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EXISTENCE THEOREMS FOR GENERALIZED VECTOR VARIATIONAL INEQUALITIES WITH PSEUDOMONOTONICITY AND THEIR APPLICATIONS

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Abstract. The purpose of this paper is to study the solvability for a class of generalized vector variational inequalities with pseudomonotonicity in reflexive Banach spaces. Utilizing the KKM-Fan lemma and the Nadler's result, we derive the solvability results for this class of generalized vector variational inequalities with pseudomonotonicity. Utilizing these results, we also establish some existence theorems for zero points of pseudomonotone multifunctions including the characterization of the existence of zero points.

1. Introduction and Preliminaries

Vector variational inequalities were initially studied by Giannessi [3] in the setting of finite dimensional Euclidean spaces. Subsequently, vector variational inequality (VVI) theory appears to be an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including differential equations, optimization, optimal control, mathematical programming, economics and transportation. Vector variational inequalities have been widely studied and generalized in infinite dimensional spaces. The reader is refereed to [1, 4-8, 10, 22] and the references therein.

Let X and Y be two real Banach spaces, $K \subset X$ be a nonempty, closed and convex subset, and $P \subset Y$ be a closed, convex and pointed cone with apex at the

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origin and int$P \neq \emptyset$ where int$P$ denotes the interior of P. The cone P is called proper if $P \neq Y$. Recall that P is said to be a closed, convex and pointed cone with apex at the origin if P is closed and the following conditions hold:

(i) $\lambda P \subset P, \ \forall \lambda > 0$;

(ii) $P + P \subset P$;

(iii) $P \cap (-P) = \{0\}$.

Given a closed, convex and pointed cone P with apex at the origin in Y, we can define relations "$\leq_P$" and "$\not\leq_P$" as follows:

$$x \leq_P y \Leftrightarrow y - x \in P$$

and

$$x \not\leq_P y \Leftrightarrow y - x \notin P.$$

Moreover, $a \not\leq_{\text{int}P} b$ means $b - a \notin$int$P$. Clearly, "$\leq_P$" is a partial order. In this case $(Y, \leq_P)$ is called an ordered Banach space ordered by P. Let $L(X, Y)$ denote the space of all continuous linear maps from X into Y. Let $L_c(X, Y)$ be the subspace of $L(X, Y)$ which consists of all completely continuous linear maps from X into Y. Recall that a mapping $g : X \to Y$ is said to be completely continuous if the weak convergence of x_n to x in X implies the strong convergence of $g(x_n)$ to $g(x)$ in Y.

Now we recall and define the following concepts and lemmas.

Definition 1.1. Let $A : L(X, Y) \to L(X, Y)$ and $f : K \to Y$ be two mappings. Let $T : K \to 2^{L(X,Y)}$ be a nonempty valued multifunction. Then

(i) T is said to be pseudomonotone if for any $x, y \in K, \ u \in Tx$ and $v \in Ty$,

$$\langle u, y - x \rangle \not\leq_{\text{int}P} 0 \ \Rightarrow \ \langle v, y - x \rangle \not\leq_{\text{int}P} 0;$$

(ii) T is said to be pseudomonotone with respect to A if for any $x, y \in K, \ u \in Tx$ and $v \in Ty$,

$$\langle Au, y - x \rangle \not\leq_{\text{int}P} 0 \ \Rightarrow \ \langle Av, y - x \rangle \not\leq_{\text{int}P} 0;$$

(iii) T is said to be pseudomonotone with respect to A and f if for any $x, y \in K, \ u \in Tx$ and $v \in Ty$,

$$\langle Au, y - x \rangle + f(y) - f(x) \not\leq_{\text{int}P} 0 \ \Rightarrow \ \langle Av, y - x \rangle + f(y) - f(x) \not\leq_{\text{int}P} 0;$$

(iv) T is said to be monotone with respect to A if for any $x, y \in K, \ u \in Tx$ and $v \in Ty$,

$$\langle Au - Av, x - y \rangle \geq_P 0;$$

in particular, a single-valued map $T : K \to L(X, Y)$ is said to be monotone if for any $x, y \in K$,

$$\langle Tx - Ty, x - y \rangle \geq_P 0.$$

Definition 1.2. A map $f : K \to Y$ is said to be convex if

$$f(tx + (1-t)y) \leq_P tf(x) + (1-t)f(y), \quad \forall x, y \in K,\ t \in [0,1].$$

Lemma 1.1. *(See Nadler [9]) Let $(X, \|\cdot\|)$ be a normed vector space and H be a Hausdorff metric on the collection $CB(X)$ of all nonempty, closed and bounded subsets of X, induced by a metric d in terms of $d(u,v) = \|u - v\|$, which is defined by*

$$H(U,V) = \max\{\sup_{u \in U} \inf_{v \in V} \|u - v\|, \sup_{v \in V} \inf_{u \in U} \|u - v\|\},$$

for U and V in $CB(X)$. If U and V lie in $CB(X)$, then for arbitrary $\varepsilon > 0$ and each $u \in U$ there exists $v \in V$ such that

$$\|u - v\| \leq (1 + \varepsilon) H(U,V).$$

Definition 1.3.

