

LARGE DEVIATIONS FOR RANDOM POWER MOMENT PROBLEM

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We consider the set M_n of all n -truncated power moment sequences of probability measures on $[0, 1]$. We endow this set with the uniform probability. Picking randomly a point in M_n , we show that the upper canonical measure associated with this point satisfies a large deviation principle. Moderate deviation are also studied completing earlier results on asymptotic normality given by Chang, Kemperman and Studden [*Ann. Probab.* **21** (1993) 1295–1309]. Surprisingly, our large deviations results allow us to compute explicitly the $(n + 1)$ th moment range size of the set of all probability measures having the same n first moments. The main tool to obtain these results is the representation of M_n on canonical moments [see the book of Dete and Studden].

1. Introduction. In this work we will study the asymptotic behavior in large deviations of random power moment problem. Let $\mathbb{P}([0, 1])$ denote the set of all probability measures (p.m.s) on the interval $[0, 1]$. In the whole paper this set will be endowed with the weak topology [see Billingsley (1999)]. For any μ in $\mathbb{P}([0, 1])$ the k th (power) moment of μ will be denoted by $c_k(\mu)$:

$$c_k(\mu) = \int_{[0,1]} x^k d\mu(x).$$

In this paper we focus on some asymptotic properties of the finite moment space M_n :

$$M_n = \{c^{(n)}(\mu) = (c_1(\mu), \dots, c_n(\mu))^T : \mu \in \mathbb{P}([0, 1])\}, \quad n \in \mathbb{N}^*.$$

M_n is the closed convex hull of the curve

$$\{(x, x^2, \dots, x^n)^T : x \in [0, 1]\}$$

[see Karlin and Studden (1966) and Kreĭn and Nudelman (1977)]. As M_n is a compact subset of \mathbb{R}^n having nonvoid interior, we may define the uniform probability \mathbb{P}_n on M_n . M_n is a very “small” set. Indeed, its volume has order 2^{-n^2} for large n . Moreover, this set “concentrates” (in some sense) on a single point. More precisely, Chang, Kemperman and Studden (1993) have shown that for any

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fixed $k \in \mathbb{N}^*$ and under the probability \mathbb{P}_n , the k first components of $c^{(n)} \in M_n$ converge in probability to the k first moments of the arcsine law. That is,

$$(1) \quad \lim_{n \rightarrow \infty} \mathbb{P}_n \left(\left\| \begin{pmatrix} c_1^{(n)} \\ \vdots \\ c_k^{(n)} \end{pmatrix} - \begin{pmatrix} \bar{c}_1 \\ \vdots \\ \bar{c}_k \end{pmatrix} \right\| \geq \xi \right) = 0, \quad \xi > 0,$$

where $\bar{c}_j = c_j(\nu)$, $j \in \mathbb{N}^*$, and

$$(2) \quad \nu(dx) = \frac{dx}{\pi \sqrt{x(1-x)}}.$$

Hence, M_n is concentrated around the n th first moments of the arcsine law. Moreover, Chang, Kemperman and Studden (1993) have studied the fluctuation limit law in (1). They have shown that the limit distribution of the fluctuations is Gaussian:

$$(3) \quad \sqrt{n}(Z_n^{(k)} - \bar{c}^{(k)}) \xrightarrow[n \rightarrow \infty]{\mathcal{L}-\mathbb{P}_n} \mathcal{N}_k(0, \Sigma_k),$$

where $Z_n^{(k)}$ denotes the random vector built with the k first coordinates of $Z_n = c^{(n)}$ (drawn randomly with distribution \mathbb{P}_n). $\bar{c}^{(k)} = (\bar{c}_1, \bar{c}_2, \dots, \bar{c}_k)^T$ is the vector of the k first moment of ν . The covariance matrix Σ_k will be described in Section 2.5. The main result of this paper is a functional large deviations principle for the sequence $(Z_n^{(k)})$. This means that $Z_n^{(k)}$ ‘‘concentrates’’ exponentially fast. More precisely, for any $c^{(n)} \in M_n$, there exists a unique measure $\sigma_n^+(c^{(n)})$ whose n first moments are $c^{(n)}$ and maximizing the $(n + 1)$ th moment [see Section 2.3, Karlin and Studden (1966), Kreĭn and Nudelman (1977) and Dette and Studden (1997)]. In Section 2.4 we show that $\sigma_n^+(Z_n)$ satisfies a large deviations principle (LDP). Roughly speaking, this means that $\sigma_n^+(Z_n)$ concentrates exponentially fast on ν . In other words, for large n ,

$$(4) \quad \mathbb{P}_n(\sigma_n^+(Z_n) \in A) \approx \exp\left(-n \inf_{\mu \in A} I(\mu)\right),$$

where A is a Borel-measurable set of $\mathbb{P}([0, 1])$ and I is the rate function of the LDP (see Sections 2.1 and 2.4 for the more precise statement). Although we prove this main result using a limit projective approach [see Dembo and Zeitouni (1998)] we manage to compute precisely the rate function I . This is performed using Szegő asymptotic theory on orthogonal polynomials [Grenander and Szegő (1958)]. Surprisingly, I is the reversed Kullback information (or cross entropy) with respect to ν . Moreover, this computation gives quantitative evaluations of the $(n + 1)$ th moment range size of the set of all probability measures having the same prescribed n first moments in term of the reversed Kullback information. To our knowledge these results on power moment problem are new, they are developed in Section 3.

In Section 2.5, we give large and moderate deviation principles for the random vector $Z_n^{(k)}$. Hence, we study exponential rates of convergence, between (3) and (4), for this vector. All the proofs are postponed to Section 4. This last section begin with a section on canonical moments [Dette and Studden (1997)] which are the main tool to prove our results.

2. Main results.

2.1. *Large deviations.* Let us first recall what is a LDP [see, e.g., Dembo and Zeitouni (1998)]. Let (u_n) be a decreasing positive sequence of real numbers.

DEFINITION 2.1. We say that a sequence (R_n) of probability measures on a measurable Hausdorff space $(U, \mathcal{B}(U))$ satisfies a LDP with rate function I and speed (u_n) if:

- (i) I is lower semicontinuous (l.s.c.), with values in $\mathbb{R}^+ \cup \{+\infty\}$.
- (ii) For any measurable set A of U ,

$$-I(\text{int } A) \leq \liminf_{n \rightarrow \infty} u_n \log R_n(A) \leq \limsup_{n \rightarrow \infty} u_n \log R_n(A) \leq -I(\text{clo } A),$$

where $I(A) = \inf_{\xi \in A} I(\xi)$ and $\text{int } A$ (resp. $\text{clo } A$) is the interior (resp. the closure) of A .

