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STRONG WEAK CONVERGENCE THEOREMS OF IMPLICIT HYBRID STEEPEST-DESCENT METHODS FOR VARIATIONAL INEQUALITIES

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Abstract. Assume that F is a nonlinear operator on a real Hilbert space H which is strongly monotone and Lipschitzian with constants $\eta > 0$ and $\kappa > 0$, respectively on a nonempty closed convex subset C of H. Assume also that C is the intersection of the fixed point sets of a finite number of nonexpansive mappings on H. We develop an implicit hybrid steepest-descent method which generates an iterative sequence $\{u_n\}$ from an arbitrary initial point $u_0 \in H$. We characterize the weak convergence of $\{u_n\}$ to the unique solution u^* of the variational inequality:

$$\langle F(u^*), v - u^* \rangle \geq 0 \quad \forall v \in C.$$

Applications to constrained generalized pseudoinverse are included.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, and $F : H \to H$ be an operator. Stampacchia [10] initially studied the classical variational inequality problem: find $u^* \in C$ such that

$$(\mathrm{VI}(F, C)) \ \langle F(u^*), v - u^*) \rangle \geq 0, \quad \forall\ v \in C.$$

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Variational inequalities ever since have been extensively studied because they include as special cases many diverse disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance, etc.. The reader is referred to [8-10, 12, 20] and the references therein.

It is well known that if F is strong monotone and Lipschitzian on C, then $\mathrm{VI}(F, C)$ has a unique solution. See, e.g., [10]. In the study of the $\mathrm{VI}(F, C)$, one of the most important problems is how to find a solution of the $\mathrm{VI}(F, C)$ if any. A great deal of effort has gone into the problem of finding a solution of $\mathrm{VI}(F, C)$; see [8, 11].

It is also known that the $\mathrm{VI}(F, C)$ is equivalent to the fixed-point equation

$$u^* = P_C(u^* - \mu F(u^*)) \tag{1}$$

where P_C is the (nearest point) projection from H onto C; i.e., $P_C x = \mathrm{argmin}_{y \in C} \|x - y\|$ for each $x \in H$ and where $\mu > 0$ is an arbitrarily fixed constant. So if F is strongly monotone and Lipschitzian on C and $\mu > 0$ is small enough, then the mapping determined by the right-hand side of this equation is a contraction. Hence the Banach contraction principle guarantees that the Picard iterates converge in norm to the unique solution of the $\mathrm{VI}(F, C)$. Such a method is called the projection method. It has been widely extended to develop various algorithms for finding solutions of various classes of variational inequalities and complementarity problems; see, e.g., [21-23]. It is remarkable that the fixed-point equation involves the projection P_C which may not be easy to compute due to the complexity of the convex set C.

Recently Yamada ([18], see also [5]) introduced a hybrid steepest-descent method for solving the $\mathrm{VI}(F, C)$ so as to reduce the complexity probably caused by the projection P_C. His idea is stated now. Let C be the fixed-point set of a nonexpansive mapping $T : H \to H$; that is, $C = \{x \in H : Tx = x\}$. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$, and let $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$ denote the fixed-point set of T. Let F be η-strongly monotone and κ-Lipschitzian on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$ and a sequence $\{\lambda_n\}$ of real numbers in $(0, 1)$ satisfying the conditions below:

(L1) $\lim_{n \to \infty} \lambda_n = 0$,
(L2) $\sum_{n=1}^{\infty} \lambda_n = \infty$,
(L3) $\lim_{n \to \infty} (\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$.

Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence $\{u_n\}$ by the following algorithm:

$$u_{n+1} := Tu_n - \lambda_{n+1} \mu F(Tu_n), \quad n \geq 0. \tag{2}$$

Then Yamada [18] proved that $\{u_n\}$ converges strongly to the unique solution of the $\mathrm{VI}(F, C)$. An example of the sequence $\{\lambda_n\}$ which satisfies conditions (L1)-(L3) is given by $\lambda_n = 1/n^\sigma$ where $0 < \sigma < 1$.

On the other hand, if C is expressed as the intersection of the fixed-point sets of N nonexpansive mappings $T_i : H \to H$ with $N \geq 1$ an integer, Yamada [18] proposed another algorithm,

$$u_{n+1} := T_{[n+1]}u_n - \lambda_{n+1}\mu F(T_{[n+1]}u_n), \quad n \geq 0, \tag{3}$$

where $T_{[k]} := T_{k \bmod N}$, for integer $k \geq 1$, with the mod function taking values in the set $\{1, 2, ..., N\}$; that is, if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_{[k]} = T_N$ if $q = 0$ and $T_{[k]} = T_q$ if $1 \leq q < N$, where $\mu \in (0, 2\eta/\kappa^2)$ and where the sequence $\{\lambda_n\}$ of parameters satisfies conditions (L1), (L2) and (L4),

(L4) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}|$ is convergent.

Under these conditions, Yamada [18] proved the strong convergence of $\{u_n\}$ to the unique solution of the $\mathrm{VI}(F, C)$.

In 2003, Xu and Kim [17] further considered and studied the hybrid steepest-descent algorithms (2) and (3). Their major contribution is that the strong convergence of algorithms (2) and (3) holds with condition (L3) replaced by the condition

$(\mathrm{L3})'$ $\lim_{n\to\infty} \lambda_n/\lambda_{n+1} = 1$ or equivalently $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})/\lambda_{n+1} = 0$,

and with condition (L4) replaced by the condition

$(\mathrm{L4})'$ $\lim_{n\to\infty} \lambda_n/\lambda_{n+N} = 1$ or equivalently $\lim_{n\to\infty}(\lambda_n - \lambda_{n+N})/\lambda_{n+N} = 0$.