(i) A single-valued map $T : K \to L(X,Y)$ is said to be v-hemicontinuous if for any given $x, y \in K$, the mapping $t \to \langle T(x + t(y-x)), y - x \rangle$ is continuous at 0^+;

(ii) A nonempty compact valued multifunction $T : K \to 2^{L(X,Y)}$ is said to be H-hemicontinuous if for any given $x, y \in K$, the mapping $t \to H(T(x + t(y-x)), T(x))$ is continuous at 0^+ where H is the Hausdorff metric defined on $CB(L(X,Y))$;

(iii) A nonempty weakly compact valued multifunction $T : K \to 2^{L(X,Y)}$ is said to be H^*-hemicontinuous if for any given $x, y \in K$, the mapping $t \to H(T(x + t(y-x)), T(x))$ is continuous at 0^+ where H is the Hausdorff metric defined on $CB(L(X,Y))$.

It is clear that the H-hemicontinuity of T implies the H^*-hemicontinuity of T. Recently, Huang and Fang [6] considered and studied the solvability for a class of vector variational inequalities in reflexive Banach spaces. They proved the solvability for this class of vector variational inequalities with monotone mappings.

Theorem 1.1. (See [6, Theorem 3.1]) *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y be a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and int$C \neq \emptyset$. Suppose that $T : K \to L_c(X,Y)$ is a v-hemicontinuous and monotone map, and $f : K \to Y$ is a completely continuous and convex map. Then, there exists $x \in K$ such that*

$$\langle Tx, y - x \rangle + f(y) - f(x) \not\leq_{\text{int}_P} 0, \quad \forall y \in K.$$

Let $T : K \to 2^{L(X,Y)}$ be a vector multifunction. Given maps $A : L(X,Y) \to L(X,Y)$ and $f : K \to Y$, consider the following generalized vector variational inequality problem (for short, GVVI): Find $x \in K$ and $u \in T(x)$ such that

$$\langle Au, y - x\rangle + f(y) - f(x) \not\leq_{\text{int}P} 0, \quad \forall y \in K.$$

Inspired by Huang and Fang [6], Zeng and Yao [8] established the solvability result for this class of GVVIs with monotone vector multifunctions.

Theorem 1.2. (See [8, Theorem 2.1]) *Let K be a nonempty, bounded closed and convex subset of a real reflexive Banach space X and Y be a real Banach space ordered by a proper closed convex and pointed cone C with apex at the origin and int$C \neq \emptyset$. Let $A : L(X,Y) \to L_c(X,Y)$ be a continuous map, $T : K \to 2^{L(X,Y)}$ be a nonempty compact valued multifunction which is H-hemicontinuous and monotone with respect to A, and $f : K \to Y$ be a completely continuous and convex map. Then there exist $x \in K$ and $u \in T(x)$ such that*

$$\langle Au, y - x\rangle + f(y) - f(x) \not\leq_{\text{int}P} 0, \quad \forall y \in K.$$

Motivated by Theorems 1.1 and 1.2, we will continue the study of the solvability for the above class of GVVIs in reflexive Banach spaces in this paper. Assume that $f : K \to Y$ is a convex map which is completely continuous on some nonempty, bounded, closed and convex subset C of K. Let $A : L(X,Y) \to L_c(X,Y)$ be such that the map $u \mapsto \langle Au, y\rangle$ is completely continuous for each $y \in X$, and $T : K \to 2^{L(X,Y)}$ be a nonempty weakly compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to A and f. It is shown that there hold the following:

(i) there exist $x_0 \in C$ and $u_0 \in T(x_0)$ such that

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\text{int}P} 0, \quad \forall y \in C;$$

(ii) if

$$-AT(x) \subset (N_C(x) \setminus \{0\})^c, \quad \forall x \in C,$$

where $N_C(x) = \{u \in L(X,Y) : \langle u, x - y\rangle + f(y) - f(x) \not\leq_{\text{int}P} 0, \ \forall y \in C\}$ for all $x \in C$, then $(AT)^{-1}0 \neq \emptyset$.

On the other hand, if $0 \in K$, $f : K \to Y$ is a convex map which is completely continuous on $K \cap B_r$ where $B_r = \{x \in X : \|x\| \leq r\}$ for some $r > 0$, and there holds the condition:

$$\langle Av, y\rangle + f(y) - f(0) \geq_{\text{int}P} 0, \quad \forall v \in T(y), \ y \in K \text{ with } \|y\| = r,$$

then it is also shown that the following hold:

(i) there exist $x_0 \in K$ and $u_0 \in T(x_0)$ such that

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\text{int}P} 0, \quad \forall y \in K.$$

(ii) if

$$-AT(x) \subset (N_K(x) \setminus \{0\})^c, \quad \forall x \in K,$$

where $N_K(x) = \{u \in L(X, Y) : \langle u, x - y\rangle + f(y) - f(x) \not\leq_{\text{int}P} 0, \ \forall y \in K\}$ for all $x \in K$, then $(AT)^{-1}0 \neq \emptyset$.

Furthermore, we apply these results to study the existence of zero points of pseudomonotone multifunctions in reflexive Banach spaces. By virtue of these results, we first derive two existence theorems for zero points of pseudomonotone multifunctions in reflexive Banach spaces. It is worth while to point out that Matsushita and Takahashi [20] obtained the similar theorem for zero points of pseudomonotone multifunctions by using the techniques in Shih and Tan [14] and Yao [15, 16]. Also, we apply our existence theorems for zero points to obtain an existence result with a coercive condition which is related to Browder [12] and Minty [13]. Further we characterize the existence of zero points of pseudomonotone multifunctions in reflexive, strictly convex and smooth Banach spaces. Compared with the corresponding results in Matsushita and Takahashi [20], our results remove the requirement that T takes convex values. Moreover, the pseudomonotonicity of multifunctions in our results is more general than that in Matsushita and Takahashi [20]. In addition, our results are very different from those in Matsushita and Takahashi [21] because they established the existence theorems of zeros of monotone operators in reflexive Banach spaces.

