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SPACE-TIME APPROACH TO PERELMAN'S $\mathcal{L}$-GEODESICS AND AN ANALOGY BETWEEN PERELMAN'S REDUCED VOLUME AND HUISKEN'S MONOTONICITY FORMULA

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Abstract. From the viewpoint of space-time geometry and the trick of space-time track, the author would like to investigate the $\mathcal{L}$-geodesics, Perelman's reduced volume and Huisken's monotonicity formula.

1. Introduction

Perelman [5] introduces a new length (energy-like) functional for paths in the space-times of solutions of the Ricci flow, called the $\mathcal{L}$-length. As seen, the naturalness of this functional can be justified by the space-time approach. At the end of §6 in [5], Perelman also remarks that

> "The first geometric interpretation of Hamilton's Harnack expression was found by Chow and Chu [C-Chu 1,2]; ...; our construction is, in a certain sense, dual to theirs.
>
> Our formula for the reduced volume resembles the expression in Huisken monotonicity for the mean curvature flow [Hu]; ...."

This motivates the author to investigate the $\mathcal{L}$-geodesics, Perelman's reduced volume and Huisken's monotonicity formula [4] from the viewpoint of space-time geometry.

This paper is organized as follows. In section 2, for the reader's convenience we recall the definitions of the $\mathcal{L}$-length, $\mathcal{L}$-geodesics, $\mathcal{L}$-geodesic equation, reduced distance and reduced volume. In section 3, we relate Perelman's $\mathcal{L}$-geodesics and $\mathcal{L}$-geodesic equation to those defined with respect to the space-time connection defined by (11) (see also Lemma 4.3 in [2]). In section 4, by the trick of space-time track introduced in [2] we give an exact analogy between Perelman's reduced volume and Huisken's monotonicity formula [4].

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2. Basic Definitions

Let $(\mathcal{N}^n, h(t))$, $t \in (\alpha, \omega)$, be a solution to the Ricci flow. From this we can easily obtain a solution $(\mathcal{N}^n, h(\tau))$ to the **backward Ricci flow**

$$\frac{\partial}{\partial \tau} h = 2\,\mathrm{Rc}$$

by reversing time. In particular, if $\omega < +\infty$, let $\tau \doteqdot \omega - t$, so that $(\mathcal{N}, h(\tau))$ is a solution to the backward Ricci flow on the time interval $(0, \omega - \alpha)$. [1]

2.1. The $\mathcal{L}$-length and the $\mathcal{L}$-geodesic

We begin by motivating the definition of Perelman's $\mathcal{L}$-length for the Ricci flow as a renormalization of the length with respect to Perelman's potentially infinite dimensional manifold $(\widetilde{\mathcal{N}}, \tilde{h})$.

2.1.1. Potentially infinite Riemannian metric on space-time

Given $N \in \mathbb{N}$, define a metric on $\widetilde{\mathcal{N}} \doteqdot \mathcal{N}^n \times \mathcal{S}^N \times (0, T)$ by

$$\tilde{h} \doteqdot h_{ij} dx^i dx^j + \tau h_{\alpha\beta} dy^\alpha dy^\beta + \left(\frac{N}{2\tau} + R \right) d\tau^2, \tag{1}$$

where $h_{\alpha\beta}$ is the metric on $\mathcal{S}^N$ of constant sectional curvature $1/(2N)$ and R denotes the scalar curvature of the evolving metric h on $\mathcal{N}$. Here we have used the convention that $\{x^i\}_{i=1}^n$ will denote coordinates on the $\mathcal{N}$ factor, $\{y^\alpha\}_{\alpha=1}^N$ coordinates on the $\mathcal{S}^N$ factor, and $x^0 \doteqdot \tau$. Latin indices $i, j, k, \ldots$ will be on $\mathcal{N}$, Greek indices $\alpha, \beta, \gamma, \ldots$ will be on $\mathcal{S}^N$, and 0 represents the (minus) time component. Choosing N large enough so that $\frac{N}{2\tau} + R > 0$ implies that the metric $\tilde{h}$ is Riemannian, i.e., positive-definite. In local coordinates,

$$\tilde{h}_{ij} = h_{ij}, \quad \tilde{h}_{\alpha\beta} = \tau h_{\alpha\beta}, \quad \tilde{h}_{00} = \frac{N}{2\tau} + R, \quad \tilde{h}_{i0} = \tilde{h}_{i\alpha} = \tilde{h}_{\alpha 0} = 0. \tag{2}$$

Let $\tilde{\gamma}(s) \doteqdot (x(s), y(s), \tau(s))$ be a shortest geodesic, with respect to the metric $\tilde{h}$, between points $p \doteqdot (x_0, y_0, 0)$ and $q \doteqdot (x_1, y_1, \tau_q) \in \widetilde{\mathcal{N}}$. Since the fibers $\mathcal{S}^N$ pinch to a point as $\tau \to 0$, it is clear that the geodesic $\tilde{\gamma}(s)$ is orthogonal to the fibers $\mathcal{S}^N$. (To see this directly, take a sequence of geodesics from $p_k \doteqdot (x_0, y_0, 1/k)$ to q and pass to the limit as $k \to \infty$.) Therefore it suffices to consider the manifold $\bar{\mathcal{N}} \doteqdot \mathcal{N} \times (0, T)$ endowed with the Riemannian metric:

$$\bar{h} \doteqdot h_{ij} dx^i dx^j + \left(\frac{N}{2\tau} + R \right) d\tau^2. \tag{3}$$

[1] We shall consider the case where $\alpha = -\infty$ (in which case we define $\omega - \alpha \doteqdot +\infty$.) On the other hand, if $\omega = +\infty$ and $\alpha = -\infty$, we may simply take $\tau = -t$. However, for the backward Ricci flow we are not as interested in the case where $\omega = +\infty$ and $\alpha > -\infty$.