It is clear that condition (L3)' is strictly weaker than condition (L3), coupled with conditions (L1) and (L2). Moreover (L3)' includes the important and natural choice $\{1/n\}$ for $\{\lambda_n\}$, while (L3) does not. It is easy to see that if $\lim_{n\to\infty} \lambda_n/\lambda_{n+N}$ exists then condition (L4) implies condition (L4)'. However in general conditions (L4) and (L4)' are not comparable: neither of them implies the other (see [16] for details). Furthermore under conditions (L1), (L2), (L3)', and (L4)', they gave the applications of algorithms (2) and (3) to the constrained generalized pseudoinverses.

Let $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$, $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Now we introduce the following implicit hybrid steepest-descent algorithms (I) and (II) as follows:

Algorithm (I). For any initial guess $u_0 \in H$, the sequence $\{u_n\}$ is generated by the following implicit iterative scheme

$$u_n := \alpha_n u_{n-1} + (1-\alpha_n)T^{\lambda_n}u_n = \alpha_n u_{n-1} + (1-\alpha_n)[Tu_n - \lambda_n \mu F(Tu_n)], \quad n \geq 1.$$

Algorithm (II). For any initial guess $u_0 \in H$, the sequence $\{u_n\}$ is generated by the following implicit iterative scheme

$$u_n := \alpha_n u_{n-1} + (1-\alpha_n)T_{[n]}^{\lambda_n}u_n = \alpha_n u_{n-1} + (1-\alpha_n)[T_{[n]}u_n - \lambda_n \mu F(T_{[n]}u_n)], \ n \geq 1,$$

where $T_{[n]} = T_{n \bmod N}$.

For the sequence $\{u_n\}$ generated by the above algorithms (I) and (II), we discuss and characterize the weak convergence of $\{u_n\}$ to the unique solution of the $\mathrm{VI}(F, C)$ under the suitable conditions which are more convenient and more simple than Xu and Kim's ones [17]. Moreover applications to constrained generalized pseudoinverse are included.

2. Preliminaries

The following lemmas will be used for the proofs of the main results of the paper in Section 3.

Lemma 2.1. See [15, Lemma 3.1]. *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of nonnegative numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2. See [7]. *Demiclosedness Principle. Assume that T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

For a nonempty closed convex subset $C \subset H$, we denote by P_C the (nearest point) projection from H onto C. In what follows, we state some well-known properties of the projection operator which will be used in the sequel; see [19].

Lemma 2.3. *Let C be a nonempty closed convex subset of H. For any $x, y \in H$ and $z \in C$, the following statements hold:*

(i) $\langle P_C x - x, z - P_C x\rangle \geq 0$;

(ii) $\|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \|P_C x - x + y - P_C y\|^2$.

Remark 2.2. Obviously, Lemma 2.3 (i) provides also a sufficient condition for a vector u to be the projection of the vector x; i.e., $u = P_C x$ if and only if $\langle u - x, z - u\rangle \geq 0,\ \forall z \in C$.

Now recall that a Banach space E is said to satisfy Opial's condition if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{for all } y \in E, y \neq x.$$

It is well known that every Hilbert space satisfies the Opial condition. See, e.g., [13].

3. Implicit Hybrid Steepest-descent Algorithms

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $F : H \to H$ be an operator such that F is Lipschitzian and strongly monotone with constants $\kappa > 0$ and $\eta > 0$, respectively on C; that is, F satisfies the conditions

$$\|Fx - Fy\| \leq \kappa \|x - y\|, \quad \forall x, y \in C, \tag{4}$$

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C. \tag{5}$$

Under these conditions, it is well-known that the variational inequality problem

$(\mathrm{VI}(F, C))$ $\langle F(u^*), v - u^* \rangle \geq 0, \quad \forall v \in C.$
has a unique solution $u^* \in C$.

Denote by P_C the projection of H onto C. Namely, for each $x \in H$, $P_C x$ is the only element in C satisfying

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}.$$

It is known that the projection P_C is characterized by the inequality

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall y \in C.$$

Thus it follows that the $\mathrm{VI}(F, C)$ is equivalent to the fixed-point problem

$$u^* = P_C(I - \mu F(u^*)),$$

where $\mu > 0$ is a constant.

In this section, assume that $T : H \to H$ is a nonexpansive mapping with $\mathrm{Fix}(T) = C$. Note that obviously, $\mathrm{Fix}(P_C) = C$. For any given numbers $\lambda \in (0, 1)$ and $\mu \in (0, 2\eta/\kappa^2)$, associating with $T : H \to H$, we define the mapping $T^\lambda : H \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in H.$$

Lemma 3.1. *See [18]. If $0 < \lambda < 1$ and $0 < \mu < 2\eta/\kappa^2$, then there holds for $T^\lambda : H \to H$,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H, \tag{6}$$

where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in (0,1)$.

Proof. For completeness, we include a simple proof.
Observe that

$$\begin{aligned}
&\kappa \geq \eta \Leftrightarrow \eta\mu \geq 1 - \sqrt{1 - 2\eta\mu + \mu^2\kappa^2} \\
&\Leftrightarrow -2\lambda(1-\lambda)[1 - \sqrt{1 - 2\eta\mu + \mu^2\kappa^2}] + 2\lambda(1-\lambda)\eta\mu \geq 0 \\
&\Leftrightarrow (1-\lambda)^2 + 2\lambda(1-\lambda)\sqrt{1 - 2\eta\mu + \mu^2\kappa^2} + \lambda^2(1 - 2\eta\mu + \mu^2\kappa^2) \\
&\quad \geq 1 - 2\lambda\mu\eta + (\lambda\mu)^2\kappa^2 \\
&\Leftrightarrow 1 - \lambda[1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)}] \geq \sqrt{1 - 2\lambda\mu\eta + (\lambda\mu)^2\kappa^2}.
\end{aligned}$$