2. Solvability of the GVVI with Pseudomonotonicity

In this section, we will prove the solvability for GVVIs with pseudomonotone vector multifunctions in reflexive Banach spaces by using the KKM-Fan lemma and Nadler's theorem. First we recall some concepts and lemmas.

Let D be a nonempty subset of a topological vector space E. A multivalued map $G : D \to 2^E$ is called a KKM map if for each finite subset $\{x_1, x_2, ..., x_n\} \subset D$,

$$\text{conv}\{x_1, x_2, ..., x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$$

where $\text{conv}\{x_1, x_2, ..., x_n\}$ denotes the convex hull of $\{x_1, x_2, ..., x_n\}$.

Lemma 2.1. (See Fan [2]). *Let D be an arbitrary nonempty subset of a Hausdorff topological vector space E. Let the multivalued mapping $G : D \to 2^E$*

be a KKM map such that $G(x)$ is closed for all $x \in D$ and is compact for at least one $x \in D$. Then

$$\bigcap_{x \in D} G(x) \neq \emptyset.$$

Lemma 2.2. (See Chen and Yang [1]). *Let Y be a real Banach space ordered by a closed, convex and pointed cone P with apex at the origin and int$C \neq \emptyset$. Then, for any $a, b, c \in Y$, the following hold:*

(i) $c \not\leq_{\text{int}P} a$ *and* $a \geq_P b$ *imply that* $c \not\leq_{\text{int}P} b$*;*

(ii) $c \not\geq_{\text{int}P} a$ *and* $a \leq_P b$ *imply that* $c \not\geq_{\text{int}P} b$*.*

Remark 2.1. Utilizing Lemma 2.2, we conclude from Definition 1.1 that the following relations hold:

(i) [T is pseudomonotone] $\Leftrightarrow$ [T is pseudomonotone with respect to I], where I is the identity mapping of $L(X, Y)$;

(ii) [T is monotone with respect to A] $\Rightarrow$ [T is pseudomonotone with respect to A and f]

$$\Downarrow$$

[T is pseudomonotone with respect to A].

Remark 2.2. If $Y = (-\infty, \infty)$, then the definition of pseudomonotonicity for multifunction $T : K \to 2^{L(X,Y)}$ reduces to the one of pseudomonotonicity for multifunction $T : K \to 2^{X^*}$ (in the sense of Karamardian [11]); i.e., for any $x, y \in K,\ u \in T(x)$ and $v \in T(y)$

$$\langle u, y - x \rangle \geq 0 \ \Rightarrow \ \langle v, y - x \rangle \geq 0.$$

Lemma 2.3. *Let K be a nonempty, closed and convex subset of a real Banach space X and Y be a real Banach space ordered by a closed, convex and pointed cone P with apex at the origin and int$C \neq \emptyset$. Let $A : L(X, Y) \to L(X, Y)$ be such that the map $u \mapsto \langle Au, y \rangle$ is completely continuous for each $y \in X$, and $T : K \to 2^{L(X,Y)}$ be a nonempty weakly compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to A and f where $f : K \to Y$ be a convex map. Then the following statements are equivalent:*

(a) *there exists $x_0 \in K$ and $u_0 \in T(x_0)$ such that*

$$\langle Au_0, y - x_0 \rangle + f(y) - f(x_0) \not\leq_{\text{int}P} 0, \quad \forall y \in K;$$

(b) *there exists* $x_0 \in K$ *such that*

$$\langle Av, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\,\mathrm{int}P} 0, \quad \forall y \in K,\ v \in T(y).$$

Proof. Suppose that there exist $x_0 \in K$ and $u_0 \in T(x_0)$ such that

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\,\mathrm{int}P} 0, \quad \forall y \in K.$$

Since T is pseudomonotone with respect to A and f,

$$\langle Av, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\,\mathrm{int}P} 0, \quad \forall y \in K, v \in T(y).$$

Conversely, suppose that there exists $x_0 \in K$ such that

$$\langle Av, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\,\mathrm{int}P} 0, \quad \forall y \in K, v \in T(y).$$

Given any $y \in K$, we know that $y_t = ty + (1-t)x_0 \in K,\ \forall t \in (0,1)$ since K is convex. Replacing y by y_t in the left-hand side of the last inequality, one derives for each $v_t \in T(y_t)$,

$$\begin{aligned}
&\langle Av_t, y_t - x_0\rangle + f(y_t) - f(x_0)\\
&= \langle Av_t, ty + (1-t)x_0 - x_0\rangle + f(ty + (1-t)x_0) - f(x_0)\\
&\leq_{\,P} \langle Av_t, t(y - x_0)\rangle + tf(y) + (1-t)f(x_0) - f(x_0)\\
&= t[\langle Av_t, y - x_0\rangle + f(y) - f(x_0)].
\end{aligned}$$

By Lemma 2.2,

$$\text{(1)} \qquad \langle Av_t, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\,\mathrm{int}P} 0, \quad \forall v_t \in T(y_t),\ t \in (0,1).$$

