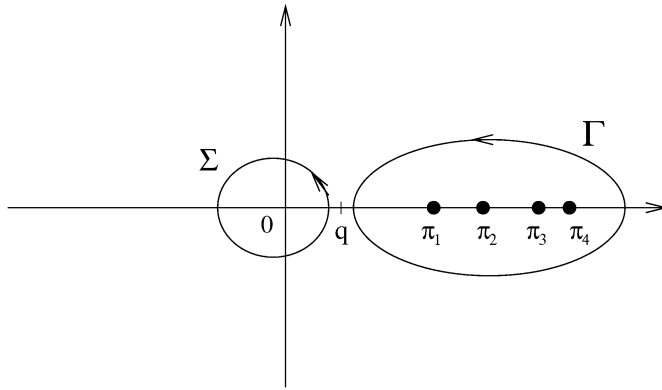


FIG. 2. Contours Γ and Σ .

where Σ is a simple closed contour enclosing 0 and lying in $\{w : \text{Re}(w) < q\}$, and Γ is a simple closed contour enclosing π_1, \dots, π_N and lying in $\{z : \text{Re}(z) > q\}$, both oriented counterclockwise (see Figure 2). Then for any $\xi \in \mathbb{R}$,

$$(63) \quad \mathbb{P}(\lambda_1 \leq \xi) = \det(1 - \mathbf{K}_{M,N}|_{(\xi, \infty)}).$$

REMARK 2.1. Note that the left-hand side of (63) does not depend on the parameter q . The Fredholm determinant on the right-hand side of (63) is also independent of the choice of q as long as $0 < q < \min\{\pi_j\}_{j=1}^N$. If we use the notation \mathbf{K}_q to denote $\mathbf{K}_{M,N}|_{(\xi, \infty)}$ for the parameter q , then $\mathbf{K}_{q'} = \mathbf{E}\mathbf{K}_q\mathbf{E}^{-1}$ where \mathbf{E} is the multiplication by $e^{(q'-q)\lambda}$; $(\mathbf{E}f)(\lambda) = e^{(q'-q)\lambda} f(\lambda)$. But determinants are invariant under conjugations as long as both $\mathbf{K}_{q'}$ and $\mathbf{E}\mathbf{K}_q\mathbf{E}^{-1}$ are in the trace class, which is the case when $0 < q, q' < \min\{\pi_j\}_{j=1}^N$. The parameter q ensures that the kernel $\mathbf{K}_{M,N}(\eta, \zeta)$ is finite when $\eta \rightarrow +\infty$ or $\zeta \rightarrow +\infty$ and the operator $\mathbf{K}_{M,N}|_{(\xi, \infty)}$ is trace class. It also helps the proof of the convergence of the operator in the next section.

PROOF. For a moment we assume that all π_j 's are distinct. Note that the density (61) is symmetric in λ_j 's. Hence using $V(\lambda) = \det(\lambda_k^{j-1})$, we find that

$$(64) \quad \begin{aligned} \mathbb{P}(\lambda_1 \leq \xi) &= \frac{1}{C'} \int_0^\infty \cdots \int_0^\infty \det(\lambda_k^{j-1}) \det(e^{-M\pi_j \lambda_k}) \prod_{k=1}^N (1 - \chi_{(\xi, \infty)}(\lambda_k)) \lambda_k^{M-N} d\lambda_k \end{aligned}$$

with some constant $C' > 0$, where $\chi_{(\xi, \infty)}$ denotes the characteristic function (indicator function). Using the fundamental identity which dates back to [1],

$$(65) \quad \int \cdots \int \det(f_j(x_k)) \det(g_j(x_k)) \prod_k d\mu(x_k) = \det\left(\int f_j(x) g_k(x) d\mu(x)\right),$$

we find

$$(66) \quad \mathbb{P}(\lambda_1 \leq \xi) = \frac{1}{C'} \det \left(\int_0^\infty (1 - \chi_{(\xi, \infty)}(\lambda)) \lambda^{j-1+M-N} e^{-M\pi_k \lambda} d\lambda \right)_{1 \leq j, k \leq N}.$$

Now set $v = M - N$, $\phi_j(\lambda) = \lambda^{j-1+v} e^{-Mq\lambda}$ and $\Phi_k(\lambda) = e^{-M(\pi_k - q)\lambda}$ for any q such that $0 < q < \min\{\pi_j\}_{j=1}^N$. Also let

$$(67) \quad A = (A_{jk})_{1 \leq j, k \leq N}, \quad A_{jk} = \int_0^\infty \phi_j(\lambda) \Phi_k(\lambda) d\lambda = \frac{\Gamma(j+v)}{(M\pi_k)^{j+v}}.$$

A direct computation shows that

$$(68) \quad \begin{aligned} \det A &= \prod_{j=1}^N \frac{\Gamma(j+v)}{(M\pi_j)^{v+1}} \cdot \det((M\pi_j)^{-(j-1)}) \\ &= \prod_{j=1}^N \frac{\Gamma(j+v)}{(M\pi_j)^{v+1}} \cdot \prod_{1 \leq j < k \leq N} ((M\pi_j)^{-1} - (M\pi_k)^{-1}). \end{aligned}$$

Thus A is invertible. Also define the operators $B : L^2((0, \infty)) \rightarrow \ell^2(\{1, \dots, N\})$, $C : \ell^2(\{1, \dots, N\}) \rightarrow L^2((0, \infty))$ by

$$(69) \quad B(j, \lambda) = \phi_j(\lambda), \quad C(\lambda, k) = \Phi_k(\lambda)$$

and let P_ξ be the projection from $(0, \infty)$ to (ξ, ∞) . Then as

$$(70) \quad \int_0^\infty \chi_{(\xi, \infty)}(\lambda) \lambda^{j-1+M-N} e^{-M\pi_k \lambda} d\lambda = (BP_\xi C)(j, k),$$

we find that

$$(71) \quad \mathbb{P}(\lambda_1 \leq \xi) = \frac{1}{C'} \det(A - BP_\xi C).$$

So,

$$(72) \quad \begin{aligned} \mathbb{P}(\lambda_1 \leq \xi) &= \frac{\det(A)}{C'} \det(1 - A^{-1}BP_\xi C) \\ &= C'' \det(1 - P_\xi CA^{-1}B) = C'' \det(1 - P_\xi CA^{-1}BP_\xi) \end{aligned}$$

for some constant C'' which does not depend on ξ . But by letting $\xi \rightarrow +\infty$ in both sides of (72), we easily find that $C'' = 1$. The kernel of the operator in the determinant is

$$(73) \quad (CA^{-1}B)(\eta, \zeta) = \sum_{j=1}^N C(\eta, j)(A^{-1}B)(j, \zeta), \quad \eta, \zeta > \xi,$$

and from Cramér's rule,

$$(74) \quad (A^{-1}B)(j, \zeta) = \frac{\det A^{(j)}(\zeta)}{\det A},$$

where $A^{(j)}(\zeta)$ is the matrix given by A with the j th column replaced by the vector $(\phi_1(\zeta), \dots, \phi_N(\zeta))^T$. To compute $A^{(j)}$, note (Hankel’s formula for Gamma function) that for a positive integer a

$$(75) \quad \frac{1}{2\pi i} \int_{\Sigma} \frac{e^w}{w^a} dw = \frac{1}{(a-1)!} = \frac{1}{\Gamma(a)},$$

where Σ is any simple closed contour enclosing the origin 0 with counterclockwise orientation. By replacing $w \rightarrow \zeta Mw$ and setting $a = j + \nu$, this implies that

$$(76) \quad \zeta^{j-1+\nu} = \frac{\Gamma(j+\nu)}{2\pi i} \int_{\Sigma} e^{\zeta Mw} \frac{M}{(Mw)^{j+\nu}} dw.$$

Substituting this formula for $\phi_j(\zeta)$ in the j th column of $A^{(j)}$, and pulling out the integrals over w ,

$$(77) \quad \det A^{(j)}(\zeta) = \frac{1}{2\pi i} \int_{\Sigma} e^{\zeta M(w-q)} \det(A'(w)) M dw,$$

where the entries of $A'(w)$ are $A'_{ab}(w) = \Gamma(a+\nu)/p_b^{a+\nu}$ where $p_b = M\pi_b$ when $b \neq j$ and $p_b = Mw$ when $b = j$. Hence

$$(78) \quad \det A^{(j)}(\zeta) = \prod_{k \neq j} \frac{1}{(M\pi_k)^{1+\nu}} \cdot \prod_{k=1}^N \Gamma(k+\nu) \\ \times \frac{1}{2\pi i} \int_{\Sigma} e^{\zeta M(w-q)} \prod_{1 \leq a < b \leq N} (p_a^{-1} - p_b^{-1}) \frac{M dw}{(Mw)^{1+\nu}},$$

and so using (68),

$$(79) \quad (A^{-1}B)(j, \zeta) = \frac{M\pi_j^{N+\nu}}{2\pi i} \int_{\Sigma} e^{\zeta M(w-q)} \prod_{k \neq j} \frac{w - \pi_k}{\pi_j - \pi_k} \frac{dw}{w^{N+\nu}}.$$

But for any simple closed contour Γ that encloses π_1, \dots, π_N but excludes w , and is oriented counterclockwise,

$$(80) \quad \frac{1}{2\pi i} \int_{\Gamma} z^M e^{-\eta Mz} \frac{1}{w-z} \prod_{k=1}^N \frac{w - \pi_k}{z - \pi_k} dz = \sum_{j=1}^N \pi_j^M e^{-M\pi_j \eta} \prod_{k \neq j} \frac{w - \pi_k}{\pi_j - \pi_k}.$$

Therefore, we find (note $N + \nu = M$)

$$(81) \quad (CA^{-1}B)(\eta, \zeta) \\ = \frac{M}{(2\pi i)^2} \int_{\Gamma} dz \int_{\Sigma} dw e^{-\eta M(z-q) + \zeta M(w-q)} \frac{1}{w-z} \prod_{k=1}^N \frac{w - \pi_k}{z - \pi_k} \cdot \left(\frac{z}{w}\right)^M,$$

which completes the proof when all π_j ’s are distinct. When some of the π_j ’s are identical, the result follows by taking proper limits and using l’Hôpital’s theorem. □

Note that for $z \in \Gamma$ and $w \in \Sigma$, $\text{Re}(w - z) < 0$. Hence using

$$(82) \quad \frac{1}{w - z} = -M \int_0^\infty e^{yM(w-q-(z-q))} dy$$

for $1/(w - z)$ in (62), the kernel $\mathbf{K}_{M,N}(\eta, \zeta)$ is equal to

$$(83) \quad \mathbf{K}_{M,N}(\eta, \zeta) = \int_0^\infty \mathbf{H}(\eta + y)\mathbf{J}(\zeta + y) dy$$

where

$$(84) \quad \mathbf{H}(\eta + y) = \frac{M}{2\pi} \int_\Gamma e^{-(\eta+y)M(z-q)} z^M \prod_{k=1}^N \frac{1}{\pi_k - z} dz$$

and

$$(85) \quad \mathbf{J}(\zeta + y) = \frac{M}{2\pi} \int_\Sigma e^{(\zeta+y)M(w-q)} w^{-M} \prod_{k=1}^N (\pi_k - w) dw.$$

2.2. *Asymptotic analysis: basic ideas.* From now on, as mentioned in the Introduction, we assume that

$$(86) \quad \pi_{r+1} = \dots = \pi_N = 1.$$

In this case, (82), (84) and (85) become

$$(87) \quad \mathbf{K}_{M,N}(\eta, \zeta) = \int_0^\infty \mathbf{H}(\eta + y)\mathbf{J}(\zeta + y) dy,$$

where

$$(88) \quad \mathbf{H}(\eta) = \frac{M}{2\pi} \int_\Gamma e^{-M\eta(z-q)} \frac{z^M}{(1-z)^{N-r}} \prod_{k=1}^r \frac{1}{\pi_k - z} dz$$

and

$$(89) \quad \mathbf{J}(\zeta) = \frac{M}{2\pi} \int_\Sigma e^{M\zeta(z-q)} \frac{(1-z)^{N-r}}{z^M} \prod_{k=1}^r (\pi_k - z) dz.$$

Set

$$(90) \quad \frac{M}{N} = \gamma^2 \geq 1.$$

For various choices of π_j , $1 \leq j \leq r$, we will consider the limit of $\mathbb{P}(\lambda_1 \leq \xi)$ when ξ is scaled as of the form (see Theorem 1.1)

$$(91) \quad \xi = \mu + \frac{\nu x}{M^\alpha}$$

for some constants $\mu = \mu(\gamma)$, $\nu = \nu(\gamma)$ and for some α , while x is a fixed real number. By translation and scaling, (63) becomes

$$(92) \quad \mathbb{P}\left(\lambda_1 \leq \mu + \frac{\nu x}{M^\alpha}\right) = \det(1 - \mathbf{K}_{M,N}|_{(\mu+\nu x/M^\alpha, \infty)}) = \det(1 - \mathcal{K}_{M,N}),$$

where $\mathcal{K}_{M,N}$ is the operator acting on $L^2((0, \infty))$ with kernel

$$(93) \quad \mathcal{K}_{M,N}(u, v) = \frac{\nu}{M^\alpha} \mathbf{K}_{M,N}\left(\mu + \frac{\nu(x+u)}{M^\alpha}, \mu + \frac{\nu(x+u)}{M^\alpha}\right).$$

Using (87), this kernel is equal to

$$(94) \quad \mathcal{K}_{M,N}(u, v) = \int_0^\infty \mathcal{H}(x+u+y) \mathcal{J}(x+v+y) dy,$$

where

$$(95) \quad \mathcal{H}(u) = \frac{\nu M^{1-\alpha}}{2\pi} \int_\Gamma e^{-\nu M^{1-\alpha} u(z-q)} e^{-M\mu(z-q)} \frac{z^M}{(1-z)^{N-r}} \prod_{\ell=1}^r \frac{1}{\pi_\ell - z} dz$$

and

$$(96) \quad \mathcal{J}(v) = \frac{\nu M^{1-\alpha}}{2\pi} \int_\Sigma e^{\nu M^{1-\alpha} v(w-q)} e^{M\mu(w-q)} \frac{(1-w)^{N-r}}{w^M} \prod_{\ell=1}^r (\pi_\ell - w) dw.$$

We need to find limits of $\mathcal{K}_{M,N}(u, v)$ for various choices of π_j 's as $M, N \rightarrow \infty$. A sufficient condition for the convergence of a Fredholm determinant is the convergence in trace norm of the operator. As $\mathcal{K}_{M,N}$ is a product of two operators, it is enough to prove the convergences of \mathcal{H} and \mathcal{J} in Hilbert–Schmidt norm. Hence in Sections 3 and 4 below, we will prove that for proper choices of μ, ν and α , there are limiting operators \mathcal{H}_∞ and \mathcal{J}_∞ acting on $L^2((0, \infty))$ such that for any real x in a compact set,

$$(97) \quad \int_0^\infty \int_0^\infty |Z_M \mathcal{H}_{M,N}(x+u+y) - \mathcal{H}_\infty(x+u+y)|^2 du dy \rightarrow 0$$

and

$$(98) \quad \int_0^\infty \int_0^\infty \left| \frac{1}{Z_M} \mathcal{J}_{M,N}(x+u+y) - \mathcal{J}_\infty(x+u+y) \right|^2 du dy \rightarrow 0$$

for some nonzero constant Z_M as $M, N \rightarrow \infty$ satisfying (90) for γ in a compact set. We will use steepest-descent analysis.