From the strong monotonicity and Lipschitz continuity of F, we obtain

$$\|T^\lambda x - T^\lambda y\|^2 \leq (1 - 2\lambda\mu\eta + (\lambda\mu)^2\kappa^2)\|Tx - Ty\|^2.$$

Therefore, it is easy to see that the conclusion holds. ■

Algorithm (I). Let $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$, $\{\lambda_n\}_{n=1}^\infty \subset (0,1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence $\{u_n\}$ by the following implicit iterative scheme

$$\begin{aligned}
u_n : &= \alpha_n u_{n-1} + (1-\alpha_n)T^{\lambda_n}u_n \\
&= \alpha_n u_{n-1} + (1-\alpha_n)[Tu_n - \lambda_n\mu F(Tu_n)], \quad n \geq 1.
\end{aligned} \tag{7}$$

Observe that by Lemma 3.1, for every $u \in H$ and $t \in (0,1)$, the operator $S_t : H \to H$ defined by $S_t x = tu + (1-t)T^\lambda x$ satisfies

$$\begin{aligned}
\langle S_t x - S_t y, x - y\rangle &= (1-t)\langle T^\lambda x - T^\lambda y, x - y\rangle \\
&\leq (1-t)(1-\lambda\tau)\|x-y\|^2 \\
&\leq (1-t)\|x-y\|^2, \quad \forall x, y \in H.
\end{aligned}$$

Hence S_t is strongly pseudocontractive (see, e.g., [3, 14]). Since S_t is also Lipschitzian, it follows from [3-4, 14] that S_t has a unique fixed point $x_t \in H$. Thus there exists a unique $x_t \in H$ such that

$$x_t = tu + (1-t)T^\lambda x_t = tu + (1-t)[Tx_t - \lambda\mu F(Tx_t)].$$

This implies that the implicit iteration scheme (7) above for generating the sequence $\{u_n\}$ of approximate solutions of the VI(F, C) is well defined.

Theorem 3.1. *Let* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *be real sequences in* $(0,1)$ *such that* $\sum_{n=0}^\infty \lambda_n < \infty$ *and* $\alpha \leq \alpha_n \leq 1-\alpha, n \geq 1$, *for some* $\alpha > 0$. *Let* $\{u_n\}$ *denote the sequence generated by Algorithm (I). Then there hold the following statements:*

(i) $\{u_n\}$ *converges weakly to an element of* $\mathrm{Fix}(T)$*;*

(ii) *if* F *is additionally sequentially continuous from the weak topology to the strong topology, then* $\{u_n\}$ *converges weakly to the unique solution* $\tilde{u}$ *of the* $\mathrm{VI}(F, C)$ *if and only if*

$$\liminf_{n\to\infty}\langle F(Tu_n), u_n - \tilde{u}\rangle \le 0.$$

Proof. At first, recall that the well known identity

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$$

holds for all $x, y \in H$ and for all $t \in [0, 1]$.

(i) Let u^* be an arbitrary element in $\mathrm{Fix}(T)$. Observe that

$$\begin{aligned}\|u_n - u^*\|^2 &= \|\alpha_n u_{n-1} + (1-\alpha_n)T^{\lambda_n}u_n - u^*\|^2\\ &= \alpha_n\|u_{n-1} - u^*\|^2 + (1-\alpha_n)\|T^{\lambda_n}u_n - u^*\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|u_{n-1} - T^{\lambda_n}u_n\|^2.\end{aligned} \tag{8}$$

Utilizing Lemma 3.1 we have

$$\begin{aligned}\|T^{\lambda_n}u_n - u^*\| &= \|T^{\lambda_n}u_n - T^{\lambda_n}u^* + T^{\lambda_n}u^* - u^*\|\\ &\le \|T^{\lambda_n}u_n - T^{\lambda_n}u^*\| + \|T^{\lambda_n}u^* - u^*\|\\ &\le (1-\lambda_n\tau)\|u_n - u^*\| + \lambda_n\mu\|F(u^*)\|,\end{aligned}$$

which hence implies that

$$\|T^{\lambda_n}u_n - u^*\|^2 \le (1-\lambda_n\tau)\|u_n - u^*\|^2 + \lambda_n\cdot\frac{\mu^2\|F(u^*)\|^2}{\tau}.$$

This together with (8) yields

$$\begin{aligned}\|u_n - u^*\|^2 &\le \alpha_n\|u_{n-1} - u^*\|^2 + (1-\alpha_n)[(1-\lambda_n\tau)\|u_n - u^*\|^2\\ &\quad +\lambda_n\cdot\frac{\mu^2\|F(u^*)\|^2}{\tau}] - \alpha_n(1-\alpha_n)\|u_{n-1} - T^{\lambda_n}u_n\|^2\\ &\le \alpha_n\|u_{n-1} - u^*\|^2 + (1-\alpha_n)\|u_n - u^*\|^2\\ &\quad +(1-\alpha_n)\lambda_n\cdot\frac{\mu^2\|F(u^*)\|^2}{\tau} - \alpha_n(1-\alpha_n)\|u_{n-1} - T^{\lambda_n}u_n\|^2,\end{aligned}$$

and so

$$\begin{aligned}\|u_n-u^*\|^2 &\le \|u_{n-1}-u^*\|^2+(1-\alpha_n)\frac{\lambda_n}{\alpha_n}\cdot\frac{\mu^2\|F(u^*)\|^2}{\tau}\\ &\quad -(1-\alpha_n)\|u_{n-1}-T^{\lambda_n}u_n\|^2\\ &\le \|u_{n-1}-u^*\|^2+\lambda_n\cdot\frac{\mu^2\|F(u^*)\|^2}{\tau\alpha}-\|u_n-u_{n-1}\|^2.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\lambda_n\cdot\frac{\mu^2\|F(u^*)\|^2}{\tau\alpha}$ converges, from Lemma 2.1 we know that $\lim_{n\to\infty}\|u_n-u^*\|$ exists. Hence we deduce that