We say that the rate function I is good if its level set $\{x \in U : I(x) \leq a\}$ is compact for any $a \geq 0$. More generally, a sequence of U -valued random variables is said to satisfy a LDP if their distributions satisfy a LDP.

To be self-contained let us recall some facts and tools on large deviations which will be useful in the paper [we refer to Dembo and Zeitouni (1998) for more on large deviations]:

- *Contraction principle.* Assume that (R_n) satisfies a LDP on $(U, \mathcal{B}(U))$ with good rate function I and speed (u_n) . Let T be a continuous mapping from U to another space V . Then, $(R_n \circ T^{-1})$ satisfies a LDP on $(V, \mathcal{B}(V))$ with good rate function

$$I'(y) = \inf_{x : T(x)=y} I(x), \quad y \in V,$$

and speed (u_n) .

- *Exponential approximation.* Assume that U is a metric space and let d denote the distance on U . Let (X_n) be a U -valued random sequence satisfying a LDP with good rate function I and speed u_n . Let (Y_n) be another U -valued random sequence. If for any $\xi > 0$,

$$\limsup_{n \rightarrow \infty} u_n \log \mathbb{P}(d(X_n, Y_n) > \xi) = -\infty,$$

then (Y_n) shares the same LDP as (X_n) .

In the sequel, talking about large deviations, if we omit the sequence (u_n) it means that $u_n = \frac{1}{n}$ ($n \in \mathbb{N}^*$). Following Section 3.7 of Dembo and Zeitouni (1998), when the sequence (u_n) satisfies $n^{-1} = o(u_n)$ we will say moderate deviations rather than large deviations with speed (u_n) .

2.2. *Kullback and reversed Kullback information.* Let $(U, \mathcal{B}(U))$ be a measurable space and P and Q be p.m.s. on $(U, \mathcal{B}(U))$. Recall that the *Kullback information* or cross entropy of P with respect to Q is defined by

$$K(P, Q) = \begin{cases} \int_U \log \frac{dP}{dQ} dP, & \text{if } P \ll Q \text{ and } \log \frac{dP}{dQ} \in L^1(P), \\ +\infty, & \text{otherwise.} \end{cases}$$

Properties of K as a function of P may be found in Bretagnolle (1979). K is the rate function for Sanov large deviations theorem [see Dembo and Zeitouni (1998)]. The rate function involved in this paper is the reversed Kullback information with respect to ν , that is,

$$I(\mu) = K(\nu, \mu), \quad \mu \in \mathbb{P}([0, 1]),$$

where ν is the arcsine law (2). Observe that I is l.s.c. [Theorem 2.7 of Borwein and Lewis (1993)]. Moreover, it is obviously a good rate function [$\mathbb{P}([0, 1])$ is a compact set].

The following property may be found in Theorem 2.1 of Gamboa and Gassiat (1997) and in Section 3 of Borwein and Lewis (1993):

PROPOSITION 2.2. (i) For $\mu \in \mathbb{P}([0, 1])$,

$$I(\mu) = \sup_{f \in C[0,1]} \left(\int_0^1 f(x) d\mu(x) + \int_0^1 \ln(1 - f(x)) d\nu(x) \right),$$

where $C[0, 1]$ is the set of all continuous functions on $[0, 1]$. (In the whole of the paper we take the convention $\ln \tau = -\infty$ whenever $\tau \leq 0$.)

(ii) Let $\Phi = (\Phi_1, \dots, \Phi_k)^T \in (C[0, 1])^k$ and for $c \in \mathbb{R}^k$,

$$\mathbb{S}_\Phi(c) = \left\{ \mu \in \mathbb{P}([0, 1]) : \int_0^1 \Phi_j(x) d\mu(x) = c_j, j = 1, \dots, k \right\}.$$

Then,

$$\inf_{\mu \in \mathbb{S}_\Phi(c)} I(\mu) = \sup_{(\lambda_0, \lambda) \in \mathbb{R}^{k+1}} \left\{ \lambda_0 + \langle \lambda, c \rangle + \int_0^1 \ln(1 - \lambda_0 - \langle \lambda, \Phi(x) \rangle) d\nu(x) \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on \mathbb{R}^k .

We will come back on the second point in Section 2.5. This property will be helpful to show the results of Section 3.

For $x \in [0, 1]$ and $k \in \mathbb{N}^*$, let $\phi_k(x) = (x, x^2, \dots, x^k)^T$. For $c \in \mathbb{R}^k$, we set $\mathbb{S}_k(c) = \mathbb{S}_{\phi_k}(c)$. In the next section we briefly recall some known facts on $\mathbb{S}_k(c)$. In Section 3 we give new results on $\mathbb{S}_k(c)$.

2.3. *Power moment problem.* Let M denote the set of all infinite moment sequences, that is,

$$M = \{c^{(\infty)}(\mu) = (c_i(\mu))_{i \in \mathbb{N}} : \mu \in \mathbb{P}([0, 1])\}.$$

For $n \in \mathbb{N}^*$, let $\Pi_n : M \rightarrow M_n$ denote the natural projection map. So, we have $M_n = \Pi_n(M)$.

Let recall some useful facts on the power moments problems [see Karlin and Studden (1966) and Kreĭn and Nudelman (1977) for an exhaustive overview on this problem]. Because $[0, 1]$ is a compact interval, the relation between the elements of M and $\mathbb{P}([0, 1])$ is bijective. In general, for $c^{(n)} \in M_n$, there exists an infinite number of probability measures such that $c^{(n)} = c^{(n)}(\mu)$. More precisely, for any $n \in \mathbb{N}^*$ and $c^{(n)} \in \mathbb{R}^n$:

- (i) $\#\mathbb{S}_n(c^{(n)}) = +\infty \Leftrightarrow c^{(n)} \in \text{int } M_n$,
- (ii) $\#\mathbb{S}_n(c^{(n)}) = 1 \Leftrightarrow c^{(n)} \in \partial M_n$ (the boundary of M_n),
- (iii) $\#\mathbb{S}_n(c^{(n)}) = 0 \Leftrightarrow c^{(n)} \notin M_n$,

where $\#A$ is the number of elements lying in A . These results come from the fact that (ϕ_n) is a Tchebycheff system [see Karlin and Studden (1966) and Kreĭn and Nudelman (1977)]. The elements of ∂M_n satisfy an extremal property. For $c^{(k)} \in M_k$, let $c^+, c^- : M_k \rightarrow \mathbb{R}$ be defined by

$$c^+(c^{(k)}) = \max\{c \in \mathbb{R} : (c_1, c_2, \dots, c_k, c)^T \in M_{k+1}\},$$

$$c^-(c^{(k)}) = \min\{c \in \mathbb{R} : (c_1, c_2, \dots, c_k, c)^T \in M_{k+1}\}.$$

Then, $c^{(k+1)} = ((c^{(k)})^T, c_{k+1})^T \in \partial M_{k+1}$ if, and only if, $c_{k+1} = c^+(c^{(k)})$ [or $c^-(c^{(k)})$].

