The Annals of Statistics
2006, Vol. 34, No. 1, 42–77
DOI: 10.1214/009053605000000868

EXTENDED STATISTICAL MODELING UNDER SYMMETRY; THE LINK TOWARD QUANTUM MECHANICS

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We derive essential elements of quantum mechanics from a parametric structure extending that of traditional mathematical statistics. The basic setting is a set $\mathcal{A}$ of incompatible experiments, and a transformation group G on the cartesian product Π of the parameter spaces of these experiments. The set of possible parameters is constrained to lie in a subspace of Π, an orbit or a set of orbits of G. Each possible model is then connected to a parametric Hilbert space. The spaces of different experiments are linked unitarily, thus defining a common Hilbert space $\mathbf{H}$. A state is equivalent to a question together with an answer: the choice of an experiment $a \in \mathcal{A}$ plus a value for the corresponding parameter. Finally, probabilities are introduced through Born's formula, which is derived from a recent version of Gleason's theorem. This then leads to the usual formalism of elementary quantum mechanics in important special cases. The theory is illustrated by the example of a quantum particle with spin.

1. Introduction. Both statistics and quantum theory deal with prediction using the concept of probability. Historically, the difference between the two disciplines has been large, but in the last few years it has diminished, not in the least due to the recent work by Barndorff-Nielsen, Gill and Jupp [7].

The lack of contact between the two disciplines is of course related to the difference in foundation, but one of the aims of the present paper is to argue that to a certain extent, this difference in foundation can be overcome. This may perhaps at first be difficult to believe: In statistics, the state of a given system is given simply by a probability measure on some measurable space. In quantum theory in its most common formulation the state of a system is given by a vector v in some abstract Hilbert space. As a continuation of this formal theory, each observable is linked to a self-adjoint operator T on the same Hilbert space in such a way that the expectation of this observable in the state v is given by (v, Tv). Associated with this is Born's formula: The transition probability from state u to state v is of the form $|(v, u)|^2$. Also, in the absence of what physicists call superselection rules, linear combinations of statevectors form new statevectors, which lead to interference phenomena unknown to classical statistics.

Received March 2003; revised March 2005.

AMS 2000 subject classifications. Primary 62A01; secondary 81P10, 62B15.

Key words and phrases. Born's formula, complementarity, complete sufficient statistics, Gleason's theorem, group representation, Hilbert space, model reduction, quantum mechanics, quantum theory, symmetry, transition probability.

The Born formula allows physicists to compute probabilities for sets of outcomes, perhaps as a function of certain parameters. Statistical methods can then be used for inference about these parameters, as discussed in [7]. By contrast, the present paper aims at giving a statistical interpretation of the vectors v themselves. If parameters are introduced as in op. cit., the total model will be similar to the hierarchical models used in Bayesian statistics. We will not use these latter kinds of parameters in the present paper. Our parametric models will be of the simplest kind, but we will emphasize that the choice between different experimental questions to focus upon also may imply a choice between different parametric models.

The quantum formalism as such is the result of a long development within physics, starting with discoveries by Max Planck, and where contributions have been made by Bohr, Pauli, Schrödinger, Heisenberg and many others. There are many good books on quantum theory, for instance, [39], where also some of the philosophical background is discussed.

Many authors have tried to find deeper foundations leading to the formalism of quantum theory. Several mathematical approaches are discussed in [60]. One such approach is quantum logic, treated in detail by Beltrametti and Cassinelli [12].

The earliest book on the mathematical foundation of quantum mechanics is [58]; in English translation, [59]. This book has had great influence; in its time it constituted a very important mathematical synthesis of the theory of quantum phenomena. The book can also be considered to be a forerunner of quantum probability. For physicists, von Neumann's book was supplemented by the book of Dirac [24], which started the development leading to modern quantum field theory.

The development of quantum probability as a mathematical discipline, continuing the more formal development of quantum theory, was started in the 1970's. A first important topic was to develop a noncommutative analogue of the notion of stochastic processes; see [1] and references therein. Other topics were noncommutative conditional expectations and quantum filtering and prediction theory ([10] and references therein).

Quantum probability was made popular among ordinary probabilists by Meyer [45]. A related book is [49], which discusses the quantum stochastic calculus founded by Hudson and Parthasarathy, but also many other themes related to the mathematics of current quantum theory. An example of a symposium proceeding aiming at covering both conventional probability theory and quantum probability is [2].

There are also links between quantum theory and statistical inference theory. A systematic treatment of quantum hypothesis testing and quantum estimation theory was first given by Helstrom [37]. In [38] several aspects of quantum inference are discussed in depth; among other things the book contains a chapter on symmetry groups. A survey paper on quantum inference is Malley and Hornstein [43].

As an example of a particular statistical topic of interest, consider that of Fisher information. Since a quantum state ordinarily allows several experiments, this concept can be generalized in a natural way. A quantum information measure due to

Helstrom can be shown to give the maximal Fisher information over all possible experiments; for a recent discussion see [6].

One can thus point to several links between ordinary probability and statistics on the one hand and their quantum counterparts on the other hand. However, a general theory encompassing both sides, based on a reasonably intuitive foundation, has until now been lacking.

The main purpose of the present paper is indeed to suggest a new approach to the statistical foundation of quantum mechanics based on elementary concepts such as choice of experiment, probability model, complementarity, symmetry and model reduction. I claim that this approach leads to a conceptual basis which is more intuitive than the usual one. This is of course a very bold statement, knowing how well established the ordinary quantum formalism is, especially since the program started here also needs further development. Nevertheless, I will claim that for readers knowing statistical theory and some group theory, the present approach will probably be more enlightening than the usual formalism.

In addition to the implications for quantum theory, the concepts needed to complete this program, and also concepts learned directly from quantum theory, may at the same time turn out to lead to an enrichment of current statistical theory.

An example is the concept of complementarity; in our approach this denotes the situation where two parameters cannot both be estimated accurately in a given context, but it can also be given a wider content. In our opinion this concept should not be confined to the microworld. This view is also in line with Bohr [16], who gave talks explaining the concept of complementarity to, among others, biologists and sociologists.

A related generalization of the ordinary statistical paradigm will in fact be basic to our main setting: Before we look at the parameter of a concrete experiment, we consider all questions that can be addressed in any experiment in a given context. Thus there is a total parameter ϕ, which is a vector containing all theoretical quantities that can be imagined for a given system. Any experiment which is chosen has a parameter that is a function of ϕ, but ϕ itself has too rich a content to be estimated. Some ordinary statistical situations that can be fit into this pattern are:

EXAMPLE 1. Consider all quantities of relevance that are contemplated at the experimental design phase. This can be made concrete in many different directions.

EXAMPLE 2. A questionnaire is designed for a statistical investigation with a fixed number of alternatives for each question. Some respondents insist on giving unexpected but informative answers, say, comments in addition to the fixed questions. The total parameter ϕ may contain some such possibilities.

EXAMPLE 3. More generally: A statistical investigation on some group of humans is performed, say, through a questionnaire. Let ϕ contain all possible information about these humans which may have some relevance to the concrete questions posed.

EXAMPLE 4. There is a fragile apparatus for some specific length measurement which is destroyed after one measurement. Let μ be the length which is to be measured. Assume furthermore that the standard deviation of measurement σ can only be estimated by destroying the apparatus. Let then $\phi = (\mu, \sigma)$.

EXAMPLE 5. Assume that a particular patient has an expected survival time λ^1 if he gets treatment 1 at a specific time t, and expected survival time λ^2 if he gets treatment 2 at that time. Here "expected" is not primarily meant in relation to a probability model, but may at this point be related to what is expected by the medical experts taking into account all knowledge they have about the patient and about the treatments. Then $\phi = (\lambda^1, \lambda^2)$ can never be estimated.

EXAMPLE 6. Let there be two questions which are to be asked of an individual, where we know that the answer will depend on the order in which the questions are posed. Let (λ_1, λ_2) be the expected answer when the questions are posed in one order, and (λ_3, λ_4) when the questions are posed in the other order. Then $\phi = (\lambda_1, \ldots, \lambda_4)$ cannot be estimated from one individual.

Many more realistic, moderately complicated, examples exist, like the behavioral parameters of a rat taken together with parameters of the brain structure which can only be measured if the rat is killed.

We will concentrate much on the statistical parameter space. An essential point of the statistical paradigm is that, before the experiment, the parameter λ is unknown; afterward it is as a rule fairly accurately determined. In this way the focus is shifted from what the value of the parameter "is" to the knowledge we have about the parameter. In a physical context this can easily be made consistent with the point of view expressed by Niels Bohr, cited from [51]: "It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature." This statement is also in agreement with current views of quantum theory, as expressed, for instance, by Fuchs [27].

It is well known that there exists in the literature a large number of suggestions for interpretations of quantum theory; a very incomplete list is given by the references [13, 15, 20, 25]. Most of these interpretations include the ordinary minimalistic interpretation of Niels Bohr (the Copenhagen school or pragmatic interpretation concentrating on interpreting the outcomes of concrete experiments; for more details see [39]). The present article also implies a particular statistical interpretation related to the Niels Bohr interpretation, but it is beyond the scope of this paper to discuss in detail relations to other interpretation given in the literature.

There are also a few related papers in the recent literature. Bohr and Ulfbeck [14] discuss a foundation of quantum mechanics which is based upon irreducible representation of groups, and thus uses symmetry in a way which is similar to ours. Caves, Fuchs and Schack [19] proposes a Bayesian approach to quantum theory based upon Gleason's powerful Hilbert space theorem. Here we will avoid taking

an abstract Hilbert space as a point of departure, but we will arrive at it from a rather concrete setting. Finally, Hardy [32] derives quantum theory and probability theory from a few reasonable axioms, without going into any details concerning the state concept.

Sections 2–7 below are preparatory: In Section 2 group actions on the sample space and on the parameter space of an experiment are discussed, and the concept of permissibility is introduced. In Section 3 it is shown that permissibility always can be achieved by going to a subgroup; such a subgroup connected to an experimental parameter will be important later. In Section 4 the relation to causal inference, in particular to the concept of counterfactuals, is discussed, while in Section 5 the main quantum-mechanical example, electron spin, is treated. Section 6 gives the starting point sketched in the abstract above: reduction of the cartesian product of the parameter spaces of complementary experiments, while Section 7 treats model reduction in general and introduces the concept of group representation.

Then in Sections 8–10 the basic Hilbert space is introduced, first for a single experiment and then tied together for several complementary experiments. The treatment in these sections could have been simplified considerably by concentrating on the parameter space. The full discussion involving the sample space is included mainly for three reasons, however: First, this paves the way for further generalizations. Second, the context of an experiment is related to the limitation of the data that can be obtained, and this context is felt to play a role in the quantization. Third, a discussion of the full experiment is needed later in Section 12.