$$\|u_n-u_{n-1}\|^2\le\|u_{n-1}-u^*\|^2-\|u_n-u^*\|^2+\lambda_n\cdot\frac{\mu^2\|F(u^*)\|^2}{\tau\alpha}\to 0 \quad \text{as } n\to\infty,$$

that is, $\lim_{n\to\infty}\|u_n-u_{n-1}\|=0$. Now, observe that

$$\alpha\|u_{n-1}-T^{\lambda_n}u_n\|\le(1-\alpha_n)\|u_{n-1}-T^{\lambda_n}u_n\|=\|u_n-u_{n-1}\|\to 0 \quad \text{as } n\to\infty.$$

Also note that it follows from the boundedness of $\{u_n\}$ that $\{Tu_n\}$ and $\{F(Tu_n)\}$ are bounded. Thus we infer that

$$\begin{aligned}\|u_{n-1}-Tu_n\| &\le \|u_{n-1}-T^{\lambda_n}u_n\|+\|T^{\lambda_n}u_n-Tu_n\|\\ &\le \|u_{n-1}-T^{\lambda_n}u_n\|+\lambda_n\mu\|F(Tu_n)\|\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

This together with (7) implies that

$$\begin{aligned}\|u_n-Tu_n\| &\le \|\alpha_n(u_{n-1}-Tu_n)-(1-\alpha_n)\lambda_n\mu F(Tu_n)\|\\ &\le \|u_{n-1}-Tu_n\|+\lambda_n\mu\|F(Tu_n)\|\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Furthermore since $\{u_n\}$ is bounded, it has a subsequence $\{u_{n_j}\}_{j=1}^{\infty}$ which converges weakly to some $\hat{u}\in H$, and hence we have $\lim_{j\to\infty}\|u_{n_j}-Tu_{n_j}\|=0$. Note that from Lemma 2.2 it follows that $I-T$ is demiclosed at zero. Thus $\hat{u}\in \mathrm{Fix}(T)$. If there exists another subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}$ which converges weakly to $\bar{u}$, then we must have $\bar{u}\in\mathrm{Fix}(T)$. Since $\lim_{n\to\infty}\|u_n-\bar{u}\|$ exists and H is an Opial space, it follows from a standard argument that $\hat{u}=\bar{u}$. Consequently $\{u_n\}$ converges weakly to $\hat{u}\in\mathrm{Fix}(T)$.

(ii) Suppose additionally that F is sequentially continuous from the weak topology to the strong topology. Moreover let $\tilde{u}$ denote the unique solution of the $\mathrm{VI}(F,C)$. We observe the following facts: $\lim_{n\to\infty}\|u_n-Tu_n\|=0$, and weak-$\lim_{n\to\infty}u_n=\hat{u}\in\mathrm{Fix}(T)=:C$. Now let $\{u_n\}$ be weakly convergent to the unique solution $\tilde{u}$ of the $\mathrm{VI}(F,C)$. Then $\|F(u_n)-F(\tilde{u})\|\to 0$ as $n\to\infty$. Since

$\lim_{n\to\infty} \|u_n - Tu_n\| = 0$, we have $\lim_{n\to\infty} \|F(Tu_n) - F(\tilde{u})\| = 0$. Hence that

$$\begin{aligned}\liminf_{n\to\infty}\langle F(Tu_n), u_n - \tilde{u}\rangle &= \liminf_{n\to\infty}\langle F(Tu_n), u_n - \tilde{u}\rangle - \lim_{n\to\infty}\langle F(\tilde{u}), u_n - \tilde{u}\rangle \\ &= \liminf_{n\to\infty}\langle F(Tu_n) - F(\tilde{u}), u_n - \tilde{u}\rangle \\ &\le \liminf_{n\to\infty} \|F(Tu_n) - F(\tilde{u})\|\|u_n - \tilde{u}\| \\ &= 0\end{aligned}$$

which implies that

$$\liminf_{n\to\infty}\langle F(Tu_n), u_n - \tilde{u}\rangle \le 0. \tag{9}$$

Conversely suppose that (9) holds. Then by using the strong monotonicity of F and the weak lower semicontinuity of $\|\cdot\|$, we conclude that

$$\begin{aligned}\eta\|\hat{u} - \tilde{u}\|^2 &\le \eta \liminf_{n\to\infty} \|u_n - \tilde{u}\|^2 \\ &\le \eta \liminf_{n\to\infty}[\|Tu_n - \tilde{u}\| + \|u_n - Tu_n\|]^2 \\ &= \eta \liminf_{n\to\infty} \|Tu_n - \tilde{u}\|^2 \\ &\le \liminf_{n\to\infty}\langle F(Tu_n) - F(\tilde{u}), Tu_n - \tilde{u}\rangle \\ &= \liminf_{n\to\infty}[\langle F(Tu_n) - F(\tilde{u}), u_n - \tilde{u}\rangle + \langle F(Tu_n) - F(\tilde{u}), Tu_n - u_n\rangle] \\ &= \liminf_{n\to\infty}\langle F(Tu_n) - F(\tilde{u}), u_n - \tilde{u}\rangle \\ &= \liminf_{n\to\infty}\langle F(Tu_n), u_n - \tilde{u}\rangle - \lim_{n\to\infty}\langle F(\tilde{u}), u_n - \tilde{u}\rangle \\ &\le -\langle F(\tilde{u}), \hat{u} - \tilde{u}\rangle \\ &\le 0.\end{aligned}$$

This shows that $\|\hat{u} - \tilde{u}\|^2 = 0$ and hence $\hat{u} = \tilde{u}$. Therefore, $\{u_n\}$ converges weakly to the unique solution $\tilde{u}$ of the $\mathrm{VI}(F, C)$. This completes the proof. ∎

Next we consider a more general case where

$$C = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i),$$

with $N \ge 1$ an integer and $T_i : H \to H$ being nonexpansive for each $1 \le i \le N$.