Let $d^{(n)} \in M_n$. We will denote by $\sigma_n^+(d^{(n)})$ the measure μ such that $c^{(n)}(\mu) = d^{(n)}$ and $c_{n+1}(\mu) = c^+(d^{(n)})$. This measure is the so-called upper canonical representation of the finite moments sequence $d^{(n)}$ [see Karlin and Studden (1966) and Kreĭn and Nudelman (1977)]. In the next section we will study large deviations properties of $\sigma_n^+(Z_n)$.

2.4. *Large deviations for $\sigma_n^+(Z_n)$.* Recall that \mathbb{P}_n denotes the normalized Lebesgue probability measure on M_n and Z_n denotes a random vector having distribution \mathbb{P}_n . For $k \leq n$, let $\Pi_k^n : M_n \rightarrow M_k$ be the projection map taking the k first coordinates of an element of M_n . Let $Z_n^{(k)}$ denote the random vector $\Pi_k^n(Z_n)$ and μ_n be the random measure $\sigma_n^+(Z_n)$. Let Q_n be the law of μ_n . Our main result is the large deviations principle for (μ_n) .

THEOREM 2.3. (μ_n) satisfies a LDP with convex good rate function I .

COROLLARY 2.4. (μ_n) converges in probability to ν .

We can also study the LDP associated to the so called tilted measures. Changing a little the measures Q_n on $\mathbb{P}([0, 1])$, we obtain the new limit law for the sequence of random probability measures μ_n . Namely, consider on $\mathbb{P}([0, 1])$ the new probability measure \tilde{Q}_n defined by

$$(5) \quad \tilde{Q}_n(B) = \frac{\mathbb{E}_{Q_n}(\exp(nF(\mu_n))\mathbb{1}_B(\mu_n))}{\mathbb{E}_{Q_n} \exp(nF(\mu_n))},$$

where B runs over the Borelian sets of $\mathbb{P}([0, 1])$ and $F: \mathbb{P}([0, 1]) \rightarrow \mathbb{R}$ is the continuous functional defined by

$$F(\mu) = \int_{[0,1]} f_0(x) d\mu(x),$$

where $f_0 \in C([0, 1])$.

We have the following results.

THEOREM 2.5. $(\tilde{Q}_n)_{n \in \mathbb{N}}$ satisfies a LDP on $\mathbb{P}([0, 1])$ with good rate function

$$I_F(\mu) = I(\mu) - F(\mu) + K_F,$$

where $K_F := \sup_{\mu' \in \mathbb{P}([0, 1])} \{F(\mu') - I(\mu')\}$.

REMARK 2.1. Let $\tilde{\mu}_n$ denote a random measure of $\mathbb{P}([0, 1])$ having distribution \tilde{Q}_n . The existence of an unique minimum point for I_F implies the convergence (in probability) of $\tilde{\mu}_n$ toward this minimum point. Under certain conditions over f_0 we can characterize the minimum points of I_F . Let $\underline{\mu} \in \mathbb{P}([0, 1])$ be a minimum point of I_F . Let $\underline{\mu} = \underline{\mu}_a + \underline{\mu}_s$ be the Lebesgue decomposition of $\underline{\mu}$ with respect to ν ($\underline{\mu}_a \ll \nu$). Then, from Theorem 3.5 of Borwein and Lewis (1993) there exists $\lambda^* \in \mathbb{R}$ with:

- (i) $g_{\lambda^*} = \frac{d\underline{\mu}_a}{d\nu} = \frac{1}{\lambda^* - f_0}$ a.s.,
- (ii) $\text{supp } \underline{\mu}_s \subset \{x \in [0, 1] : f_0(x) = \lambda^*\}$.

We now give two particular cases where the sequence $(\tilde{\mu}_n)$ has a limit.

Let $\bar{\lambda} := \max_{x \in [0, 1]} f_0(x)$ and $\chi_0 := \int_{[0, 1]} (\bar{\lambda} - f_0)^{-1} d\nu$.

- (i) Assume $\chi_0 \geq 1$, then there exist $\lambda^* \geq \bar{\lambda}$ such that

$$\int_{[0, 1]} g_{\lambda^*}(x) d\nu(x) = \int_{[0, 1]} \frac{d\nu(x)}{\lambda^* - f_0(x)} = 1$$

and $(\tilde{\mu}_n)$ converges in probability to $g_{\lambda^*} \nu$.

- (ii) If $\chi_0 < 1$ and $\{x \in [0, 1] : f_0 = \bar{\lambda}\}$ reduces to the singleton $\{x_0\}$, then $(\tilde{\mu}_n)$ converges in probability to

$$g_{\bar{\lambda}} \nu + (1 - \chi_0)\delta_{x_0}.$$

2.5. *Large and moderate deviations for finite moments sequence.* Although we may obtain a LDP for $(Z_n^{(k)})$ using Theorem 2.3 and the contraction principle, we will show the following theorem.

THEOREM 2.6. $(Z_n^{(k)})$ satisfies a LDP with convex good rate function

$$(6) \quad I_k(c^{(k)}) = \begin{cases} -\ln(c^+(c^{(k)}) - c^-(c^{(k)})) - k \ln 4, & \text{if } c^{(k)} \in \text{int } M_k, \\ +\infty, & \text{otherwise.} \end{cases}$$

Theorem 2.3 will be shown by first proving Theorem 2.6 and a projective limit argument [see Section 4.6 in Dembo and Zeitouni (1998)].

REMARK 2.2. I_k achieves the value 0 only at $\bar{c}^{(k)}$. Hence, using Borel Cantelli lemma, that yields to the almost sure convergence of $Z_n^{(k)}$ to $\bar{c}^{(k)}$.