3. Proofs of Theorem 1.1(a) and Theorem 1.2. We first consider the proof of Theorem 1.1(a). The proof of Theorem 1.2 will be very similar (see Section 3.4 below). We assume that for some $0 \leq k \leq r$,

$$(99) \quad \pi_1^{-1} = \dots = \pi_k^{-1} = 1 + \gamma^{-1}$$

and $\pi_{k+1}^{-1} \geq \dots \geq \pi_r^{-1}$ are in a compact subset of $(0, 1 + \gamma^{-1})$.

For the scaling (91), we take

$$(100) \quad \alpha = 2/3$$

and

$$(101) \quad \mu = \mu(\gamma) := \left(\frac{1 + \gamma}{\gamma}\right)^2, \quad v = v(\gamma) := \frac{(1 + \gamma)^{4/3}}{\gamma}$$

so that

$$(102) \quad \xi = \mu + \frac{vx}{M^{2/3}}.$$

The reason for such choices will be made clear during the following asymptotic analysis. There is still an arbitrary parameter q . It will be chosen in (118) below.

The functions (95) and (96) are now

$$(103) \quad \mathcal{H}(u) = \frac{vM^{1/3}}{2\pi} \int_{\Gamma} e^{-vM^{1/3}u(z-q)} e^{Mf(z)} \frac{1}{(p_c - z)^k g(z)} dz$$

and

$$(104) \quad \mathcal{J}(v) = \frac{vM^{1/3}}{2\pi} \int_{\Sigma} e^{vM^{1/3}v(z-q)} e^{-Mf(z)} (p_c - z)^k g(z) dz$$

with

$$(105) \quad p_c := \frac{\gamma}{\gamma + 1},$$

and

$$(106) \quad f(z) := -\mu(z - q) + \log(z) - \frac{1}{\gamma^2} \log(1 - z),$$

where we take the principal branch of \log [i.e., $\log(z) = \ln |z| + i \arg(z)$, $-\pi < \arg(z) < \pi$], and

$$(107) \quad g(z) := \frac{1}{(1 - z)^r} \prod_{\ell=k+1}^r (\pi_{\ell} - z).$$

Now we find the critical point of f . As

$$(108) \quad f'(z) = -\mu + \frac{1}{z} - \frac{1}{\gamma^2(z - 1)},$$

$f'(z) = 0$ is a quadratic equation. But with the choice (101) of μ , there is a double root at $z = p_c = \frac{\gamma}{\gamma+1}$. Note that for γ in a compact subset of $[1, \infty)$, p_c is strictly less than 1. Being a double root,

$$(109) \quad f'(p_c) = f''(p_c) = 0,$$

where

$$(110) \quad f''(z) = -\frac{1}{z^2} + \frac{1}{\gamma^2(z-1)^2}.$$

It is also direct to compute

$$(111) \quad f^{(3)}(z) = \frac{2}{z^3} - \frac{2}{\gamma^2(z-1)^3}, \quad f^{(3)}(p_c) = \frac{2(\gamma+1)^4}{\gamma^3} = 2v^3$$

and

$$(112) \quad f(p_c) = -\mu(p_c - q) + \log\left(\frac{\gamma}{\gamma+1}\right) - \frac{1}{\gamma^2} \log\left(\frac{1}{\gamma+1}\right).$$

As $f^{(3)}(p_c) > 0$, the steepest-descent curve of $f(z)$ comes to the point p_c with angle $\pm\pi/3$ to the real axis. Once the contour Γ is chosen to be the steepest-descent curve near p_c and is extended properly, it is expected that the main contribution to the integral of \mathcal{H} comes from a contour near p_c . There is, however, a difficulty since at the critical point $z = p_c$, the integral (103) blows up due to the term $(p_c - z)^k$ in the denominator when $k \geq 1$. Nevertheless, this can be overcome if we choose Γ to be close to p_c , but not exactly pass through z_0 . Also as the contour Γ should contain all π_j 's, some of which may be equal to p_c , we will choose Γ to intersect the real axis *to the left* of p_c . By formally approximating the function f by a third-degree polynomial and the function $g(z)$ by $g(p_c)$, we expect

$$(113) \quad \mathcal{H}(u) \sim \frac{vM^{1/3}}{2\pi g(p_c)} \times \int_{\Gamma} e^{-vM^{1/3}u(z-q)} e^{M(f(z_c)+(f^{(3)}(p_c)/3!)(z-p_c)^3)} \frac{1}{(p_c - z)^k} dz$$

for some contour Γ_{∞} . Now taking the intersection point of Γ with the real axis to be on the left of p_c of distance of order $M^{-1/3}$, and then changing the variables by $vM^{1/3}(z - p_c) = a$, we expect

$$(114) \quad \mathcal{H}(u) \sim \frac{(-vM^{1/3})^k e^{Mf(p_c)}}{2\pi g(p_c)} e^{-vM^{1/3}u(p_c-q)} \int_{\Gamma_{\infty}} e^{-ua+(1/3)a^3} \frac{1}{a^k} da.$$

Similarly, we expect that

$$(115) \quad \mathcal{J}(v) \sim \frac{g(p_c)e^{-Mf(p_c)}}{2\pi(-vM^{1/3})^k} e^{vM^{1/3}v(p_c-q)} \int_{\Sigma_{\infty}} e^{va-(1/3)a^3} a^k da$$

for some contour Σ_{∞} . When multiplying $\mathcal{H}(u)$ and $\mathcal{J}(v)$, the constant prefactors

$$(116) \quad Z_M := \frac{g(p_c)}{(-vM^{1/3})^k e^{Mf(p_c)}}$$

and $1/Z_M$ cancel each other out. However, note that there are still functions $e^{-\nu M^{1/3}u(p_c-q)}$ and $e^{\nu M^{1/3}v(p_c-q)}$, one of which may become large when $M \rightarrow \infty$ [cf. (97) and (98)]. This trouble can be avoided if we can simply take $q = p_c$, but since Γ should be on the left of q , this simple choice is excluded. Nevertheless, we can still take q to be p_c minus some positive constant of order $M^{-1/3}$. Fix

$$(117) \quad \varepsilon > 0$$

and set

$$(118) \quad q := p_c - \frac{\varepsilon}{\nu M^{1/3}}.$$

We then take Γ to intersect the real axis at $p_c - c/(\nu M^{1/3})$ for some $0 < c < \varepsilon$. With this choice of q and Γ, Σ , we expect that

$$(119) \quad Z_M \mathcal{H}(u) \sim \mathcal{H}_\infty(u),$$

where

$$(120) \quad \mathcal{H}_\infty(u) := \frac{e^{-\varepsilon u}}{2\pi} \int_{\Gamma_\infty} e^{-ua+(1/3)a^3} \frac{1}{a^k} da,$$

and

$$(121) \quad \frac{1}{Z_M} \mathcal{J}(v) \sim \mathcal{J}_\infty(v),$$

where

$$(122) \quad \mathcal{J}_\infty(v) := \frac{e^{\varepsilon v}}{2\pi} \int_{\Sigma_\infty} e^{va-(1/3)a^3} a^k da.$$

Here the contour Γ_∞ is, as in the left-hand side of Figure 3, from $\infty e^{i\pi/3}$ to $\infty e^{-i\pi/3}$, passes the real axis on the left of the origin and lies in the region $\text{Re}(a + \varepsilon) > 0$, is symmetric about the real axis and is oriented from top to bottom. The contour Σ_∞ is, as in the right-hand side picture of Figure 3, from $\infty e^{-i2\pi/3}$ to $\infty e^{i2\pi/3}$, lies in the region $\text{Re}(a + \varepsilon) < 0$, is symmetric about the real axis and is oriented from bottom to top.

This argument should be justified and the following is a rigorous estimate.

PROPOSITION 3.1. *Fix $\varepsilon > 0$ and set q by (118). Define $\mathcal{H}_\infty(u)$ and $\mathcal{J}_\infty(v)$ by (120) and (122), respectively. Then the following hold for $\mathcal{H}(u)$ and $\mathcal{J}(v)$ in (103) and (104) for $M/N = \gamma^2$ with γ in a compact subset of $[1, \infty)$.*

(i) *For any fixed $U \in \mathbb{R}$, there are constants $C, c > 0, M_0 > 0$ such that*

$$(123) \quad |Z_M \mathcal{H}(u) - \mathcal{H}_\infty(u)| \leq \frac{C e^{-cu}}{M^{1/3}}$$

for $u \geq U$ when $M \geq M_0$.

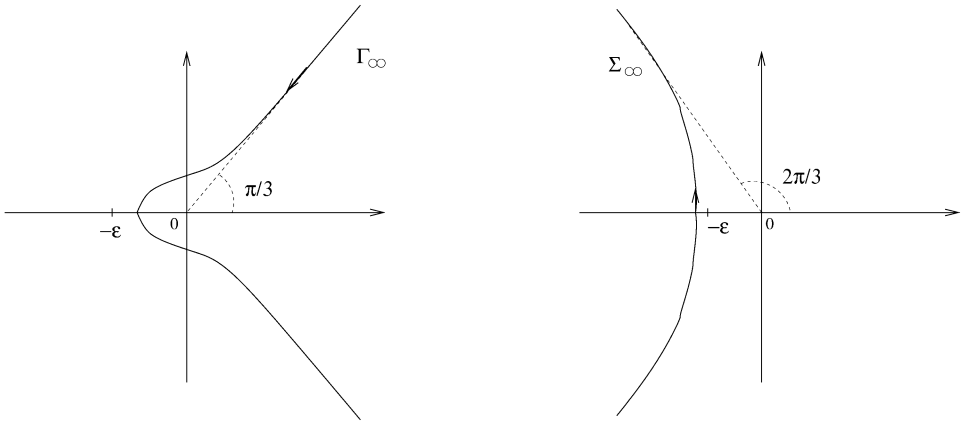


FIG. 3. Contours Γ_∞ and Σ_∞ of $\mathcal{H}(u)$ and $\mathcal{J}(v)$.

(ii) For any fixed $V \in \mathbb{R}$, there are constants $C, c > 0, M_0 > 0$ such that

$$(124) \quad \left| \frac{1}{Z_M} \mathcal{J}(v) - \mathcal{J}_\infty(v) \right| \leq \frac{C e^{-cv}}{M^{1/3}}$$

for $v \geq V$ when $M \geq M_0$.

The proof of this result is given in the following two subsections.

3.1. *Proof of Proposition 3.1(i).* The steepest-descent curve of f will depend on γ and M . Instead of controlling uniformity of the curve in γ and M , we will rather explicitly choose Γ which will be a steep-descent (though not the steepest-descent) curve of $f(z)$. Fix $R > 0$ such that $1 + R > \max\{1, \pi_{r+1}, \dots, \pi_k\}$. Define

$$(125) \quad \Gamma_0 := \left\{ p_c + \frac{\varepsilon}{2\nu M^{1/3}} e^{i\theta} : \pi/3 \leq \theta \leq \pi \right\},$$

$$(126) \quad \Gamma_1 := \left\{ p_c + t e^{i\pi/3} : \frac{\varepsilon}{2\nu M^{1/3}} \leq t \leq 2(1 - p_c) \right\},$$

$$(127) \quad \Gamma_2 := \{ p_c + 2(1 - p_c) e^{i\pi/3} + x : 0 \leq x \leq R \},$$

$$(128) \quad \Gamma_3 := \{ 1 + R + iy : 0 \leq y \leq \sqrt{3}(1 - p_c) \}.$$

Set

$$(129) \quad \Gamma = \left(\bigcup_{k=0}^3 \Gamma_k \right) \cup \overline{\left(\bigcup_{k=0}^3 \Gamma_k \right)}$$

and choose the orientation of Γ counterclockwise. See Figure 4.

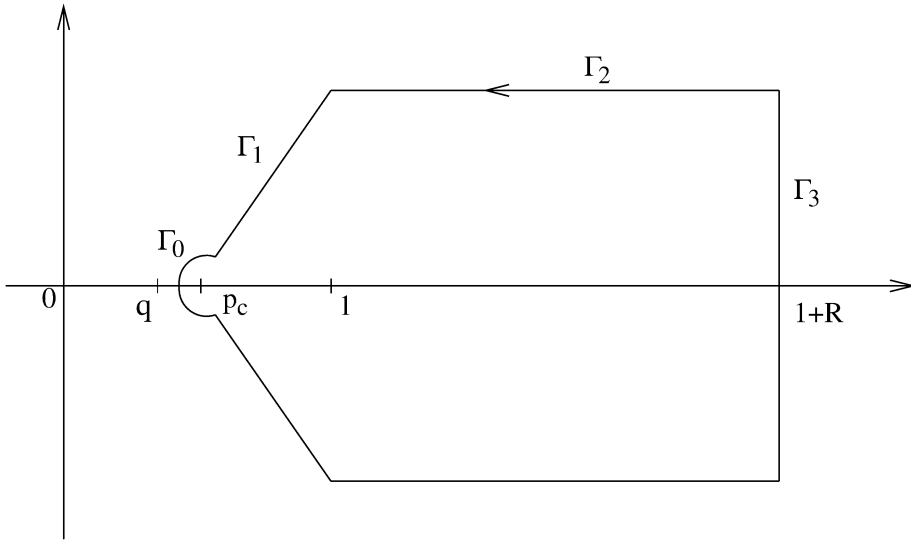


FIG. 4. Contour Γ .

Note that all the singular points of the integrand of \mathcal{H} are inside of Γ and hence the deformation to this new Γ is allowed. Direct calculations show the following properties of Γ . Recall (106),

$$(130) \quad f(z) := -\mu(z - q) + \log(z) - \frac{1}{\gamma^2} \log(1 - z).$$

LEMMA 3.1. For $\gamma \geq 1$, $\text{Re}(f(z))$ is decreasing for $z \in \Gamma_1 \cup \Gamma_2$ as $\text{Re}(z)$ increases. Also for γ in a compact subset of $[1, \infty)$, we can take $R > 0$ large enough such that

$$(131) \quad \max_{z \in \Gamma_3} \text{Re}(f(z)) \leq \text{Re}(f(p_*)),$$

where $p_* := p_c + 2(1 - p_c)e^{i\pi/3}$ is the intersection of Γ_1 and Γ_2 .