We propose the following implicit hybrid steepest-descent algorithm for solving the $\mathrm{VI}(F, C)$.

Algorithm (II). Let $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence $\{u_n\}$ by the following implicit iterative scheme

$$
\begin{aligned}
u_n &:= \alpha_n u_{n-1} + (1-\alpha_n) T_{[n]}^{\lambda_n} u_n \\
&= \alpha_n u_{n-1} + (1-\alpha_n)[T_{[n]} u_n - \lambda_n \mu F(T_{[n]} u_n)], \quad n \geq 1,
\end{aligned} \tag{10}
$$

where $T_{[n]} = T_{n \bmod N}$.

We remark that as in Algorithm (I), the implicit iteration scheme (10) above for generating the sequence $\{u_n\}$ of approximate solutions of the $\mathrm{VI}(F, C)$ is well defined.

We are now in a position to prove the main result of this paper.

Theorem 3.2. *Let $\{\alpha_n\}$ and $\{\lambda_n\}$ be real sequences in $(0,1)$ such that $\sum_{n=0}^{\infty} \lambda_n < \infty$ and $\alpha \leq \alpha_n \leq 1-\alpha, n \geq 1$, for some $\alpha > 0$. Assume that*

$$
\begin{aligned}
C &= \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 T_2 ... T_N) = \mathrm{Fix}(T_N T_1 ... T_{N-1}) \\
&= \cdots = \mathrm{Fix}(T_2 T_3 ... T_N T_1).
\end{aligned} \tag{11}
$$

Let $\{u_n\}$ denote the sequence generated by Algorithm (II). Then there hold the following statements:

(i) *$\{u_n\}$ converges weakly to an element of $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$;*

(ii) *if F is additionally sequentially continuous from the weak topology to the strong topology, then $\{u_n\}$ converges weakly to the unique solution $\tilde{u}$ of the $\mathrm{VI}(F, C)$ if and only if*

$$\liminf_{n \to \infty} \langle F(T_{[n]} u_n), u_n - \tilde{u} \rangle \leq 0.$$

Proof.

(i) Let u^* be an arbitrary element in $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$. Observe that

$$
\begin{aligned}
\|u_n - u^*\|^2 &= \|\alpha_n u_{n-1} + (1-\alpha_n) T_{[n]}^{\lambda_n} u_n - u^*\|^2 \\
&= \alpha_n \|u_{n-1} - u^*\|^2 + (1-\alpha_n) \|T_{[n]}^{\lambda_n} u_n - u^*\|^2 \\
&\quad -\alpha_n (1-\alpha_n) \|u_{n-1} - T_{[n]}^{\lambda_n} u_n\|^2.
\end{aligned} \tag{12}
$$

Utilizing Lemma 3.1, we have

$$
\begin{aligned}
\|T_{[n]}^{\lambda_n} u_n - u^*\| &= \|T_{[n]}^{\lambda_n} u_n - T_{[n]}^{\lambda_n} u^* + T_{[n]}^{\lambda_n} u^* - u^*\| \\
&\leq \|T_{[n]}^{\lambda_n} u_n - T_{[n]}^{\lambda_n} u^*\| + \|T_{[n]}^{\lambda_n} u^* - u^*\| \\
&\leq (1 - \lambda_n \tau) \|u_n - u^*\| + \lambda_n \mu \|F(u^*)\|
\end{aligned}
$$

which implies that

$$\|T_{[n]}^{\lambda_n} u_n - u^*\|^2 \le (1-\lambda_n \tau)\|u_n - u^*\|^2 + \lambda_n \cdot \frac{\mu^2 \|F(u^*)\|^2}{\tau}.$$

This together with (12) yields

$$\begin{aligned}\|u_n - u^*\|^2 &\le \alpha_n \|u_{n-1} - u^*\|^2 + (1-\alpha_n)[(1-\lambda_n\tau)\|u_n - u^*\|^2 \\ &\quad + \lambda_n \cdot \frac{\mu^2\|F(u^*)\|^2}{\tau}] - \alpha_n(1-\alpha_n)\|u_{n-1} - T_{[n]}^{\lambda_n} u_n\|^2 \\ &\le \alpha_n\|u_{n-1} - u^*\|^2 + (1-\alpha_n)\|u_n - u^*\|^2 \\ &\quad + (1-\alpha_n)\lambda_n \cdot \frac{\mu^2\|F(u^*)\|^2}{\tau} - \alpha_n(1-\alpha_n)\|u_{n-1} - T_{[n]}^{\lambda_n} u_n\|^2\end{aligned}$$

and so

$$\begin{aligned}\|u_n - u^*\|^2 &\le \|u_{n-1} - u^*\|^2 + (1-\alpha_n)\frac{\lambda_n}{\alpha_n} \cdot \frac{\mu^2\|F(u^*)\|^2}{\tau} \\ &\quad -(1-\alpha_n)\|u_{n-1} - T_{[n]}^{\lambda_n} u_n\|^2 \\ &\le \|u_{n-1} - u^*\|^2 + \lambda_n \cdot \frac{\mu^2\|F(u^*)\|^2}{\tau\alpha} - \|u_n - u_{n-1}\|^2.\end{aligned}$$