Remark. The components of the Levi-Civita connection ${}^N\tilde{\nabla}$ of $(\bar{\mathcal{N}}, \bar{h})$ are defined by

$$ {}^N\tilde{\nabla}_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b} = \sum_{c=0}^{n} {}^N\tilde{\Gamma}_{ab}^c \frac{\partial}{\partial x^c}, $$

where $x^0 = \tau$. By direct computation, we have that

$$ \begin{aligned} {}^N\tilde{\Gamma}_{ij}^k &= \Gamma_{ij}^k, \\ {}^N\tilde{\Gamma}_{i0}^k &= R_i^k, \\ {}^N\tilde{\Gamma}_{00}^k &= -\frac{1}{2}\nabla^k R \end{aligned} $$

and

$$ \begin{aligned} {}^N\tilde{\Gamma}_{ij}^0 &= -\left(\frac{N}{2\tau} + R\right)^{-1} R_{ij}, \\ {}^N\tilde{\Gamma}_{i0}^0 &= \left(\frac{N}{2\tau} + R\right)^{-1} \frac{1}{2}\nabla_i R, \\ {}^N\tilde{\Gamma}_{00}^0 &= \left(\frac{N}{2\tau} + R\right)^{-1} \frac{1}{2}\left(\frac{\partial R}{\partial \tau} + \frac{R}{\tau}\right) - \frac{1}{2\tau}. \end{aligned} $$

In particular, ${}^N\tilde{\Gamma}_{ab}^k$ are independent of N, whereas

$$ \begin{aligned} \lim_{N\to\infty} {}^N\tilde{\Gamma}_{ij}^0 &= 0, \\ \lim_{N\to\infty} {}^N\tilde{\Gamma}_{i0}^0 &= 0, \\ \lim_{N\to\infty} {}^N\tilde{\Gamma}_{00}^0 &= -\tfrac{1}{2\tau}. \end{aligned} $$

For convenience, denote $x(s) \doteqdot \gamma(s)$. Now we use $s = \tau$ as the parameter of the curve. Let $\dot{\gamma}(\tau) \doteqdot \frac{d\gamma}{d\tau}(\tau)$. The length of a path $\bar{\gamma}(\tau) \doteqdot (\gamma(\tau), \tau)$, with respect to the metric $\bar{h}$, is given by the following:

$$ \begin{aligned} &\operatorname{Length}_{\bar{h}}(\bar{\gamma}) \\ &= \int_0^{\tau_q} \sqrt{\frac{N}{2\tau} + R + |\dot{\gamma}(\tau)|^2} d\tau \\ &= \int_0^{\tau_q} \sqrt{\frac{N}{2\tau}} \sqrt{1 + \frac{2\tau}{N}\left(R + |\dot{\gamma}(\tau)|^2\right)} d\tau \\ &= \int_0^{\tau_q} \sqrt{\frac{N}{2\tau}} \left(1 + \frac{\tau}{N}\left(R + |\dot{\gamma}(\tau)|^2\right) + O\left(N^{-2}\right)\right) d\tau \\ &= \int_0^{\tau_q} \sqrt{\frac{N}{2\tau}} d\tau + \int_0^{\tau_q} \sqrt{\frac{\tau}{2N}} \left(R + |\dot{\gamma}(\tau)|^2\right) d\tau + \int_0^{\tau_q} \sqrt{\frac{1}{2\tau}} O\left(N^{-3/2}\right) d\tau \end{aligned} $$

$$= \sqrt{2N\tau_q} + \frac{1}{\sqrt{2N}} \int_0^{\tau_q} \sqrt{\tau} \left(R + |\dot{\gamma}(\tau)|^2 \right) d\tau + \sqrt{2\tau_q} O \left(N^{-3/2} \right).$$

The calculation indicates that as $N \to \infty$, a shortest geodesic should approach a minimizer of the following length functional:

$$\int_0^{\tau_q} \sqrt{\tau} \left(R(\gamma(\tau), \tau) + |\dot{\gamma}(\tau)|^2_{h(\tau)} \right) d\tau.$$

Note that the functional only depends on the data of $(\mathcal{N}, h)$.

A natural geometry on space-time (in the sense of lengths, distances and geodesics) is given by the following.

Definition. Let $(\mathcal{N}^n, h(\tau))$, $\tau \in (A, \Omega)$, be a solution to the backward Ricci flow $\frac{\partial}{\partial \tau} h = 2\,\mathrm{Rc}$, and let $\gamma : [\tau_1, \tau_2] \to \mathcal{N}$ be a piecewise C^1-path, where $[\tau_1, \tau_2] \subset (A, \Omega)$ and $\tau_1 \geq 0$. The **$\mathcal{L}$-length** of γ is defined by

(4) $$\mathcal{L}(\gamma) \doteqdot \mathcal{L}_h(\gamma) \doteqdot \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left(R(\gamma(\tau), \tau) + \left| \frac{d\gamma}{d\tau}(\tau) \right|^2_{h(\tau)} \right) d\tau.$$

It is clear that the $\mathcal{L}$-length is defined only for paths defined on a subinterval of the time interval where the solution to the backward Ricci flow exists.