PROOF. For Γ_1 , by setting $z = p_c + te^{i\pi/3}$, $0 \leq t \leq 2(1 - p_c)$,

$$(132) \quad \begin{aligned} F_1(t) &:= \text{Re}(f(p_c + te^{i\pi/3})) \\ &= -\mu\left(p_c + \frac{1}{2}t - q\right) + \frac{1}{2} \ln(p_c^2 + pct + t^2) \\ &\quad - \frac{1}{2\gamma^2} \ln((1 - p_c)^2 - (1 - p_c)t + t^2) \end{aligned}$$

and

$$(133) \quad F_1'(t) = -\frac{t^2((\gamma + 1)^2 t^2 - (\gamma^2 - 1)t + 2\gamma)}{2\gamma^2(p_c^2 + pct + t^2)((1 - p_c)^2 - (1 - p_c)t + t^2)}.$$

The denominator is equal to $2\gamma^2|z|^2|1 - z|^2$, and hence is positive. To show that the numerator is positive, set

$$(134) \quad T_1(t) := (\gamma + 1)^2 t^2 - (\gamma^2 - 1)t + 2\gamma.$$

A simple calculus shows that

$$(135) \quad \min_{t \in [0, 2(1-p_c)]} T_1(t) = \begin{cases} T_1\left(\frac{\gamma^2 - 1}{2(\gamma + 1)^2}\right), & 1 \leq \gamma \leq 5, \\ T_1(2(1 - p_c)), & \gamma \geq 5. \end{cases}$$

But $T_1\left(\frac{\gamma^2 - 1}{2(\gamma + 1)^2}\right) \geq 2$ for $1 \leq \gamma \leq 5$, and $T_1(2(1 - p_c)) = 6$ for $\gamma \geq 5$, and hence we find that $T_1(t) > 0$ for $t \in [0, 2(1 - p_c)]$ and for all $\gamma \geq 1$. Thus we find that $F_1(t)$ is an increasing function in $t \in [0, 2(1 - p_c)]$.

For Γ_2 , by setting $z = p_c + 2(1 - p_c)e^{i\pi/3} + x, x \geq 0$,

$$(136) \quad \begin{aligned} F_2(x) &:= \operatorname{Re}(f(p_c + 2(1 - p_c)e^{i\pi/3} + x)) \\ &= -\mu(1 + x - q) + \frac{1}{2} \ln((1 + x)^2 + 3(1 - p_c)^2) \\ &\quad - \frac{1}{2\gamma^2} \ln(x^2 + 3(1 - p_c)^2) \end{aligned}$$

and

$$(137) \quad F_2'(x) = -\frac{T_2(x)}{\gamma^2(\gamma + 1)^2((1 + x)^2 + 3(1 - p_c)^2)(x^2 + 3(1 - p_c)^2)},$$

where

$$(138) \quad \begin{aligned} T_2(x) &= (\gamma + 1)^4 x^4 + (\gamma^4 + 6\gamma^3 + 12\gamma^2 + 10\gamma + 3)x^3 \\ &\quad + (2\gamma^3 + 13\gamma^2 + 20\gamma + 9)x^2 \\ &\quad + 2(2\gamma^2 + 7\gamma + 5)x + 6(\gamma + 2) > 0, \quad x \geq 0. \end{aligned}$$

Hence $F_2(x)$ is decreasing for $x \geq 0$.

For Γ_3 , setting $z = 1 + R + iy, 0 \leq y \leq \sqrt{3}(1 - p_c)$,

$$(139) \quad \begin{aligned} F_3(y) &= \operatorname{Re}(f(1 + R + iy)) \\ &= -\frac{\mu\varepsilon}{\nu M^{1/3}} - \mu(R + 1 - p_c) \\ &\quad + \frac{1}{2} \ln((1 + R)^2 + y^2) - \frac{1}{2\gamma^2} \ln(R^2 + y^2). \end{aligned}$$

As $1 \leq \mu = (\gamma + 1)^2/\gamma^2 \leq 4$ for $\gamma \geq 1$ and $1 - p_c = 1/(\gamma + 1) \leq 1$, $F_3(y)$ can be

made arbitrarily small when R is taken to be large. But

$$\begin{aligned}
 \operatorname{Re}(f(p_*)) &= \operatorname{Re}(f(p_c + 2(1 - p_c)e^{i\pi/3})) \\
 (140) \quad &= -\frac{\mu\varepsilon}{\nu M^{1/3}} - \mu(1 - p_c) \\
 &\quad + \frac{1}{2} \ln(p_c^2 + 2p_c(1 - p_c) + 4(1 - p_c)^2) - \frac{1}{2\gamma^2} \ln(5(1 - p_c)^2)
 \end{aligned}$$

is bounded for γ in a compact subset of $[1, \infty)$. Thus the result follows. \square

As γ is in a compact subset of $[1, \infty)$, we assume from here on that

$$(141) \quad 1 \leq \gamma \leq \gamma_0$$

for a fixed $\gamma_0 \geq 1$. Now we split the contour $\Gamma = \Gamma' \cup \Gamma''$ where Γ' is the part of Γ in the disk $|z - p_c| < \delta$ for some $\delta > 0$ which will be chosen in the next paragraph, and Γ'' is the rest of Γ . Let Γ'_∞ be the image of the contour Γ' under the map $z \mapsto \nu M^{1/3}(z - p_c)$ and let $\Gamma''_\infty = \Gamma_\infty \setminus \Gamma'_\infty$. Set

$$(142) \quad \mathcal{H}(u) = \mathcal{H}'(u) + \mathcal{H}''(u), \quad \mathcal{H}_\infty(u) = \mathcal{H}'_\infty(u) + \mathcal{H}''_\infty(u),$$

where $\mathcal{H}'(u)$ [resp. $\mathcal{H}'_\infty(u)$] is the part of the integral formula of $\mathcal{H}(u)$ [resp. $\mathcal{H}_\infty(u)$] integrated only over the contour Γ' (resp. Γ'_∞).

Fix δ such that

$$(143) \quad 0 < \delta < \min\left\{\frac{\nu^3}{6C_0}, \frac{1}{2(1 + \gamma_0)}\right\}, \quad C_0 := 4^3 + 4(1 + \gamma_0)^4.$$

For $|s - p_c| \leq \delta$,

$$\begin{aligned}
 (144) \quad \left|\frac{1}{4!}f^{(4)}(s)\right| &= \left|\frac{1}{4}\left|\frac{-1}{s^4} + \frac{\gamma^{-2}}{(1-s)^4}\right|\right| \leq \frac{1}{4}\left(\frac{1}{(p_c - \delta)^4} + \frac{\gamma^{-2}}{(1 - p_c - \delta)^4}\right) \\
 &\leq \frac{1}{4}(4^4 + 2^4(1 + \gamma_0)^4) = C_0.
 \end{aligned}$$

Hence by Taylor's theorem, for $|z - p_c| \leq \delta$,

$$\begin{aligned}
 (145) \quad &\left|\operatorname{Re}\left(f(z) - f(p_c) - \frac{f^{(3)}(p_c)}{3!}(z - p_c)^3\right)\right| \\
 &\leq \left|f(z) - f(p_c) - \frac{f^{(3)}(p_c)}{3!}(z - p_c)^3\right| \\
 &\leq \left(\max_{|s-p_c|\leq\delta} \frac{|f^{(4)}(s)|}{4!}\right)|z - p_c|^4 \\
 &\leq C_0|z - p_c|^4 \leq \frac{\nu^3}{6}|z - p_c|^3.
 \end{aligned}$$

Therefore, by recalling (111), we find for $0 \leq t \leq \delta$,

$$(146) \quad \operatorname{Re}(f(p_c + te^{i\pi/3})) - f(p_c) \leq -\frac{\nu^3}{6}t^3.$$

Especially, from Lemma 3.1 [note that $\operatorname{Re}(f(z)) = \operatorname{Re}(f(\bar{z}))$],

$$(147) \quad \max_{z \in \Gamma''} \operatorname{Re}(f(z)) \leq \operatorname{Re}(f(p_c + \delta e^{i\pi/3})) \leq f(p_c) - \frac{\nu^3}{6}\delta^3.$$

In the following subsections, we consider two cases separately: first when u is in a compact subset of \mathbb{R} , and the other when $u > 0$.

3.1.1. *When u is in a compact subset of \mathbb{R} .* Suppose that $|u| \leq u_0$ for some $u_0 > 0$. First we estimate

$$(148) \quad \begin{aligned} |Z_M \mathcal{H}''(u)| &\leq \frac{|g(p_c)|}{2\pi(\nu M^{1/3})^{k-1}} \\ &\times \int_{\Gamma''} e^{-\nu M^{1/3}u \operatorname{Re}(z-q)} e^{M \operatorname{Re}(f(z)-f(p_c))} \frac{1}{|p_c - z|^k |g(z)|} |dz|. \end{aligned}$$

Using (147) and $\operatorname{Re}(z - q) \leq \sqrt{(1 + R)^2 + 3}$ for $z \in \Gamma''$,

$$(149) \quad |Z_M \mathcal{H}''(u)| \leq \frac{|g(p_c)|}{2\pi(\nu M^{1/3})^{k-1}} e^{\nu M^{1/3}u_0 \sqrt{(1+R)^2+3}} e^{-M(\nu^3/6)\delta^3} \frac{L_\Gamma C_g}{\delta^k},$$

where L_Γ is the length of Γ and $C_g > 0$ is a constant such that

$$(150) \quad \frac{1}{C_g} \leq \min_{z \in \Gamma''} |g(z)| \leq C_g.$$

For $\gamma \in [1, \gamma_0]$, L_Γ , C_g and $|g(p_c)|$ are uniformly bounded, and hence

$$(151) \quad |Z_M \mathcal{H}''(u)| \leq e^{-(\nu^3/12)\delta^3 M}$$

when M is sufficiently large.

On the other hand, for $a \in \Gamma''_\infty$, we have $a = te^{\pm i\pi/3}$, $\delta \nu M^{1/3} \leq t < +\infty$, and hence

$$(152) \quad \begin{aligned} |\mathcal{H}''_\infty(u)| &= \left| \frac{e^{-\varepsilon u}}{2\pi} \int_{\Gamma''_\infty} \frac{e^{-ua} e^{(1/3)a^3}}{a^k} da \right| \leq \frac{e^{\varepsilon u_0}}{2\pi} \int_{\Gamma''_\infty} \frac{e^{u_0|a|} e^{(1/3)\operatorname{Re}(a^3)}}{|a|^k} |da| \\ &\leq \frac{e^{\varepsilon u_0}}{\pi} \int_{\delta \nu M^{1/3}}^\infty \frac{e^{u_0 t - (1/3)t^3}}{t^k} dt \leq e^{-(\nu^3/6)\delta^3 M} \end{aligned}$$

when M is sufficiently large.

Now we estimate $|Z_M \mathcal{H}'(u) - e^{-\varepsilon u} \mathcal{H}'_\infty(u)|$. Using the change of variables $a = \nu M^{1/3}(z - p_c)$ for the integral (120) for $\mathcal{H}'_\infty(u)$,

$$\begin{aligned}
 & |Z_M \mathcal{H}'(u) - \mathcal{H}'_\infty(u)| \\
 (153) \quad & \leq \frac{\nu M^{1/3}}{2\pi} \\
 & \quad \times \int_{\Gamma'} \frac{e^{-\nu M^{1/3} u \operatorname{Re}(z-q)}}{|\nu M^{1/3}(z - p_c)|^k} \left| e^{M(f(z)-f(p_c))} \frac{g(p_c)}{g(z)} - e^{M(\nu^3/3)(z-p_c)^3} \right| |dz|.
 \end{aligned}$$

We split the integral into two parts. Let $\Gamma' = \Gamma'_1 \cup \Gamma'_2$ where $\Gamma'_1 = \Gamma_0 \cup \overline{\Gamma_0}$ and $\Gamma'_2 = \Gamma' \setminus \Gamma'_1$.

For $z \in \Gamma'_1$, $|z - p_c| = \varepsilon/(2\nu M^{1/3})$. Hence using (145),

$$\begin{aligned}
 & |e^{M(f(z)-f(p_c))} - e^{M(\nu^3/3)(z-p_c)^3}| \\
 & \leq \max(|e^{M(f(z)-f(p_c))}|, |e^{M(\nu^3/3)(z-p_c)^3}|) \\
 (154) \quad & \quad \times M \left| f(z) - f(p_c) - \frac{\nu^3}{3}(z - p_c)^3 \right| \\
 & \leq e^{M \operatorname{Re}((\nu^3/3)(z-p_c)^3 + (\nu^3/6)|z-p_c|^3)} M C_0 |z - p_c|^4 \\
 & \leq e^{(1/16)\varepsilon^3} \frac{C_0 \varepsilon^4}{16\nu^4 M^{1/3}}.
 \end{aligned}$$

Also

$$(155) \quad \left| \frac{g(p_c)}{g(z)} - 1 \right| \leq \frac{1}{|g(z)|} \max_{|s-p_c| \leq \varepsilon/(2\nu M^{1/3})} |g'(s)| \cdot |z - p_c| \leq \frac{C_0 \overline{C} \varepsilon}{2\nu M^{1/3}},$$

where

$$(156) \quad \overline{C} := \max \left\{ |g'(s)| : |s - p_c| \leq \frac{\varepsilon}{2\nu M^{1/3}}, s \in \Gamma'_2 \right\},$$

which is uniformly bounded as Γ is uniformly away from the singular points of g . Hence using

$$\begin{aligned}
 & \left| e^{M(f(z)-f(p_c))} \frac{g(p_c)}{g(z)} - e^{M(\nu^3/3)(z-p_c)^3} \right| \\
 & = \left| (e^{M(f(z)-f(p_c))} - e^{M(\nu^3/3)(z-p_c)^3}) \frac{g(p_c)}{g(z)} \right. \\
 (157) \quad & \quad \left. + e^{M(\nu^3/3)(z-p_c)^3} \left(\frac{g(p_c)}{g(z)} - 1 \right) \right| \\
 & \leq \left(\frac{C_0 \varepsilon^4}{16\nu^4} e^{(1/16)\varepsilon^3} + \frac{C_0 \overline{C}}{2\nu} \right) \cdot \frac{1}{M^{1/3}},
 \end{aligned}$$

and the fact that the length of Γ'_1 is $4\pi R_0/(3M^{1/3})$, we find that the part of the integral in (153) over Γ'_1 is less than or equal to some constant divided by $M^{1/3}$.