Since $\sum_{n=0}^{\infty} \lambda_n \cdot \frac{\mu^2\|F(u^*)\|^2}{\tau\alpha}$ converges, from Lemma 2.1 we know that $\lim_{n\to\infty}\|u_n - u^*\|$ exists. Hence we deduce that

$$\|u_n - u_{n-1}\|^2 \le \|u_{n-1} - u^*\|^2 - \|u_n - u^*\|^2 + \lambda_n \cdot \frac{\mu^2\|F(u^*)\|^2}{\tau\alpha} \to 0 \quad \text{as } n \to \infty,$$

that is, $\lim_{n\to\infty}\|u_n - u_{n-1}\| = 0$. Obviously it is easy to see that $\|u_n - u_{n+i}\| \to 0$ as $n \to \infty, \forall i = 1, 2, ..., N$. Now observe that

$$\alpha\|u_{n-1} - T_{[n]}^{\lambda_n} u_n\| \le (1-\alpha_n)\|u_{n-1} - T_{[n]}^{\lambda_n} u_n\| = \|u_n - u_{n-1}\| \to 0 \quad \text{as } n \to \infty.$$

Also, note that it follows from the boundedness of $\{u_n\}$ that $\{T_{[n]}u_n\}$ and $\{F(T_{[n]} u_n)\}$ are bounded. Thus we infer that

$$\begin{aligned}\|u_{n-1} - T_{[n]}u_n\| &\le \|u_{n-1} - T_{[n]}^{\lambda_n} u_n\| + \|T_{[n]}^{\lambda_n} u_n - T_{[n]}u_n\| \\ &\le \|u_{n-1} - T_{[n]}^{\lambda_n} u_n\| + \lambda_n\mu\|F(T_{[n]}u_n)\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This together with (10) implies that

$$\begin{aligned}\|u_n - T_{[n]}u_n\| &\le \|\alpha_n(u_{n-1} - T_{[n]}u_n) - (1-\alpha_n)\lambda_n\mu F(T_{[n]}u_n)\| \\ &\le \|u_{n-1} - T_{[n]}u_n\| + \lambda_n\mu\|F(T_{[n]}u_n)\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Consequently, we have for each $i = 1, 2, ..., N$

$$\begin{aligned}\|u_n - T_{[n+i]}u_n\| &\le \|u_n - u_{n+i}\| \\ &\quad + \|u_{n+i} - T_{[n+i]}u_{n+i}\| + \|T_{[n+i]}u_{n+i} - T_{[n+i]}u_n\| \\ &\le 2\|u_n - u_{n+i}\| + \|u_{n+i} - T_{[n+i]}u_{n+i}\| \to 0 \quad \text{as } n \to \infty\end{aligned}$$

and so

$$\lim_{n\to\infty} \|u_n - T_{[n+i]}u_n\| = 0, \quad \forall i = 1, 2, ..., N. \tag{13}$$

It follows from (13) (see also [7, 21]) that $\lim_{n\to\infty} \|u_n - T_l u_n\| = 0, \forall l = 1, 2, ..., N$. Furthermore since $\{u_n\}$ is bounded, it has a subsequence $\{u_{n_j}\}_{j=1}^{\infty}$ which converges weakly to some $\hat{u} \in H$ and hence we have $\lim_{j\to\infty} \|u_{n_j} - T_l u_{n_j}\| = 0$. Note that from Lemma 2.2 it follows that $I - T_l$ is demiclosed at zero. Thus $\hat{u} \in \mathrm{Fix}(T_l)$. Since l is an arbitrary element in the finite set $\{1, 2, ..., N\}$, we get $\hat{u} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. If there exists another subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}$ which converges weakly to $\bar{u}$, then we must have $\bar{u} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Since $\lim_{n\to\infty} \|u_n - \bar{u}\|$ exists and H is an Opial space, it follows from a standard argument that $\hat{u} = \bar{u}$. Therefore $\{u_n\}$ converges weakly to $\hat{u} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$.

(ii) Suppose additionally that F is sequentially continuous from the weak topology to the strong topology. Moreover let $\tilde{u}$ denote the unique solution of the $\mathrm{VI}(F, C)$. Again we observe $\lim_{n\to\infty} \|u_n - T_{[n]}u_n\| = 0$ and weak-$\lim_{n\to\infty} u_n = \hat{u} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i) =: C$. Now let $\{u_n\}$ be weakly convergent to the unique solution $\tilde{u}$ of the $\mathrm{VI}(F, C)$. Then $\|F(u_n) - F(\tilde{u})\| \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty} \|u_n - T_{[n]}u_n\| = 0$, we have $\lim_{n\to\infty} \|F(T_{[n]}u_n) - F(\tilde{u})\| = 0$. Hence

$$\begin{aligned}\liminf_{n\to\infty} \langle F(T_{[n]}u_n), u_n - \tilde{u}\rangle &= \liminf_{n\to\infty} \langle F(T_{[n]}u_n), u_n - \tilde{u}\rangle - \lim_{n\to\infty} \langle F(\tilde{u}), u_n - \tilde{u}\rangle \\ &= \liminf_{n\to\infty} \langle F(T_{[n]}u_n) - F(\tilde{u}), u_n - \tilde{u}\rangle \\ &\le \liminf_{n\to\infty} \|F(T_{[n]}u_n) - F(\tilde{u})\| \|u_n - \tilde{u}\| \\ &= 0\end{aligned}$$

which implies that

$$\liminf_{n\to\infty} \langle F(T_{[n]}u_n), u_n - \tilde{u}\rangle \le 0. \tag{14}$$