We now turn on moderate deviation properties of $(Z_n^{(k)})$. For $i, j \in \mathbb{N}^*$, define

$$(7) \quad a_{ij} = \begin{cases} 2^{-2i+1} \binom{2i}{i-j}, & \text{if } 1 \leq j \leq i, \\ 0, & \text{if } j > i. \end{cases}$$

For $k \in \mathbb{N}^*$, let $A_k = (a_{ij})_{i,j=1}^k$ and set

$$\Sigma_k = \frac{1}{2} A_k A_k^T$$

and

$$J_k(\bar{x}) = \frac{1}{2} \bar{x}^T \Sigma_k^{-1} \bar{x}, \quad \bar{x} \in \mathbb{R}^k.$$

THEOREM 2.7. Let (u_n) be a sequence decreasing to 0 such that $n^{-1} = o(u_n)$ and let $\tilde{Z}_n^{(k)} := \sqrt{nu_n}(Z_n^{(k)} - \bar{c}^{(k)})$. Then $(\tilde{Z}_n^{(k)})$ satisfies a moderate deviations principle with good rate function J_k .

3. On the $(n + 1)$ th moment range size of the probability measures having the same n first moments. In this section, for $c^{(k)} \in M_k$ ($k \in \mathbb{N}^*$), we set

$$r_{k+1}(c^{(k)}) = \sup_{\mu_1, \mu_2 \in \mathbb{S}_k(c^{(k)})} \left\{ \int_0^1 x^{k+1} d\mu_1(x) - \int_0^1 x^{k+1} d\mu_2(x) \right\}.$$

Using the large deviations properties of the previous section and the contraction principle, we obviously obtain the following:

THEOREM 3.1. Let $c^{(k)} \in M_k$. Then:

(i)

$$(8) \quad r_{k+1}(c^{(k)}) = \exp\left(- \inf_{\mu \in \mathbb{S}_k(c^{(k)})} I(\mu) - k \ln 4\right).$$

(ii) Let $\mu \in \mathbb{P}([0, 1])$ and $c^{(k)} = c^{(k)}(\mu)$, then

$$\lim_{k \rightarrow \infty} \frac{1}{k} \ln[4^k r_{k+1}(c^{(k)})] = -I(\mu).$$

REMARK 3.1. (i) Using Proposition 2.2, (8) may be expressed as the supremum of a concave function. Indeed, we have

$$r_{k+1}(c^{(k)}) = \exp\left(-k \ln 4 - \sup_{(\lambda_0, \lambda) \in \mathbb{R}^{n+1}} H_k(\lambda_0, \lambda, c^{(k)})\right),$$

where for $(\lambda_0, \lambda) \in \mathbb{R}^{k+1}$,

$$H_k(\lambda_0, \lambda, c^{(k)}) = \lambda_0 + \langle \lambda, c^{(k)} \rangle + \int_0^1 \ln(1 - \lambda_0 - \langle \lambda, \phi_k(x) \rangle) d\nu(x).$$

So for all $(\lambda_0, \lambda) \in \mathbb{R}^{k+1}$,

$$r_{k+1}(c^{(k)}) \leq \exp[-k \ln 4 - H_k(\lambda_0, \lambda, c^{(k)})].$$

This last inequality is helpful to study the superresolution rate in the power moment problem. This will be done in a forthcoming paper of Gamboa and Lozada. Superresolution occurs for $c^{(k)} \in \partial M_k$. In this case as we saw in Section 2.3, $\mathbb{S}_k(c^{(k)})$ reduces to a single p.m. Superresolution rate is the concentration rate of the set $\mathbb{S}_k(c^{(k)} + \eta)$ when $\eta \in \mathbb{R}^k$ is a small perturbation [see Gamboa and Gassiat (1996) and Doukhan and Gamboa (1996) for more on this problem].

(ii) Let P be a polynomial having degree $k \in \mathbb{N}^*$. Assume that P is positive on $[0, 1]$ and satisfies the normalizing condition

$$\int_0^1 \frac{d\nu(x)}{P(x)} = 1.$$

Set, for $j \in \mathbb{N}^*$, $d_j = c_j(\frac{\nu}{P})$ and, as usual, $d^{(j)} = (d_i)_{i=1, \dots, j}$. Then, using optimization results developed in Theorem 2.1 of Gamboa and Gassiat (1997) or Theorem 4.1 of Borwein and Lewis (1993) we have, for $j \geq k$,

$$I\left(\frac{\nu}{P}\right) = \inf_{\mu \in \mathbb{S}_j(d^{(j)})} I(\mu)$$

(see also Section 2.5 for related results). Therefore, in this frame Theorem 3.1 gives

$$r_{j+1}(d^{(j)}) = \exp\left(-j \ln 4 - \int_0^1 \ln P(x) d\nu(x)\right), \quad j \geq k.$$

(iii) Obviously, using the contraction principle and Theorem 2.6, we obtain that the sequence $(r_{k+1}(Z_n^{(k)}))$ satisfies a LDP with good rate function,

$$R(r) = \begin{cases} -\ln(4^k r), & \text{if } r \in]0, 4^{-k}[, \\ +\infty, & \text{otherwise.} \end{cases}$$

4. Proofs.

4.1. *Canonical moments.* This section is devoted to canonical moments which are the basic tool for our results [as those obtained in Chang, Kemperman and Studden (1993)]. We will present few properties of canonical moments. An exhaustive study of canonical moments and their applications can be found in the very nice book of Dette and Studden (1997). Let p_i denote the i th canonical moments defined in $\text{int } M_k, k \in \mathbb{N}^*$, by

$$\begin{aligned} p_i(c^{(k)}) &= p_i(c^{(i)}) \\ &= \frac{c_i - c^-(c^{(i-1)})}{c^+(c^{(i-1)}) - c^-(c^{(i-1)})}, \quad 1 \leq i \leq k, \end{aligned}$$

where c^+ and c^- have been defined in Section 2.3. For $k \in \mathbb{N}^*$, we will denote by \mathbf{p}_k the map from $\text{int } M_k$ to $]0, 1[^k$,

$$c^{(k)} \mapsto p^{(k)}(c^{(k)}) = (p_1(c_1), \dots, p_2(c^{(2)}), \dots, p_k(c^{(k)}))^T.$$

The map \mathbf{p}_k is a continuous one-to-one onto correspondence between the interior of the k th moment space and $]0, 1[^k$ [see, Skibinsky (1968) and Dette and Studden (1997)]. Obviously, the sequence (\mathbf{p}_k) induces a one-to-one correspondence \mathbf{p}_∞ between $\text{int } M$ and $]0, 1[^{\mathbb{N}}$. A very interesting property of the canonical moments is that they are invariants under linear transformations of the interval. Therefore, as the reader will easily check, all the results obtained on $[0, 1]$ are also valid for any other bounded closed interval of the real line. The canonical moments of the arcsine law ν are all equal to $\frac{1}{2}$ [see Skibinsky (1969)]. We will denote them by \bar{p}_j .