Now that we have defined the $\mathcal{L}$-length we may mimic basic Riemannian comparison geometry in the space-time setting for the Ricci flow. We compute the first variation of the $\mathcal{L}$-length and find the equation for the critical points of $\mathcal{L}$ (the $\mathcal{L}$-geodesic equation). We shall also compare this equation with the geodesic equation for the space-time graph (with respect to a natural space-time connection) in Section 3.

Let $(\mathcal{N}^n, h(\tau))$, $\tau \in (A, \Omega)$, be a solution to the backward Ricci flow. Consider a variation of the C^2-path $\gamma : [\tau_1, \tau_2] \to \mathcal{N}$; that is, let

$$G : [\tau_1, \tau_2] \times (-\varepsilon, \varepsilon) \to \mathcal{N}$$

be a C^2-map such that

$$G|_{[\tau_1, \tau_2] \times \{0\}} = \gamma.$$

We say that a variation $G(\cdot, \cdot)$ of a C^2-path γ is C^2 if $G\left(\frac{\sigma^2}{4}, s \right)$ is C^2 in (σ, s). Define

$$\gamma_s \doteqdot G|_{[\tau_1, \tau_2] \times \{s\}} : [\tau_1, \tau_2] \to \mathcal{N} \quad \text{for} - \varepsilon < s < \varepsilon.$$

Let

$$X(\tau, s) \doteqdot \frac{\partial G}{\partial \tau}(\tau, s) = \frac{\partial \gamma_s}{\partial \tau}(\tau) \text{ and } Y(\tau, s) \doteqdot \frac{\partial G}{\partial s}(\tau, s) = \frac{\partial \gamma_s}{\partial s}(\tau)$$

be the tangent vector field and variation vector field along $\gamma_s(\tau)$, respectively. The first variation formula for $\mathcal{L}$ is given by

Lemma. (Equation 7.1, Perelman [5]) *Given a C^2-family of curves $\gamma_s : [\tau_1, \tau_2] \to \mathcal{N}$, the first variation of its $\mathcal{L}$-length is given by*

$$\begin{aligned}\frac{1}{2}(\delta_Y \mathcal{L})(\gamma_s) &\doteqdot \frac{1}{2}\frac{d}{ds}\mathcal{L}(\gamma_s) = \sqrt{\tau} Y \cdot X\Big|_{\tau_1}^{\tau_2} \\ &+ \int_{\tau_1}^{\tau_2} \sqrt{\tau} Y \cdot \left(\frac{1}{2}\nabla R - \frac{1}{2\tau}X - \nabla_X X - 2\operatorname{Rc}(X)\right) d\tau,\end{aligned} \tag{5}$$

where the covariant derivative ∇ is with respect to $h(\tau)$.

Proof. For a proof we refer the reader to [5]. ■

The $\mathcal{L}$-first variation formula (5) leads us to the following.

Definition. If γ is a critical point of the $\mathcal{L}$-length functional among all C^2-paths with fixed endpoints, then γ is called an **$\mathcal{L}$-geodesic**.

It follows from the $\mathcal{L}$-first variation formula that a C^2-path $\gamma : [\tau_1, \tau_2] \to (\mathcal{N}, h)$ is an $\mathcal{L}$-geodesic if and only if it satisfies the **$\mathcal{L}$-geodesic equation**:

$$\nabla_X X - \frac{1}{2}\nabla R + 2\operatorname{Rc}(X) + \frac{1}{2\tau}X = 0, \tag{6}$$

where $X(\tau) \doteqdot \frac{d\gamma}{d\tau}(\tau)$.

Remark. Let $(\mathcal{M}, g(\tau))$ be a complete solution to the backward Ricci flow with bounded sectional curvature. (1) Given a space-time point $(p, \tau_1) \in \mathcal{M} \times [0, T)$ and a tangent vector $V \in T_p\mathcal{M}$, there exists a unique $\mathcal{L}$-geodesic $\gamma : [\tau_1, T) \to \mathcal{M}$ with

$$\lim_{\tau \to \tau_1} \sqrt{\tau} X(\tau) = V.$$

(2) Given two points $p, q \in \mathcal{M}$ and $0 \leq \tau_1 < \tau_2 < T$, there exists a smooth path $\gamma : [\tau_1, \tau_2] \to \mathcal{M}$ from p to q such that γ has the minimal $\mathcal{L}$-length among all such paths. Furthermore, all $\mathcal{L}$-length minimizing paths are smooth $\mathcal{L}$-geodesics. For more details, we refer the reader to [3, 6].

2.2. The reduced distance and the reduced volume

We motivate the definition of Perelman's reduced volume by computing the volume of geodesic spheres in the potentially infinite-dimensional manifold.

Let $p=(x_0,y_0,0)$, $\bar{\tau}\in(0,T)$, and

$$B_{\tilde{g}}\left(p,\sqrt{2N\bar{\tau}}\right)\subset\widetilde{\mathcal{M}}\doteqdot\mathcal{M}\times\mathcal{S}^N\times(0,T)$$

denote the ball centered at p with radius $\sqrt{2N\bar{\tau}}$ with respect to the metric:

$$\tilde{g}\doteqdot g_{ij}dx^idx^j+\tau g_{\alpha\beta}dy^\alpha dy^\beta+\left(\frac{N}{2\tau}+R\right)d\tau^2,$$

where $g_{\alpha\beta}$ is the metric on $\mathcal{S}^N$ of constant sectional curvature $1/(2N)$. For any point $w=(x,y,\tau_w)\in\partial B_{\tilde{g}}(p,\sqrt{2N\bar{\tau}})$, because of the factor τ in $\tau g_{\alpha\beta}dy^\alpha dy^\beta$, we have

$$\begin{aligned}\sqrt{2N\bar{\tau}}&=d_{\tilde{g}}(w,p)=d_{\tilde{g}}((x,y,\tau_w),(x_0,y_0,0))\\&=d_{\tilde{g}}((x,y,\tau_w),(x_0,y,0)).\end{aligned}$$