Before that, in Section 11, operators and states are introduced.

An important result is proved in Section 12: Born's formula for the transition probability between experiments. From this, the basic formalism of elementary quantum mechanics is derived in Section 13.

In what follows, we will make several explicit assumptions; most of them are relatively weak and fairly natural in a statistical setting. The exceptions to this are Assumption 5, which is a simple assumption about the connection between the parameter spaces associated with different choices of experiments; Assumption 7, which through a limitation of the parameter space serves to restrict us to a discussion of elementary quantum theory; and finally, Assumption 8, which gives the symmetry assumption needed to derive Born's formula and from this the formalism of elementary quantum mechanics.

2. Statistical models and groups. In general the total parameter space Φ—the range of the total parameter ϕ—can have almost any structure; in this paper we will assume:

ASSUMPTION 1. Φ is a locally compact topological space. There is a transformation group G acting on Φ which satisfies certain weak technical requirements (see Appendix A.1) so that Φ can be given a *right invariant measure* ν, that is, a measure which satisfies $\nu((d\phi)g) = \nu(d\phi)$.

Note that in this paper, group actions will always be written to the right: $\phi \mapsto \phi g$. The reason for this is simply that it facilitates the introduction of the right invariant measure, which from several points of view [34] in the case of a single parameter can be argued to be the best choice of a noninformative prior under symmetry in ordinary Bayesian statistical inference.

The right invariant measure is unique (up to a fixed constant) for *transitive* transformation groups, that is, group actions where the space consists of one single orbit. An orbit is defined as a set of the form $\{\phi : \phi = \phi_0 g : g \in G\}$. In general the space Φ can be divided into several orbits, and the invariant measure is unique on each orbit; it must be supplemented by some measure on the orbit indices in order to give a measure on the whole space Φ.

When a group G is defined on the (total) parameter space Φ, an important property that an experimental parameter may or may not have is the following (cf. McCullagh [44], who chose to call this concept natural):

DEFINITION 1. The parameter λ is called *permissible* as a function $\lambda(\phi)$ if it satisfies:

$$\text{If } \lambda(\phi_1) = \lambda(\phi_2) \qquad \text{then } \lambda(\phi_1 g) = \lambda(\phi_2 g) \qquad \text{for all } g \in G.$$

The most important argument for this restriction is that it leads to a uniquely defined action of the group G on the image space Λ of $\lambda(\phi)$:

$$(\lambda g)(\phi) = \lambda(\phi g). \tag{1}$$

Several general arguments for permissibility are given in [33, 34]: When this property holds, the best equivariant estimator, which essentially is the Bayes estimator under prior ν, is conserved under model reduction using functions of λ. Also, in the transitive case credibility intervals under the invariant prior turn out to be identical to confidence intervals, and certain paradoxes related to Bayes estimation are avoided.

Trivially, the total parameter $\lambda = \phi$ itself is permissible. Also, the vector parameter $(\lambda^1, \ldots, \lambda^k)$ is permissible if each λ^i is permissible.

As will be shown in the next section, if λ is not permissible with respect to G, one can always define a maximal subgroup with respect to which λ is permissible. This will be the usual case in our setting.

Let now a general group D of transformations be defined on the parameter space Λ—the range of λ. This transformation group D will be kept fixed, being thought of as a part of the specification of the problem in addition to the statistical model.

Sometimes a group D of transformations on the sample space is defined first, and then the actions on the parameter space are introduced via the statistical model by defining probability measures $\mathrm{P}^{\lambda g}$ for $g \in D$ on the sample space X by

$$\mathrm{P}^{\lambda g}(B) = \mathrm{P}^{\lambda}(Bg^{-1}) \qquad \text{for sets } B. \tag{2}$$

Then the connection between these two transformation groups is a *homomorphism*: If g_1 and g_2 are taken to act on the two spaces X and Λ, then g_i^{-1} and $g_1 g_2$ act on both spaces in the same way. The concept of homomorphism will be fundamental to this paper. It means that we have very similar group actions: The identity element, inverses and subgroups are mapped as they should be between the two transformation groups; that is, the essential structure is inherited. This is the reason why the same symbol D can and will be used for both transformation groups. If g is mapped by (2) into the identity e only when $g = e$, then the homomorphism will be an *isomorphism*: The structures of the two groups are then essentially identical. If in addition a one-to-one correspondence can be established between the spaces upon which the groups act, everything will be equivalent.

A further discussion of symmetry groups in statistics is given in [34] and in Appendix A.1. Note that the existence of a group D acting on the parameter space Λ in fact requires very few explicit invariance properties. What is needed is basically: (i) The sample space and the parameter space should both be closed under the transformations in the group. (ii) If the problem is formulated in terms of a loss function, this should be unchanged when observations and parameters are transformed conformably by the group. (iii) If a noninformative prior on Λ is needed, the right invariant distribution ν on this space should be used.

3. Experimental parameters and permissibility. Assuming that a parameter or total parameter ϕ is used to model some given part of reality, there are usually many questions that can be investigated in such a setting. Very often different such questions are addressed performing different experiments on the specific part of reality in question. (A related case is when different questions are addressed within the same experiment, e.g., when statisticians consider different sets of orthogonal contrasts in an analysis of variance experiment.)

Let $\mathcal{A}$ be the set of such questions from now on in this paper assumed to be connected to different experiments.

ASSUMPTION 2. For each $a \in \mathcal{A}$ there is a parameter $\lambda^a = \lambda^a(\phi)$, for which we assume that a probability model $\mathrm{P}^{\lambda^a}(\cdot)$ exists corresponding to experiment a. It is assumed that each experiment is maximal, that is, that there exists no possible experiment with parameter μ^a such that λ^a is a proper function of μ^a.

In a physical context, $\mathrm{P}^{\lambda^a}(\cdot)$ should be the probability measure for the measurement apparatus, at the present moment left unspecified.

When we in the sequel talk about choice of experiment/question a, we really mean a choice of (a, λ^a). But the probability measure $\mathrm{P}^{\lambda^a}(\cdot)$ is thought to be connected to the measurement apparatus, and is not at the outset included in this choice. Quantum probabilities are first introduced in Theorem 5.

When a transformation group G is defined on the (total) parameter space Φ, an important property of the experimental parameter λ^a is whether it is a permissible

function $\lambda^a(\phi)$. As already said, the most important argument for this restriction is that it leads to a uniquely defined transformation group G^a on the image space Λ^a of $\lambda^a(\phi)$, so that $(\lambda^a g^a)(\phi) = \lambda^a(\phi g^a)$ for $g^a \in G^a$.

As a simple illustration of a group connected to a parameter space or the total parameter space, look at the (total) parameter $\phi = (\mu, \sigma)$ with the translation/scale group $(\mu, \sigma) \mapsto (a + b\mu, b\sigma)$ where $b > 0$. The following one-dimensional parameters are permissible: μ, σ, μ^3, $\mu + \sigma$, $\mu + 3\sigma$, and if a such parameter is asked for some reason, say as a focus parameter, all these give valid candidates.

On the other hand, the following parameters are not permissible, and would according to McCullagh [44] lead to absurd focus parameters under this group: $\mu + \sigma^2$, σe^{μ}, $\tan(\mu)/\sin(\sigma)$.

A further example is given by the coefficient of variation σ/μ. This is not permissible. (The location part of the transformation does not make sense here.) But it will be permissible if the group is reduced to the pure scale group $(\mu, \sigma) \mapsto (b\mu, b\sigma)$, $b > 0$. This points at an important general

PRINCIPLE. If a focus parameter $\lambda^a(\phi)$ is not permissible with respect to the basic group G, then take a subgroup G^a so that it becomes permissible with respect to this subgroup.

LEMMA 1. *Given a parameter λ^a, there is always a maximal subgroup G^a of G such that λ^a is permissible with respect to G^a.*

PROOF. Let G^a be the set of all $g \in G$ such that for all $\phi_1, \phi_2 \in \Phi$ we have that $\lambda^a(\phi_1) = \lambda^a(\phi_2)$ if and only if $\lambda^a(\phi_1 g) = \lambda^a(\phi_2 g)$. Then G^a contains the identity. Furthermore, using the definition with ϕ_1, ϕ_2 replaced by $\phi_1 g_1, \phi_2 g_1$, it follows that $g_1 g_2 \in G^a$ when $g_1 \in G^a$ and $g_2 \in G^a$. Using the definition with ϕ_1, ϕ_2 replaced by $\phi_1 g^{-1}, \phi_2 g^{-1}$, it is clear that it contains inverses. Hence G^a is a group. It follows from the construction that it is maximal. □

From this it follows that the group G^a also acts on $\Lambda^a = \lambda^a(\Phi)$, by a simple homomorphism determined as in (1).

4. Experimental parameters and counterfactuals. In our view this choice of experiment can also be related to the literature on causal inference, in particular to the concept of counterfactuals, which has a central place there. A counterfactual question is a question of the form: "What would the result have been if ...?". A counterfactual variable, in the way this concept is used in the literature, is a hypothetical variable giving the result of performing an experiment under some specific condition a, when this condition a is known not to hold. A typical example is when several treatments can be allocated to some given experimental unit at some fixed time, and then in reality only one of these treatments can be chosen.

The use of such a concept goes back to Neyman [48], and has in recent decades been discussed by, among others, Rubin [54], Robins [52, 53], Pearl [50] and Gill and Robins [29]. On the other hand, Dawid [21] is skeptical of an extensive use of counterfactuals. The discussion of the last paper shows some of the positions taken by several prominent scientists on this issue.

In our setting, we choose and perform one experiment a, and then any other experiment b imagined at the same time must be regarded as a counterfactual experiment. However, instead of introducing counterfactual variables, I use counterfactual parameters λ^a, which in my view is a more useful concept. Parameters are hypothetical entities that usually cannot be observed directly. Nevertheless they may be useful in our mental modeling of phenomena and in our discussion of them. In the last decades, such mental models in causal inference have been developed to great sophistication, among other ways by using various graphical tools [41, 50]. In the present paper we will limit mental models to scalar and vector parameters, some counterfactual, leading to what we have called a total parameter, but this model concept can in principle be generalized.

When it is decided to perform one particular experiment $a \in \mathcal{A}$, the λ^a becomes the parameter of this specific experiment, an experiment which then also may include a technical or experimental error. In any case, the experiment will give an estimate $\hat{\lambda}^a$. If the technical error can be neglected, we have a *perfect* experiment, implying $\hat{\lambda}^a = \lambda^a$.