In Lemma 1.4 of Chang, Kemperman and Studden (1993) the first Taylor expansion of \mathbf{p}_k around $\bar{p}^{(k)}$ is given,

$$(9) \quad c_m = \bar{c}_m + 2 \sum_{j=1}^m a_{mj} (p_j - \frac{1}{2}) + \mathbf{O}\left(\sum_{j=1}^m |p_j - \frac{1}{2}|^2\right),$$

where, for $m, j \in \mathbb{N}^*, c^{(m)} = (c_1, \dots, c_m)^T \in \text{int } M_m, p^{(m)}(c^{(m)}) = (p_1, \dots, p_m) \in]0, 1[^m$ and a_{mj} has been defined in Section 2.5.

We recall that the beta distribution $(\beta(a, b))$ of parameters $a, b > 0$, has density with respect to the Lebesgue measure on $[0, 1]$,

$$[B(a, b)]^{-1} x^{a-1} (1-x)^{b-1},$$

where $B(a, b) = \int_{[0,1]} x^{a-1} (1-x)^{b-1} dx$.

We have the following lemma.

LEMMA 4.1 [Theorem 1.3 of Chang, Kemperman and Studden (1993)]. *If M_n is endowed with \mathbb{P}_n , then the random vector $p^{(n)} = (p_1, p_2, \dots, p_n)^T$ satisfies:*

- (i) $(p_i)_{i=1}^n$ are independent random variables,
- (ii) $p_i \sim \beta(n - i + 1, n - i + 1), i = 1, \dots, n.$

Now, we display an equation linking canonical and ordinary power moments given in Skibinsky (1967), for $c^{(n)} \in \text{int } M_n,$

$$(10) \quad r_{n+1}(c^{(n)}) = c^+(c^{(n)}) - c^-(c^{(n)}) = \prod_{i=1}^n p_i(c^{(n)})(1 - p_i(c^{(n)})).$$

In (15) we will express $r_{n+1}(c^{(n)})$ as a function of Hankel determinants.

4.2. Proof of Theorem 2.6.

LEMMA 4.2 [Exercise 4.2.7 of Dembo and Zeitouni (1998)]. *Let $(X_{n,1})$ [resp. $(X_{n,2})$] be a sequence of random variables taking its values in a regular space \mathcal{X}_1 (resp. \mathcal{X}_2) satisfying a LDP with good rate function I_1 (resp. I_2). If $X_{n,1}$ and $X_{n,2}$ are independent, then $X_n = (X_{n,1}, X_{n,2})$ satisfies a LDP in $\mathcal{X}_1 \times \mathcal{X}_2$ with good rate function $I(x_1, x_2) = I_1(x_1) + I_2(x_2).$*

LEMMA 4.3. *The sequence of p.m.s $(\beta(n, n))_{n \in \mathbb{N}}$ satisfies on $[0, 1]$ a LDP with good rate function*

$$\hat{I}(x) = -\ln(x - x^2) - \ln 4.$$

PROOF. Let X_n be a random variable having $\beta(n, n)$ distribution, then we may write

$$X_n \stackrel{\mathcal{L}}{=} \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^{2n} Y_i},$$

where the random variables (Y_i) are independent with standard exponential distribution [see, e.g., Bartoli and Del Moral (2001), page 71]. Consider the bidimensional random vector

$$V_n = \left(\frac{1}{n} \sum_{i=1}^n Y_i, \frac{1}{n} \sum_{i=n+1}^{2n} Y_i \right)^T.$$

From Lemma 4.2 and Cramér’s theorem [Theorem 2.2.3 in Dembo and Zeitouni (1998)], we obtain a LDP for (V_n) with good rate function

$$I_V(x, y) = x + y - 2 + \log xy, \quad x > 0, y > 0.$$

Hence, the LDP for (X_n) follows from the one for (V_n) and the contraction principle with the continuous map $(x, y) \mapsto x/(x + y).$ \square

From Lemmas 4.3 and 4.1, we have that, for $j \in \mathbb{N}^*$, $(p_j(Z_n))_{n \in \mathbb{N}}$ ($j \in \mathbb{N}$, fixed) satisfies a LDP with good rate function \hat{I} . Thanks to Lemma 4.2, for $k \in \mathbb{N}$, we also have a LDP for $(p^{(k)}(Z_n))_{n \in \mathbb{N}}$ with good rate function

$$\hat{I}_k(p^{(k)}) = - \sum_{i=1}^k \ln(p_i - p_i^2) - k \ln 4,$$

where $p^{(k)} = (p_1, p_2, \dots, p_k)^T \in]0, 1[^k$.

We have $Z_n^{(k)} = \mathbf{p}_k^{-1}(p^{(k)}(Z_n))$. The contraction principle yields the LDP for $(Z_n^{(k)})_{n \in \mathbb{N}}$ with good rate function,

$$I_k(c^{(k)}) = \hat{I}_k(p^{(k)}(c^{(k)})).$$

Using (10), we may write I_k as in (6).

It is obvious that the function \hat{I}_k achieves its minimum value 0 at $\bar{p}^{(k)}$ (and only at this point). Indeed, remember that $\bar{p}_j = \frac{1}{2}$ for $1 \leq j \leq k$. Consequently, the function I_k achieves its minimum value at $\bar{c}^{(k)}$.

4.3. Proof of Theorem 2.7.

LEMMA 4.4. *Let (u_n) be a sequence decreasing to 0 and $n^{-1} = o(u_n)$. Let Y_n be a random variable having $\beta(n - l, n - l)$ distribution ($n \in \mathbb{N}^*, l \in \mathbb{N}$). Set*

$$X_n := \sqrt{nu_n}(Y_n - \frac{1}{2}),$$

then (X_n) satisfies a LDP with good rate function $J_1(x) = 4x^2$ and speed (u_n) .

PROOF. It is well known that $B_n = ((n - 1)!)^2 / (2n - 1)!$. Using Stirling's formula for $n!$, we have

$$(11) \quad B_n = 4^{-n} \left(\frac{\pi}{n}\right)^{1/2} (2 + \xi_n)$$

with $\xi_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $\lim_n \frac{1}{n} \ln B_n = -\ln 4$.