Hence, letting $\gamma(\tau)\doteqdot(\gamma_{\mathcal{M}}(\tau),y,\tau)$, $\tau\in[0,\tau_w]$, with $\gamma(0)=(x_0,y,0)$ and $\gamma_{\mathcal{M}}(\tau_w)=w$, we have

$$\begin{aligned}\sqrt{2N\bar{\tau}}&=\inf_{\gamma}\operatorname{Length}_{\tilde{g}}(\gamma)\\&=\inf_{\gamma_{\mathcal{M}}}\begin{pmatrix}\dfrac{1}{\sqrt{2N}}\displaystyle\int_0^{\tau_w}\sqrt{\tau}\left(R+|\dot{\gamma}_{\mathcal{M}}(\tau)|^2\right)d\tau\\+\sqrt{2N\tau_w}+O\left(N^{-3/2}\right)\end{pmatrix}\\&=\sqrt{2N\tau_w}+\frac{1}{\sqrt{2N}}L(x,\tau_w)+O\left(N^{-3/2}\right),\end{aligned}\tag{7}$$

where

$$L(x,\tau_w)\doteqdot\inf_{\gamma_{\mathcal{M}}}\int_0^{\tau_w}\sqrt{\tau}\left(R+|\dot{\gamma}_{\mathcal{M}}(\tau)|^2\right)d\tau$$

and the infimum is taken over $\gamma_{\mathcal{M}}:[0,\tau_w]\to\mathcal{M}$ with $\gamma_{\mathcal{M}}(0)=x_0$ and $\gamma_{\mathcal{M}}(\tau_w)=x$. Therefore for any $w=(x,y,\tau_w)\in\partial B_{\tilde{g}}(p,\sqrt{2N\bar{\tau}})$,

$$\sqrt{\tau_w}=\sqrt{\bar{\tau}}-\frac{1}{2N}L(x,\tau_w)+O\left(N^{-2}\right).$$

This implies that the geodesic sphere $\partial B_{\tilde{g}}\left(p,\sqrt{2N\bar{\tau}}\right)$, with respect to $\tilde{g}$, is $O(N^{-1})$-close to the hypersurface $\mathcal{M}\times\mathcal{S}^N\times\{\bar{\tau}\}$.

Note that since the fibers $\mathcal{S}^N$ pinch to a point as $\tau\to0$, if $w=(x,y,\tau_w)\in\partial B_{\tilde{g}}(p,\sqrt{2N\bar{\tau}})$, then any point in $\{x\}\times\mathcal{S}^N\times\{\tau_w\}$ also lies on the sphere $\partial B_{\tilde{g}}\left(p,\sqrt{2N\bar{\tau}}\right)$. We have that the volume of $\partial B_{\tilde{g}}\left(p,\sqrt{2N\bar{\tau}}\right)$ is roughly (since the sphere has small curvature for N large) the volume of the hypersurface $\mathcal{M} \times \mathcal{S}^N \times \{\bar{\tau}\}$ in $\widetilde{\mathcal{M}}$ and its volume can be computed as:

$$\begin{aligned}
&\mathrm{Vol}_{\tilde{g}}\, \partial B_{\tilde{g}}\left(p, \sqrt{2N\bar{\tau}}\right) \\
&\approx \int_{\partial B_{\tilde{g}}\left(p,\sqrt{2N\bar{\tau}}\right)} d\mu_{g_{\mathcal{M}}(\tau_w)}(x) \wedge \tau_w^{N/2} d\mu_{\mathcal{S}^N}(y) \\
&\approx \mathrm{Vol}(\mathcal{S}^N, g_{\mathcal{S}^N}) \int_{\mathcal{M}} \left(\sqrt{\bar{\tau}} - \frac{1}{2N} L(x,\tau_w) + O(N^{-2})\right)^N d\mu_{g_{\mathcal{M}}(\bar{\tau})} \\
&\approx \omega_N \left(\sqrt{2N\bar{\tau}}\right)^N \int_{\mathcal{M}} \left(1 - \frac{1}{2N\sqrt{\bar{\tau}}} L(x,\bar{\tau}) + O(N^{-2})\right)^N d\mu_{g_{\mathcal{M}}(\bar{\tau})},
\end{aligned}$$

where ω_N is the volume of the unit sphere $\mathcal{S}^N$ (recall that $g_{\mathcal{S}^N}$ has constant sectional curvature $1/(2N)$, i.e., radius $\sqrt{2N}$). We observe that

$$\begin{aligned}
&\lim_{N\to\infty} \left(1 - \frac{1}{2N\sqrt{\bar{\tau}}} L(x,\bar{\tau}) + O(N^{-2})\right)^N \\
=& \lim_{N\to\infty} \left(1 - \frac{1}{N}\frac{1}{2\sqrt{\bar{\tau}}} L(x,\bar{\tau})\right)^N \\
=& \quad e^{-\frac{1}{2\sqrt{\bar{\tau}}} L(x,\bar{\tau})}.
\end{aligned}$$