We are here at a crucial point for understanding the whole theory of this paper, namely the transition from the unobserved parameter to the observed variable. Let us again look at a single patient at some given time who can be given two different treatments. Define λ^a as the expected survival time of this patient under treatment a. Then make a choice of treatment, say $a = 1$. Ultimately, we then observe a survival time t^1 for this patient. There is no technical error involved here, so we might say that we then have $\lambda^1 = \hat{\lambda}^1 = t^1$. *And this is in fact true.* Per definition, λ^1 is connected to the single patient, the definite treatment time and a definite choice of treatment. So even though λ^1 is defined at the outset as an unknown parameter, its definition is such that, once the experiment is carried out, the parameter must by definition take the value t^1.

This simple, but crucial phenomenon, which is related to how a concept can be defined in a given situation, is in my view of quantum mechanics closely connected to what physicists call "the collapse of the wave packet" when an observation is undertaken.

5. A quantum particle with spin. Perhaps the most simple quantum-mechanical system is an electron with its spin. The spin component λ can be measured in any space direction a, and λ always takes one of the values -1 or $+1$. Given such a (perfect) measurement, this defines in the usual quantum formalism a certain state vector v in a complex two-dimensional vector space $\mathbf{H}$, formally as the eigenvector of an operator corresponding to the given measurement with the

given measurement value as eigenvalue. And given this state vector v, quantum mechanics offers formulae, versions of which will be discussed later, for predicting the results of further measurements. This quantum-mechanical model for the electron also has several applications to other systems. The setup itself is generally called a qubit in the literature.

As a contrast to this formalism, and to illustrate the general theory of this paper, we give a nonstandard description of a particle with spin, a description which will turn out in the end to be essentially equivalent to the one given by ordinary quantum theory.

The total parameter ϕ corresponding to electron spin may be defined as a vector in three-dimensional space; the direction of the vector gives the spin axis, the norm gives the spinning speed. The associated group G is then the group of all rotations of this vector in R^3 around the origin. At the outset, ϕ is a model quantity and hence unknown. As indicated before, we will assume throughout that such a total parameter can never assume a definite value in the sense that it never can be estimated. Nevertheless, such an abstract quantity turns out to be useful in model discussions.

Now let the electron have such a total parameter ϕ attached to it. Assume first that the system defines a context such that it is only possible to estimate some given component of ϕ. From this point of view, the most that we can hope to be able to measure is the angular momentum component $\theta^a(\phi) = |\phi|\cos(\alpha)$ in some direction given by a unit vector a, where α is the angle between ϕ and a.

The function $\theta^a(\cdot)$ is easily seen to be nonpermissible for fixed a. This is simply because two vectors with the same component along a in general will have different such components after a rotation. The maximal possible choice of the group G^a with respect to which $\theta^a(\cdot)$ is permissible is the group of rotations of the unit vector around the axis a, possibly together with a 180^{o} rotation around any axis perpendicular to a.

The group G^a also acts on the image space for θ^a. This group action has several orbits: For each $\kappa \in (0, 1]$, one orbit is given by the two-point set $\{-\kappa, \kappa\}$ in Θ^a. In addition there is an orbit for $\kappa = 0$.

We want in general that any reduction of the parameter space should be to an orbit or to a set of orbits. Since the value of κ may be considered to be arbitrary, we concentrate on $\lambda^a = \mathrm{sign}(\theta^a)$, taking the two values -1 and $+1$. This also implies that the function $\lambda^a(\phi)$ is permissible with respect to the group G^a, and that this group acts upon λ^a by exchanging its two values. Assume now that the electron in itself defines such a context that only λ^a can be measured, an assumption which is consistent with experience. The apparatus usually used to measure such a discretized spin component is called a Stern–Gerlach device.

The unconditional prior probability for λ^a is $1/2$ for each of the values ± 1 by symmetry. Assume now that we know that $\lambda^a = +1$, and that we afterward will measure the spin component in another direction b. We assume for simplicity that

we have an ideal measurement apparatus in the direction b, so that what we seek is the transition probability in parameter space,

$$\mathrm{P}(\lambda^b = +1 | \lambda^a = +1).$$

The formal quantum-mechanical solution of this is well known in the physics literature. Let the components of the (unit) a-vector be (a_x, a_y, a_z), and let σ_x, σ_y and σ_z be the three Pauli spin operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Calculate the eigenvector v^a for the operator $a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$ corresponding to the eigenvalue $+1$, and do a similar thing in the b-direction. Then the formalism of quantum mechanics (see Section 14 below) says that

$$\mathrm{P}(\lambda^b = +1 | \lambda^a = +1) = |v^{a\dagger} v^b|^2. \tag{4}$$

A straightforward calculation then gives

$$\mathrm{P}(\lambda^b = +1 | \lambda^a = +1) = \big(1 + \cos(u)\big)/2, \tag{5}$$

where u is the angle between the a-vector and the b-vector.

A general statistical approach to transition probabilities is given in Theorem 5 below.

6. Parameters of several statistical experiments. Up to now, we have assumed the existence of a total parameter. This section gives a very general alternative way to arrive at this concept.

Consider a set $\mathcal{A}$ of mutually exclusive experiments, each of the ordinary statistical kind, but we will concentrate on the parameter spaces $\Lambda^a; a \in \mathcal{A}$. The whole set of parameters of the experiments is given by points in the big space

$$\Pi = \bigtimes_a \Lambda^a,$$

a Cartesian product. If all parameter spaces have the same structure Λ, this can be considered to be the set of functions from $\mathcal{A}$ to Λ.

Let there be defined a transformation group G on Π.

EXAMPLE 7 (Compare Example 5). Let $\pi = (\lambda^1, \lambda^2)$, where λ^1 and λ^2 are the expected lifelengths of a single patient under two mutually exclusive treatments. Let G be the joint set of time scale transformations together with the exchange $\lambda^1 \leftrightarrow \lambda^2$.

EXAMPLE 8. Consider again the electron spin. Let $\pi = (\lambda^a; a \in \mathcal{A})$, where λ^a is the spin component ± 1 of a perfect measurement in the direction a of an electron. Let G be the group generated by the transformations:

(i) Inversions: $\lambda^a \mapsto -\lambda^a$.

(ii) Rotations of experiments: If $a \mapsto ao$ under a rotation o, replace each λ^a with λ^{ao}. This gives a permutation within the cartesian product.

Note in general that the points of Π make sense mathematically, but not directly physically, hence it does not make sense in a physical context to give values to the individual points of this space. The space Π will hence not be called a state space.

So what operations are meaningful with the spaces Π? I have mentioned group operations. One can also adjoin such spaces corresponding to different systems, and adjoin π with some other parameter. Finally, one can look at subspaces.

Assume that the experiments are related in some way. Then it may be reasonable to try to reduce the space Π. The purpose of this reduction may be to achieve parsimony. This should not be thought of as an approximation, however, but may be a result of some physical theory. Note that theories are formulated not in terms of observations, but in terms of parameters, the theoretical language behind observations.

Let Π be reduced to a subspace Ψ with the property:

PROPERTY 1. Ψ is an orbit, that is, a set of the form $\{\pi : \pi = \pi_0 g : g \in G\}$, or a set of orbits for the group G. Use the notation G also for this group acting on Ψ.

This is a necessary condition in order that G should be a transformation group on the reduced space. It is also consistent with the discussion elsewhere in this paper. In [34] there are given several examples of model reductions connected to single experiments where the reduced space is an orbit or a set of orbits of an associated transformation group.

It is natural in certain situations to demand also:

PROPERTY 2. Each section $\{\pi \in \Pi : \lambda^a(\pi) = \lambda_0\}$ has a nonzero intersection with Ψ for a set of specified values λ_0.

In fact, this will always be true for some values λ_0. In a future publication we hope to use this fact together with some group representation theory to discuss quantization itself.

Let now the model reduction be associated with some function ϕ on Π which is one-to-one on the subset Ψ and undefined elsewhere. It follows then from Property 1 that the group G is well defined on the range of ϕ.

DEFINITION 2. If such a function exists, call $\Phi = \phi(\Psi)$ the total parameter space. Any function with the above properties is called a total parameter.

A total parameter ϕ can in principle be replaced by any other total parameter in one-to-one correspondence with ϕ. But it is important to have a simple representation.

If Property 2 holds, then each λ^a can be regarded as a function on Φ.

EXAMPLE 8 (continued). Restrict Π to the subset Ψ, the set of all π such that there exists a vector ϕ that gives each λ^a equal to $\text{sign}(a \cdot \phi)$. Let $\phi(\pi)$ be this direction normed as a unit vector.

– Taken as a unit vector $\phi(\pi)$ is a unique function of π.

PROOF. Suppose that there is a π which corresponds to two different unit vectors ϕ_1 and ϕ_2. Then $a = \phi_1 - \phi_2$, normalized gives $\lambda^a = +1$ corresponding to ϕ_1 and $\lambda^a = -1$ corresponding to ϕ_2, a contradiction. □

– The set Ψ is an orbit of G.

PROOF. It is easy to see that Ψ is closed under inversions and rotations. □

– All sections $\{\pi : \lambda^a(\pi) = \pm 1\}$ have nonzero intersections with Ψ.

PROOF. Obvious. □

From this, we are back to the situation discussed in Section 5.

7. Experiment, model reduction and group representation. Now let the experimentalist have the choice between different experiments $a \in \mathcal{A}$ on the same unit(s), where the experiment a consists of measuring some y^a, with $y^a = y^a(\omega)$ being a function on some sample space S, and where the measurement process is modeled with a parameter λ^a. This parameter is a part of the model description of the units, and all the model parameters may be seen as functions $\lambda^a(\phi)$ of a total parameter ϕ.

We use a common sample space S for all experiments a, since this space can be imagined in terms of a common measurement apparatus or some set of apparatus. Specifically we assume:

ASSUMPTION 3. There is a common sample space S. The reduced model probability measures P^{λ^a} are jointly dominated, that is, absolutely continuous with respect to a fixed probability measure P on the sample space S.

In the electron case this simply means that one in principle can assume that the same or the same kind of Stern–Gerlach apparatus can be used for every measurement. The measure P can be assumed to be Bernoulli$(1/2)$.

In the previous section, a global model reduction was introduced by reducing the large space Π to one or a few orbits of the basic group G. As in the electron spin example, it may also be natural or necessary to reduce the original parameter θ^a to a new parameter λ^a. All such model reduction is done by selecting one or a few orbits of the relevant group G^a.

The most important theoretical argument for model reduction associated with orbits of the group is the following: All models should have a parameter space which is invariant under the group. For the reduced model this is only possible when the parameter space in question is composed of orbits of the relevant group.