Let $\mathcal{A} :=]a, b[: a < b$. \mathcal{A} is a base of the usual topology on \mathbb{R} . Let $]a, b[\in \mathcal{A}$. For n large enough,

$$\begin{aligned} \mathbb{P}(X_n \in]a, b[) &= \mathbb{P}\left(Y_n \in \left] \frac{1}{2} + \frac{a}{\sqrt{nu_n}}, \frac{1}{2} + \frac{b}{\sqrt{nu_n}} \right[\right) \\ &= B_{n-l-1}^{-1} \int_{1/2+a/\sqrt{nu_n}}^{1/2+b/\sqrt{nu_n}} x^{n-l-1} (1-x)^{n-l-1} dx \\ &\leq B_{n-l-1}^{-1} m^+ \left(\frac{1}{2} + \frac{a}{\sqrt{nu_n}}, \frac{1}{2} + \frac{b}{\sqrt{nu_n}} \right)^{n-l-1} \frac{b-a}{\sqrt{nu_n}}, \end{aligned}$$

where $m^+(a, b) := \sup\{t - t^2 : t \in]a, b[\}$ and $m^-(a, b) := \inf\{t - t^2 : t \in]a, b[\}$. The function $t \mapsto t^2 - t$ is strictly convex in $]0, 1[$ with minimum value at $\frac{1}{2}$, so

$$m^+\left(\frac{1}{2} + \frac{a}{\sqrt{nu_n}}, \frac{1}{2} + \frac{b}{\sqrt{nu_n}}\right) = \begin{cases} \frac{1}{4} - \frac{a^2}{nu_n}, & \text{if } a > 0, \\ \frac{1}{4} - \frac{b^2}{nu_n}, & \text{if } b < 0, \\ 0, & \text{if } a < 0 < b. \end{cases}$$

Using (11),

$$\limsup_n u_n \ln \mathbb{P}(X_n \in]a, b[) \leq \begin{cases} -4a^2, & \text{if } a > 0, \\ -4b^2, & \text{if } b < 0, \\ 0, & \text{if } a < 0 < b. \end{cases}$$

This leads to

$$(12) \quad \sup_{\{A \in \mathcal{A} : x \in A\}} \left\{ -\limsup_n u_n \ln \mathbb{P}(X_n \in A) \right\} \geq 4x^2.$$

Now, analogously, we have

$$\mathbb{P}(X_n \in A) \geq m^-\left(\frac{1}{2} + \frac{a}{\sqrt{nu_n}}, \frac{1}{2} + \frac{b}{\sqrt{nu_n}}\right)^{n-l-1} \frac{b-a}{\sqrt{nu_n}} \frac{1}{B_{n-l-1}}.$$

But,

$$m^-\left(\frac{1}{2} + \frac{a}{\sqrt{nu_n}}, \frac{1}{2} + \frac{b}{\sqrt{nu_n}}\right) = \begin{cases} \frac{1}{4} - \frac{b^2}{nu_n}, & \text{if } a > 0, \\ \frac{1}{4} - \frac{a^2}{nu_n}, & \text{if } b < 0, \\ \frac{1}{4} - \frac{\max\{a^2, b^2\}}{nu_n}, & \text{if } a < 0 < b. \end{cases}$$

So, using (11), we obtain

$$\limsup_n u_n \ln \mathbb{P}(X_n \in]a, b[) \leq \begin{cases} -4b^2, & \text{if } a > 0, \\ -4a^2, & \text{if } b < 0, \\ -4 \max\{a^2, b^2\}, & \text{if } a < 0 < b. \end{cases}$$

Consequently,

$$(13) \quad \sup_{\{A \in \mathcal{A} : x \in A\}} \left\{ -\liminf_n u_n \ln \mathbb{P}(X_n \in A) \right\} \leq 4x^2.$$

Using Theorem 4.1.11 of Dembo and Zeitouni (1998), (12) and (13), we get a weak LDP for X_n with rate function J_1 . The full LDP is then a consequence of the

exponential tightness of the laws of Y_n . Indeed, for $K > 0$ and n large enough, we have

$$\begin{aligned} \mathbb{P}(X_n \in [-K, K]^c) &= 2\mathbb{P}\left(Y_n \in \left] \frac{1}{2} + \frac{K}{\sqrt{nu_n}}, 1 \right[\right) \\ &= 2B_{n-l-1}^{-1} \int_{1/2+K/\sqrt{nu_n}}^1 x^{n-l-1} (1-x)^{n-l-1} dx \\ &\leq B_{n-l-1}^{-1} \left(\frac{1}{4} - \frac{K^2}{nu_n} \right)^{n-l-1} \left(\frac{1}{2} - \frac{K}{\sqrt{nu_n}} \right). \end{aligned}$$

Thus, using (11) and the expansion

$$\ln\left(\frac{1}{4} - \frac{K^2}{nu_n}\right) = -\ln 4 + 1 - \frac{4K^2}{nu_n} + o\left(\frac{1}{nu_n}\right),$$

we obtain

$$\limsup_n u_n \ln \mathbb{P}(X_n \in [-K, K]^c) \leq -4K^2,$$

which implies the exponential tightness. \square

LEMMA 4.5. *For every sequence (u_n) decreasing to 0 with $n^{-1} = o(u_n)$ and $k \in \mathbb{N}^*$, the sequence of random vectors $Z'_n := 2\sqrt{nu_n}A_k(p^{(k)}(Z_n) - \bar{p}^{(k)})$ satisfies a LDP with good rate function J_k and speed (u_n) .*

PROOF. Using the two previous lemmas, we can state a LDP for the random vectors $W_n := \sqrt{nu_n}(p^{(k)}(Z_n) - \bar{p}^{(k)})$ with good rate function

$$J'_k(\bar{x}) = 4\bar{x}^T \bar{x}.$$

So, using the contraction principle, we obtain a LDP for $Z'_n = 2A_k W_n$ with good rate function

$$\begin{aligned} J_k(\bar{x}) &= \inf\{J'_k(\bar{y}) : \bar{x} = 2A_k \bar{y}\} = \frac{1}{4}\bar{x}^T (A_k^{-1})^T A_k^{-1} \bar{x} \\ &= \frac{1}{2}\bar{x}^T \Sigma_k^{-1} \bar{x}. \end{aligned} \quad \square$$

LEMMA 4.6. *The random vector sequences $(\tilde{Z}_n^{(k)})$ and (Z'_n) are exponentially equivalent.*

PROOF. It is obvious that there exists a constant C_0 such that

$$|\mathbf{O}(|p^{(k)} - \bar{p}^{(k)}|^2)| \leq C_0 |p^{(k)} - \bar{p}^{(k)}|^2,$$

where $\mathbf{O}(|p^{(k)} - \bar{p}^{(k)}|^2)$ is the function which appears in (9). So,

$$\begin{aligned} \{|\tilde{Z}_n^{(k)} - Z'_n| > \varepsilon\} &= \{\sqrt{nu_n}|\mathbf{O}(|p^{(k)} - \bar{p}^{(k)}|^2)| > \varepsilon\} \\ &\subset \left\{ \sqrt{nu_n} |p^{(k)} - \bar{p}^{(k)}|^2 > \frac{\varepsilon}{C_0} \right\}. \end{aligned}$$