For convenience, denote the quantity $\frac{1}{2\sqrt{\bar{\tau}}} L(x,\bar{\tau})$ by the **reduced distance** ℓ, i.e.,

$$\ell(x,\bar{\tau}) \doteqdot \frac{1}{2\sqrt{\bar{\tau}}} L(x,\bar{\tau}). \tag{8}$$

Therefore, we have

$$\lim_{N\to\infty} \left(1 - \frac{1}{2N\sqrt{\bar{\tau}}} L(x,\bar{\tau}) + O(N^{-2})\right)^N = e^{-\ell(x,\bar{\tau})}.$$

It is easy to see that

$$\begin{aligned}
&\frac{\mathrm{Vol}_{\tilde{g}}\left(\partial B_{\tilde{g}}\left(p, \sqrt{2N\bar{\tau}}\right)\right)}{\left(\sqrt{2N\bar{\tau}}\right)^{N+n}} \\
&= (2N)^{-n/2}\, \omega_N \left(\int_{\mathcal{M}} \bar{\tau}^{-n/2} e^{-\ell(x,\bar{\tau})} d\mu_{g_{\mathcal{M}}(\bar{\tau})} + O(N^{-1})\right).
\end{aligned} \tag{9}$$

In particular, we obtain the geometric invariant

$$\int_{\mathcal{M}} \bar{\tau}^{-n/2} e^{-\ell(x,\bar{\tau})} d\mu_{g_{\mathcal{M}}(\bar{\tau})}$$

for $\bar{\tau} \in (0, T)$.

Thus we are led to the following.

Definition. Let $(\mathcal{M}^n, g(\tau))$, $\tau \in [0, T]$, be a complete solution to the backward Ricci flow with bounded curvature. The **reduced volume** functional is defined by

$$\tilde{V}(\tau) \doteqdot \int_{\mathcal{M}} (4\pi\tau)^{-n/2} e^{-\ell(x,\tau)} d\mu_{g(\tau)}(x) \tag{10}$$

for $\tau \in (0, T)$.

3. SPACE-TIME APPROACH TO PERELMAN'S $\mathcal{L}$-GEODESIC EQUATION

We now compare the $\mathcal{L}$-geodesic equation for γ with the geodesic equation for the graph $\bar{\gamma}(\tau) = (\gamma(\tau), \tau)$ with respect to the following space-time connection (see also Lemma 4.3 in [1]):

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k, \quad \tilde{\Gamma}_{i0}^k = \tilde{\Gamma}_{0i}^k = R_i^k, \quad \tilde{\Gamma}_{00}^k = -\frac{1}{2}\nabla^k R, \quad \tilde{\Gamma}_{00}^0 = -\frac{1}{2\tau}, \tag{11}$$

where $i, j, k \geq 1$ (above and below), and the rest of the components are zero. It is instructive to compare the Christoffel symbols $\tilde{\Gamma}$ above with the the symbols ${}^N\tilde{\Gamma}$ of the Levi-Civita connection ${}^N\tilde{\nabla}$ for the metric $\bar{h}$ introduced in subsection 2.1. For $k \geq 1$, note that $\tilde{\Gamma}_{ab}^k = {}^N\tilde{\Gamma}_{ab}^k$ is independent of N, whereas $\tilde{\Gamma}_{ab}^0 = \lim_{N\to\infty} {}^N\tilde{\Gamma}_{ab}^0$ for all $a, b \geq 0$.

Let $\tau = \tau(\sigma) \doteqdot \sigma^2/4$, i.e., $\sigma \doteqdot 2\sqrt{\tau}$. We look for a geodesic, with respect to the space-time connection defined above, of the form

$$\tilde{\beta}(\sigma) \doteqdot (\gamma(\tau(\sigma)), \sigma^2/4),$$

where $\gamma : [\tau_1, \tau_2] \to \mathcal{M}$ is a path. For convenience, let $\beta(\sigma) \doteqdot \gamma(\tau(\sigma))$, $\tilde{\beta}^i \doteqdot x^i \circ \beta \doteqdot \beta^i$ for $i = 1, \ldots, n$, and $\tilde{\beta}^0 \doteqdot x^0 \circ \tilde{\beta}$ (so that $\tilde{\beta}^0(\sigma) = \sigma^2/4$).

The motivation for change of time-variable is given by the following.

Claim. If $\tilde{\beta} : [0, \bar{\sigma}] \to \mathcal{N} \times [0, T]$ is a geodesic, with respect to the connection $\tilde{\nabla}$, with $\tilde{\beta}^0(0) = 0$ and $\frac{d\tilde{\beta}^0}{d\sigma}(\sigma) \neq 0$ for $\sigma > 0$, then $\tilde{\beta}^0(\sigma) = A\sigma^2$ for some positive constant A.

Proof. If $\tilde{\beta}^0(\sigma) = \tau(\sigma)$, then the time-component of the geodesic equation with respect to $\tilde{\nabla}$ is:

$$\begin{aligned} 0 &= \frac{d^2\tilde{\beta}^0}{d\sigma^2} + \sum_{0\le i,j\le n} \left(\tilde{\Gamma}_{ij}^0 \circ \tilde{\beta}\right) \frac{d\tilde{\beta}^i}{d\sigma}\frac{d\tilde{\beta}^j}{d\sigma} \\ &= \frac{d^2\tau}{d\sigma^2} - \frac{1}{2\tau}\left(\frac{d\tau}{d\sigma}\right)^2 \end{aligned}$$

since $\tilde{\Gamma}_{ij}^0 = 0$ when $i \ge 1$ or $j \ge 1$, and $\tilde{\Gamma}_{00}^0 = -\frac{1}{2\tau}$. Hence, assuming $\tau(\sigma) > 0$ and $\frac{d\tau}{d\sigma}(\sigma) > 0$ for $\sigma > 0$, we have