For $z \in \Gamma'_2$, we have $z = p_c + te^{\pm i\pi/3}$, $\varepsilon/(2\nu M^{1/3}) \leq t \leq \delta$. From (145) [cf. (154)]

$$\begin{aligned}
 & \left| e^{M(f(z)-f(p_c))} - e^{M(v^3/3)(z-p_c)^3} \right| \\
 (158) \quad & \leq e^{M \operatorname{Re}((v^3/3)(z-p_c)^3 + (v^3/6)|z-p_c|^3)} M C_0 |z - p_c|^4 \\
 & \leq e^{-M(v^3/6)t^3} \cdot C_0 M t^4.
 \end{aligned}$$

Also

$$(159) \quad \left| \frac{g(p_c)}{g(z)} - 1 \right| \leq \frac{1}{|g(z)|} \max_{s \in \Gamma'_2} |g'(s)| \cdot |p_c - z| \leq C_0 \bar{C} t,$$

and hence

$$\begin{aligned}
 & \left| e^{M(f(z)-f(p_c))} \frac{g(p_c)}{g(z)} - e^{M(v^3/3)(z-p_c)^3} \right| \\
 & = \left| (e^{M(f(z)-f(p_c))} - e^{M(v^3/3)(z-p_c)^3}) \frac{g(p_c)}{g(z)} \right| \\
 (160) \quad & + \left| e^{M(v^3/3)(z-p_c)^3} \left(\frac{g(p_c)}{g(z)} - 1 \right) \right| \\
 & \leq e^{-(v^3/6)Mt^3} C_0^3 M t^4 + e^{-(v^3/3)Mt^3} C_0 \bar{C} t \\
 & \leq (C_0^3 + C_0 \bar{C}) e^{-(v^3/6)Mt^3} (M t^4 + t).
 \end{aligned}$$

Using

$$(161) \quad e^{-\nu M^{1/3} u \operatorname{Re}(z-q)} \leq e^{\nu M^{1/3} u_0 (|p_c-q| + |z-p_q|)} = e^{\varepsilon u_0 + \nu u_0 M^{1/3} t},$$

we find by substituting $z = p_c + te^{\pm i\pi/3}$ into (153), the part of the integral in (153) over Γ'_2 is less than or equal to

$$(162) \quad \frac{\nu M^{1/3}}{\pi} (C_0^3 + C_0 \bar{C}) \int_{\varepsilon/(2\nu M^{1/3})}^{\delta} \frac{e^{\varepsilon u_0 + \nu u_0 M^{1/3} t}}{(\nu M^{1/3} t)^k} e^{-(v^3/6)Mt^3} (M t^4 + t) dt.$$

Then by the change of variables $s = \nu M^{1/3} t$, the last integral is less than or equal to

$$(163) \quad \frac{(C_0^3 + C_0 \bar{C}) e^{\varepsilon u_0}}{\pi M^{1/3}} \int_{\varepsilon/2}^{\infty} \frac{e^{u_0 s - (1/6)s^3}}{s^k} \left(\frac{1}{\nu^4} s^4 + \frac{1}{\nu} s \right) ds,$$

which is a constant divided by $M^{1/3}$. Therefore, together with the estimate for the part over Γ'_1 , this implies that

$$(164) \quad |Z_M \mathcal{H}'(u) - \mathcal{H}'_{\infty}(u)| \leq \frac{C_1}{M^{1/3}}$$

for some positive constant $C_1 > 0$ for any $M > 0$. Now combining (151), (152) and (164), we find that for any $u_0 > 0$, there are constants $C > 0, M_0 > 0$ which may depend on u_0 such that

$$(165) \quad |Z_M \mathcal{H}(u) - \mathcal{H}_\infty(u)| \leq \frac{C}{M^{1/3}}$$

for $|u| \leq u_0$ and $M \geq M_0$. Hence we obtain (123).

3.1.2. *When $u > 0$.* We first estimate $|Z_M \mathcal{H}''(u)|$ using (148). For $z \in \Gamma''$, $\text{Re}(z - q) = \text{Re}(p_c - q) + \text{Re}(z - p_c) \geq \frac{\varepsilon}{vM^{1/3}} + \frac{1}{2}\delta$, and hence as $u > 0$,

$$(166) \quad |Z_M \mathcal{H}''(u)| \leq \frac{|g(p_c)|}{2\pi(vM^{1/3})^{k-1}} e^{-\varepsilon u} e^{-(1/2)vM^{1/3}\delta u} e^{-M(v^3/6)\delta^3} \frac{L\Gamma}{\delta^k C_g}.$$

On the other hand, for $a \in \Gamma''_\infty$, by estimating as in (152) but now using $u > 0$ and $\text{Re}(a) \geq \frac{1}{2}\delta vM^{1/3}$ for $a \in \Gamma''_\infty$, we find

$$(167) \quad \begin{aligned} |\mathcal{H}''_\infty(u)| &\leq \frac{e^{-\varepsilon u}}{2\pi} \int_{\Gamma''_\infty} \frac{e^{-\text{Re}(a)+(1/3)\text{Re}(a^3)}}{|a|^k} |da| \\ &\leq \frac{e^{-\varepsilon u} e^{-(1/2)\delta vM^{1/3}u}}{2\pi} \int_{\Gamma''_\infty} \frac{e^{(1/3)\text{Re}(a^3)}}{|a|^k} |da| \\ &\leq e^{-\varepsilon u} e^{-(1/2)\delta vM^{1/3}u} e^{-M(v^3/6)\delta^3} \end{aligned}$$

when M is sufficiently large.

In order to estimate $|Z_M \mathcal{H}'(u) - \mathcal{H}'_\infty(u)|$, we note that as $u > 0$, for $z \in \Gamma'$,

$$(168) \quad e^{-vM^{1/3}u \text{Re}(z-q)} \leq e^{-(1/2)\varepsilon u}$$

which is achieved at $z = p_c - \frac{\varepsilon}{2vM^{1/3}}$. Using the same estimates as in the case when u is in a compact set for the rest of the terms of (153), we find that

$$(169) \quad |Z_M \mathcal{H}'(u) - \mathcal{H}'_\infty(u)| \leq \frac{C_2 e^{-(1/2)\varepsilon u}}{M^{1/3}}$$

for some constant $C_2 > 0$ when M is large enough. Now (166), (167) and (169) yield (123) for the case when $u > 0$.

3.2. *Proof of Proposition 3.1(ii).* We choose the contour Σ which will be a steep-descent curve of $-f(z)$, explicitly as follows. Let $R > 0$. Define

$$(170) \quad \Sigma_0 := \left\{ p_c + \frac{3\varepsilon}{vM^{1/3}} e^{i(\pi-\theta)} : 0 \leq \theta \leq \frac{\pi}{3} \right\},$$

$$(171) \quad \Sigma_1 := \left\{ p_c + t e^{2i\pi/3} : \frac{3\varepsilon}{vM^{1/3}} \leq t \leq 2p_c \right\},$$

$$(172) \quad \Sigma_2 := \{ p_c + 2p_c e^{2i\pi/3} - x : 0 \leq x \leq R \},$$

$$(173) \quad \Sigma_3 := \{ -R + i(\sqrt{3}p_c - y) : 0 \leq y \leq \sqrt{3}p_c \},$$

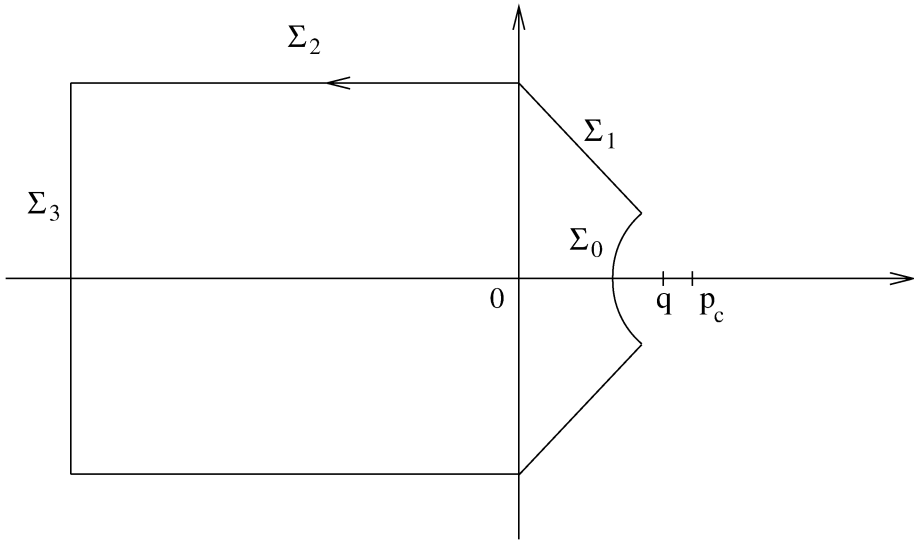


FIG. 5. Contour Σ .

and set

$$(174) \quad \Sigma = \left(\bigcup_{k=0}^3 \Sigma_k \right) \cup \overline{\left(\bigcup_{k=0}^3 \Sigma_k \right)}.$$

The orientation of Σ is counterclockwise. See Figure 5. We first prove decay properties of $\text{Re}(-f(z))$ analogous to Lemma 3.1.

LEMMA 3.2. *For $\gamma \geq 1$, $\text{Re}(-f(z))$ is decreasing for $z \in \Sigma_1 \cup \Sigma_2$ as $\text{Re}(z)$ decreases. Also for γ in a compact subset of $[1, \infty)$, we can take large $R > 0$ (independent of γ) such that*

$$(175) \quad \max_{z \in \sigma_3} \text{Re}(-f(z)) \leq \text{Re}(-f(p_*)),$$

where $p_* = p_c + 2p_c e^{2i\pi/3}$ is the intersection of Σ_1 and Σ_2 .

PROOF. For $z \in \Sigma_1$, by setting $z = p_c + te^{2i\pi/3}$, $0 \leq t \leq 2p_c$,

$$(176) \quad \begin{aligned} F_1(t) &:= \text{Re}(-f(p_c + te^{2i\pi/3})) \\ &= \mu \left(p_c - \frac{1}{2}t - c \right) - \frac{1}{2} \ln(p_c^2 - p_c t + t^2) \\ &\quad + \frac{1}{2\gamma^2} \ln((1 - p_c)^2 + (1 - p_c)t + t^2) \end{aligned}$$

and hence

$$(177) \quad F'_1(t) = -\frac{t^2((\gamma + 1)^2t^2 + (\gamma^2 - 1)t + 2\gamma)}{2\gamma^2(p_c^2 - p_c t + t^2)((1 - p_c)^2 + (1 - p_c)t + t^2)},$$

which is nonnegative for all $t \geq 0$.

For Σ_2 , by setting $z = p_c + 2p_c e^{2i\pi/3} - x, x \geq 0$,

$$(178) \quad \begin{aligned} F_2(x) &:= \operatorname{Re}(-f(p_c + 2p_c e^{2i\pi/3} - x)) \\ &= \mu(-x - c) - \frac{1}{2} \ln(x^2 + 3p_c^2) + \frac{1}{2\gamma^2} \ln((1 + x)^2 + 3p_c^2). \end{aligned}$$

A direct computation shows that

$$(179) \quad F'_2(x) = -\frac{T_2(x)}{\gamma^2(\gamma + 1)^2(x^2 + 3p_c^2)((1 + x)^2 + 3p_c^2)},$$

where

$$(180) \quad \begin{aligned} T_2(x) &= (\gamma + 1)^4 x^4 + (3\gamma^4 + 10\gamma^3 + 12\gamma^2 + 6\gamma + 1)x^3 \\ &\quad + (9\gamma^4 + 20\gamma^3 + 13\gamma^2 + 2\gamma)x^2 \\ &\quad + 2(5\gamma^4 + 7\gamma^3 + 2\gamma^2)x + 6(2\gamma^4 + \gamma^3). \end{aligned}$$

Hence $F_2(x)$ is decreasing for $x \geq 0$.

For Σ_3 , setting $z = -R + i(\sqrt{3}p_c - y), 0 \leq y \leq \sqrt{3}p_c$,

$$(181) \quad \begin{aligned} F_3(y) &= \operatorname{Re}(-f(-R + i(\sqrt{3}p_c - y))) \\ &= \mu(-R - c) - \frac{1}{2} \ln(R^2 + (\sqrt{3}p_c - y)^2) \\ &\quad + \frac{1}{2\gamma^2} \ln((1 + R)^2 + (\sqrt{3}p_c - y)^2). \end{aligned}$$

When $R \rightarrow +\infty, F_3(y)$ can be made arbitrarily small. But

$$(182) \quad \operatorname{Re}(-f(p_c + 2p_c e^{2i\pi/3})) = -\mu c - \ln(\sqrt{3}p_c) + \frac{1}{2\gamma^2} \ln(1 + 3p_c^2)$$

is bounded for γ in a compact subset of $[1, \infty)$. Thus the result follows. \square

Let δ be given in (143). Let $\Sigma = \Sigma' \cup \Sigma''$ where Σ' is the part of Σ that lies in the disk $|z - p_c| < \delta$, and let $\Sigma'' = \Sigma \setminus \Sigma'$. Let Σ'_∞ be the image of Σ' under the map $z \mapsto \nu M^{1/3}(z - p_c)$ and let $\Sigma''_\infty = \Sigma_\infty \setminus \Sigma'_\infty$. Set

$$(183) \quad \mathcal{J}(v) = \mathcal{J}'(v) + \mathcal{J}''(v), \quad \mathcal{J}_\infty = \mathcal{J}'_\infty(v) + \mathcal{J}''_\infty(v),$$

where $\mathcal{J}'(v)$ [resp. $\mathcal{J}'_\infty(v)$] is the part of the integral formula of $\mathcal{J}(v)$ [resp. $\mathcal{J}_\infty(v)$] integrated over the contour Σ' (resp. Σ'_∞).

As before, we consider two cases separately: the first case when v is in a compact subset of \mathbb{R} , and the second case when $v > 0$.

3.2.1. *When v is in a compact subset of \mathbb{R} .* There is $v_0 > 0$ such that $|v| \leq v_0$. First, we estimate

$$(184) \quad \left| \frac{1}{Z_M} \mathcal{J}''(v) \right| \leq \frac{(vM^{1/3})^{k+1}}{2\pi |g(p_c)|} \times \int_{\Sigma''} e^{\nu M^{1/3} v_0 |z-q|} e^{M \operatorname{Re}(-f(z)+f(p_c))} |p_c - z|^k |g(z)| |dz|.$$

From Lemma 3.2 and (145), following the proof of (147), we find

$$(185) \quad \max_{z \in \Sigma''} \operatorname{Re}(-f(z)) \leq \operatorname{Re}(-f(p_c + \delta e^{2i\pi/3})) \leq f(p_c) - \frac{v^3}{6} \delta^3.$$

Hence using $|z - q| \leq \sqrt{(R_1 + 1)^2 + 3}$ for $z \in \Sigma''$,

$$(186) \quad \left| \frac{1}{Z_M} \mathcal{J}''(v) \right| \leq \frac{(vM^{1/3})^{k+1}}{2\pi |g(p_c)|} e^{\nu M^{1/3} v_0 \sqrt{(R_1+1)^2+3}} e^{-(v^3/6)\delta^3 M} \delta^k \tilde{C}_g L_\Sigma,$$

where \tilde{C}_g is the maximum of $|g(z)|$ over $z \in \Sigma''$ and L_Σ is the length of Σ'' , both of which are uniformly bounded. Hence we find that when M is sufficiently large,

$$(187) \quad \left| \frac{1}{Z_M} \mathcal{J}''(v) \right| \leq e^{-(v^3/12)\delta^3 M}.$$

When $a \in \Sigma''_\infty$, $a = te^{\pm 2i\pi/3}$, $\delta \nu M^{1/3} \leq t < +\infty$, and

$$(188) \quad \begin{aligned} |\mathcal{J}''_\infty(v)| &= \left| \frac{e^{\varepsilon v}}{2\pi} \int_{\Sigma''_\infty} e^{\nu a - (1/3)a^3} a^k da \right| \\ &\leq \frac{e^{\varepsilon v_0}}{\pi} \int_{\delta \nu M^{1/3}}^\infty e^{\nu_0 t - (1/3)t^3} t^k dt \leq e^{-(v^3/6)\delta^3 M} \end{aligned}$$

when M is sufficiently large.

Finally we estimate

$$(189) \quad \begin{aligned} &\left| \frac{1}{Z_M} \mathcal{J}'(v) - \mathcal{J}'_\infty(v) \right| \\ &\leq \frac{\nu M^{1/3}}{2\pi} \int_{\Sigma'} |e^{\nu M^{1/3} v(z-q)}| |\nu M^{1/3} (z - p_c)|^k \\ &\quad \times \left| e^{-M(f(z)-f(p_c))} \frac{g(z)}{g(p_c)} - e^{-M(v^3/3)(z-p_c)^3} \right| |dz|. \end{aligned}$$

As before, we split the contour $\Sigma' = \Sigma'_1 \cup \Sigma'_2$ where $\Sigma'_1 = \Sigma_0 \cup \overline{\Sigma_0}$ and $\Sigma'_2 = \Sigma' \setminus \Sigma'_1$, and by following the steps of (154)–(163), we arrive at

$$(190) \quad \left| \frac{1}{Z_M} \mathcal{J}'(v) - \mathcal{J}'_\infty(v) \right| \leq \frac{C_3}{M^{1/3}},$$

for some constant $C_3 > 0$ when M is large enough. From (187), (188) and (190), we obtain (124).