Let $C_1 > 0$. For $\sqrt{nu_n} > C_1$,

$$\left\{ \sqrt{nu_n} |p^{(k)} - \bar{p}^{(k)}|^2 > \frac{\varepsilon}{kC_0} \right\} \subset \left\{ \sqrt{nu_n} |p^{(k)} - \bar{p}^{(k)}| > \left(\frac{\varepsilon C_1}{kC_0} \right)^{1/2} \right\}.$$

This leads to

$$\begin{aligned} & \limsup_n u_n \ln \mathbb{P}(|\tilde{Z}_n^{(k)} - Z'_n| > \varepsilon) \\ & \leq \limsup_n u_n \ln \mathbb{P}\left(\sqrt{nu_n} |p^{(k)} - \bar{p}^{(k)}| > \left(\frac{\varepsilon C_1}{kC_0}\right)^{1/2}\right) \leq -4 \frac{\varepsilon C_1}{kC_0}, \end{aligned}$$

where the last inequality follows from the LDP for the random vectors W_n (Lemma 4.5). Therefore, taking $C_1 \rightarrow \infty$, we obtain

$$\limsup_n u_n \ln \mathbb{P}(|\tilde{Z}_n^{(k)} - Z'_n| > \varepsilon) = -\infty.$$

Then $(\tilde{Z}_n^{(k)})$ and (Z'_n) are exponentially equivalent. \square

Theorem 2.7 is obtained directly using the two previous lemmas.

4.4. Proof of Theorem 2.3.

4.4.1. *Statement of the LDP.* Consider on M the product topology and the product algebra. Let M_n^+ be the subset of M defined by

$$M_n^+ = \{c^{(\infty)} \in M : c_{n+1} = c^+(c^{(n)})\}.$$

We define on M the following sequence of p.m.s $(\bar{P}_n)_{n \in \mathbb{N}}$:

$$\bar{P}_n(B) = P_n(\Pi_n(B \cap M_n^+)), \quad B \text{ measurable set of } M.$$

THEOREM 4.7. $(\bar{P}_n)_{n \in \mathbb{N}}$ satisfies a LDP with rate function

$$I_M(c^{(\infty)}) = \lim_n -\ln(4^n (c^+(c^{(n)}) - c^-(c^{(n)}))).$$

PROOF. Let $k \in \mathbb{N}$ and B_k be a measurable set of M_k . We have, for $n \geq k$,

$$\begin{aligned} \bar{P}_n \circ \Pi_k^{-1}(B_k) &= \bar{P}_n(B_k \times \mathbb{R}^{\mathbb{N}}) = P_n(\Pi_n(B_k \times \mathbb{R}^{\mathbb{N}} \cap M_n^+)) = P_n(B_k \times \mathbb{R}^{n-k}) \\ &= \mathbb{P}(Z_n^{(k)} \in B_k). \end{aligned}$$

Therefore, the family $\{\bar{P}_n \circ \Pi_k^{-1}\}_{n \in \mathbb{N}}$ satisfies a LDP with rate function I_k . The Dawson–Gärtner theorem [Theorem 4.6.1 of Dembo and Zeitouni (1998)] leads to a LDP for \bar{P}_n on the projective limit of the spaces M_n which is M . The good rate function is $I_M(c^{(\infty)}) = \sup_n I_n(\Pi_n c^{(\infty)})$. For $c^{(\infty)} \in M$, $I_n(\Pi_n c^{(\infty)})$ is nonincreasing in n , thus, $\sup_n I_n = \lim_n I_n$ [see (10)]. \square

Let G denote the one-to-one correspondence between M and $\mathbb{P}([0, 1])$. It is easy to see that it is continuous. Therefore, using the contraction principle, we obtain a LDP for the probability measure family $\{\bar{P}_n \circ G^{-1}\}_{n \in \mathbb{N}}$. But, for B Borelian of $\mathbb{P}([0, 1])$,

$$Q_n(B) = (\bar{P}_n \circ G^{-1})(B) = \bar{P}_n(G^{-1}(B)).$$

We have $c^{(\infty)} \in G^{-1}(B) \cap M_n^+$ if, and only if, $G(c^{(\infty)}) = \sigma_n^+(c^{(n)}) \in B$. So,

$$\begin{aligned} (\bar{P}_n \circ G^{-1})(B) &= \mathbb{P}(Z_n \in \Pi_n(G^{-1}(B) \cap M_n^+)) \\ &= \mathbb{P}(\sigma_n^+(Z_n) \in B). \end{aligned}$$

Therefore, the contraction principle gives the LDP for $(Q_n)_{n \in \mathbb{N}}$ with good rate function

$$(14) \quad \tilde{I}(\mu) = \lim_n -\ln[4^{n-1}(c_n^+(\mu) - c_n^-(\mu))],$$

where $c_n^+(\mu)$ and $c_n^-(\mu)$ denote $c^+(c^{(n-1)}(\mu))$ and $c^-(c^{(n-1)}(\mu))$, respectively. Obviously, from (10), we may conclude that \tilde{I} achieves the value 0 only at ν .

LEMMA 4.8. *The function \tilde{I} in (14) is convex.*

PROOF. Let $\mu = \alpha\mu_1 + \beta\mu_2$ with $\mu, \mu_1, \mu_2 \in \mathbb{P}([0, 1])$ and α, β positive real numbers such that $\alpha + \beta = 1$. Thus,

$$c^{(n)}(\mu) = \alpha c^{(n)}(\mu_1) + \beta c^{(n)}(\mu_2).$$

Thanks to convexity of M_{n+1} ,

$$\alpha c^+(c^{(n)}(\mu_1)) + \beta c^+(c^{(n)}(\mu_2)) \in \{c : (c^{(n)}(\mu), c) \in M_{n+1}\}$$

and by the definition of c^+ , we have

$$c^+(c^{(n)}(\mu)) \geq \alpha c^+(c^{(n)}(\mu_1)) + \beta c^+(c^{(n)}(\mu_2)).$$

Analogously, we obtain that $c^-(c^{(n)}(\mu)) \leq \alpha c^-(c^{(n)}(\mu_1)) + \beta c^-(c^{(n)}(\mu_2))$. Thus,

$$c_n^+(\mu) - c_n^-(\mu) \geq \alpha(c_n^+(\mu_1) - c_n^-(\mu_1)) + \beta(c_n^+(\mu_2) - c_n^-(\mu_2)).$$

Since $t \mapsto -\ln 4^n t$ is a nonincreasing convex function in $t > 0$ for all $n \in \mathbb{N}^*$, we obtain that \tilde{I} is convex as the supremum of convex functions. \square