Since $T : K \to 2^{L(X,Y)}$ be a nonempty weakly compact valued multifunction, $T(y_t)$ and $T(x_0)$ are weakly compact and hence lie in $CB(L(X,Y))$. From Lemma 1.1 it follows that for each $t \in (0,1)$ and each fixed $v_t \in T(y_t)$ there exists an $u_t \in T(x_0)$ such that

$$\|v_t - u_t\| \leq (1+t)H(T(y_t), T(x_0)).$$

Since $T(x_0)$ is weakly compact, without loss of generality, we may assume that $u_t \rightharpoonup u_0 \in T(x_0)$ as $t \to 0^+$. Since T is H^*-hemicontinuous, $H(T(y_t), T(x_0)) \to 0$ as $t \to 0^+$. Thus one has for each $\hbar \in (L(X,Y))^*$,

$$\begin{aligned}
\langle \hbar, v_t - u_0\rangle &= \langle \hbar, v_t - u_t\rangle + \langle \hbar, u_t - u_0\rangle\\
&\leq \|\hbar\|\|v_t - u_t\| + \langle \hbar, u_t - u_0\rangle\\
&\leq \|\hbar\|(1+t)H(T(y_t), T(x_0)) + \langle \hbar, u_t - u_0\rangle \to 0 \quad \text{as } t \to 0^+.
\end{aligned}$$

This implies that $v_t \rightharpoonup u_0$ as $t \to 0^+$. Note that the map $u \mapsto \langle Au, y\rangle$ is completely continuous for each $y \in X$. Hence we deduce that for each $y \in K$

$$\langle Av_t, y - x_0\rangle \to \langle Au_0, y - x_0\rangle \quad \text{as } t \to 0^+.$$

Also from (1) we know that

$$\langle Av_t, y - x_0\rangle + f(y) - f(x_0) \in Y \setminus (-\text{int}P).$$

Since $Y \setminus (-\text{int}P)$ is closed, we have

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \in Y \setminus (-\text{int}P),$$

and so

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\text{int}P} 0, \quad \forall y \in K.$$

Finally, let us show that the vector u_0 in the last inequality is not dependent on y, that is,

$$\langle Au_0, z - x_0\rangle + f(z) - f(x_0) \not\leq_{\text{int}P} 0, \quad \forall z \in K.$$

Indeed, take a fixed $z \in K$ arbitrarily and define $z_t = x + t(z - x)$ for all $t \in (0, 1)$. Utilizing Lemma 1.1, for each $u_t \in T(x_0)$ where $t \in (0, 1)$ there exists $w_t \in T(z_t)$ such that $\|u_t - w_t\| \leq (1 + t)H(T(x_0), T(z_t))$. Since T is H^*-hemicontinuous, $H(T(x_0), T(z_t)) \to 0$ as $t \to 0^+$. Thus one has for each $\hbar \in (L(X, Y))^*$,

$$\begin{aligned}\langle \hbar, w_t - u_0\rangle &= \langle \hbar, w_t - u_t\rangle + \langle \hbar, u_t - u_0\rangle \\ &\leq \|\hbar\|\|u_t - w_t\| + \langle \hbar, u_t - u_0\rangle \\ &\leq \|\hbar\|(1 + t)H(T(x_0), T(z_t)) + \langle \hbar, u_t - u_0\rangle \to 0 \quad \text{as } t \to 0^+.\end{aligned}$$

This implies that $w_t \rightharpoonup u_0$ as $t \to 0^+$. Note that the map $u \mapsto \langle Au, y\rangle$ is completely continuous for each $y \in X$. Hence we deduce that for $z \in K$

$$\langle Aw_t, z - x_0\rangle \to \langle Au_0, z - x_0\rangle \quad \text{as } t \to 0^+.$$

Also from (1) we know that

$$\langle Aw_t, z - x_0\rangle + f(z) - f(x_0) \in Y \setminus (-\text{int}P), \quad \forall t \in (0, 1).$$

Since $Y \setminus (-\text{int}P)$ is closed, we have

$$\langle Au_0, z - x_0\rangle + f(z) - f(x_0) \in Y \setminus (-\text{int}P),$$

and so

$$\langle Au_0, z - x_0\rangle + f(z) - f(x_0) \not\leq_{\text{int}P} 0, \quad \forall z \in K.$$

The proof is complete. ■

Theorem 2.1. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X and Y be a real Banach space ordered by a proper closed convex and pointed cone P with apex at the origin and int$P \neq \emptyset$. Assume that $f : K \to Y$ is a convex map which is completely continuous on some nonempty, bounded, closed and convex subset C of K. Let $A : L(X,Y) \to L_c(X,Y)$ be such that the map $u \mapsto \langle Au, y\rangle$ is completely continuous for each $y \in X$, and $T : K \to 2^{L(X,Y)}$ be a nonempty weakly compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to A and f. Then the following statements hold:*

(i) *there exist $x_0 \in C$ and $u_0 \in T(x_0)$ such that*

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\,\mathrm{int}P} 0, \quad \forall y \in C;$$

(ii) *if*

$$-AT(x) \subset (N_C(x) \setminus \{0\})^c, \quad \forall x \in C, \tag{2}$$

where $N_C(x) = \{u \in L(X,Y) : \langle u, x - y\rangle + f(y) - f(x) \not\leq_{\,\mathrm{int}P} 0,\ \forall y \in C\}$ for all $x \in C$, then $(AT)^{-1}0 \neq \emptyset$.