3.2.2. *When $v > 0$.* The proof in this case is again very similar to the estimate of $\mathcal{H}(u)$ when $u > 0$. The only change is the following estimates:

$$(191) \quad \operatorname{Re}(z - q) \leq \operatorname{Re}(p_c + \delta e^{2i\pi/3} - q) = \frac{\varepsilon}{\nu M^{1/3}} - \frac{1}{2}\delta, \quad z \in \Sigma'',$$

$$(192) \quad \operatorname{Re}(z - q) \leq \operatorname{Re}\left(p_c + \frac{3\varepsilon}{\nu M^{1/3}} e^{2i\pi/3} - q\right) = -\frac{\varepsilon}{2\nu M^{1/3}}, \quad z \in \Sigma',$$

and

$$(193) \quad \operatorname{Re}(z - q) = -\frac{1}{2}|z - p_c| + \frac{\varepsilon}{\nu M^{1/3}}, \quad z \in \Sigma_1.$$

Then for large enough $M > 0$,

$$(194) \quad \left| \frac{1}{Z_M} \mathcal{J}''(v) \right| \leq e^{\varepsilon v} e^{-(1/2)\delta \nu M^{1/3} v} e^{-(v^3/12)\delta^3 M},$$

$$(195) \quad |\mathcal{J}''_\infty(v)| \leq e^{\varepsilon v} e^{-(1/2)\delta \nu M^{1/3} v} e^{-(v^3/6)\delta^3 M}$$

and

$$(196) \quad \left| \frac{1}{Z_M} \mathcal{J}'(v) - \mathcal{J}'_\infty(v) \right| \leq \frac{C}{M^{1/3}} e^{-(1/2)\varepsilon v}$$

for some constant $C > 0$. We skip the detail.

3.3. *Proof of Theorem 1.1(a).* From Proposition 3.1, the discussion in Section 2.2 implies that under the assumption of Theorem 1.1(a),

$$(197) \quad \mathbb{P}\left((\lambda_1 - (1 + \gamma^{-1})^2) \cdot \frac{\gamma}{(1 + \gamma)^{4/3}} M^{2/3} \leq x\right)$$

converges, as $M \rightarrow \infty$, to the Fredholm determinant of the operator acting on $L^2((0, \infty))$ whose kernel is

$$(198) \quad \int_0^\infty \mathcal{H}_\infty(x + u + y) \mathcal{J}_\infty(x + v + y) dy.$$

From the integral representation (10) of the Airy function, by simple changes of variables,

$$(199) \quad \operatorname{Ai}(u) = \frac{-1}{2\pi i} \int_{\Gamma_\infty} e^{-ua + (1/3)a^3} da = \frac{1}{2\pi i} \int_{\Sigma_\infty} e^{ub - (1/3)b^3} db,$$

and hence a simple algebra shows that

$$\begin{aligned}
 & \int_0^\infty \mathcal{H}_\infty(u+y) \mathcal{J}_\infty(v+y) dy \\
 & - \int_0^\infty e^{-\varepsilon(u+y)} \text{Ai}(u+y) \text{Ai}(v+y) e^{\varepsilon(v+y)} dy \\
 & = \frac{e^{-\varepsilon(u-v)}}{(2\pi)^2} \int_0^\infty dy \int_{\Gamma_\infty} da \int_{\Sigma_\infty} db \\
 & \quad \times e^{-(u+y)a+(1/3)a^3} e^{(v+y)b-(1/3)b^3} \left(\left(\frac{b}{a} \right)^k - 1 \right) \\
 (200) \quad & = \sum_{m=1}^k \frac{e^{-\varepsilon(u-v)}}{(2\pi)^2} \int_{\Gamma_\infty} da \int_{\Sigma_\infty} db \\
 & \quad \times e^{-ua+(1/3)a^3} e^{vb-(1/3)b^3} \frac{(b-a)b^{m-1}}{a^m} \int_0^\infty e^{-(a-b)y} dy \\
 & = - \sum_{m=1}^k \frac{e^{-\varepsilon(u-v)}}{(2\pi)^2} \int_{\Gamma_\infty} da \int_{\Sigma_\infty} db \\
 & \quad \times e^{-ua+(1/3)a^3} e^{vb-(1/3)b^3} \frac{b^{m-1}}{a^m} \\
 & = \sum_{m=1}^k e^{-\varepsilon u} s^{(m)}(u) t^{(m)}(v) e^{\varepsilon v},
 \end{aligned}$$

where the choice of the contours Σ_∞ and Γ_∞ ensures that $\text{Re}(a-b) > 0$ which is used in the third equality.

Let \mathbf{E} be the multiplication operator by $e^{-\varepsilon u}$; $(\mathbf{E}f)(u) = e^{-\varepsilon u} f(u)$. The computation (200) implies that (197) converges to

$$(201) \quad \det \left(1 - \mathbf{E} \mathbf{A}_x \mathbf{E}^{-1} - \sum_{m=1}^k \mathbf{E} s_x^{(m)} \otimes t_x^{(m)} \mathbf{E}^{-1} \right).$$

The general formula of the Fredholm determinant of a finite-rank perturbation of an operator yields that this is equal to

$$(202) \quad \det(1 - \mathbf{E} \mathbf{A}_x \mathbf{E}^{-1}) \cdot \det \left(\delta_{mn} - \left\langle \frac{1}{1 - \mathbf{E} \mathbf{A}_x \mathbf{E}^{-1}} \mathbf{E} s_x^{(m)}, t_x^{(n)} \mathbf{E}^{-1} \right\rangle \right)_{1 \leq m, n \leq k},$$

which is equal to (17) due to the proof of the following lemma. This completes the proof.

LEMMA 3.3. *The function $F_k(x)$ in Definition 1.1 is well defined. Also $s^{(m)}(u)$ defined in (13) can be written as*

$$(203) \quad s^{(m)}(u) = \sum_{\ell+3n=m-1} \frac{(-1)^n}{3^n \ell! n!} u^\ell + \frac{1}{(m-1)!} \int_\infty^u (u-y)^{m-1} \text{Ai}(y) dy.$$

PROOF. It is known that \mathbf{A}_x has norm less than 1 and is trace class (see, e.g., [39]). The only thing we need to check is that the product $\langle \frac{1}{1-\mathbf{A}_x} s^{(m)}, t^{(n)} \rangle$ is finite. By using the standard steepest-descent analysis,

$$(204) \quad \text{Ai}(u) \sim \frac{1}{2\sqrt{\pi}u^{1/4}} e^{-(2/3)u^{3/2}}, \quad u \rightarrow +\infty,$$

and

$$(205) \quad t^{(m)}(v) \sim \frac{v^{m/2}}{2\sqrt{\pi}v^{3/4}} e^{-(2/3)v^{3/2}}, \quad v \rightarrow +\infty.$$

But for $s^{(m)}$, since the critical point $a = i$ is above the pole $a = 0$, the residue at $a = 0$ contributes to the asymptotics and $s^{(m)}(u)$ grows in powers of u as $u \rightarrow +\infty$:

$$(206) \quad s^{(m)}(u) \sim \sum_{\ell+3n=m-1} \frac{(-1)^n}{3^n \ell! n!} u^\ell + \frac{(-1)^m}{2\sqrt{\pi}u^{m/2}u^{1/4}} e^{-(2/3)u^{3/2}}, \quad u \rightarrow +\infty.$$

But the asymptotics (204) of the Airy function as $u \rightarrow \infty$ implies that for any $U, V \in \mathbb{R}$, there is a constant $C > 0$ such that

$$(207) \quad |\mathbf{A}(u, v)| \leq C e^{-(2/3)(u^{3/2}+v^{3/2})}, \quad u \geq U, v \geq V,$$

which, together with (205), implies that the inner product $\langle \frac{1}{1-\mathbf{A}_x} s^{(m)}, t^{(n)} \rangle$ is finite.

Also $s^{(m)}(u)$ defined in (13) satisfies $\frac{d^m}{du^m} s^{(m)}(u) = \text{Ai}(u)$. Hence $s^{(m)}(u)$ is m -folds integral of $\text{Ai}(u)$ from ∞ to u plus a polynomial of degree $m - 1$. But the asymptotics (206) determines the polynomial and we obtain the result. \square

3.4. *Proof of Theorem 1.2.* The analysis is almost identical to that of proof of Theorem 1.1(a) with the only change of the scaling

$$(208) \quad \pi_j^{-1} = 1 + \gamma^{-1} - \frac{w_j}{M^{1/3}}.$$

We skip the detail.

4. Proof of Theorem 1.1(b). We assume that for some $1 \leq k \leq r$,

$$(209) \quad \pi_1^{-1} = \dots = \pi_k^{-1} > 1 + \gamma^{-1}$$

are in a compact subset of $(1 + \gamma^{-1}, \infty)$, and $\pi_{k+1}^{-1}, \dots, \pi_r^{-1}$ are in a compact subset of $(0, \pi_1^{-1})$.

For the scaling (91), we take

$$(210) \quad \alpha = 1/2$$

and

$$(211) \quad \begin{aligned} \mu = \mu(\gamma) &:= \frac{1}{\pi_1} + \frac{\gamma^{-2}}{(1 - \pi_1)}, \\ \nu = \nu(\gamma) &:= \sqrt{\frac{1}{\pi_1^2} - \frac{\gamma^{-2}}{(1 - \pi_1)^2}}, \end{aligned}$$

so that

$$(212) \quad \xi = \mu + \frac{\nu x}{\sqrt{M}}.$$

It is direct to check that the term inside the square-root of ν is positive from the condition (209). Again, the reason for such a choice will be clear during the subsequent asymptotic analysis.

The functions (95) and (96) are now

$$(213) \quad \mathcal{H}(u) = \frac{\nu M^{1/2}}{2\pi} \int_{\Gamma} e^{-\nu M^{1/2} u(z-q)} e^{Mf(z)} \frac{1}{(\pi_1 - z)^k g(z)} dz$$

and

$$(214) \quad \mathcal{J}(v) = \frac{\nu M^{1/2}}{2\pi} \int_{\Sigma} e^{\nu M^{1/2} v(z-q)} e^{-Mf(z)} (\pi_1 - z)^k g(z) dz,$$

where

$$(215) \quad f(z) := -\mu(z - q) + \log(z) - \frac{1}{\gamma^2} \log(1 - z),$$

where \log is the principal branch of logarithm, and

$$(216) \quad g(z) := \frac{1}{(1 - z)^r} \prod_{\ell=k+1}^r (\pi_\ell - z).$$

The arbitrary parameter q will be chosen in (220) below. Now as

$$(217) \quad f'(z) = -\mu + \frac{1}{z} - \frac{1}{\gamma^2(z - 1)},$$

with the choice (211) of μ , two critical points of f are $z = \pi_1$ and $z = \frac{1}{\mu\pi_1}$. From the condition (209), it is direct to check that

$$(218) \quad \pi_1 < \frac{\gamma}{1 + \gamma} < \frac{1}{\mu\pi_1} < 1.$$

Also a straightforward computation shows that

$$(219) \quad \begin{aligned} f''(\pi_1) &= -v^2 < 0, \\ f''\left(\frac{1}{\mu\pi_1}\right) &= (\gamma v \mu \pi_1 (1 - \pi_1))^2 > 0. \end{aligned}$$

Due to the nature of the critical points, the point $z = \pi_1$ is suitable for the steepest-descent analysis for $\mathcal{J}(v)$ and standard steepest-descent analysis will yield a good leading term of the asymptotic expansion of $\mathcal{J}(v)$. However, for $\mathcal{H}(u)$, the appropriate critical point is $z = 1/(\mu\pi_1)$, and in order to find the steepest-descent curve passing the point $z = 1/(\mu\pi_1)$, we need to deform the contour Γ through the pole $z = \pi_1$ and possibly some of $\pi_{k+1}, \pi_{k+2}, \dots, \pi_r$. In the following, we will show that the leading term of the asymptotic expansion of $\mathcal{H}(u)$ comes from the pole $z = \pi_1$. Before we state precise estimates, we first need some definitions.

Given any fixed $\varepsilon > 0$, we set

$$(220) \quad q := \pi_1 - \frac{\varepsilon}{v\sqrt{M}}.$$

Set

$$(221) \quad \begin{aligned} \mathcal{H}_\infty(u) &:= i e^{-\varepsilon u} \cdot \text{Res}_{a=0} \left(\frac{1}{a^k} e^{-(1/2)a^2 - ua} \right), \\ \mathcal{J}_\infty(v) &:= \frac{1}{2\pi} e^{\varepsilon v} \int_{\Sigma_\infty} s^k e^{(1/2)s^2 + vs} ds, \end{aligned}$$

where Σ_∞ is the imaginary axis oriented from the bottom to the top, and let

$$(222) \quad Z_M := \frac{(-1)^k e^{-Mf(\pi_1)} g(\pi_1)}{v^k M^{k/2}}.$$

PROPOSITION 4.1. *Fix $\varepsilon > 0$ and set q by (220). The following hold for $M/N = \gamma^2$ with γ in a compact subset of $[1, \infty)$.*

(i) *For any fixed $V \in \mathbb{R}$, there are constants $C, c > 0, M_0 > 0$ such that*

$$(223) \quad \left| \frac{1}{Z_M} \mathcal{J}(v) - \mathcal{J}_\infty(v) \right| \leq \frac{C e^{-cv}}{\sqrt{M}}$$

for $v \geq V$ when $M \geq M_0$.

(ii) For any fixed $U \in \mathbb{R}$, there are constants $C, c > 0, M_0 > 0$ such that

$$(224) \quad |Z_M \mathcal{H}(u) - \mathcal{H}_\infty(u)| \leq \frac{C e^{-cu}}{\sqrt{M}}$$

for $u \geq U$ when $M \geq M_0$.

We prove this result in the following two subsections.

4.1. *Proof of Proposition 4.1(i).* Let $R > 0$ and define

$$(225) \quad \Sigma_1 := \left\{ \pi_1 - \frac{2\varepsilon}{v\sqrt{M}} + iy : 0 \leq y \leq 2 \right\},$$

$$(226) \quad \Sigma_2 := \left\{ \pi_1 + 2i - x : \frac{2\varepsilon}{v\sqrt{M}} \leq x \leq R \right\},$$

$$(227) \quad \Sigma_3 := \{ \pi_1 - R + i(2 - y) : 0 \leq y \leq 2 \},$$

and set

$$(228) \quad \Sigma = \left(\bigcup_{k=1}^3 \Sigma_k \right) \cup \overline{\left(\bigcup_{k=1}^3 \Sigma_k \right)}.$$

The orientations of $\Sigma_j, j = 1, 2, 3$, and Σ are indicated in Figure 6.