Here is another argument: The Pitman estimator is equal to the Bayes estimator under right invariant prior, and this estimator is important in many applications. In order that this shall make sense for the reduced model, the parameter space of this reduced model must be constructed from orbits of the parameter group actions.

A further discussion of model reduction under symmetry in statistics and in quantum mechanics will be given elsewhere, and we then also hope to relate the discussion to the concept of group representation, which is very useful in quantum theory.

Generally (see also Appendix A.2), a group representation is a class of operators $\{U(g);\, g \in G\}$ on a vector space space V, where G is a group, such that the operators satisfy the property $U(gh) = U(g)U(h)$. This gives a group of operators homomorphic to the group G, and, as the name says, it is used to represent the group in a specific way. There is a large mathematical literature on group representations.

Specifically, the regular representation $U(G)$ on $\mathrm{L}^2(\Phi, \nu)$, where ν is a right invariant measure for the basic group G, is given by

$$U(g)f(\phi) = f(\phi g). \tag{6}$$

Explicitly, this implies that $U(G)$ is a group of linear operators acting on $\mathrm{L}^2(\Phi, \nu)$. The group property of $U(G)$ is well known and easily verified. The same formula (6) is valid for any subspace V of $\mathrm{L}^2(\Phi, \nu)$ which is invariant under the group of operators $U(G)$, that is, such that $U(g)f \in V$ when $f \in V$ and $g \in G$.

We will also consider group representation spaces of the group G^a acting on ϕ. Let λ^a be a permissible function of ϕ. Then

$$V_\lambda^a = \{f \in \mathrm{L}^2(\Phi, \nu) : f(\phi) = \tilde{f}(\lambda^a(\phi))\}$$

is an invariant subspace of $\mathrm{L}^2(\Phi, \nu)$ under the regular representation $U(G^a)$.

8. Experimental basis and the Hilbert space of a single experiment. Up to now the discussion has been largely in terms of models and abstract parameters. Now we introduce observations in more detail. We have already stressed that in a given situation we have a choice between different experiments/questions a. In this section we give a general discussion fixing this experiment, and hence fixing the parametric function $\lambda(\phi)$. Given a measurement instrument, this will lead to a statistical model P^λ.

In this section we will need to introduce some statistical concepts; for a more thorough treatment, see, for example, [42].

We use the ordinary concept of sufficiency, repeated for convenience:

DEFINITION 3. A random variable $t = t(\omega)$; $\omega \in S$ connected to a model P^λ is called sufficient if the conditional distribution of each other variable y, given t, is independent of the parameter λ.

A sufficient statistic t is minimal if all other sufficient statistics are functions of t. It is complete if

$$E^\lambda(h(t)) = 0 \qquad \text{for all } \lambda \text{ implies } h(t) \equiv 0. \tag{7}$$

It is well known that a minimal sufficient statistic always exists and is unique except for invertible transformations, and that every complete sufficient statistic is minimal. If the statistical model has a density belonging to an exponential class

$$b(y)d(\lambda)e^{c(\lambda)'t(y)},$$

and if $c(\Lambda) = \{c(\lambda) : \lambda \in \Lambda\}$ contains some open set, then the statistic t is complete sufficient.

Recall that a function $\xi(\lambda)$ is called unbiasedly estimable if $E^\lambda(y) = \xi(\lambda)$ for some y. Given a complete sufficient statistic t, every unbiasedly estimable function $\xi(\lambda)$ has one and only one unbiased estimator that is a function of t. This is the unique unbiased estimator with minimum risk under weak conditions [42]. Thus complete sufficiency leads to efficient estimation.

ASSUMPTION 4. For each $a \in \mathcal{A}$ the experiment can be chosen in such a way that there is a complete sufficient statistic t^a under the model P^{λ^a}.

For the rest of this section we fix such an experiment and drop the index a. We write D for G^a, which will be a fixed group on the common sample space S, but also acts on the selected parameter space.

DEFINITION 4. The Hilbert space $\mathbf{K}$ is defined as the set of all functions $h(t)$ such that $h(t) \in L^2(S, P)$ and $f(\phi) = E^{\lambda(\phi)}(h(t)) \in L^2(\Phi, \nu)$.

In this definition the function h is assumed to be complex-valued. It is easy to see that (7) holds for complex functions if and only if it holds for real-valued functions.

A sufficient condition for $f \in L^2(\Phi, \nu)$ is that $\int E^{\lambda(\phi)}(|h(t)|^2)\nu(d\phi) < \infty$. Since it is defined as a closed subspace of a Hilbert space, the Hilbert space property of $\mathbf{K}$ is seen to hold.

Let then the group D be acting upon the sample space S, on the parameter space Λ and on the total parameter space Φ. Recall the brief discussion of group representations in Section 7. In particular, recall the definition of the space V_λ, an invariant space under the regular representation of the group D on $L^2(\Phi, \nu)$.

PROPOSITION 1. *Each space* $\mathbf{K}$ *is an invariant space for the regular representation of the observational group* D *on* $\mathrm{L}^2(S, P)$, *that is, under* $U(g)h(t) = h(tg)$; $g \in D$.

PROOF. If t is sufficient under the model P^λ, and D is the group acting on the sample space, then tg given by $(tg)(\omega) = t(\omega g)$ is sufficient for all $g \in D$. This is proved by a simple exercise using (2). Also, if t is complete, then tg must be complete; hence the two must be equivalent. The norm conditions are easy to verify. Therefore $\mathbf{K}$ is invariant under D. □

Consider now the operator A from $\mathbf{K}$ to $V_\lambda \subset \mathrm{L}^2(\Phi, \nu)$ defined by

$$(Ay)(\lambda(\phi)) = \int y(\omega)\mathrm{P}^{\lambda(\phi)}(d\omega) = \mathrm{E}^{\lambda(\phi)}(y), \tag{8}$$

using again the (reduced) model $\mathrm{P}^\lambda(d\omega)$ corresponding to the experiment a. In the following it will be important to use $\mathbf{K}$ to construct a Hilbert space related to the parameter space.

DEFINITION 5. Define the space $\mathbf{L}$ by $\mathbf{L} = A\mathbf{K}$.

By the definition of a complete sufficient statistic, the operator A will have a trivial kernel as a mapping from $\mathbf{K}$ onto $A\mathbf{K}$. Hence this mapping is one-to-one. It is also continuous and has a continuous inverse. (See below.) Hence $\mathbf{L}$ is a closed subspace of $\mathrm{L}^2(\Phi, \nu)$, and therefore a Hilbert space. Note also that $\mathbf{L}$ is the space in $\mathrm{L}^2(\Phi, \nu)$ of unbiasedly estimable functions with estimators in $\mathrm{L}^2(S, \mathrm{P})$. It is in general included in the space V_λ of all functions of the parameter λ.

PROPOSITION 2. *The space* $\mathbf{L}$ *is an invariant subspace of* $\mathrm{L}^2(\Phi, \nu)$ *for the regular representation of the group* D *on* $\mathrm{L}^2(\Phi, \nu)$.

PROOF. Assume that $\xi(\lambda) = \mathrm{E}^\lambda(y)$ is unbiasedly estimable. Then also $\eta(\lambda) = \xi(\lambda g) = \mathrm{E}^{\lambda g}(y) = \mathrm{E}^\lambda(yg)$ is unbiasedly estimable, so $\mathbf{L}$ is an invariant space under the regular representation $\overline{U}$ of D, defined by $\overline{U}(g)\tilde{f}(\lambda) = \tilde{f}(\lambda g)$. □

A main result is now:

THEOREM 1. *The spaces* $\mathbf{K} \subset \mathrm{L}^2(S, \mathrm{P})$ *and* $\mathbf{L} \subset \mathrm{L}^2(\Phi, \nu)$ *are unitarily related. Also, the regular representations of the group* D *properly defined on these spaces are unitarily related.*

PROOF. We will show that the mapping A can be replaced by a unitary map in the relation $\mathbf{L} = A\mathbf{K}$.

Recall that the connection from the observation group to the parameter group D is given from the model by

$$\mathrm{P}^{\lambda g}(B) = \mathrm{P}^{\lambda}(Bg^{-1}); \qquad g \in D. \tag{9}$$

Using the definition (8) and the connection (9), we find the following relationships. We assume that the random variable $y(\cdot)$ belongs to $\mathbf{K} \subset \mathrm{L}^2(S, \mathrm{P})$ and that $\overline{U}$ is chosen as a representation on the invariant space $\mathbf{L}$. Then

$$\begin{aligned} \overline{U}(g)Ay(\lambda) &= \int y(\omega)\mathrm{P}^{\lambda g}(d\omega) \\ &= \int y(\omega)\mathrm{P}^{\lambda}(d\omega g^{-1}) \\ &= \int y(\omega g)\mathrm{P}^{\lambda}(d\omega) = A\dot{U}(g)y(\lambda), \end{aligned} \tag{10}$$

where $\dot{U}$ is the representation on $\mathbf{K}$ given by $\dot{U}y(\omega) = y(\omega g)$, that is, the regular representation on $\mathrm{L}^2(S, \mathrm{P})$ restricted to this space.

Thus $\overline{U}(g)A = A\dot{U}(g)$ on $\mathbf{K}$.

Hence

$$U(g) = \overline{U}(g) = A\dot{U}(g)A^{-1}; \qquad g \in D.$$

Recall that the action of D on Λ is defined by $(\lambda g)(\phi) = \lambda(\phi g)$, and that $\overline{U}(g) = U(g)$ on V_λ. Here $U(g)f(\phi) = f(\phi g)$ when $f \in V_\lambda$ and $g \in D$.

By Naimark and Štern ([47], page 48), if two representations of a group are equivalent, they are unitarily equivalent. (The result there is formulated for the finite-dimensional case, but the proof is valid in general.) Hence for some unitary C we have

$$\overline{U}(g) = C\dot{U}(g)C^{\dagger}. \tag{11}$$

Since the unitary operators in this proof are defined on $\mathbf{K}$ and $\mathbf{L}$, respectively, it follows that these spaces are related by $\mathbf{L} = C\mathbf{K}$.

Definition 4 may also be coupled to the operator A and to an arbitrary Hilbert space $\mathbf{K}'$ of sufficient statistics, which for instance may be the whole space $\mathrm{L}^2(S, \mathrm{P})$. First let

$$\mathbf{M} = \{y \in \mathbf{K}' : \mathrm{E}^{\lambda} y = 0 \text{ for all } \lambda\}. \tag{12}$$

Then $\mathbf{K}$ may be considered as the factor space $\mathbf{K}'/\mathbf{M}$, that is, the equivalence classes of the old $\mathbf{K}'$ with respect to the linear subspace $\mathbf{M}$ (cf. [47], I.2.10IV).