Proof. We define two multivalued maps $F, G : C \to 2^C$ as follows:

$$F(y) = \{x \in C : \langle Au, y-x\rangle + f(y) - f(x) \not\leq_{\,\mathrm{int}P} 0 \text{ for some } u \in T(x)\}, \quad \forall y \in C$$

and

$$G(y) = \{x \in C : \langle Av, y-x\rangle + f(y) - f(x) \not\leq_{\,\mathrm{int}P} 0 \text{ for all } v \in T(y)\}, \quad \forall y \in C.$$

Then $F(y)$ and $G(y)$ are nonempty due to $y \in G(y) \cap F(y)$. We claim that F is a KKM mapping. If this is false, then there exist a finite set $\{x_1, ..., x_n\} \subset C$ and $t_i \geq 0,\ i = 1, 2, ..., n$ with $\sum_{i=1}^n t_i = 1$ such that

$$x = \sum_{i=1}^n t_i x_i \notin \bigcup_{i=1}^n F(x_i).$$

Hence for any $u \in T(x)$ one has

$$\langle Au, x_i - x\rangle + f(x_i) - f(x) \leq_{\,\mathrm{int}P} 0, \quad i = 1, 2, ..., n.$$

It follows that

$$\begin{aligned} 0 &= \langle Au, x - x\rangle + f(x) - f(x) \\ &\geq_P \sum_{i=1}^{n} t_i \langle Au, x - x_i\rangle + f(x) - \sum_{i=1}^{n} t_i f(x_i) \\ &= \sum_{i=1}^{n} t_i [\langle Au, x - x_i\rangle + f(x) - f(x_i)] \\ &\geq_{\mathrm{int}P} 0 \end{aligned}$$

which leads to a contradiction since P is proper. So F is a KMM mapping. Furthermore we can prove that $F(y) \subset G(y)$ for every $y \in C$. Indeed, let $x \in F(y)$. Then for some $u \in T(x)$ one has

$$\langle Au, y - x\rangle + f(y) - f(x) \not\leq_{\mathrm{int}P} 0.$$

Since T is pseudomonotone with respect to A and f, one has

$$\langle Av, y - x\rangle + f(y) - f(x) \not\leq_{\mathrm{int}P} 0, \quad \forall y \in C,\ v \in T(y).$$

Hence $F(y) \subset G(y)$ for all $y \in C$, and so G is also a KMM mapping. Now we claim that for each $y \in C$, $G(y) \subset C$ is closed in the weak topology of X. Indeed, suppose $\bar{x} \in \overline{G(y)}^w$, the weak closure of $G(y)$. Then there exists a sequence $\{x_n\}$ in $G(y)$ such that $\{x_n\}$ converges weakly to $\bar{x} \in C$. So we derive for each $v \in T(y)$

$$\langle Av, y - x_n\rangle + f(y) - f(x_n) \not\leq_{\mathrm{int}P} 0$$

which implies that

$$\langle Av, y - x_n\rangle + f(y) - f(x_n) \in Y \setminus (-\mathrm{int}P).$$

Since Av and f are completely continuous and $Y \setminus (-\mathrm{int}P)$ is closed, so

$$\langle Av, y - x_n\rangle + f(y) - f(x_n) \to \langle Av, y - \bar{x}\rangle + f(y) - f(\bar{x})$$

and $\langle Av, y - \bar{x}\rangle + f(y) - f(\bar{x}) \in Y \setminus (-\mathrm{int}P)$. Thus we get

$$\langle Av, y - \bar{x}\rangle + f(y) - f(\bar{x}) \not\leq_{\mathrm{int}P} 0,$$

and so $\bar{x} \in G(y)$. This shows that $G(y)$ is weakly closed for each $y \in C$. Since X is reflexive and $C \subset K$ is nonempty, bounded, closed and convex, C is a weakly compact subset of X and so $G(y)$ is also weakly compact. According to Lemma 2.1,

$$\bigcap_{y \in C} G(y) \neq \emptyset.$$

This implies that there exists $x_0 \in C$ such that

$$\langle Av, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\mathrm{int}P} 0, \quad \forall y \in C,\ v \in T(y).$$

Therefore by applying Lemma 2.3, we conclude that there exist $x_0 \in C$ and $u_0 \in T(x_0)$ such that

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\text{int}P} 0, \quad \forall y \in C. \tag{3}$$

This shows that the conclusion (i) is valid.