4.4.2. *Identification of the rate function.* From Corollary 1.4.6 of Dette and Studden (1997), we may write the following equation:

$$(15) \quad r_n(c^{(n-1)}) = \frac{\underline{H}_{n-1}\bar{H}_{n-1}}{\underline{H}_{n-2}\bar{H}_{n-2}}, \quad c^{(n-1)} = (c_1, \dots, c_{n-1})^T \in M_{n-1},$$

where, for $c^{(2m+1)} \in M_{2m+1}$ ($m \in \mathbb{N}^*$),

$$\begin{aligned} \underline{H}_{2m} &= \begin{vmatrix} c_0 & \cdots & c_m \\ \vdots & & \vdots \\ c_m & \cdots & c_{2m} \end{vmatrix}, \\ \overline{H}_{2m} &= \begin{vmatrix} c_1 - c_2 & \cdots & c_m - c_{m+1} \\ \vdots & & \vdots \\ c_m - c_{m+1} & \cdots & c_{2m-1} - c_{2m} \end{vmatrix}, \\ \underline{H}_{2m+1} &= \begin{vmatrix} c_1 & \cdots & c_{m+1} \\ \vdots & & \vdots \\ c_{m+1} & \cdots & c_{2m+1} \end{vmatrix}, \\ \overline{H}_{2m+1} &= \begin{vmatrix} c_0 - c_1 & \cdots & c_m - c_{m+1} \\ \vdots & & \vdots \\ c_m - c_{m+1} & \cdots & c_{2m} - c_{2m+1} \end{vmatrix}. \end{aligned}$$

These determinants are called *Hankel determinants*. Within the moment problem they play an important role. The conditions under which a sequence is a moment sequence can be expressed on these Hankel determinants [see Karlin and Studden (1966), Kreĭn and Nudelman (1977) and Dette and Studden (1997)].

We have the following relations between Hankel determinants. Let $\mu \in \mathbb{P}([0, 1])$. Let $\mu' \in \mathbb{P}([0, 1])$ be defined by

$$\mu'(dx) = \frac{x - x^2}{c_1(\mu) - c_2(\mu)} \mu(dx).$$

Obviously, μ' verifies

$$c_k(\mu') = \frac{c_{k+1}(\mu) - c_{k+2}(\mu)}{c_1(\mu) - c_2(\mu)}.$$

Hence,

$$\overline{H}_{2m}(\mu) = (c_1(\mu) - c_2(\mu))^m \underline{H}_{2(m-1)}(\mu').$$

Moreover,

$$\frac{\overline{H}_{2m}(\mu)}{\overline{H}_{2(m-1)}(\mu)} = (c_1(\mu) - c_2(\mu)) \frac{\underline{H}_{2(m-1)}(\mu')}{\underline{H}_{2(m-2)}(\mu')}.$$

In what follows we set $c^{(n)} = c^{(n)}(\mu)$, $c_n^+ = c^+(c^{(n-1)})$, and $c_n^- = c^-(c^{(n-1)})$. The Hankel matrices depending on μ' are tagged with a prime. We have

$$\begin{aligned} (c_{n+1}^+ - c_{n+1}^-)(c_n^+ - c_n^-) &= \frac{\overline{H}_n \underline{H}_n}{\overline{H}_{n-2} \underline{H}_{n-2}} \\ &= (c_1 - c_2) \frac{\underline{H}'_{n-2}}{\underline{H}'_{n-4}} \frac{\underline{H}_n}{\underline{H}_{n-2}}, \end{aligned}$$

where the last equality is only true if n is even.

As the limit in (14) exists, we can calculate it taking n even,

$$\begin{aligned} & \lim_m \ln(4^{2m-1}(c_{2m}^+ - c_{2m}^-)) \\ &= \frac{1}{2} \lim_m \ln(4^{2m}(c_{2m+1}^+ - c_{2m+1}^-)) \\ & \quad + \frac{1}{2} \lim_m \ln(4^{2m-1}(c_{2m}^+ - c_{2m}^-)) \\ &= \frac{1}{2} \lim_m \ln(4^{4m-1}(c_{2m+1}^+ - c_{2m+1}^-)(c_{2m}^+ - c_{2m}^-)) \\ &= \frac{1}{2} \lim_m \ln\left(4^{4m-1}(c_1 - c_2) \frac{\underline{H}'_{2(m-1)}}{\underline{H}'_{2(m-2)}} \frac{\underline{H}_{2m}}{\underline{H}_{2(m-1)}}\right). \end{aligned}$$

Grenander and Szegö (1958) have showed a general limit theorem for the quotient of Hankel determinants (see Theorem in Section 5.2 and Section 6.3 of this book). In our framework it can be written in the following way.

THEOREM 4.9. *Let $\mu \in \mathbb{P}([0, 1])$. Call $f(x)$ its Radon–Nikodym derivative with respect to the Lebesgue measure, then*

$$\lim_n 4^{2n+1} \frac{\underline{H}_{2n}(\mu)}{\underline{H}_{2n-2}(\mu)} = 2\pi \exp\left(\int_0^1 \ln f(x) d\nu(x)\right).$$

Now we may write

$$\begin{aligned} & \lim_m \ln(4^{2m-1}(c_{2m}^+ - c_{2m}^-)) \\ &= \frac{1}{2} \lim_m \ln\left(4^{2m+1} \frac{\underline{H}_{2m}}{\underline{H}_{2(m-1)}}\right) + \frac{1}{2} \lim_m \ln\left(4^{2m-1} \frac{\underline{H}'_{2(m-1)}}{\underline{H}'_{2(m-2)}}\right) \\ & \quad + \frac{1}{2} \lim_m \ln(4^{-1}(c_1 - c_2)) \\ &= \frac{1}{2} \int_0^1 \ln f(x) d\nu(x) + \frac{1}{2} \int_0^1 \ln \frac{(x - x^2)f(x)}{c_1 - c_2} d\nu(x) \\ & \quad + \frac{1}{2} \ln(c_1 - c_2) + \frac{1}{2} \ln(\pi) \\ &= \frac{1}{2} \int_0^1 \ln[\pi(x - x^2)f^2(x)] d\nu(x) = -I(\mu). \end{aligned}$$

□

4.5. *Proof of Theorem 2.5.* The LDP for (\tilde{Q}_n) is a direct consequence of Theorem III.17 of den Hollander (2000). The rate function which controls the LDP is

$$I_F(\mu) = I(\mu) - F(\mu) + K_F,$$

where $K_F := \sup_{\mu' \in \mathbb{P}([0,1])} \{F(\mu') - I(\mu')\}$. \square

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