Here is a proof of this fact: Let $\xi \in A\mathbf{K}'$, such that $\xi(\lambda) = \mathrm{E}^{\lambda}(y)$ for some $y \in \mathbf{K}'$. Then y is an unbiased estimator of the function $\xi(\lambda)$. By Lehmann and Casella ([42], Lemma 1.10), $\xi(\lambda)$ has one and only one unbiased estimator which is a function $h(t)$ of t. Then every unbiased estimator of $\xi(\lambda)$ is of the form $y = h(t) + x$, where $x \in \mathbf{M}$; this constitutes an equivalence class. On the other hand, every $h(t)$ can be taken as such a y. □

9. The parametric Hilbert space of a selected experiment. Return to the situation where one selects an experiment a among a class of experiments $\mathcal{A}$. Corresponding to this choice we now have a parametric Hilbert space $\mathbf{L}^a$ and an observational Hilbert space $\mathbf{K}^a$. This models a certain measurement apparatus, and in many cases one would expect that the parameter space, and hence the space $\mathbf{L}^a$, will represent some intrinsic property of nature, and therefore be independent of the choice of measurement apparatus.

However, to cover all cases, and to get a unique definition, we will define the parametric Hilbert space connected to question $a \in \mathcal{A}$ through a special choice of measurement apparatus.

DEFINITION 6. (i) Before any experiment is done, λ^a is just the name of some parameter. After the experiment, we have some estimate $\hat{\lambda}^a$ of this parameter. The experiment is called perfect if experimental error can be neglected, so that $\hat{\lambda}^a$ is the realized value of the parameter in this experiment.

(ii) Define the Hilbert space $\mathbf{H}^a$ connected to question $a \in \mathcal{A}$ as the space $\mathbf{L}^a$ for a perfect experiment with parameter λ^a.

One remark is that even in the perfect case it may be important to distinguish between a parameter and its realized value. In the electron spin case, a perfect measurement means simply that the Stern–Gerlach apparatus functions without any error.

We will see later that under natural assumptions a nonperfect experiment may be related to the same space $\mathbf{H}^a$.

PROPOSITION 3. *With the above definitions the space* $\mathbf{H}^a$ *is just the space* V_λ^a *of functions* $\tilde{f}$ *of* $\lambda^a(\cdot)$ *such that* $f(\phi) = \tilde{f}(\lambda^a(\phi)) \in \mathrm{L}^2(\Phi, \nu)$.

PROOF. If $\tilde{f}$ is arbitrary and the experiment is perfect, then $\int |\tilde{f}(\hat{\lambda}^a)|^2 \, d\mathrm{P} = |\tilde{f}(\lambda^a(\phi))|^2$ is finite. This then follows from Definitions 4, 5 and 6. □

As an example, in the electron spin case, the total parameter ϕ is the spin vector and $\mathrm{L}^2(\Phi, \nu)$ corresponds to a measure ν which is uniform on any shell, and where any measure on $|\phi|$ can be used. Let $\lambda^a(\phi) = \mathrm{sign}(a \cdot \phi)$. Then $\mathbf{H}^a$ is simply the space of functions of $\lambda^a(\phi)$, a two-dimensional space. Specifically, $\mathbf{H}^a$ is the space of functions of ϕ which are constant on the two half-spaces separated by a plane through the origin perpendicular to the vector a.

All this indicates that our discussion could have been simplified by concentrating on the parameter space. Our reasons for nevertheless giving a full treatment involving the sample space have been given in the Introduction.

10. The quantum-theoretical Hilbert space. Our task in this section is to tie the spaces $\mathbf{H}^a$ together. Our essential point of departure here is that the parameter spaces of the different experiments have a similar structure. Then it is not unreasonable to assume that they can be transformed over to each other by some element of the basic group G. This will not give the most general case of the quantum-mechanical formalism, but gives a treatment which includes qubits, higher spins, several particles and the most important cases of entanglement, a phenomenon which is much discussed in the quantum-mechanical literature.

ASSUMPTION 5. For each pair of experiments $a, b \in \mathcal{A}$ there is an element g_{ab} of the basic group G which induces a correspondence between the respective parameters,

$$\lambda^b = \lambda^a g_{ab} \quad \text{or} \quad \lambda^b(\phi) = \lambda^a(\phi g_{ab}). \tag{13}$$

This assumption is fairly strong, and it makes the task of connecting the spaces really simple. On the other hand, it seems to be satisfied in concrete cases. The same assumption will be needed in Section 12.

In the electron spin case Φ was a space of vectors, and G was the rotation group together with changes of scale. Then (13) holds if g_{ab} is any rotation transforming a to b.

If (13) holds for transformations on some component spaces, it also holds for the cartesian product of these spaces when the relevant cartesian product of groups is used.

Another interesting relation is connected to Assumption 5 in the following way: (13) implies that one ought to have $\lambda^b g^b = \lambda^a g^a g_{ab}$ for some $g^b \in G^b$. Hence it follows that $\lambda^a g_{ab} g^b = \lambda^a g^a g_{ab}$, so g^a and $g_{ab} g^b g_{ab}^{-1}$ act in the same way on λ^a. One can give many examples of group transformations where $g^a = g_{ab} g^b g_{ab}^{-1}$ holds in general, giving an isomorphism between the groups G^a and G^b.

Assumption 5 will be crucial in connecting the Hilbert spaces $\mathbf{H}^a$ for the different experiments. First, from the construction of the Hilbert spaces, $\mathbf{H}^a$ is a space of functions of $\lambda^a(\phi)$, and $\mathbf{H}^b$ is a space of functions of $\lambda^b(\phi)$. Furthermore, the spaces are constructed in the same way. Specifically, if $f^a(\phi) = \tilde{f}(\lambda^a(\phi))$ and $f^b(\phi) = \tilde{f}(\lambda^b(\phi))$, then by (13) we have

$$f^b(\phi) = f^a(\phi g_{ab}) = U(g_{ab}) f^a(\phi). \tag{14}$$

This implies:

THEOREM 2. (a) *There is a connection between the spaces* $\mathbf{H}^a$ *and* $\mathbf{H}^b$ *given by*

$$\mathbf{H}^b = U(g_{ab}) \mathbf{H}^a. \tag{15}$$

(b) *There are a Hilbert space* $\mathbf{H}$ *and for each* $a \in \mathcal{A}$ *a unitary transformation* E^a *such that* $\mathbf{H}^a = E^a\mathbf{H}$.

(c) *For any experiment satisfying Assumption* 4 *and such that the parametric Hilbert space* $\mathbf{L}^a$ *is equal to* $\mathbf{H}^a$, *there are unitary transformations* F^a *such that the observational Hilbert spaces satisfy* $\mathbf{K}^a = F^a\mathbf{H}$.

PROOF. (a) Proved above.

(b) Obvious from (15). The space $\mathbf{H}$ can be chosen as any fixed $\mathbf{H}^c$.

(c) From (a) and Theorem 1. □

Now introduce:

ASSUMPTION 6. The group G is the smallest group containing all the subgroups G^a.

From this we get:

THEOREM 3. $\mathbf{H}$ *is an invariant space for some abstract representation* W *of the whole group* G.

PROOF. It follows from Proposition 2 that $\mathbf{H}^a$ is an invariant space for the group G^a.

This can now be extended. Observe first that

$$W(g_1g_2g_3) = E^{a\dagger}U^a(g_1)E^aE^{b\dagger}U^b(g_2)E^bE^{c\dagger}U^c(g_3)E^c \tag{16}$$

gives a representation on $\mathbf{H}$ of the set of elements in G that can be written as a product $g_1g_2g_3$ with $g_1 \in G^a$, $g_2 \in G^b$ and $g_3 \in G^c$.

Continuing in this way, using Assumption 6, implying that the group G is generated by $\{G^a; a \in A\}$, we are able to construct a representation W of the whole group G on the space $\mathbf{H}$. In particular, one is able to take $\mathbf{H}$ as an invariant space for a representation of this group. □

As an example, the two-dimensional Hilbert space of a particle with spin is always an (irreducible) invariant space for the rotation group. This determines to a large extent $\mathbf{H}$, if we in addition assume $\mathbf{H}$ to be as small as possible. In general, the requirement that $\mathbf{H}$ should be a representation space for G may put a constraint on the dimension of $\mathbf{H}$.

The construction above gives a concrete representation of the quantum-mechanical Hilbert space. Since all Hilbert spaces of the same dimension are unitarily equivalent, other representations—or just an abstract representation—may be used in practice. This is sufficient to give the Born formula as proved below, and through this the ordinary quantum formalism. But the concrete representation facilitates interpretation.

For our construction, the unitary connection (15) between the Hilbert spaces for single experiments is the most important premise. This can easily also be related to the space-time issue. Say, let ξ be the theoretical position, π the theoretical momentum, and let $\mathbf{H}^1$ and $\mathbf{H}^2$ be the corresponding L^2-spaces of parametric functions. Then we can consider the unitary transformation from $\mathbf{H}^1$ to $\mathbf{H}^2$ given for some constant $\hbar$ by

$$f_2(\pi) = \frac{1}{\sqrt{2 \cdot 3.14\hbar}} \int e^{\pi \xi i/\hbar} f_1(\xi) \overline{v}(d\xi),$$

and in this way introduce a common Hilbert space. This can be connected to the relevant group, namely the group of space translations together with the Lorentz group, and it can be argued that $\hbar$ should be a universal constant. This will be further discussed in [36]. From physics it is known that $\hbar = 1.055 \cdot 10^{-34} Js$.

11. Operators and states. So, by what has just been proved, for each a the Hilbert space $\mathbf{H}^a$ of unbiasedly estimable functions of λ^a can be put in unitary correspondence with a common Hilbert space $\mathbf{H}$. From now on we shall make an assumption which is common in elementary quantum mechanics, but which is very restrictive from a statistical point of view.

ASSUMPTION 7. Each reduced (maximal) parameter λ^a takes only a finite or denumerably infinite number of values λ_k^a.

LEMMA 2. *These values can be arranged such that each $\lambda_k^a = \lambda_k$ is the same for all a $(k = 1, 2, \dots)$.*

PROOF. By Assumption 5

$$\{\phi : \lambda^b(\phi) = \lambda_k^b\} = \{\phi : \lambda^a(\phi g_{ab}) = \lambda_k^b\} = \{\phi : \lambda^a(\phi) = \lambda_k^b\} g_{ba}.$$

The sets in brackets on the left-hand side here are disjoint with union Φ. But then the sets on the right-hand side are disjoint with union $\Phi g_{ba} = \Phi$, and this implies that $\{\lambda_k^b\}$ gives all possible values of λ^a. □

In spite of Lemma 2, since in any statistical model a parameter can be changed to any one-to-one function of it, we may sometimes use the notation λ_k^a in order to have the most general treatment.