Further, in terms of (3) and the definition of $N_C(x)$, i.e.,

$$N_C(x) = \{u \in L(X,Y) : \langle u, x - y\rangle + f(y) - f(x) \not\leq_{\text{int}P} 0,\ \forall y \in C\}$$

for all $x \in C$, we conclude that

$$-Au_0 \in N_C(x_0) = \{u \in L(X,Y) : \langle u, x_0 - y\rangle + f(y) - f(x_0) \not\leq_{\text{int}P} 0,\ \forall y \in C\}.$$

Suppose $-Au_0 \neq 0$. Then

$$-Au_0 \in N_C(x_0) \setminus \{0\}.$$

This shows that there exists $x_0 \in C$ such that

$$(-AT(x_0)) \cap (N_C(x_0) \setminus \{0\}) \neq \emptyset,$$

which hence contradicts (2). Consequently, $Au_0 = 0$ and so $(AT)^{-1}0 \neq \emptyset$. The proof is complete. ■

Corollary 2.1. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y be a real Banach space ordered by a proper closed convex and pointed cone P with apex at the origin and int$P \neq \emptyset$. Assume that $f : K \to Y$ is a convex and completely continuous map. Let $A : L(X,Y) \to L_c(X,Y)$ be such that the map $u \mapsto \langle Au, y\rangle$ is completely continuous for each $y \in X$, and $T : K \to 2^{L(X,Y)}$ be a nonempty weakly compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to A and f. Then there exist $x_0 \in K$ and $u_0 \in T(x_0)$ such that*

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\text{int}P} 0, \quad \forall y \in K.$$

Proof. Putting $K = C$, by Theorem 2.1 we obtain the desired result. ■

Corollary 2.2. *Let K be a nonempty, bounded, closed and convex subset of $X = R^n$ and Y be a real Banach space ordered by a proper closed convex and pointed cone P with apex at the origin and int$P \neq \emptyset$. Assume that $f : K \to Y$ is a convex and continuous map. Let $A : L(R^n, Y) \to L(R^n, Y)$ be such that the map $u \mapsto \langle Au, y\rangle$ is completely continuous for each $y \in X$, and $T : K \to 2^{L(R^n,Y)}$ be a nonempty weakly compact valued multifunction which is H^*-hemicontinuous and* *pseudomonotone with respect to* A *and* f. *Then there exist* $x_0 \in K$ *and* $u_0 \in T(x_0)$ *such that*

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\text{int}P} 0, \quad \forall y \in K.$$

If the boundedness of K in Corollary 2.1 is dropped off, then we have the following theorem under certain coercivity condition:

Theorem 2.2. *Let* K *be a nonempty, closed and convex subset of a real reflexive Banach space* X *with* $0 \in K$ *and* Y *be a real Banach space ordered by a proper closed convex and pointed cone* P *with apex at the origin and int*$P \neq \emptyset$. *Assume that* $f : K \to Y$ *is a convex map which is completely continuous on* $K \cap B_r$ *where* $B_r = \{x \in X : \|x\| \leq r\}$ *for some* $r > 0$. *Let* $A : L(X,Y) \to L_c(X,Y)$ *be such that the map* $u \mapsto \langle Au, y\rangle$ *is completely continuous for each* $y \in X$, *and* $T : K \to 2^{L(X,Y)}$ *be a nonempty weakly compact valued multifunction which is* H^**-hemicontinuous and pseudomonotone with respect to* A *and* f. *Suppose there holds the condition:*

$$(4) \qquad \langle Av, y\rangle + f(y) - f(0) \geq_{\text{int}P} 0, \quad \forall v \in T(y),\ y \in K \text{ with } \|y\| = r.$$

Then the following statements hold:

(i) *there exist* $x_0 \in K$ *and* $u_0 \in T(x_0)$ *such that*

$$\langle Au_0, y - x_0\rangle + f(y) - f(x_0) \not\leq_{\text{int}P} 0, \quad \forall y \in K.$$

(ii) *if*

$$-AT(x) \subset (N_K(x) \setminus \{0\})^c, \quad \forall x \in K,$$

where $N_K(x) = \{u \in L(X,Y) : \langle u, x - y\rangle + f(y) - f(x) \not\leq_{\text{int}P} 0,\ \forall y \in K\}$ *for all* $x \in K$, *then* $(AT)^{-1}0 \neq \emptyset$.

Proof. Put $C = K \cap B_r$ where $B_r = \{x \in X : \|x\| \leq r\}$. By Theorem 2.1, there exist $x_r \in C$ and $u_r \in T(x_r)$ such that

$$(5) \qquad \langle Au_r, y - x_r\rangle + f(y) - f(x_r) \not\leq_{\text{int}P} 0, \quad \forall y \in C.$$

Putting $y = 0$ in the above inequality, one has

$$(6) \qquad \langle Au_r, x_r\rangle + f(x_r) - f(0) \not\geq_{\text{int}P} 0.$$

Combining (4) with (6), we know that $\|x_r\| < r$. For any $z \in K$ choose $t \in (0, 1)$ enough small such that $(1-t)x_r + tz \in C$. Putting $y = (1-t)x_r + tz$ in (5), one has

$$\langle Au_r, (1-t)x_r + tz - x_r\rangle + f((1-t)x_r + tz) - f(x_r) \not\leq_{\text{int}P} 0.$$

Since f is convex,

$$\begin{aligned}&\langle Au_r,(1-t)x_r+tz-x_r\rangle+f((1-t)x_r+tz)-f(x_r)\\ \le\ & _Ct\langle Au_r,z-x_r\rangle+(1-t)f(x_r)+tf(z)-f(x_r)\\ =\ & t[\langle Au_r,z-x_r\rangle+f(z)-f(x_r)].\end{aligned}$$

By Lemma 2.2,

$$\langle Au_r,z-x_r\rangle+f(z)-f(x_r)\not\leq_{\,\mathrm{int}P}0,\quad \forall z\in K.$$