Conversely, suppose that (14) holds. Then by using the strong monotonicity of F

and the weak lower semicontinuity of $\|\cdot\|$, we conclude that

$$\begin{aligned}
\eta\|\hat{u}-\tilde{u}\|^2 &\le \eta \liminf_{n\to\infty} \|u_n-\tilde{u}\|^2 \\
&\le \eta \liminf_{n\to\infty} [\|T_{[n]}u_n-\tilde{u}\| + \|u_n - T_{[n]}u_n\|]^2 \\
&= \eta \liminf_{n\to\infty} \|T_{[n]}u_n-\tilde{u}\|^2 \\
&\le \liminf_{n\to\infty} \langle F(T_{[n]}u_n)-F(\tilde{u}), T_{[n]}u_n-\tilde{u}\rangle \\
&= \liminf_{n\to\infty} [\langle F(T_{[n]}u_n)-F(\tilde{u}), u_n-\tilde{u}\rangle + \langle F(T_{[n]}u_n)-F(\tilde{u}), T_{[n]}u_n-u_n\rangle] \\
&= \liminf_{n\to\infty} \langle F(T_{[n]}u_n)-F(\tilde{u}), u_n-\tilde{u}\rangle \\
&= \liminf_{n\to\infty} \langle F(T_{[n]}u_n), u_n-\tilde{u}\rangle - \lim_{n\to\infty}\langle F(\tilde{u}), u_n-\tilde{u}\rangle \\
&\le -\langle F(\tilde{u}), \hat{u}-\tilde{u}\rangle \\
&\le 0.
\end{aligned}$$

This shows that $\|\hat{u}-\tilde{u}\|^2 = 0$ and hence $\hat{u}=\tilde{u}$. Therefore $\{u_n\}$ converges weakly to the unique solution $\tilde{u}$ of the $\mathrm{VI}(F,C)$. This completes the proof. ■

Remark 3.1. Recall that a self-mapping of a nonempty closed convex subset K of a Hilbert space H is said to be attracting nonexpansive [1-2] if T is nonexpansive and if $\|Tx-p\| < \|x-p\|$ for $x,p\in K$ with $x\notin \mathrm{Fix}(T)$ and $p\in\mathrm{Fix}(T)$. Recall also that T is firmly nonexpansive [1-2] if $\langle x-y, Tx-Ty\rangle \ge \|Tx-Ty\|^2$ for all $x,y\in K$. It is known that assumption (11) in Theorem 3.2 is automatically satisfied if each T_i is attracting nonexpansive. Since a projection is firmly nonexpansive, we have the following consequence of Theorem 3.2.

Corollary 3.1. *Let $\mu\in(0,2\eta/\kappa^2)$ and let $\{\alpha_n\}$ and $\{\lambda_n\}$ be real sequences in $(0,1)$ such that $\sum_{n=0}^{\infty}\lambda_n<\infty$ and $\alpha\le\alpha_n\le 1-\alpha, n\ge 1$, for some $\alpha>0$. Let $u_0\in H$ and let the sequence $\{u_n\}$ be generated by the implicit iterative algorithm*

$$u_n := \alpha_n u_{n-1} + (1-\alpha_n)[P_{[n]}u_n - \lambda_n\mu F(P_{[n]}u_n)], \quad n\ge 1,$$

where $P_k = P_{C_k}, 1\le k\le N$. Then there hold the following statements:

(i) $\{u_n\}$ converges weakly to an element of $C := \bigcap_{k=1}^{N} C_k$;

(ii) if F is additionally sequentially continuous from the weak topology to the strong topology, then $\{u_n\}$ converges weakly to the unique solution $\tilde{u}$ of the $\mathrm{VI}(F,C)$ if and only if

$$\liminf_{n\to\infty}\langle F(P_{[n]}u_n), u_n-\tilde{u}\rangle \le 0.$$

4. Applications to Constrained Generalized Pseudoinverse

Let K be a nonempty closed convex subset of a real Hilbert space H. Let A be a bounded linear operator on H. Given an element $b \in H$, consider the minimization problem

$$\min_{x \in K} \|Ax - b\|^2. \tag{15}$$

Let S_b denote the solution set. Then, S_b is closed convex. It is known that S_b is nonempty if and only if $P_{\overline{A(K)}}(b) \in A(K)$. In this case, S_b has a unique element with minimum norm; that is, there exists a unique point $\hat{x} \in S_b$ satisfying

$$\|\hat{x}\|^2 = \min\{\|x\|^2 : x \in S_b\}. \tag{16}$$

Definition 4.1. See [6]. The K-constrained pseudoinverse of A (symbol $\hat{A}_K$) is defined as

$$D(\hat{A}_K) = \{b \in H : P_{\overline{A(K)}}(b) \in A(K)\}, \quad \hat{A}_K(b) = \hat{x}, \quad \text{and} \quad b \in D(\hat{A}_K),$$

where $\hat{x} \in S_b$ is the unique solution to (16).

Now we recall the K-constrained generalized pseudoinverse of A; [17-18].

Let $\theta : H \to R$ be a differentiable convex function such that θ' is a κ-Lipschitzian and η-strongly monotone operator for some $\kappa > 0$ and $\eta > 0$. Under these assumptions, there exists a unique point $\hat{x}_0 \in S_b$ for $b \in D(\hat{A}_K)$ such that

$$\theta(\hat{x}_0) = \min\{\theta(x) : x \in S_b\}. \tag{17}$$

Definition 4.2. See [17]. The K-constrained generalized pseudoinverse of A associated with θ (symbol $\hat{A}_{K,\theta}$) is defined as

$$D(\hat{A}_{K,\theta}) = D(\hat{A}_K), \quad \hat{A}_{K,\theta}(b) = \hat{x}_0, \quad \text{and} \quad b \in D(\hat{A}_{K,\theta}),$$

where $\hat{x}_0 \in S_b$ is the unique solution to (17). Note that if $\theta(x) = \|x\|^2/2$, then the K-constrained generalized pseudoinverse $\hat{A}_{K,\theta}$ of A associated with θ reduces to the K-constrained pseudoinverse $\hat{A}_K$ of A in Definition 4.1.