LEMMA 4.1. For $\gamma \geq 1, \operatorname{Re}(-f(z))$ is decreasing for $z \in \Sigma_1 \cup \Sigma_2$ as z travels on the contour along the prescribed orientation. Also when γ is in a compact

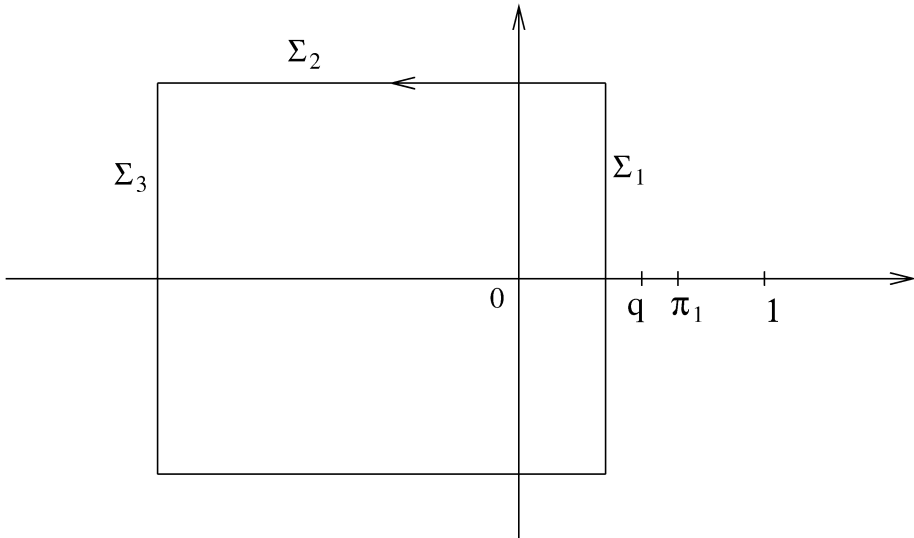


FIG. 6. Contour Σ .

subset of $[1, \infty)$, we can take $R > 0$ large enough so that

$$(229) \quad \max_{z \in \Sigma_3} \operatorname{Re}(-f(z)) \leq \operatorname{Re}(-f(p_*)),$$

where $p_* = \pi_1 + 2i$ is the intersection of Σ_1 and Σ_2 .

PROOF. Any $z \in \Sigma_1$ is of the form $z = x_0 + iy$, $0 \leq y \leq 2$, $x_0 := \pi_1 - \frac{2\varepsilon}{v\sqrt{M}}$. Set for $y \geq 0$

$$(230) \quad \begin{aligned} F_1(y) &:= \operatorname{Re}(-f(x_0 + iy)) \\ &= \mu(x_0 - q) - \frac{1}{2} \ln(x_0^2 + y^2) + \frac{1}{2\gamma^2} \ln((1 - x_0)^2 + y^2). \end{aligned}$$

Then

$$(231) \quad F'_1(y) = \frac{-y((\gamma^2 - 1)y^2 + \gamma^2(1 - x_0)^2 - x_0^2)}{\gamma^2(x_0^2 + y^2)((1 - x_0)^2 + y^2)}.$$

But as $0 < x_0 < \pi_1 < \frac{\gamma}{1+\gamma}$, a straightforward computation shows that $\gamma^2(1 - x_0)^2 - x_0^2 > 0$. Therefore, $\operatorname{Re}(-f(z))$ decreases as z moves along Σ_1 .

For $z \in \Sigma_2$, we have $z = \pi_1 - x + 2i$, $\frac{2\varepsilon}{v\sqrt{M}} \leq x \leq R$. Set

$$(232) \quad \begin{aligned} F_2(x) &:= \operatorname{Re}(-f(\pi_1 - x + 2i)) \\ &= \mu(\pi_1 - q - x) - \frac{1}{2} \ln((\pi_1 - x)^2 + y^2) \\ &\quad + \frac{1}{2\gamma^2} \ln((1 - \pi_1 + x)^2 + y^2). \end{aligned}$$

Then

$$(233) \quad F'_2(x) = -\mu - \frac{x - \pi_1}{(x - \pi_1)^2 + 4} + \frac{x + 1 - \pi_1}{\gamma^2((x + 1 - \pi_1)^2 + 4)}.$$

As the function $g(s) = \frac{s}{s^2+4}$ satisfies $-\frac{1}{4} \leq g(s) \leq \frac{1}{4}$ for all $s \in \mathbb{R}$, we find that for all $x \in \mathbb{R}$,

$$(234) \quad F'_2(x) \leq -\mu + \frac{1}{4} + \frac{1}{4\gamma^2} = -\frac{4 - \pi_1}{4\pi_1} - \frac{3 + \pi_1}{4\gamma^2(1 - \pi_1)}$$

using the definition (211) of μ . But as $0 < \pi_1 < \frac{\gamma}{\gamma+1} < 1$, $F'_2(x) < 0$ for all $x \in \mathbb{R}$, and we find that $\operatorname{Re}(-f(z))$ decreases as z moves on Σ_2 .

For $z \in \Sigma_3$, $z = \pi_1 - R + i(2 - y)$, $0 \leq y \leq 2$. Then for γ in a compact subset

of $[1, \infty)$, we can take $R > 0$ sufficiently large so that

$$\begin{aligned}
 F_3(y) &:= \operatorname{Re}(-f(\pi_1 - R + i(2 - y))) \\
 (235) \quad &= \mu(\pi_1 - R - q) - \frac{1}{2} \ln((\pi_1 - R)^2 + (2 - y)^2) \\
 &\quad + \frac{1}{2\gamma^2} \ln((1 - \pi_1 + R)^2 + (2 - y)^2)
 \end{aligned}$$

can be made arbitrarily small. However,

$$(236) \quad \operatorname{Re}(-f(\pi_1 + 2i)) = \mu(\pi_1 - q) - \frac{1}{2} \ln(\pi_1^2 + 4) + \frac{1}{2\gamma^2} \ln((1 - \pi_1)^2 + 4)$$

is bounded for all $\gamma \geq 1$. Hence the result (229) follows. \square

As γ is in a compact subset of $[1, \infty)$, we assume that

$$(237) \quad 1 \leq \gamma \leq \gamma_0$$

for some fixed $\gamma_0 \geq 1$. Also as π_1 is in a compact subset of $(0, \frac{\gamma}{\gamma+1})$, we assume that there is $0 < \Pi < 1/2$ such that

$$(238) \quad \Pi \leq \pi_1.$$

Fix δ such that

$$(239) \quad 0 < \delta < \min\left\{\frac{\Pi}{2}, \frac{1}{2(1 + \gamma_0)^3}, \frac{\nu^2}{4C_1}\right\}, \quad C_1 := \frac{8}{3}\left(\frac{1}{\Pi^3} + (1 + \gamma_0)^3\right).$$

Then for $|z - \pi_1| \leq \delta$, by using the general inequality

$$\begin{aligned}
 (240) \quad &\left| \operatorname{Re}(-f(z) + f(\pi_1)) + \frac{1}{2} f''(\pi_1)(z - \pi_1) \right| \\
 &\leq \left(\max_{|s - \pi_1| \leq \delta} \frac{1}{3!} |f^{(3)}(s)| \right) |z - \pi_1|^3
 \end{aligned}$$

and the simple estimate for $|s - \pi_1| \leq \delta$,

$$\begin{aligned}
 (241) \quad |f^{(3)}(s)| &= \left| \frac{2}{s^3} - \frac{2}{\gamma^2(s-1)^3} \right| \\
 &\leq \frac{2}{(\pi_1 - \delta)^3} + \frac{2}{\gamma_0^2(1 - \pi_1 - \delta)^3} \\
 &\leq \frac{16}{\Pi^3} + \frac{128}{\gamma_0^2} = 6C_1,
 \end{aligned}$$

we find that

$$\begin{aligned}
 & \left| \operatorname{Re}(-f(z) + f(\pi_1)) + \frac{1}{2}f''(\pi_1)(z - \pi_1) \right| \\
 (242) \quad & \leq C_1|z - \pi_1|^3 \\
 & \leq \frac{\nu^2}{4}|z - \pi_1|^2, \quad |z - \pi_1| \leq \delta.
 \end{aligned}$$

We split the contour $\Sigma = \Sigma' \cup \Sigma''$ where Σ' is the part of Σ in the disk $|z - \pi| \leq \delta$, and Σ'' is the rest of Σ . Let Σ'_∞ be the image of Σ' under the map $z \mapsto \nu\sqrt{M}(z - \pi_1)$ and let $\Sigma''_\infty = \Sigma_\infty \setminus \Sigma'_\infty$. Set

$$(243) \quad \mathcal{J}(v) = \mathcal{J}'(v) + \mathcal{J}''(v), \quad \mathcal{J}_\infty(v) = \mathcal{J}'_\infty(v) + \mathcal{J}''_\infty(v),$$

where $\mathcal{J}'(v)$ [resp. $\mathcal{J}'_\infty(v)$] is the part of the integral formula of $\mathcal{J}(v)$ [resp. $\mathcal{J}_\infty(v)$] integrated over the contour Σ' (resp. Σ'_∞).

Lemma 4.1 and inequality (242) imply that

$$\begin{aligned}
 & \max_{z \in \Sigma''} \operatorname{Re}(-f(z) + f(\pi_1)) \leq \operatorname{Re}(-f(z_0) + f(\pi_1)) \\
 (244) \quad & \leq \operatorname{Re}\left(-\frac{1}{2}f''(\pi_1)(z_0 - \pi_1)^2\right) + \frac{\nu^2}{4}|z_0 - \pi_1|^2 \\
 & = \operatorname{Re}\left(\frac{1}{2}\nu^2(z_0 - \pi_1)^2\right) + \frac{\nu^2}{4}\delta^2,
 \end{aligned}$$

where z_0 is the intersection in the upper half-plane of the circle $|s - \pi_1| = \delta$ and the line $\operatorname{Re}(s) = \pi_1 - \frac{2\varepsilon}{\nu\sqrt{M}}$. As $M \rightarrow \infty$, z_0 becomes close to $\pi_1 + i\delta$. Therefore when M is sufficiently large,

$$(245) \quad \max_{z \in \Sigma''} \operatorname{Re}(-f(z) + f(\pi_1)) \leq -\frac{\nu^2}{12}\delta^2.$$

Using this estimate and the fact that $\operatorname{Re}(z - \pi_1) < 0$ for $z \in \Sigma''$, an argument similar to that in Section 3.2 yields (223). We skip the detail.

4.2. *Proof of Proposition 4.1(ii).* By using Cauchy’s residue theorem, for a contour Γ' that encloses all the zeros of g but π_1 , we find

$$\begin{aligned}
 (246) \quad \mathcal{H}(u) &= i\nu\sqrt{M} \operatorname{Res}_{z=\pi_1} \left(e^{-\nu\sqrt{M}u(z-q)} e^{Mf(z)} \frac{1}{(\pi_1 - z)^k g(z)} \right) \\
 &+ \frac{\nu\sqrt{M}}{2\pi} \int_{\Gamma'} e^{-\nu\sqrt{M}u(z-q)} e^{Mf(z)} \frac{1}{(\pi_1 - z)^k g(z)} dz.
 \end{aligned}$$

Using the choice (220) of q and setting $z = \pi_1 + \frac{a}{v\sqrt{M}}$ for the residue term, we find that

$$\begin{aligned}
 Z_M \mathcal{H}(u) &= \mathcal{H}_1(u) \\
 (247) \quad &+ \frac{g(\pi_1)e^{-\varepsilon u}}{2\pi(v\sqrt{M})^{k-1}} \\
 &\times \int_{\Gamma'} e^{-v\sqrt{M}u(z-\pi_1)} e^{M(f(z)-f(\pi_1))} \frac{1}{(z-\pi_1)^k g(z)} dz,
 \end{aligned}$$

where

$$\begin{aligned}
 (248) \quad \mathcal{H}_1(u) &:= i e^{-\varepsilon u} \operatorname{Res}_{a=0} \left(\frac{1}{a^k} e^{-ua} e^{M(f(\pi_1+a/(v\sqrt{M}))-f(\pi_1))} \right. \\
 &\quad \left. \times \frac{g(\pi_1)}{g(\pi_1+a/(v\sqrt{M}))} \right).
 \end{aligned}$$

We first show that $\mathcal{H}_1(u)$ is close to $\mathcal{H}_\infty(u)$. Note that all the derivatives $f^{(\ell)}(\pi_1)$ and $g^{(\ell)}(\pi_1)$ are bounded and $|g(\pi_1)|$ is strictly positive for γ and π_1 under our assumptions. The function

$$(249) \quad e^{M(f(\pi_1+a/(v\sqrt{M}))-f(\pi_1))}$$

has the expansion of the form

$$(250) \quad e^{-(1/2)a^2+a^2(c_1(a/(\sqrt{M}))+c_2(a/(\sqrt{M}))^2+\dots)}$$

for some constants c_j 's when a is close to 0. On the other hand, the function

$$(251) \quad \frac{g(\pi_1)}{g(\pi_1+a/(v\sqrt{M}))}$$

has the Taylor expansion of the form

$$(252) \quad 1 + c_1 \left(\frac{a}{\sqrt{M}} \right) + c_2 \left(\frac{a}{\sqrt{M}} \right)^2 + \dots$$

for different constants c_j 's. Hence we find the expansion

$$\begin{aligned}
 (253) \quad &e^{-ua} e^{M(f(\pi_1+a/(v\sqrt{M}))-f(\pi_1))} \frac{g(\pi_1)}{g(\pi_1+a/(v\sqrt{M}))} \\
 &= e^{-ua-(1/2)a^2} \left(1 + \sum_{\ell,m=1}^{\infty} c_\ell a^{2\ell} \left(\frac{a}{\sqrt{M}} \right)^m + \sum_{\ell=1}^{\infty} d_\ell \left(\frac{a}{\sqrt{M}} \right)^\ell \right)
 \end{aligned}$$

for some constants c_ℓ, d_ℓ . Now as

$$(254) \quad \operatorname{Res}_{a=0} \left(\frac{1}{a^\ell} e^{-au-(1/2)a^2} \right)$$

is a polynomial of degree at most $\ell - 1$ in u , we find that

$$\begin{aligned}
 (255) \quad & \operatorname{Res}_{a=0} \left(\frac{1}{a^k} e^{-ua} e^{M(f(\pi_1+a/(v\sqrt{M}))-f(\pi_1))} \frac{g(\pi_1)}{g(\pi_1+a/(v\sqrt{M}))} \right) \\
 &= \operatorname{Res}_{a=0} \left(\frac{1}{a^k} e^{-ua-(1/2)a^2} \right) + \sum_{j=1}^{k-1} \frac{q_j(u)}{(\sqrt{M})^j}
 \end{aligned}$$

for some polynomials q_j . Therefore, due to the factor $e^{-\varepsilon u}$ in \mathcal{H}_1 , for any fixed $U \in \mathbb{R}$, there are constants $C, c, M_0 > 0$ such that

$$(256) \quad \left| \mathcal{H}_1(u) - i e^{-\varepsilon u} \operatorname{Res}_{a=0} \left(\frac{1}{a^k} e^{-ua-(1/2)a^2} \right) \right| \leq \frac{C e^{-cu}}{\sqrt{M}}$$

for all $u \geq U$ when $M \geq M_0$.