Further, in terms of the last inequality and the definition of $N_K(x)$, i.e.,

$$N_K(x)=\{u\in L(X,Y):\langle u,x-y\rangle+f(y)-f(x)\not\leq_{\,\mathrm{int}P}0,\ \forall y\in K\}$$

for all $x\in K$, we conclude that

$$-Au_r\in N_K(x_r)=\{u\in L(X,Y):\langle u,x_r-y\rangle+f(y)-f(x_r)\not\leq_{\,\mathrm{int}P}0,\ \forall y\in K\}.$$

Suppose $-Au_r\neq 0$. Then

$$-Au_r\in N_K(x_r)\setminus\{0\}.$$

This shows that there exists $x_r\in K$ such that

$$(-AT(x_r))\cap(N_K(x_r)\setminus\{0\})\neq\emptyset,$$

which hence leads to a contradiction. Consequently, $Au_r=0$ and so $(AT)^{-1}0\neq\emptyset$. The proof is complete. ■

Corollary 2.3. *Let K be a nonempty, closed and convex subset of $X=R^n$ with $0\in K$ and Y be a real Banach space ordered by a proper closed convex and pointed cone P with apex at the origin and int$P\neq\emptyset$. Assume that $f:K\to Y$ is a convex map which is continuous on $K\cap B_r$ where $B_r=\{x\in R^n:\|x\|\le r\}$ for some $r>0$. Let $A:L(R^n,Y)\to L(R^n,Y)$ be such that the map $u\mapsto\langle Au,y\rangle$ is completely continuous for each $y\in X$, and $T:K\to 2^{L(R^n,Y)}$ be a nonempty weakly compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to A and f. If the following holds:*

$$\langle Av,y\rangle+f(y)-f(0)\geq_{\mathrm{int}P}0,\quad \forall v\in T(y),\ y\in K \text{ with } \|y\|=r,$$

then there exist $x_0\in K$ and $u_0\in T(x_0)$ such that

$$\langle Au_0,y-x_0\rangle+f(y)-f(x_0)\not\leq_{\,\mathrm{int}P}0,\quad \forall y\in K.$$

3. Applications

Utilizing Theorems 2.1 and 2.2 in Section 2, we will derive some new results on the existence of zero points of pseudomonotone multifunctions in real reflexive Banach spaces.

Theorem 3.1. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X. Assume that $f : K \to (-\infty, \infty)$ is a convex function which is weakly sequential continuous on some nonempty, bounded, closed and convex subset C of K. Let $A : X^* \to X^*$ be such that the map $u \mapsto \langle Au, y\rangle$ is weak* sequentially continuous for each $y \in X$, and $T : K \to 2^{X^*}$ be a nonempty weak* compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to A and f. If*

$$-AT(x) \subset (N_C(x) \setminus \{0\})^c, \quad \forall x \in C, \tag{7}$$

where $N_C(x) = \{u \in X^* : \langle u, x - y\rangle + f(y) - f(x) \geq 0,\ \forall y \in C\}$ for all $x \in C$, then $(AT)^{-1}0 \neq \emptyset$.

Proof. Putting $Y = (-\infty, \infty)$ and $P = [0, \infty)$, we know that int$P = (0, \infty)$, $L(X, Y) = X^*$ the dual of X and

$$L_c(X, Y) = L(X, Y) = X^*.$$

In this case, we also know that $(L(X, Y))^* = X^{**} = X$ and

$$\sigma(L(X, Y), (L(X, Y))^*) = \sigma(X^*, X) \text{ the weak}^* \text{ topology of } X^*.$$

Obviously, it is easy to see that $f : K \to (-\infty, \infty)$ is a convex function which is completely continuous on the nonempty, bounded, closed and convex subset C of K and that $A : X^* \to X^*$ is such that the map $u \mapsto \langle Au, y\rangle$ is completely continuous for each $y \in X$. Therefore, by Theorem 2.1 we obtain the desired result. ∎

Corollary 3.1. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X. Assume that $f : K \to (-\infty, \infty)$ is a convex function which is weakly sequentially continuous on some nonempty, bounded, closed and convex subset C of K. Let $T : K \to 2^{X^*}$ be a nonempty weak* compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to I and f. If*

$$-T(x) \subset (N_C(x) \setminus \{0\})^c, \quad \forall x \in C,$$

where $N_C(x) = \{u \in X^* : \langle u, x - y\rangle + f(y) - f(x) \geq 0,\ \forall y \in C\}$ for all $x \in C$, then $T^{-1}0 \neq \emptyset$.

Proof. Put $A = I$ the identity mapping of X^*. Then it is clear that $I : X^* \to X^*$ is such that the map $u \mapsto \langle I(u), y\rangle$ is weak* sequentially continuous for each $y \in X$. Hence by Theorem 3.1 we obtain the desired result. ■

Theorem 3.2. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with $0 \in K$. Assume that $f : K \to (-\infty, \infty)$ is a convex function which is weakly sequential continuous on $K \cap B_r$ where $B_r = \{x \in X : \|x\| \leq r\}$ for some $r > 0$. Let $A : X^* \to X^*$ be such that the map $u \mapsto \langle Au, y\rangle$ is weak* sequentially continuous for each $y \in X$, and $T : K \to 2^{X^*}$ be a nonempty weak* compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to A and f. Suppose there holds the condition:*

$$\langle Av, y\rangle + f(y) - f(0) > 0, \quad \forall v \in T(y),\ y \in K \text{ with } \|y\| = r.$$

If

$$-AT(x) \subset (N_K(x) \setminus \{0\})^c, \quad \forall x \in K,$$

where $N_K(x) = \{u \in L(X, Y) : \langle u, x - y\rangle + f(y) - f(x) \geq 0,\ \forall y \in K\}$ for all $x \in K$, then $(AT)^{-1}0 \neq \emptyset$.