In the finite case Assumption 7 implies that G^a, as acting upon λ^a, is a group of permutations, and that the corresponding invariant measure is the counting measure.

Recall that the Hilbert space $\mathbf{H}$ is chosen as one fixed space $\mathbf{H}^c$. In this space let $f_j^c(\phi)$ be defined as the trivial function which equals 1 when $\lambda^c(\phi) = \lambda_j$, otherwise 0. These are eigenfunctions of the operator S^c defined by $S^c f(\phi) =$

$\lambda^c(\phi) f(\phi)$. In a different space $\mathbf{H}^a$ these functions correspond to $f_j^a(\phi) = f_j^c(\phi g_{ca}) = U(g_{ca}) f_j^c(\phi)$. Now define vectors in $\mathbf{H}$ by

$$v_j^a = W(g_{ca}) f_j^c, \tag{17}$$

where W is the representation defined by (16). These are eigenvectors of the self-adjoint operator $T^a = W(g_{ca}) S^c W(g_{ac})$ with eigenvalues λ_j.

An eigenvector v_j^a represents the statement that the parameter λ^a has been measured with a perfect measurement that has given the value λ_j.

In general it is not true that all unit vectors of $\mathbf{H}$ can be given such an interpretation. Among other things one has to take into account what are called superselection rules: For an absolutely conserved quantity μ, the linear combinations of eigenvectors corresponding to different eigenvalues of the operator associated to μ are not possible state vectors. Superselection rules are well known among physicists, but they are not always stressed in textbooks in quantum mechanics.

In [35], Theorem 6 and Lemma 2, we proved the following under the assumption that the unitary group generated by $\{W(g)\}$ and the phase factors is transitive on the component spaces $\mathbf{H}_r$ below:

THEOREM 4. *There is a decomposition of* $\mathbf{H}$ *of the form* $\mathbf{H}_1 \oplus \mathbf{H}_2 \oplus \cdots$, *where each* $\mathbf{H}_r$ *is an irreducible invariant space under the group* G. *Assume that the unitary group generated by* $\{W(g)\}$ *and the phase factors* $e^{i\alpha}$ *is transitive on each component* $\mathbf{H}_r$. *Then all unit vectors of each* $\mathbf{H}_r$ *are unitarily equivalent to some* f_i^b, *an indicator of an event* $\lambda^b = \lambda_i^b$. *On the other hand, if two such indicators, say* f_i^b *and* f_j^c, *are unitarily equivalent to the same* $v \in \mathbf{H}_r$, *and the relevant unitary transformation can be considered as a subrepresentation of the regular representation, then there is a one-to-one function* F *such that* $\lambda^c = F(\lambda^b)$ *and* $\lambda_j^c = F(\lambda_i^b)$.

In simple terms a state is characterized by the fact that a (maximal) perfect measurement is performed, and this has led to some value of the corresponding maximal parameter. Concretely: A perfect experiment $a \in \mathcal{A}$ has led us to consider the Hilbert space $\mathbf{H}^a$, and the result $\lambda^a = \lambda_k$ is exactly characterized by the indicator function f_k^a. Translated to the $\mathbf{H}$-space, the state given by the information $\lambda^a = \lambda_k$ is then characterized by the vector v_k^a.

COROLLARY 1. *Under the Assumptions of Theorem* 4, *all unit vectors of each irreducible space* $\mathbf{H}_r$ *can be taken as state vectors with the following interpretation*: *A question* $a \in \mathcal{A}$ (*or more precisely*: *What is the value of* λ^a?) *has been asked, and the answer is given by the realized value* $\lambda^a = \lambda_k$, *or in other words*: *A perfect measurement corresponding to the reduced parameter* λ^a *has been performed, and the result is* $\hat{\lambda}^a = \lambda_k$.

This is consistent with the well-known quantum-mechanical interpretation of a state vector. In our treatment, this interpretation of a state as a question-answer pair is crucial.

The operator T^a may be written

$$T^a = \sum_k \lambda_k v_k^a v_k^{a\dagger}. \tag{18}$$

These operators are self-adjoint, and they satisfy the trivial relation $v_k^{a\dagger} T^a v_k^a = \lambda_k$.

Using the results of this section to construct the joint state vector for a system consisting of several partial systems, with symmetries only within the partial systems, one follows the recipe $v_{i_1 i_2 i_3}^{a_1 a_2 a_3} = v_{i_1}^{a_1} \otimes v_{i_2}^{a_2} \otimes v_{i_3}^{a_3}$, where it is assumed that system k is in state $\lambda^{a_k} = \lambda_{i_k}$ for $k = 1, 2, 3$. By time development under interaction, as described by the Schrödinger equation, or by other means, other, entangled, multicomponent states will occur. This will be further discussed in [36] and elsewhere.

12. Born's formula. We have now obtained a statistical interpretation of the quantum-mechanical Hilbert space: Under the assumptions of Theorem 4 all vectors in that space can be equivalently characterized as question-answer pairs and, furthermore, the Hilbert space is invariant under a suitable representation of the basic group G.

To complete the derivation of the formalism of quantum mechanics from the statistical parameter approach, the most important task left is to arrive at the Born formula, which gives the probability of transition from one state to another. The fact that such a formula exists is amazing, and must be seen as a result of the symmetry of the situation together with the limitation imposed by the Hilbert space. Even though I use a different approach, my own result is related to recent attempts to link the formula to general decision theory: An interesting development which goes in this direction was recently initiated by Deutsch [22]. The approach of Deutsch has been criticized by Finkelstein [26], by Barnum et al. [8] and by Gill [28], who gave a constructive set of arguments using three reasonable assumptions.

In this section I will concentrate on the case with one irreducible component in the Hilbert space, that is, I will neglect superselection rules. This is really no limitation, since transitions between different components are impossible.

What I am going to prove is a result connecting two different perfect experiments in the same system. Assume that we know from the first perfect experiment that $\lambda^a = \lambda_k$. Next assume that we perform another perfect experiment $b \in \mathcal{A}$. In both cases, the notion of perfect measurement means that measurement error can be neglected. More realistic experiments are treated in Theorems 7 and 8 below. In the perfect case it turns out that we can find a formula for

$$P(\hat{\lambda}^b = \lambda_i | \lambda^a = \lambda_k) = P(\lambda^b = \lambda_i | \lambda^a = \lambda_k)$$

which depends only upon the state vectors v_k^a and v_i^b.

This formula has a large number of important consequences in quantum mechanics and, as already said, it can be argued for in different ways. I will prove it from the following:

ASSUMPTION 8. (i) The transition probabilities exist in the sense that the probabilities above do not depend upon anything else.

(ii) The transition probability from $\lambda^a = \lambda_k$ in the first perfect experiment to $\lambda^a = \lambda_k$ in the second perfect experiment is 1.

(iii) For all a, b, c we have that $\mu(\phi) = \lambda^a(\phi g_{bc})$ is a valid experimental parameter.

(iv) For all a, b, c, i, k we have

$$P\big(\lambda^b(\phi) = \lambda_i | \lambda^a(\phi) = \lambda_k\big) = P\big(\lambda^b(\phi g_{bc}) = \lambda_i | \lambda^a(\phi g_{bc}) = \lambda_k\big).$$

REMARK. (1) Assumption 8 is an important instance where the symmetry group setting is used in an essential way to derive a result that does not itself involve the symmetry group G.

(2) Crucial assumptions will also be Assumption 3, that a common sample space can be used in all experiments, and Assumption 5.

(3) We have $\lambda^b(\phi g_{bc}) = \lambda^c(\phi)$, so three experimental parameters are included in Assumption 8.

(4) In the proof below we transform a single experiment by some element of G. The use of the transformation g on t is then justified by:

LEMMA 3. *Consider the homomorphism from the sample space transformations to the parameter space transformations given by*

$$P^{\lambda g}(y \in B) = P^{\lambda}(y \in Bg^{-1}) = P^{\lambda}(yg \in B).$$

When $y = t$ is a complete sufficient statistic, this is an isomorphism, so that one can let g be defined on the parameter space to begin with.

PROOF. Assume that there are group elements g_1 and g_2 of two different sample group transformations such that

$$P^{\lambda g}(t \in B) = P^{\lambda}(tg_1 \in B) = P^{\lambda}(tg_2 \in B).$$

Then for all λ and for all functions h we have

$$E^{\lambda}(h(tg_1)) = E^{\lambda}(h(tg_2)).$$

By the definition of a complete sufficient statistic it then follows that $tg_1 = tg_2$. □

Born's formula is given by:

THEOREM 5. *Under the assumptions above and the assumptions of Theorem* 4 *the transition formula is as follows*:

$$\mathrm{P}(\lambda^b = \lambda_i | \lambda^a = \lambda_k) = |v_k^{a\dagger} v_i^b|^2. \tag{19}$$

The proof will depend upon a recent variant [17, 18] of a well-known mathematical result given by Gleason [30]. One advantage of this recent variant is that it also is valid for dimension 2, when the ordinary Gleason theorem fails.

THE BUSCH–GLEASON THEOREM. *Consider any Hilbert space* $\mathbf{H}$. *Define the set of* effects *as the set of operators on this Hilbert space with eigenvalues in the interval* $[0, 1]$. *Assume that there is a generalized probability measure* π *on these effects, that is, a set function satisfying*

$$\begin{aligned} \pi(E) &\geq 0 \qquad \text{for all } E, \\ \pi(I) &= 1, \\ \sum_i \pi(E_i) &= \pi(E) \qquad \text{for effects } E_i \text{ with sum } E. \end{aligned}$$

Then π *is necessarily of the form* $\pi(E) = \mathrm{tr}(\rho E)$ *for some positive, self-adjoint, trace* 1 *operator* ρ.

The effects involved in the Busch–Gleason theorem turn out to have a rather straightforward statistical interpretation. Look at an experiment b, corresponding to a parameter λ^b which can take the values λ_i. Let the result of this experiment be given by a discrete complete sufficient statistic t, thus allowing for an experimental error. Let t have a likelihood

$$p_i(t) = P(t | \lambda^b = \lambda_i).$$

The choice of experiment b, the set of possible parameter values $\{\lambda_i\}$ and the result t again constitute a question-and-answer set, but now in a more advanced form. The point is that the answer is uncertain, so that all these elements together with the likelihood function must be included to specify the question-and-answer.

PROPOSITION 4. *Exactly this information, the experiment* b, *the possible answers and the statistic* t *can be recovered from the effect defined by*

$$E = \sum_i p_i(t) v_i^b v_i^{b\dagger}. \tag{20}$$

On the other hand, for fixed t *every effect* E *can be written in the spectral form* (20).