Now we estimate the integral over Γ' in (247). We will choose Γ' properly so that the integral is exponentially small when $M \rightarrow \infty$. Let $\pi_* := \min\{\pi_{k+1}, \dots, \pi_r, 1, \frac{1}{\mu\pi_1}\}$. Then $\pi_1 = \dots = \pi_k < \pi_*$. Let $\delta > 0$ and $R > \max\{\pi_{k+1}, \dots, \pi_r, 1\}$ be determined in Lemma 4.2 below. Define

$$(257) \quad \Gamma_1 := \left\{ \frac{\pi_1 + \pi_*}{2} + iy : 0 \leq y \leq \delta \right\},$$

$$(258) \quad \Gamma_2 := \left\{ x + i\delta : \frac{\pi_1 + \pi_*}{2} \leq x \leq x_0 \right\},$$

$$(259) \quad \Gamma_3 := \left\{ 1 + \frac{1}{1+\gamma} e^{i(\pi-\theta)} : \theta_0 \leq \theta \leq \frac{\pi}{2} \right\},$$

$$(260) \quad \Gamma_4 := \left\{ x + i \frac{1}{1+\gamma} : 1 \leq x \leq R \right\},$$

$$(261) \quad \Gamma_5 := \left\{ R + i \left(\frac{1}{1+\gamma} - y \right) : 0 \leq y \leq \frac{1}{1+\gamma} \right\},$$

where x_0 and θ_0 are defined by the relation

$$(262) \quad x_0 + i\delta = 1 + \frac{1}{1+\gamma} e^{i(\pi-\theta_0)}.$$

Set

$$(263) \quad \Gamma' = \left(\bigcup_{j=1}^5 \Gamma_j \right) \cup \overline{\left(\bigcup_{j=1}^5 \Gamma_j \right)}.$$

See Figure 7 for Γ' and its orientation.

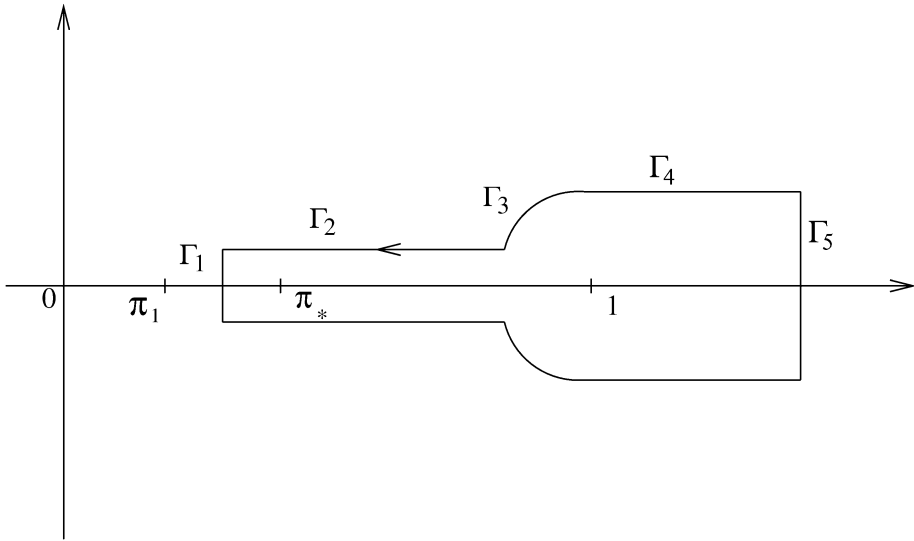


FIG. 7. Contour Σ' .

LEMMA 4.2. For γ in a compact subset of $[1, \infty)$ and for π_1 in a compact subset of $(0, \frac{\gamma}{1+\gamma})$, there exist $\delta > 0$ and $c > 0$ such that

$$(264) \quad \operatorname{Re}(f(z) - f(\pi_1)) \leq -c, \quad z \in \Gamma_1 \cup \Gamma_2.$$

Also $\operatorname{Re}(f(z))$ is a decreasing function in $z \in \Gamma_3 \cup \Gamma_4$, and when $R > \max\{\pi_{k+1}, \dots, \pi_r, 1\}$ is sufficiently large,

$$(265) \quad \operatorname{Re}(f(z) - f(\pi_1)) \leq \operatorname{Re}\left(f\left(1 + i\frac{1}{1+\gamma}\right) - f(\pi_1)\right), \quad z \in \Gamma_5.$$

PROOF. Note that

$$(266) \quad |f'(z)| = \left| -\mu + \frac{1}{z} - \frac{\gamma^{-2}}{z-1} \right|$$

is bounded for z in the complex plane minus union of two compact disks centered at 0 and 1. For γ and π_1 under the assumption, π_1 and $\frac{1}{\mu\pi_1}$ are uniformly away from 0 and 1 (and also from $\frac{\gamma}{1+\gamma}$). Therefore, in particular, there is a constant $C_1 > 0$ such that for $z = x + iy$ such that $x \in [\pi_1, \frac{1}{\mu\pi_1}]$ and $y > 0$,

$$(267) \quad \begin{aligned} |\operatorname{Re}(f(x + iy) - f(x))| &\leq |f(x + iy) - f(x)| \\ &\leq \max_{0 \leq s \leq 1} |f'(x + isy)| \cdot |y| \leq C_1 |y|. \end{aligned}$$

On the other hand, a straightforward calculation shows that when $z = x$ is real, the function

$$(268) \quad \operatorname{Re}(f(x) - f(\pi_1)) = -\mu(x - q) + \ln|x| - \gamma^{-2} \ln|1 - x| - f(\pi_1)$$

decreases as x increases when $x \in (\pi_1, \frac{1}{\mu\pi_1})$. [Recall that π_1 and $\frac{1}{\mu\pi_1}$ are the two roots of $F'(z) = 0$ and $0 < \pi_1 < \frac{\gamma}{1+\gamma} < \frac{1}{\mu\pi_1} < 1$.] Therefore we find that for $z = x + iy$ such that $x \in [(\pi_1 + \pi_{k+1})/2, \frac{1}{\mu\pi_1}]$,

$$(269) \quad \begin{aligned} \operatorname{Re}(f(z) - f(\pi_1)) &\leq \operatorname{Re}(f(x) - f(\pi_1)) + C_1|y| \\ &\leq \operatorname{Re}\left(f\left(\frac{\pi_1 + \pi_*}{2}\right) - f(\pi_1)\right) + C_1|y|. \end{aligned}$$

As π_* is in a compact subset of $(\pi_1, 1)$, we find that there is $c_1 > 0$ such that

$$(270) \quad \operatorname{Re}(f(z) - f(\pi_1)) \leq -c_1 + C_1|y|$$

for above z , and hence there are $\delta > 0$ and $c > 0$ such that for $z = x + iy$ satisfying $|y| \leq \delta, \frac{\pi_1 + \pi_*}{2} \leq x \leq \frac{1}{\mu\pi_1}$,

$$(271) \quad \operatorname{Re}(f(z) - f(\pi_1)) \leq -c.$$

Note that as $\frac{1}{\mu\pi_1}$ is in a compact subset of $(\frac{\gamma}{\gamma+1}, 1)$ under our assumption, we can take δ small enough such that x_0 defined by (262) is uniformly left to the point $\frac{1}{\mu\pi_1}$. Therefore (271) holds for $z \in \Sigma_1 \cup \Sigma_2$.

For $z = 1 + \frac{1}{1+\gamma}e^{i(\pi-\theta)} \in \Gamma_3$,

$$(272) \quad \begin{aligned} F_3(\theta) &:= \operatorname{Re}\left(f\left(1 + \frac{1}{1+\gamma}e^{i(\pi-\theta)}\right)\right) \\ &= -\mu\left(1 + \frac{1}{1+\gamma}\cos(\pi - \theta) - q\right) \\ &\quad + \frac{1}{2}\ln\left(1 + \frac{2}{1+\gamma}\cos(\pi - \theta) + \frac{1}{(1+\gamma)^2}\right) - \frac{1}{\gamma^2}\ln\left(\frac{1}{1+\gamma}\right). \end{aligned}$$

We set $t = \cos(\pi - \theta)$ and define

$$(273) \quad \begin{aligned} G(t) &:= F_3(\theta) = -\mu\left(1 + \frac{1}{1+\gamma}t - q\right) \\ &\quad + \frac{1}{2}\ln\left(1 + \frac{2}{1+\gamma}t + \frac{1}{(1+\gamma)^2}\right) + \frac{1}{\gamma^2}\ln(1+\gamma). \end{aligned}$$

Then

$$(274) \quad G'(t) = -\mu\frac{1}{1+\gamma} + \frac{(1+\gamma)^{-1}}{1+2(1+\gamma)^{-1}t+(1+\gamma)^{-2}}$$

is a decreasing function in $t \in [-1, 1]$ and hence

$$(275) \quad G'(t) \leq G'(-1) = \frac{1}{1+\gamma}\left(-\mu + \left(\frac{1+\gamma}{\gamma}\right)^2\right) \leq 0$$

as the function $\mu = \frac{1}{\pi_1} + \frac{\gamma^{-2}}{(1-\pi_1)}$ in $\pi_1 \in (0, \frac{\gamma}{1+\gamma}]$ takes the minimum value $\frac{(1+\gamma)^2}{\gamma^2}$ at $\pi = \frac{\gamma}{1+\gamma}$. Therefore, $G(t)$ is a decreasing function in $t \in [-1, 1]$ and $\text{Re}(f(z))$ is a decreasing function in $z \in \Gamma_3$.

Set $y_1 = \frac{1}{1+\gamma}$. For $z \in \Gamma_4$, $z = x + iy_1$, $x \geq 1$. Let

$$(276) \quad \begin{aligned} F_4(x) &:= \text{Re}(f(x + iy_1)) \\ &= -\mu(x - q) + \frac{1}{2} \ln(x^2 + y_1^2) - \frac{1}{2\gamma^2} \ln((x - 1)^2 + y_1^2). \end{aligned}$$

Then

$$(277) \quad F'_4(x) = -\mu + \frac{x}{x^2 + y_1^2} - \frac{x - 1}{\gamma^2((x - 1)^2 + y_1^2)}.$$

But the last term is nonnegative and the middle term is less than 1 as $x \geq 1$. Also by the computation of (275), $\mu \geq \frac{(1+\gamma)^2}{\gamma^2} \geq 1$. Therefore we find that $F'_4(x) \leq 0$ for $x \geq 0$, and $F_4(x)$ decreases as $x \geq 1$ increases.

Finally, for $z = R + iy$, $0 \leq y \leq \frac{1}{1+\gamma}$,

$$(278) \quad \text{Re}(f(R + iy)) = -\mu(R - q) + \frac{1}{2} \ln(x^2 + y^2) - \frac{1}{2\gamma^2} \ln((x - 1)^2 + y^2)$$

can be made arbitrarily small when R is taken large enough, while

$$(279) \quad \begin{aligned} &\text{Re}\left(f\left(1 + i\frac{1}{1+\gamma}\right)\right) \\ &= -\mu(1 - q) + \frac{1}{2} \ln\left(1 + \frac{1}{(1+\gamma)^2}\right) - \frac{1}{\gamma^2} \ln\left(\frac{1}{1+\gamma}\right) \end{aligned}$$

is bounded. \square

This lemma implies that

$$(280) \quad \text{Re}(f(z) - f(\pi_1)) \leq -c$$

for all $z \in \Gamma'$. Also note that $\text{Re}(z - \pi_1) > 0$ for $z \in \Gamma'$. Therefore for any fixed $U \in \mathbb{R}$, there are constants $C, c, M_0 > 0$ such that

$$(281) \quad \left| \frac{g(\pi_1)e^{-\varepsilon u}}{2\pi(v\sqrt{M})^{k-1}} \int_{\Gamma'} e^{-v\sqrt{M}u(z-\pi_1)} e^{M(f(z)-f(\pi_1))} \frac{1}{(z - \pi_1)^k g(z)} dz \right| \leq C e^{-\varepsilon u} e^{-cM}$$

for $M > M_0$ and for $u \geq U$. Together with (256), this implies Proposition 4.1(ii).

4.3. *Proof of Theorem 1.1(b).* From Proposition 4.1 and the discussion in Section 2.2, we find that under the assumption of Theorem 1.1(b)

$$(282) \quad \mathbb{P}\left(\left(\lambda_1 - \left(\frac{1}{\pi_1} + \frac{\gamma^{-2}}{1 - \pi_1}\right)\right) \cdot \frac{\sqrt{M}}{\sqrt{1/\pi_1^2 - \gamma^{-2}/(1 - \pi_1)^2}} \leq x\right)$$

converges, as $M \rightarrow \infty$, to the Fredholm determinant of the operator acting on $L^2((0, \infty))$ given by the kernel

$$(283) \quad \int_0^\infty \mathcal{H}_\infty(x + u + y) \mathcal{J}_\infty(x + v + y) dy.$$

Now we will express the terms $\mathcal{H}_\infty(u)$ and $\mathcal{J}_\infty(v)$ in terms of the Hermite polynomials.

The generating function formula (see (1.13.10) of [24]) of Hermite polynomials H_n ,

$$(284) \quad \sum_{n=0}^\infty \frac{H_n(x)}{n!} t^n = e^{2xt - t^2},$$

implies that

$$(285) \quad \begin{aligned} -i e^{\varepsilon u} \mathcal{H}_\infty(u) &= \text{Res}_{a=0} \left(\frac{1}{a^k} e^{-(1/2)a^2 - (u+y)a} \right) \\ &= \frac{(-1)^{k-1} (2\pi)^{1/4}}{\sqrt{(k-1)!}} p_{k-1}(u), \end{aligned}$$

where the orthonormal polynomial $p_{k-1}(x)$ is defined in (31). The forward shift operator formula (see (1.13.6) of [24])

$$(286) \quad H'_n(x) = 2n H_{n-1}(x)$$

implies that

$$(287) \quad p'_k(y) = \sqrt{k} p_{k-1}(y),$$

and hence

$$(288) \quad -i e^{\varepsilon u} \mathcal{H}_\infty(u) = \text{Res}_{a=0} \left(\frac{1}{a^k} e^{-(1/2)a^2 - (u+y)a} \right) = \frac{(-1)^{k-1} (2\pi)^{1/4}}{\sqrt{k!}} p'_k(u).$$

On the other hand, the Rodrigues-type formula (see (1.13.9) of [24])

$$(289) \quad H_n(x) = e^{x^2} \left(-\frac{d}{dx} \right)^n [e^{-x^2}]$$

implies that

$$(290) \quad p_n(\xi) = \frac{(-1)^n}{(2\pi)^{1/4} \sqrt{n!}} e^{\xi^2/2} \left(\frac{d}{d\xi} \right)^n [e^{-\xi^2/2}].$$

Since the integral which appears in $\mathcal{J}_\infty(v)$ is equal to

$$(291) \quad \int_{\Sigma_\infty} s^k e^{(1/2)s^2+vs} ds = \left(\frac{d}{dv}\right)^k \int_{\Sigma_\infty} e^{(1/2)s^2+vs} ds = i \left(\frac{d}{dv}\right)^k e^{-v^2/2},$$

we find

$$(292) \quad e^{-\varepsilon v} \mathcal{J}_\infty(v) = (-1)^k i (2\pi)^{-1/4} \sqrt{k!} e^{-v^2/2} p_k(v).$$

After a trivial translation, the Fredholm determinant of the operator (283) is equal to the Fredholm determinant of the operator acting on $L^2((x, \infty))$ with the kernel

$$(293) \quad K_2(u, v) := \int_0^\infty \mathcal{H}_\infty(u+y) \mathcal{J}_\infty(v+y) dy.$$