$$\frac{d}{d\sigma}\log\frac{d\tau}{d\sigma} = \frac{\frac{d^2\tau}{d\sigma^2}}{\frac{d\tau}{d\sigma}} = \frac{\frac{d\tau}{d\sigma}}{2\tau} = \frac{d}{d\sigma}\log\sqrt{\tau},$$

so that

$$\frac{d\tau}{d\sigma} = C\sqrt{\tau}$$

for some constant $C > 0$. Since $\tau(0) = 0$, we conclude

$$\tau(\sigma) = C^2\sigma^2/4. \qquad \blacksquare$$

By direct computation, we have

$$\frac{d\beta^k}{d\sigma} = \frac{\sigma}{2}\frac{d\gamma^k}{d\tau}, \quad \frac{d\tilde{\beta}^0}{d\sigma} = \frac{\sigma}{2},$$

and

$$\begin{aligned} \frac{d^2\beta^k}{d\sigma^2} &= \frac{d}{d\sigma}\left(\frac{\sigma}{2}\frac{d\gamma^k}{d\tau}(\tau(\sigma))\right) \\ &= \left(\frac{\sigma}{2}\right)^2 \frac{d^2\gamma^k}{d\tau^2}(\tau(\sigma)) + \frac{1}{2}\left(\frac{d\gamma^k}{d\tau}(\tau(\sigma))\right). \end{aligned}$$

We justify the change of variables from τ to σ via the geodesic equation with respect to $\tilde{\Gamma}$ by showing the time-component of $\tilde{\beta}$ satisfies the geodesic equation:

$$\begin{aligned} \frac{d^2\tilde{\beta}^0}{d\sigma^2} + \sum_{0\le i,j\le n} \left(\tilde{\Gamma}_{ij}^0 \circ \tilde{\beta}\right) \frac{d\tilde{\beta}^i}{d\sigma}\frac{d\tilde{\beta}^j}{d\sigma} &= \frac{d^2}{d\sigma^2}\left(\sigma^2/4\right) + \tilde{\Gamma}_{00}^0\left(\tilde{\beta}(\sigma)\right)(\sigma/2)^2 \\ &= \frac{1}{2} - \frac{1}{2\left(\sigma^2/4\right)}(\sigma/2)^2 = 0. \end{aligned}$$

(This last equation justifies defining the time-component of $\tilde{\beta}(\sigma)$ as $\sigma^2/4$, and in particular, the change of variables $\sigma = 2\sqrt{\tau}$.) For the space components, the geodesic equation with respect to $\tilde{\Gamma}$ says that for $k = 1, ..., n$,

$$\begin{aligned} 0 &= \frac{d^2\tilde{\beta}^k}{d\sigma^2} + \sum_{0\le i,j\le n} \tilde{\Gamma}_{ij}^k \frac{d\tilde{\beta}^i}{d\sigma}\frac{d\tilde{\beta}^j}{d\sigma} \\ &= \frac{d^2\beta^k}{d\sigma^2} + \sum_{1\le i,j\le n} \Gamma_{ij}^k \frac{d\beta^i}{d\sigma}\frac{d\beta^j}{d\sigma} + 2\sum_{1\le i\le n} \tilde{\Gamma}_{i0}^k \frac{d\beta^i}{d\sigma}\frac{d\tilde{\beta}^0}{d\sigma} + \tilde{\Gamma}_{00}^k \frac{d\tilde{\beta}^0}{d\sigma}\frac{d\tilde{\beta}^0}{d\sigma}. \end{aligned}$$

This is equivalent to:

$$0 = \left(\frac{\sigma}{2}\right)^2 \frac{d^2\gamma^k}{d\tau^2}(\tau(\sigma)) + \sum_{1\le i,j\le n} \Gamma_{ij}^k \left(\frac{\sigma}{2}\frac{d\gamma^i}{d\tau}(\tau(\sigma))\right)\left(\frac{\sigma}{2}\frac{d\gamma^j}{d\tau}(\tau(\sigma))\right)$$
$$+\frac{1}{2}\left(\frac{d\gamma^k}{d\tau}(\tau(\sigma))\right) + 2\sum_{1\le i\le n} R_i^k \left(\frac{\sigma}{2}\frac{d\gamma^i}{d\tau}(\tau(\sigma))\right)\left(\frac{\sigma}{2}\right) - \frac{1}{2}\left(\frac{\sigma}{2}\right)^2 \nabla^k R,$$

which, after dividing by $\tau = \sigma^2/4$, implies

$$0 = \frac{d^2\gamma^k}{d\tau^2}(\tau(\sigma)) + \sum_{1\le i,j\le n} \Gamma_{ij}^k \frac{d\gamma^i}{d\tau}(\tau(\sigma))\frac{d\gamma^j}{d\tau}(\tau(\sigma)) + \frac{1}{2\tau}\left(\frac{d\gamma^k}{d\tau}(\tau(\sigma))\right)$$
$$+2\sum_{1\le i\le n} R_i^k \frac{d\gamma^i}{d\tau}(\tau(\sigma)) - \frac{1}{2}\nabla^k R.$$

That is, in invariant notation and with $X \doteqdot \frac{d\gamma}{d\tau}$, we have

$$\nabla_X X - \frac{1}{2}\nabla R + 2\,\mathrm{Rc}(X) + \frac{1}{2\tau}X = 0,$$

which is the same as (6). Thus $\mathcal{L}$-geodesics correspond to geodesics defined with respect to the space-time connection. In particular, $\gamma(\tau)$ is an $\mathcal{L}$-geodesic if and only if $\beta(\sigma) \doteqdot \gamma\left(\sigma^2/4\right)$ is a geodesic with respect the space-time connection $\tilde{\nabla}$. Since $\tilde{\Gamma}_{ab}^c = \lim_{N\to\infty} {}^N\tilde{\Gamma}_{ab}^c$, we also conclude that the Riemannian geodesic equation for the metric $\bar{h}$ on $\mathcal{N}^n \times (0,T)$ (defined in subsection 2.1) limits to the $\sigma = 2\sqrt{\tau}$ reparametrization of the $\mathcal{L}$-geodesic equation as $N \to \infty$.