PROOF. This is a spectral decomposition from which the eigenvalues $p_i(t)$ and the eigenvectors v_i^b can be recovered. As discussed before, the eigenvectors correspond to the question-and-answers for the case without measurement errors, and from the likelihood the minimal sufficient observator t can be recovered. The last part is obvious. □

All this was discussed from a slightly different perspective in [35] for the case of a two-dimensional Hilbert space.

Consider now the situation where a quantum system is known to be in a state given by v_k^a, that is, a perfect experiment a has been performed with result $\lambda^a = \lambda_k$. Then make a new experiment b, but let this experiment be nonperfect. We require the probability $\pi(E)$ that the result of the latter experiment shall be t, corresponding to the effect E given by (20). For this situation it is natural to define

$$\pi(E) = \sum_i p_i(t) P(\lambda^b = \lambda_i | \lambda^a = \lambda_k). \tag{21}$$

An important point in our development is that under Assumption 8, this π, when ranging over all the effects E, will be a generalized probability. The crucial result is the following:

PROPOSITION 5. *Under Assumption 8, if E_1, E_2 and $E_1 + E_2$ all are effects, then*

$$\pi(E_1 + E_2) = \pi(E_1) + \pi(E_2).$$

PROOF. Let $E_1 = E$ be given by (20), and let

$$E_2 = \sum_j q_j(t) v_j^c v_j^{c\dagger}$$

for another experiment c with another likelihood q_j.

First we remark that the relations $\pi(rE_1) = r\pi(E_1)$ and $\pi(E_1 + E_2) = \pi(E_1) + \pi(E_2)$ are trivial when E_1, E_2, rE_1 and $E_1 + E_2$ are all effects and all $v_i^c = v_i^b$.

We now turn to the general case. The statistic t may then be assumed to be sufficient and complete with respect to both likelihoods. By Assumption 5 the parameters of the two experiments are connected by a group transformation. Then by imitating the argument in the proof of Lemma 3, a complete sufficient statistic for experiment b can be transformed by an isomorphic group transformation to a complete sufficient statistic for experiment c; hence the complete sufficient statistics for the two experiments may be assumed identical.

Consider the experiment E_3 defined by selecting experiment E_1 with probability $1/2$ and experiment E_2 with probability $1/2$. Since the same measurement apparatus was used in both experiments, one can arrange things in such a way that the person reading t for experiment E_3 does not know which of the experiments E_1 or E_2 was chosen. This arrangement is necessary in order to avoid the

result that the conditionality principle should disturb our argument for this situation; see [3] and the response to these comments. We can regard E_3 as a genuinely new experiment here.

Now use Assumption 5. From this assumption there exists a group element g_{bc} such that $\lambda^c(\phi) = \lambda^b(\phi g_{bc})$. We can, and will, rotate experiment b in such a way that all final state vectors coincide with those of experiment c. Then from Assumption 8, the transition probability to experiment E_2 is the same as if a rotated initial state was chosen and the state vectors v_i^b were chosen, but with a different likelihood $q_i'(t) = q_i(tg_{bc})$.

From this perspective, the experiment E_3 can also be related to the same state vectors, but with a likelihood

$$r_i(t) = \tfrac{1}{2}\big(p_i(t) + q_i'(t)\big). \tag{22}$$

The statistic t will be sufficient relative to this likelihood, but may not be complete or minimal. However, this is not needed for our argument.

This gives

$$\pi(E_3) = \tfrac{1}{2}\pi(E_1) + \tfrac{1}{2}\pi(E_2) \tag{23}$$

for experiments transformed to have the same final states.

We can now transform back so that all three experiments have the same initial state. Since experiment E_3 in the rotated form had the same question-and-answer form as the other two experiments, only with a different likelihood (22), this experiment must also correspond to some effect. Then from (23), Assumption 8 and the fact that the same sample space is used for all three experiments both in the original and in the rotated version, the transition probability must satisfy

$$\pi(E_3) = \pi\big(\tfrac{1}{2}(E_1 + E_2)\big) = \tfrac{1}{2}\pi(E_1) + \tfrac{1}{2}\pi(E_2). \tag{24}$$

The first equality here obviously holds in the rotated case; then it also holds when we rotate back. If $E_1 + E_2$ is an effect, the factor $1/2$ can be removed throughout by suitably redefining the likelihood. □

PROPOSITION 6. *For fixed initial state $\lambda^a = \lambda_k$, the set function defined by (21) from the transition probability will under Assumption 8 be a generalized probability on the final effects.*

PROOF. The additivity property for a finite number of effects follows by induction from Proposition 5. The argument of Proposition 5 can also be used with a countable set of effects, so the additivity property for generalized effects follows for these set functions.

It is obvious that $\pi(E) \geq 0$. The limiting effect I corresponds to an experiment and experimental result with likelihood 1 on each single parameter value, and it is clear that the transition probability to this effect must be 1 from every initial state. □

PROOF OF THEOREM 5. Fix a and k and hence the state v_k^a, interpreted as $\lambda^a = \lambda_k$. Define $q_{a,k}(v) = \pi_{a,k}(E)$ to be equal to the transition probability from v_k^a to the effect $E = vv^\dagger$ for an arbitrary state vector v, assumed to exist in Assumption 8. Generalize to any E by (21). By Proposition 6 the conditions of the Busch–Gleason theorem are satisfied.

By this theorem, for any $v \in \mathbf{H}$, we have $\pi_{a,k}(vv^\dagger) = v^\dagger \rho v$ for some ρ, which is positive, self-adjoint and has trace 1. This implies $\rho = \sum_j c_j u_j u_j^\dagger$ for some orthogonal set of vectors $\{u_j\}$. Self-adjointness implies that each c_j is real-valued, and positivity demands $c_j \geq 0$ for each j. The trace 1 condition implies $\sum_j c_j = 1$.

Inserting this gives $\pi_{a,k}(vv^\dagger) = \sum_j c_j |v^\dagger u_j|^2$. Specialize now to the particular case given by $v = v_k^a$ for some k. For this case one must have $\sum_j c_j |v_k^{a\dagger} u_j|^2 = 1$, and thus

$$\sum_j c_j (1 - |v_k^{a\dagger} u_j|^2) = 0.$$

This implies for each j that either $c_j = 0$ or $|v_k^{a\dagger} u_j| = 1$. Since the last condition implies $u_j = v_k^a$ (modulus an irrelevant phase factor), and this is a condition which only can be true for one j, it follows that $c_j = 0$ for all j other than the one leading to $u_j = v_k^a$, and $c_j = 1$ for this particular j. Summarizing all this, we get $\rho = v_k^a v_k^{a\dagger}$ and Theorem 5 follows. □

A new challenge is of course to investigate to what extent this result, in fact all the results here from Section 11 onward, generalize to the case of parameters taking more than a countable set of values. This will possibly require more advanced mathematical tools, but in that case it also seems quite certain that one can draw on known advanced results from quantum probability.

The results above are valid and have relevance also outside quantum theory. In Section 12.5 of [35] a large-scale example is sketched where, using Born's formula, the prior probability of a second experiment is found, given the result of a first experiment.

By the same proof, Born's formula can be generalized to $P(E|\lambda^a = \lambda_k) = v_k^{a\dagger} E v_k^a$ for an arbitrary final effect E [also Theorem 7(i) below]. This gives a transition probability from any state vector $v_k^a \in \mathbf{H}$.

Recall that $\mathbf{H}$ was originally defined using perfect experiments. Using Born's formula, it can be seen that a large class of experiments take the same Hilbert space as a point of departure.

13. Basic formulae of quantum mechanics and of quantum statistics. Our state concept may now be summarized as follows: To the state $\lambda^a(\cdot) = \lambda_k$ there corresponds the state vector v_k^a, and these vectors determine the transition probabilities as in (19). The probability distribution (19) also implies for perfect experiments:

THEOREM 6. (a) $\mathrm{E}(\hat{\lambda}^b|\lambda^a=\lambda_k)=v_k^{a\dagger}T^b v_k^a$, *where* $T^b=\sum\lambda_j v_j^b v_j^{b\dagger}$.
(b) $\mathrm{E}(f(\hat{\lambda}^b)|\lambda=\lambda_k)=v_k^{a\dagger}f(T^b)v_k^a$, *where* $f(T^b)=\sum f(\lambda_j)v_j^b v_j^{b\dagger}$.

Thus, in ordinary quantum-mechanical terms, the expectation of every observable in any state is given by the familiar formula.

It follows from Theorem 6(a) and from the preceding discussion that the first three rules of Isham ([39], page 71), taken there as a basis for quantum mechanics, are satisfied. The fourth rule, the Schrödinger equation, will be discussed in [36].

Now turn to nonperfect experiments. In ordinary statistics, a measurement is a probability measure $\mathrm{P}^\theta(dy)$ depending upon a parameter θ. Assume now that such a measurement depends upon the parameter λ^b, while the current state is given by $\lambda^a=\lambda_k$. Then as in Theorem 6(b):

THEOREM 7. (a) *Corresponding to the experiment* $b\in\mathcal{A}$ *one can define an operator-valued measure* M *by* $M(dy)=\sum_j \mathrm{P}^{\lambda_j^b}(dy)v_j^b v_j^{b\dagger}$. *Then, given the initial state* $\lambda^a=\lambda_k$, *the probability distribution of the result of experiment* b *is given by* $\mathrm{P}[dy|\lambda^a=\lambda_k]=v_k^{a\dagger}M(dy)v_k^a$.
(b) *These operators satisfy* $M[S]=I$ *for the whole sample space* S, *and furthermore* $\sum M(A_i)=M(A)$ *for any finite or countable sequence of disjoint elements* $\{A_1,A_2,\ldots\}$ *with* $A=\bigcup_i A_i$.

Theorem 7(b) is easily checked directly.

A more general state assumption is a Bayesian one corresponding to this setting. From Theorem 7(a) we easily find:

THEOREM 8. *Let the current state be given by probabilities* $\pi(\lambda_k)$ *for different values of* λ_k. *Then, defining* $\rho=\sum\pi(\lambda_k)v_k^a v_k^{a\dagger}$, *we get* $\mathrm{P}[dy]=\mathrm{tr}[\rho M(dy)]$.

A density operator ρ of such a kind is often used in quantum mechanics; the definition above gives a precise interpretation. In fact, these results are the basis for much of quantum theory, in particular for the quantum-statistical inference in [7]; for a formulation, see also [39].