By (288) and (292),

$$(294) \quad \begin{aligned} & (u-v) K_2(u, v) e^{\varepsilon(u-v)} \\ &= \int_0^\infty (u+y) \cdot p'_k(u+y) p_k(v+y) e^{-(v+y)^2/2} dy \\ & \quad - \int_0^\infty p'_k(u+y) p_k(v+y) \cdot (v+y) e^{-(v+y)^2/2} dy. \end{aligned}$$

Note that p_k satisfies the differential equation

$$(295) \quad p''_k(y) - yp'_k(y) + kp_k(y) = 0,$$

which follows from the differential equation

$$(296) \quad H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$$

for the Hermite polynomial (see (1.13.5) of [24]). Now use (295) for the first integral of (294) and integrate by parts the second integral by noting that $(v+y)e^{-(v+y)^2/2} = \frac{d}{dy} e^{-(v+y)^2/2}$ to obtain

$$(297) \quad \begin{aligned} & (u-v) K_2(u, v) e^{\varepsilon(u-v)} \\ &= \int_0^\infty kp_k(u+y) p_k(v+y) e^{-(v+y)^2/2} dy \\ & \quad - p'_k(u) p_k(v) e^{-v^2/2} - \int_0^\infty p'_k(u+y) p'_k(v+y) e^{-(v+y)^2/2} dy. \end{aligned}$$

Note that the terms involving $p''_k(u+y)$ are cancelled out. Then integrating by parts the second integral and noting that $p'_k(u+y) = \frac{d}{dy} p_k(u+y)$, we obtain

$$(298) \quad (u-v) K_2(u, v) e^{\varepsilon(u-v)} = -p'_k(u) p_k(v) e^{-v^2/2} + p_k(u) p'_k(v) e^{-v^2/2}.$$

By using (287), this implies that

$$(299) \quad K_2(u, v) = \sqrt{k} e^{-\varepsilon u} \frac{p_k(u) p_{k-1}(v) - p_{k-1}(u) p_k(v)}{u-v} e^{-v^2/2} e^{\varepsilon v}.$$

Therefore, upon conjugations, the Fredholm determinant of K_2 is equal to the Fredholm determinant $\det(1 - \mathbf{H}_x)$ in (33). This completes the proof of Theorem 1.1(b).

5. Samples of finitely many variables. In this section we prove Proposition 1.1.

We take $M \rightarrow \infty$ and fix $N = k$. We suppose that

$$(300) \quad \pi_1 = \dots = \pi_k.$$

Then from (60), the density of the eigenvalues is

$$(301) \quad p(\lambda) = \frac{1}{C} V(\lambda)^2 \prod_{j=1}^k e^{-M\pi_1 \lambda_j} \lambda_j^{M-k},$$

and hence

$$(302) \quad \mathbb{P}(\lambda_1 \leq t) = \frac{1}{C} \int_0^t \dots \int_0^t V(y)^2 \prod_{j=1}^k e^{-M\pi_1 y_j} y_j^{M-k} dy_j,$$

where

$$(303) \quad C = \frac{\prod_{j=0}^{k-1} (1+j)!(M-k+j)!}{(M\pi_1)^{Mk}}.$$

By using the change of the variables $y_j = \frac{1}{\pi_1} (1 + \frac{\xi_j}{\sqrt{M}})$,

$$(304) \quad \begin{aligned} & \mathbb{P}\left(\lambda_1 \leq \frac{1}{\pi_1} + \frac{x}{\pi_1 \sqrt{M}}\right) \\ &= \frac{e^{-kM}}{\pi_1^{kM} M^{k^2/2} C} \\ & \quad \times \int_{-\sqrt{M}}^x \dots \int_{-\sqrt{M}}^x V(\xi)^2 \prod_{j=1}^k e^{-\sqrt{M}\xi_j} \left(1 + \frac{\xi_j}{\sqrt{M}}\right)^{M-k} d\xi_j. \end{aligned}$$

As $M \rightarrow \infty$ while k is fixed, $\pi_1^{kM} M^{k^2/2} e^{kM} C \rightarrow (2\pi)^{k/2} \prod_{j=0}^{k-1} (1+j)!$. By using the dominated convergence theorem,

$$(305) \quad \lim_{M \rightarrow \infty} \mathbb{P}\left(\left(\lambda_1 - \frac{1}{\pi_1}\right) \pi_1 \sqrt{M} x\right) = G_k(x).$$

The result (51) follows from (305) and the fact that G_k is a distribution function.

6. Last passage percolation, queuing theory and heuristic arguments.

There is a curious connection between complex Gaussian sample covariance matrices and a last passage percolation model.

Consider the lattice points $(i, j) \in \mathbb{Z}^2$. Suppose that to each (i, j) , $i = 1, \dots, N$, $j = 1, \dots, M$, an independent random variable $X(i, j)$ is associated. Let $(1, 1) \nearrow (N, M)$ be the set of “up/right paths” $\pi = \{(i_k, j_k)\}_{k=1}^{N+M-1}$ where $(i_{k+1}, j_{k+1}) - (i_k, j_k)$ is either $(1, 0)$ or $(0, 1)$, and $(i_1, j_1) = (1, 1)$ and $(i_{N+M-1}, j_{N+M-1}) = (N, M)$. There are $\binom{N+M-2}{N-1}$ such paths. Define

$$(306) \quad L(N, M) := \max_{\pi \in (1,1) \nearrow (N,M)} \sum_{(i,j) \in \pi} X(i, j).$$

If $X(i, j)$ is interpreted as time spent to pass through the site (i, j) , $L(N, M)$ is the *last passage time* to travel from $(1, 1)$ to (N, M) along an admissible up/right path.

Let π_1, \dots, π_N be positive numbers. When $X(i, j)$ is the exponential random variable of mean $\frac{1}{\pi_i M}$ [the density function of $X(i, j)$ is $\pi_i M e^{-\pi_i M x}$, $x \geq 0$], it is known that $L(N, M)$ has the same distribution as the largest eigenvalue of the complex Gaussian sample covariance matrix of M sample vectors of N variables [see (61)]: for $M \geq N$,

$$(307) \quad \mathbb{P}(L(N, M) \leq x) = \frac{1}{C} \int_0^x \cdots \int_0^x \frac{\det(e^{-M\pi_i \xi_j})_{1 \leq i, j \leq N}}{V(\pi)} V(\xi) \prod_{j=1}^N \xi_j^{M-N} d\xi_j.$$

We emphasize that $X(i, j)$, $j = 1, 2, \dots, M$, are identically distributed for each fixed i . As a consequence, we have the following. Recall that

$$(308) \quad \pi_j^{-1} = \ell_j.$$

PROPOSITION 6.1. *Let $L(M, N)$ be the last passage time in the above percolation model with exponential random variables at each site. Let λ_1 be the largest eigenvalue of M (complex) samples of $N \times 1$ vectors as in the Introduction. Then for any $x \in \mathbb{R}$,*

$$(309) \quad \mathbb{P}(L(M, N) \leq x) = \mathbb{P}(\lambda_1(M, N) \leq x).$$

Formula (307) for the case of $\pi_1 = \cdots = \pi_N$ was obtained in Proposition 1.4 of [21]. The general case follows from a suitable generalization. Indeed, let $x_i, y_j \in [0, 1)$ satisfy $0 \leq x_i y_j < 1$ for all i, j . When the attached random variable, denoted by $Y(i, j)$, is the geometric random variable of parameter $x_i y_j$ [i.e., $\mathbb{P}(X(i, j) = k) = (1 - x_i y_j)(x_i y_j)^k$, $k = 0, 1, 2, \dots$], the last passage time, $G(N, M)$, from $(1, 1)$ to (N, M) defined as in (306) is known to satisfy

$$(310) \quad \mathbb{P}(G(N, M) \leq n) = \prod_{i,j \geq 1} (1 - x_i y_j) \cdot \sum_{\lambda: \lambda_1 \leq n} s_\lambda(x) s_\lambda(y),$$

where the sum is over all partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ such that the first part $\lambda_1 \leq n$, and s_λ denotes the Schur function, $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. This identity was obtained by using the Robinson–Schensted–Knuth correspondence between generalized permutations (matrices of nonnegative integer entries) and pairs of semistandard Young tableaux (see e.g. [21, 30] and (7.30) of [6]). The normalization constant follows from the well-known Cauchy identity (see, e.g., [36])

$$(311) \quad \sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y) = \prod_{i,j \geq 1} (1 - x_i y_j),$$

where the sum is over all partitions. Now set $x_i = 1 - \frac{M\pi_i}{L}$, $i = 1, \dots, N$, $x_i = 0$, $i > N$, and $y_j = 1$, $j = 1, \dots, M$, $y_j = 0$, $j > M$, and $n = xL$. By taking $L \rightarrow \infty$, it is easy to compute that $\frac{1}{L}Y(i, j)$ converges to the exponential random variable $X(i, j)$, while one can check that the summation on the right-hand side of (310) converges to the right-hand side of (307), and hence the identity (307) follows. There are determinantal formulas for the right-hand side of (310) (see, e.g., [16, 21, 30]), some of which, by taking the above limit, would yield an alternative derivation of Proposition 2.1.

Another equivalent model is a queueing model. Suppose that there are N tellers and M customers. Suppose that initially all M customers are on the first teller in a queue. The first customer will be served from the first teller and then go to the second teller. Then the second customer will come forward to the first teller. If the second customer finishes his/her business before the first customer finishes his/her business from the second teller, the second customer will line up a queue on the second teller, and so on. At any instance, only one customer can be served at a teller and all customers should be served from all tellers in the order. The question is the total exit time $E(M, N)$ for M customers to exit from N queues. We assume that the service time at teller i is given by the exponential random variable of mean $\frac{1}{\pi_i M}$. Assuming the independence, consideration of the last customer in the last queue will yield the recurrence relation

$$(312) \quad E(M, N) = \max\{E(M - 1, N), E(M, N - 1)\} + e(N),$$

where $e(N)$ denotes the service time at the teller N . But note that the last passage time in the percolation model also satisfies the same recurrence relation

$$(313) \quad L(M, N) = \max\{L(M - 1, N), L(M, N - 1)\} + X(M, N),$$

where $X(M, N)$ is the same exponential random variable as $e(N)$. Therefore we find that $E(M, N)$ and $L(M, N)$ have the same distribution. Thus all the results in the Introduction also applied to $E(M, N)$.

Now we indicate how the critical value $\ell_j = \pi_j^{-1} = 1 + \gamma^{-1}$ of Theorem 1.1 can be predicted in the last passage percolation model.

First, when $X(i, j)$ are all identical exponential random variables of mean 1, Theorem 1.6 of [21] shows that, as $M, N \rightarrow \infty$ such that M/N is in a compact subset of $(0, \infty)$, $L(N, M)$ is approximately

$$(314) \quad L(N, M) \sim L(M, N) \sim (\sqrt{M} + \sqrt{N})^2 + \frac{(\sqrt{M} + \sqrt{N})^{4/3}}{(MN)^{1/6}} \chi_0,$$

where χ_0 denotes the random variable of the GUE Tracy–Widom distribution. On the other hand, note that when $N = 1$, $X(1, j)$ are independent, identically distributed exponential random variables of mean $\frac{1}{M\pi_1}$, and hence the classical central limit theorem implies that the last passage time from $(1, 1)$ to $(1, xM)$ is approximately

$$(315) \quad \pi_1^{-1}x + \frac{x\pi_1^{-1}}{\sqrt{M}}g,$$

where g denotes the standard normal random variable. Note the different fluctuations which are due to different dimensions of two models.

Now consider the case when $r = 1$ in Theorem 1.1, that is, $\pi_2 = \pi_3 = \dots = \pi_N = 1$; $X(i, j)$, $1 \leq j \leq M$, is exponential of mean $\frac{1}{M\pi_1}$ and $X(i, j)$, $2 \leq i \leq N$, $1 \leq j \leq M$, is exponential of mean $\frac{1}{M}$. We take $M/N = \gamma^2 \geq 1$. An up/right path consists of two pieces: a piece on the first column $(1, j)$ and the other piece in the “bulk,” (i, j) , $i \geq 2$. Of course the first part might be empty. We will estimate how long the last passage path stays in the first column. Consider the last passage path *conditioned* such that it lies on the first column at the sites $(1, 1), (1, 2), \dots, (1, xM)$ and then enters to the bulk (i, j) , $i \geq 2$. See Figure 8. Then from (315) and (314), we expect that the (conditioned) last passage time is, to the leading order,

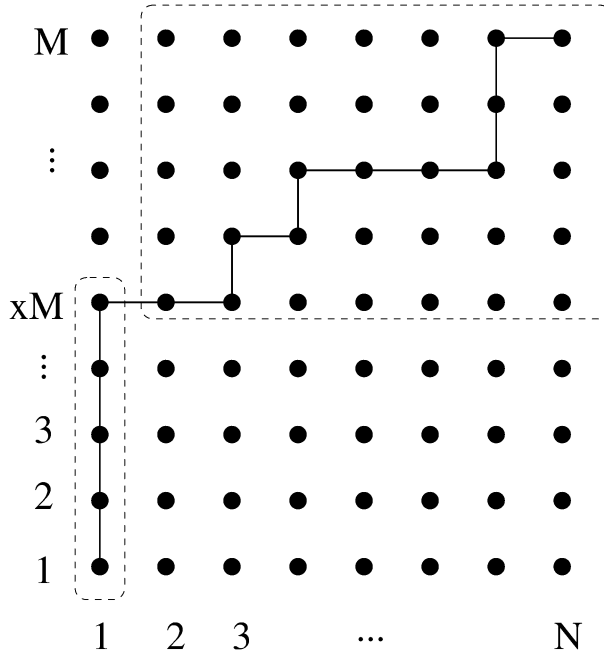
$$(316) \quad f(x) = \pi_1^{-1}x + (\sqrt{1-x} + \gamma^{-1})^2.$$

It is reasonable to expect that the last passage time is the maximum of $f(x)$ over $x \in [0, 1]$, to the leading order. An elementary calculus shows that

$$(317) \quad \max_{x \in [0, 1]} f(x) = \begin{cases} f(0) = (1 + \gamma^{-1})^2, & \text{if } \pi^{-1} \leq 1 + \gamma^{-1}, \\ f\left(1 - \frac{\gamma^{-2}}{(\pi^{-1} - 1)^2}\right) = \frac{1}{\pi_1} + \frac{\gamma^{-2}}{1 - \pi_1}, & \\ & \text{if } \pi^{-1} > 1 + \gamma^{-1}. \end{cases}$$

When $\max f$ occurs at $x = 0$, the last passage path enters directly into the bulk and hence the fluctuation of the last passage time is of $M^{-2/3}$ due to (314). But if the $\max f(x)$ occurs for some $x > 0$, then the fluctuation is $M^{-1/2}$ due to (315), which is larger than the fluctuation $M^{-2/3}$ from the bulk. Note that the value of $\max f$ in (317) agrees with the leading term in the scaling of Theorem 1.1. This provides an informal explanation of the critical value $1 + \gamma^{-1}$ of π_1^{-1} .

If one can make this kind of argument for the sample covariance matrix, one might be able to generalize it to the real sample covariance matrix.

FIG. 8. Last passage percolation when $r = 1$.

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