4. An Analogue Between Perelman's Reduced Volume and Huisken's Monotonicity Formula

Given a 1-parameter family of metrics $g(t)$, $t \in \mathcal{I}$, on a manifold M^n and functions $\beta(t) : M^n \to \mathbb{R}$, we define the metric g_β on $\tilde{M}^{n+1} \doteqdot M^n \times \mathcal{I}$ by (see [2])

$$g_\beta(x,t) \doteqdot g(x,t) + \beta^2(x,t)\,dt^2.$$

We consider the family of hypersurfaces given by the time slices $M_t \doteqdot M^n \times \{t\} \subset \tilde{M}^{n+1}$. A choice of unit normal vector field to M_t is

$$\nu \doteqdot -\frac{1}{\beta}\frac{\partial}{\partial t}.$$

The hypersurfaces M_t parametrized by the maps $X_t : M^n \to \tilde{M}^{n+1}$ defined by $X_t(x) \doteqdot (x,t)$ are evolving by the flow

$$\frac{\partial}{\partial t}X_t = -\beta\nu.$$

This implies the metrics are evolving by

$$\frac{\partial}{\partial t} g_{ij} = -2\beta h_{ij},$$

where h_{ij} is the second fundamental form of $M_t \subset \tilde{M}^{n+1}$. One way of seeing this formula is from

$$\frac{1}{\beta} h_{ij} = (\Gamma_\beta)_{ij}^0 = -\frac{1}{2} (g_\beta)^{00} \frac{\partial}{\partial x^0} (g_\beta)_{ij} = -\frac{1}{2\beta^2} \frac{\partial}{\partial t} g_{ij},$$

where $x^0 = t$. Hence

$$\beta h_{ij} = R_{ij}. \tag{12}$$

Consider the special case where $\beta(t)^2 = R(t)$ is the scalar curvature of $g(t)$. Tracing (12) we get $\beta H = R$ so that $\beta = H$ and the hypersurfaces M_t satisfy the mean curvature flow: $\frac{\partial}{\partial t} X_t = -H\nu$.

Now we consider the more general setting of hypersurfaces evolving in a Riemannian manifold. Given $\left(P^{n+1}, g\right)$, let $X_t : M^n \to P^{n+1}$, $t \in \mathcal{I}$, parametrize a 1-parameter family of hypersurfaces $M_t = X_t(M^n)$ evolving in their normal directions

$$\frac{\partial}{\partial t} X_t = -\beta\nu,$$

where $\beta(t) : M^n \to \mathbb{R}$ are arbitrary functions. We consider the product metric $g + N dt^2$ on $P^{n+1} \times \mathcal{I}$. The **space-time track** is defined by

$$\tilde{M}^{n+1} \doteqdot \{(x,t) : x \in M_t,\ t \in \mathcal{I}\} \subset P^{n+1} \times \mathcal{I}.$$

We parametrize this by the map

$$\tilde{X} : M^n \times \mathcal{I} \to P^{n+1} \times \mathcal{I}$$

defined by

$$\tilde{X}(p,t) \doteqdot (X_t(p), t).$$

Let ${}^N\hat{g}$ denote the induced metric on $\tilde{M}^{n+1}$. Its components

$${}^N\hat{g}_{ab} \doteqdot \left\langle \frac{\partial \tilde{X}}{\partial x^a}, \frac{\partial \tilde{X}}{\partial x^b} \right\rangle_{g+Ndt^2} = \left\langle \frac{\partial X_t}{\partial x^a}, \frac{\partial X_t}{\partial x^b} \right\rangle_g + N\delta_{a0}\delta_{b0},$$

where $a, b \geq 0$ are given by

$${}^N\hat{g}_{ij} = g_{ij}, \quad {}^N\hat{g}_{i0} = 0, \quad {}^N\hat{g}_{00} = \beta^2 + N,$$

where $i, j \geq 1$.

Now, following Perelman, we renormalize length function associated to the metric (similar to what we did in section 2) on $M^n \times \mathcal{J}$ (we switch from $\mathcal{I}$ to $\mathcal{J}$ when we consider the time parameter to be τ instead of t)

$$^N\breve{g}(x,\tau) \doteqdot g(x,\tau) + \left(\beta^2(x,\tau) + \frac{N}{2\tau}\right) d\tau^2,$$

where $\frac{d\tau}{dt} = -1$ and $g(\tau) = g(t(\tau))$ is the pulled back metric on M^n by X_τ of the induced metric on $M_\tau \doteqdot X_\tau(M^n) \subset P^{n+1}$. We may also think of this metric as defined on an open subset of P^{n+1} by pushing forward by the diffeomorphism $(x,\tau) \mapsto X_\tau(x)$. Let $\gamma : [0,\tau_0] \to M^n$ be a path and define the path $\bar{\gamma} : [0,\tau_0] \to P^{n+1}$ by

$$\bar{\gamma}(\tau) \doteqdot X_\tau(\gamma(\tau)) \in M_\tau$$

so that $(\gamma(\tau), \tau) \in M^n \times \mathcal{J}$ corresponds to the point $\bar{\gamma}(\tau) \in M_\tau \subset P^{n+1}$. We have

$$\mathrm{L}_{(^N\breve{g})}(\bar{\gamma}) = \int_0^{\tau_0} \left(\left| \frac{d\gamma}{d\tau} \right|^2_{g(\tau)} + \beta^2 + \frac{N}{2\tau} \right)^{1/2} d\tau.$$