Note that the density matrix $v_k^a v_k^{a\dagger}$ is equivalent to the pure state v_k^a; similarly, a density matrix $v_j^b v_j^{b\dagger}$ is equivalent to the statement that a perfect measurement giving $\lambda^b=\lambda_j$ has just been performed. By straightforward application of Born's formula one gets:

THEOREM 9. *Assume an initial state* v_k^a, *and assume that a perfect measurement of* λ^b *has been performed without knowing that value. Then this state is described by a density matrix* $\sum_j|v_k^{a\dagger}v_j^b|^2 v_j^b v_j^{b\dagger}$.

This is related to the celebrated and much discussed projection postulate of von Neumann. Writing $P_j = v_j^b v_j^{b\dagger}$ and $\rho = v_k^a v_k^{a\dagger}$ here, the jth term in the last formula can be written $P_j \rho P_j$, which corresponds to a special case of the Dirac–von Neumann formula [57].

In general we have assumed for simplicity in this section that the state vectors are nondegenerate eigenvectors of the corresponding operators, meaning that the parameter λ^a contains all relevant information about the system. This can be generalized, however.

14. The electron revisited. The electron spin is in a way the simplest possible quantum-mechanical system. The Hilbert space **H** is two-dimensional. **H** can fruitfully be regarded as an irreducible representation space of the rotation group. This group can be generated by the matrices σ_x, σ_y and σ_z given by (3).

In the standard quantum-mechanical formulation these three matrices are taken as basic quantities, observables corresponding to the spin in the x-, y- and z-directions, respectively. They have all eigenvalues ± 1, corresponding to the values of these spin observables. The corresponding eigenvectors are then taken as state vectors for these (perfect) measurement results.

As a generalization, the observable $T^a = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$ for a real-valued unit vector $a = (a_x, a_y, a_z)$ also has eigenvalues ± 1, and the eigenvectors have a similar state vector interpretation, corresponding to a spin vector in the direction a.

The transition probabilities between states defined by spin in different directions are found from the Born formula, from which (5) is derived.

A more direct representation of the spin state of an electron was discussed in [35]. In agreement with the alternative representation of quantum mechanics proposed in the present paper, start with a spin vector ϕ and choose a direction a in which the spin component shall be measured. As in Section 6 it is only possible to measure $\lambda^a = \operatorname{sign}(\theta^a) = \operatorname{sign}(\phi \cdot a)$.

Define the 3-vector $u = \lambda^a a$. We claim that this vector gives a unique representation of the spin state of the electron. As has now been stressed repeatedly, we regard the state as a question-and-answer pair. The question (what is the spin component in direction a?) is given by the chosen vector a; the answer is given by λ^a. We can recover both these elements uniquely from the vector u, since a spin component -1 in the direction a is equivalent to a spin component $+1$ in the direction $-a$.

For those knowing some quantum mechanics, the spin state can also be represented by the Bloch sphere or Poincaré sphere matrix

$$\rho = \tfrac{1}{2}(I + u \cdot \sigma),$$

where σ is a formal 3-vector with components given by the 2-by-2 matrices σ_x, σ_y and σ_z above. Obviously, specifying ρ is equivalent to specifying u.

Finally, by conventional quantum mechanics we have $\rho = vv^\dagger$, where v is the ordinary complex two-dimensional Hilbert space state vector, only defined modulo

an arbitrary phase factor for an isolated system. Thus the spin state can be given in any of four different ways:

(1) as a question a together with an answer λ^a;
(2) by the 3-vector u;
(3) by the Bloch sphere matrix ρ;
(4) by the Hilbert space state vector v.

The discussion here can be generalized to other density matrices and further to the effects of Section 12; see [35].

15. Discussion. The treatment of quantum theory given in this paper, is of course still not complete. In [36] two further themes will be discussed from the present point of view, namely the spacetime structure (including transformations related to Planck's constant) and the Schrödinger equation, which gives the time development of the state vector.

Our point of departure here is that both quantum theory and statistical theory deal with prediction, both using probability models of some kind. In our view, what we have arrived at seems to point at a general theory from which both traditional statistical theory and quantum theory emerge as special cases.

A basic premise is that the states of quantum mechanics are related to the parameter space of statistical models. This is an assumption that we have in common with other authors, for instance, Caves, Fuchs and Schack [19]. Hidden variable models for quantum mechanics have been criticized in many contexts. In my view, a hidden (total) *parameter* model is a more flexible and useful concept. A hidden parameter does not in general have a value; in a given situation it can be looked upon more as part of the conceptual framework needed to describe the situation. Only by focusing on some given function of the hidden total parameter can we obtain a concrete parameter on which inference can be made from specific experiments.

We allow the choice between several complementary experiments/questions on the same units. Furthermore, we impose symmetry conditions of the form often done in statistics, but more complicated because of the choice of experiment. Finally, we allow model reduction using the orbit index of the experimental symmetry group. This leads to essential parts of quantum theory, and we find that the set of functions of complete sufficient statistics for the experiments essentially determines the Hilbert space needed for the quantum formulation.

Large parts of the present theory should in principle be valid on a macroscopic scale, too. This leads to the question of whether large-scale situations can be found which can be related in some way to this theory. Some brief examples of related applications can be mentioned.

As an example of partly complementary parameters, look at different sets of orthogonal contrasts in an analysis of variance situation. In randomized experiments

we have a symmetry group on the sample space leading to calculations [4] which in fact have some formal resemblance to those of quantum theory.

With moderately complicated issues for a statistical investigation, it is always wise to elucidate the issue in question from several angles. This may involve performing experiments with different, but related parameters and making inference on different, but related parameters. A related case is conditioning on different ancillary statistics, where a connection to quantum theory was hinted at in [5].

In [33] it is shown that existing chemometric prediction methods can be related to rotational symmetry combined with a model reduction of the kind discussed in this paper.

Thus the theory developed here may seem to have something to say to current applied statistics. These questions must wait for further developments, however.

John von Neumann once said: "In mathematics you don't understand things. You just get used to them" (cited from [11]). By now, generations of physicists and mathematicians have got ten used to the formal Hilbert space approach to quantum theory. And important results have followed from this, both applied and theoretical; some of the latter are mentioned in the Introduction. This gives overwhelming evidence that quantum theory is important and useful. But this in itself does not prove that the ordinary logical foundation for the theory is the simplest one. Our claim is the following: Physics is basically an empirical science, and hence one should work for, instead of a logical foundation suggested by formal mathematics, one that is related to quantitative methodology used by other empirical sciences. This has been some of the motivation behind the present work, and the results obtained seem to confirm that such a link is possible.

APPENDIX

A.1. Further properties of group actions. Adding a group to a statistical model specification is often of interest, and does have consequences; see [42]. First let a group G act on a measurable sample space S. Measurability questions are ignored here, as is common when discussing transformation groups; a full account of this aspect is given in [56].

The *orbits* of a group G acting on S are the sets of the form $\omega_0 g$, where ω_0 is fixed and g runs through G. The orbits of the parameter group induced from G by (2) are defined similarly. Under conditions as given below, each set of orbits can be given an index. The orbit index in the sample space will always have a distribution which depends only upon the orbit index in the parameter space.

Concentrate now on the group G acting on the total parameter space Φ. Similar concepts can be defined for the other group actions discussed above. The group G is also assumed to have a topology.

We assume, as is commonly done, that the group operations $(g_1, g_2) \mapsto g_1 g_2$ and $g \mapsto g^{-1}$ are continuous. Furthermore, we will assume that the action $(g, \phi) \mapsto \phi g$ is continuous for $\phi \in \Phi$. An additional condition, discussed in [61],

is that every inverse image of compact sets under the function $(g, \phi) \mapsto (\phi g, \phi)$ should be compact. A continuous action by a group G on a space Φ satisfying this condition is called *proper*. This technical condition turns out to have useful properties and is assumed throughout this paper. When the group action is proper, the orbits of the group can be proved to be closed sets relative to the topology of Φ.

For fixed $\phi \in \Phi$, a *stability subgroup* H of G is defined as $\{h : \phi h = \phi\}$. These are transformed within orbits of G as $H \mapsto g^{-1} H g$.

Every locally compact group possesses a right-invariant Haar measure ν satisfying $\nu(Dg) = \nu(D)$ for $D \subset G$ [46]. This induces a right-invariant measure on Φ itself if each stability group H is compact, which is the case if the action G on Φ is proper and the group is locally compact. The last assertion is proved in ([61], Theorem 2.3.13(c)). A right-invariant measure ν on Φ satisfies by definition $\nu(Fg) = \nu(F)$ for all (measurable) $F \subset \Phi$ and $g \in G$.

A.2. On group representation theory. A matrix representation of a group G is defined as a function U from the group to the set of (here complex) matrices satisfying $U(gh) = U(g)U(h)$ for all $g, h \in G$. In other words, a representation is a homomorphism from G to the multiplicative group of square matrices of a fixed dimension. Any representation U and any fixed nonsingular matrix K of the same size can be used to construct another representation $S(g) = KU(g)K^{-1}$. If the group is compact (and also in some other cases), we can always find such S of minimal block diagonal form, and at the same time we can take S to be unitary $[S(g)^{\dagger} S(g) = I]$. If (and only if) the group is Abelian, each minimal block will be one-dimensional.

An important aspect of this reduction appears if we look upon the matrices as operators on a vector space: Then each collection of blocks gives an invariant vector space under the multiplicative group of matrices, and each single minimal block gives an irreducible invariant vector space. For compact groups, the irreducible invariant vector spaces will be finite-dimensional. The minimal matrices in the blocks are called irreducible representations of the group.

More generally, a class of operators $\{U(g); g \in G\}$ (where G is a group) on a, possibly infinite-dimensional, vector space is a representation if $U(gh) = U(g)U(h)$ for all g, h. A representation of a compact group always has a complete reduction in minimal matrix representations as described above. In particular, this holds for the unitary regular representation defined on a Hilbert space $\mathrm{L}^2(\Phi, \nu)$ by $U_R(g) f(\phi) = f(\phi g)$. Here ν is the right-invariant measure for G on Φ.

A useful result is Schur's lemma:

If U and U' are irreducible representations, and A is a bounded linear map such that $U(g)A = AU'(g)$ for all g, then either U and U' are isomorphic or $A = 0$. If $U(g)A = AU(g)$ for all g, then necessarily $A = \lambda I$ for some scalar λ.

More on group representations can be found in [9, 23, 31, 40, 47, 55, 62].

Acknowledgments. First of all I want to thank Richard Gill for numerous discussions, for thoughtful comments and for exchange of ideas in general. Also, advice and helpful remarks by Chris Isham, Peter Jupp, Peter McCullagh and Erling Størmer are appreciated.

Much of the inspiration behind this paper was provided by three conferences arranged by Andrei Khrennikov in Växjö, Sweden.

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