We now apply the results in Section 3 to construct the K-constrained generalized pseudoinverse $\hat{A}_{K,\theta}$ of A. But first, observe that $\widetilde{x} \in K$ solves the minimization problem (15) if and only if there holds the following optimality condition:

$$\langle A^*(A\widetilde{x} - b), x - \widetilde{x}\rangle \geq 0, \quad x \in K,$$

where A^* is the adjoint of A. This is equivalent to, for each $\lambda > 0$,

$$\langle [\lambda A^* b + (I - \lambda A^* A)\widetilde{x}] - \widetilde{x}, x - \widetilde{x} \rangle \geq 0, \quad x \in K,$$

or

$$P_K(\lambda A^* b + (I - \lambda A^* A)\widetilde{x}) = \widetilde{x}. \tag{18}$$

Define a mapping $T : H \to H$ by

$$Tx = P_K(A^* b + (I - \lambda A^* A)x), \quad x \in H. \tag{19}$$

Lemma 4.1. See [17]. *If $\lambda \in (0, 2\|A\|^{-2})$ and if $b \in D(\hat{A}_K)$, then T is attracting nonexpansive and $Fix(T) = S_b$.*

Theorem 4.1. *Let $\mu \in (0, 2\eta/\kappa^2)$ and let $\{\alpha_n\}$ and $\{\lambda_n\}$ be real sequences in $(0,1)$ such that $\sum_{n=0}^{\infty} \lambda_n < \infty$ and $\alpha \leq \alpha_n \leq 1 - \alpha, n \geq 1$, for some $\alpha > 0$. Given an initial guess $u_0 \in H$, let u_n be the sequence generated by the implicit iterative algorithm*

$$u_n = \alpha_n u_{n-1} + (1 - \alpha_n)[Tu_n - \lambda_n \mu \theta'(Tu_n)], \quad n \geq 1, \tag{20}$$

where T is given in (19). Then there hold the following statements:

- (i) *$\{u_n\}$ converges weakly to an element of S_b;*
- (ii) *if θ' is additionally sequentially continuous from the weak topology to the strong topology, then $\{u_n\}$ converges weakly to $\hat{A}_{K,\theta}(b)$ if and only if*

$$\liminf_{n \to \infty} \langle \theta'(Tu_n), u_n - \hat{A}_{K,\theta}(b) \rangle \leq 0.$$

Proof. The minimization problem (17) is equivalent to the following variational inequality problem:

$$\langle \theta'(\hat{x}_0), x - \hat{x}_0 \rangle \geq 0, \quad \forall x \in S_b, \tag{21}$$

where $\hat{x}_0 = \hat{A}_{K,\theta}(b)$. Since $\text{Fix}(T) = S_b$ and θ' is Lipschitzian and strongly monotone with constants $\kappa > 0$ and $\eta > 0$, respectively, the statements follow immediately from Theorem 3.1 with $F := \theta'$. ■

Lemma 4.2. *See [1-2]. Assume that N is a positive integer and assume that $\{T_i\}_{i=1}^N$ are N attracting nonexpansive mappings on H having a common fixed point. Then*

$$\bigcap_{i=1}^{N} \text{Fix}(T_i) = \text{Fix}(T_1 T_2 \ldots T_N).$$

Now assume that $\{S_b^1, ..., S_b^N\}$ *is a family of* N *closed convex subsets of* K *such that* $S_b = \bigcap_{i=1}^N S_b^i$. *For each* $1 \le i \le N$, *we define* $T_i : H \to H$ *by*

$$T_i x = P_{S_b^i}(A^*b + (I - \lambda A^*A)x), \quad x \in H,$$

where $P_{S_b^i}$ *is the projection from* H *onto* S_b^i.

Theorem 4.2. *Let* $\mu \in (0, 2\eta/\kappa^2)$ *and let* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *be real sequences in* $(0,1)$ *such that* $\sum_{n=0}^\infty \lambda_n < \infty$ *and* $\alpha \le \alpha_n \le 1 - \alpha, n \ge 1$, *for some* $\alpha > 0$. *Given an initial guess* $u_0 \in H$, *let* u_n *be the sequence generated by the implicit iterative algorithm*

$$u_n = \alpha_n u_{n-1} + (1 - \alpha_n) T_{[n]}^{\lambda_n} u_n, \quad n \ge 1, \tag{22}$$

where $T_{[n]}^{\lambda_n} u_n = T_{[n]} u_n - \lambda_n \mu \theta'(T_{[n]} u_n), n \ge 1$. *Then there hold the following statements:*

(i) $\{u_n\}$ *converges weakly to an element of* S_b;

(ii) *if* θ' *is additionally sequentially continuous from the weak topology to the strong topology, then* $\{u_n\}$ *converges weakly to* $\hat{A}_{K,\theta}(b)$ *if and only if*

$$\liminf_{n\to\infty} \langle \theta'(T_{[n]} u_n), u_n - \hat{A}_{K,\theta}(b) \rangle \le 0.$$

Proof. The minimization problem (17) is equivalent to the following variational inequality problem:

$$\langle \theta'(\hat{x}_0), x - \hat{x}_0 \rangle \ge 0, \quad \forall x \in S_b,$$

where $\hat{x}_0 = \hat{A}_{K,\theta}(b)$. In the proof of [17, Theorem 4.2], Xu and Kim have proved that

$$S_b = \mathrm{Fix}(T) = \bigcap_{i=1}^N \mathrm{Fix}(T_i). \tag{23}$$

By Lemmas 4.1 and 4.2, we see that assumption (11) in Theorem 3.2 holds. Since $\bigcap_{i=1}^N \mathrm{Fix}(T_i) = S_b$ and θ' is Lipschitzian and strongly monotone with constants $\kappa > 0$ and $\eta > 0$, respectively, the statements follow immediately from Theorem 3.2 with $F := \theta'$. ∎

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