Again, motivated by carrying out the expansion of $\mathrm{L}_{(^N\breve{g})}(\bar{\gamma})$ in powers of N, and considering highest order non-trivial term, we define the **$\mathcal{L}$-length** of γ by

$$\begin{aligned} \mathcal{L}(\gamma) &\doteqdot \int_0^{\tau_0} \sqrt{\tau} \left(\left| \frac{d\gamma}{d\tau}(\tau) \right|^2_{g(\tau)} + \beta^2(\gamma(\tau), \tau) \right) d\tau \\ &= \int_0^{\tau_0} \sqrt{\tau} \left| \frac{d\bar{\gamma}}{d\tau}(\tau) \right|^2_g d\tau. \end{aligned}$$

(The equality holds since $\iota^* g = g_\beta$, where $\iota : M^n \times \mathcal{J} \to P^{n+1}$ is defined by $\iota(x,\tau) \doteqdot X_\tau(x)$.) Making the change of variables $\sigma = 2\sqrt{\tau}$, we have

$$\mathcal{L}(\gamma) = \int_0^{2\sqrt{\tau_0}} \left| \frac{d\bar{\gamma}}{d\tau}(\sigma) \right|^2_g d\sigma.$$

This is the energy of the path $\bar{\gamma}(\sigma)$ and assuming that τ_0, $\gamma(0) = p$ and $\gamma(\tau_0) = q$ are fixed, $\mathcal{L}(\gamma)$ is minimized by a constant speed geodesic and

$$\breve{L}(q,\tau_0) \doteqdot \inf_\gamma \mathcal{L}(\gamma) = \frac{d_g(p,q)^2}{2\sqrt{\tau_0}}.$$

Let $\breve{\ell}(q,\tau_0) \doteqdot \frac{1}{2\sqrt{\tau_0}} \breve{L}(q,\tau_0)$. Then

$$\breve{\ell}(q,\tau_0) = \frac{d_g(p,q)^2}{4\tau_0}.$$

Recall that Perelman's **reduced volume** for a solution to the backward Ricci flow is defined by

$$\tilde{V}(\tau) \doteqdot \int_M (4\pi\tau)^{-n/2} e^{-\ell(x,\tau)} d\mu_{g(\tau)}(x), \tag{10}$$

where ℓ is defined in (8). From the above considerations, we see that Huisken's monotonicity formula for the mean curvature flow (see [4]) is the analogue of the monotonicity of $\tilde{V}(\tau)$. In particular, if $P^{n+1} = \mathbb{R}^{n+1}$, then Huisken's monotone quantity is

$$\int_{X_t} (4\pi\tau)^{-n/2} e^{-\frac{|x|^2}{4\tau}} d\mu = \int_{M^n} (4\pi\tau)^{-n/2} e^{-\check{\ell}} d\mu.$$

Remark. The above can perhaps be seen more clearly and simply in the case of a fixed Riemannian metric g on a manifold M^n. Define on $M \times \mathcal{J}$, where $\mathcal{J}$ is an interval, the metric

$${}^N\mathring{g}(x,\tau) \doteqdot g(x) + \frac{N}{2\tau} d\tau^2.$$

Then given $\gamma : [\tau_1, \tau_2] \to M^n$, the length of $\tilde{\gamma} : [\tau_1, \tau_2] \to M^n \times \mathcal{J}$ defined by $\tilde{\gamma}(\tau) \doteqdot (\gamma(\tau), \tau)$ is

$$\begin{aligned} \mathrm{L}_{({}^N\mathring{g})}(\tilde{\gamma}) &= \int_{\tau_1}^{\tau_2} \left(\left| \frac{d\gamma}{d\tau} \right|_{g(\tau)}^2 + \frac{N}{2\tau} \right)^{1/2} d\tau \\ &= \sqrt{N} \left(\sqrt{2\tau_2} - \sqrt{2\tau_1} \right) + \frac{1}{\sqrt{2N}} \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left| \frac{d\gamma}{d\tau} \right|_{g(\tau)}^2 d\tau + O\left(N^{-3/2} \right). \end{aligned}$$

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References

1. B. Chow and S.-C. Chu, A geometric interpretation of Hamilton's Harnack inequality for the Ricci flow, *Math. Res. Lett.* **2** (1995), 701-718.
2. B. Chow and S.-C. Chu, Spacetime formulation of Harnack inequalities for curvature flows of hypersurfaces, *J. Geom. Anal.*, **11** (2001), 219–231.
3. B. Chow, P. Lu and L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, Vol. 77, AMS, 2006.

4. G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, *J. Diff. Geom.*, **31(1)** (1990), 285-299.

5. G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159.

6. B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo and L. Ni, *The Ricci flow: Techniques and Applications*, Mathematical Surveys and Monographs, Vol. 135, AMS, 2007.

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