Proof. Putting $Y = (-\infty, \infty)$ and $P = [0, \infty)$, by Theorem 2.2 we obtain the desired result.

Corollary 3.2. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with $0 \in K$. Assume that $f : K \to (-\infty, \infty)$ is a convex function which is weakly sequential continuous on $K \cap B_r$ where $B_r = \{x \in X : \|x\| \leq r\}$ for some $r > 0$. Let $T : K \to 2^{X^*}$ be a nonempty weak* compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to I and f. Suppose there holds the condition:*

$$\langle v, y\rangle + f(y) - f(0) > 0, \quad \forall v \in T(y),\ y \in K \text{ with } \|y\| = r.$$

If

$$-T(x) \subset (N_K(x) \setminus \{0\})^c, \quad \forall x \in K,$$

where $N_K(x) = \{u \in L(X, Y) : \langle u, x - y\rangle + f(y) - f(x) \geq 0,\ \forall y \in K\}$ for all $x \in K$, then $T^{-1}0 \neq \emptyset$.

Let $T : X \to 2^{X^*}$ and $f : X \to (-\infty, \infty)$. For $r > 0$ and $x_0 \in X$, we consider the following generalized variational inequality (for short, GVI($T, f, B_r[x_0]$)): Find $\hat{x} \in B_r[x_0]$ and $u^* \in T(\hat{x})$ such that

$$\langle u^*, y - \hat{x}\rangle + f(y) - f(\hat{x}) \geq 0, \quad \forall y \in B_r[x_0],$$

where $B_\delta[x_0] = \{x \in X : \|x - x_0\| \leq \delta\}$. We say that the pair $(\hat{x}, u^*)$ solves the GVI$(T, f, B_r[x_0])$. Next utilizing Theorem 3.1 we establish an existence result under the assumptions that T satisfies coercivity condition and X is reflexive; for related coercivity condition, see Yao [16].

Theorem 3.3. *Let X be a real reflexive Banach space and let $f : X \to (-\infty, \infty)$ be a convex and weakly sequentially continuous function. Let $T : X \to 2^{X^*}$ be a nonempty weak* compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to I and f. Suppose there exists $\hat{x} \in X$ such that the following hold:*

(i)

$$\lim_{y \to \hat{x}} \Big(\inf_{y^* \in T(y)} \frac{\langle y^*, y - \hat{x}\rangle + f(y) - f(\hat{x})}{\|y - \hat{x}\|} \Big) > 0; \tag{8}$$

(ii) *If for some $r > 0$ and some $u^* \in T(\hat{x})$, $(\hat{x}, u^*)$ solves the GVI$(T, f, B_r[\hat{x}])$, then $u^* = 0$.*
Then $T^{-1}0 \neq \emptyset$.

Proof. From condition (8), there exists $\delta > 0$ such that

$$\inf_{y^* \in T(y)} \frac{\langle y^*, y - \hat{x}\rangle + f(y) - f(\hat{x})}{\|y - \hat{x}\|} > 0, \quad \forall y \in B_\delta[\hat{x}] \setminus \{\hat{x}\}.$$

In particular,

$$\inf_{y^* \in T(y)} \langle y^*, y - \hat{x}\rangle + f(y) - f(\hat{x}) > 0, \quad \forall y \in B_\delta[\hat{x}] \setminus \{\hat{x}\}.$$

It follows that

$$-T(y) \subset (N_{B_\delta[\hat{x}]}(y) \setminus \{0\})^c, \quad \forall y \in B_\delta[\hat{x}] \setminus \{\hat{x}\}, \tag{9}$$

where $N_{B_\delta[\hat{x}]}(y) = \{x^* \in X^* : \langle x^*, y - x\rangle + f(x) - f(y) \geq 0,\ \forall x \in B_\delta[\hat{x}]\}$ for all $y \in B_\delta[\hat{x}]$.

On the other hand, let us show that

$$-T(\hat{x}) \subset (N_{B_\delta[\hat{x}]}(\hat{x}) \setminus \{0\})^c.$$

Indeed, if this is false, then

$$-T(\hat{x}) \cap (N_{B_\delta[\hat{x}]}(\hat{x}) \setminus \{0\}) \neq \emptyset.$$

Hence there exists $u^* \in T(\hat{x})$ with $u^* \neq 0$ such that

$$\langle -u^*, \hat{x} - x\rangle + f(x) - f(\hat{x}) \geq 0, \quad \forall x \in B_\delta[\hat{x}],$$

and hence

$$\langle u^*, x-\hat{x}\rangle + f(x) - f(\hat{x}) \geq 0, \quad \forall x \in B_\delta[\hat{x}].$$

Thus, $(\hat{x}, u^*)$ solves the GVI$(T, f, B_\delta[\hat{x}])$. From condition (ii) we conclude that $u^* = 0$. This leads to a contradiction. Consequently, from (9) we know that

$$-T(y) \subset (N_{B_\delta[\hat{x}]}(y) \setminus \{0\})^c, \quad \forall y \in B_\delta[\hat{x}].$$

Note that X is reflexive and $B_\delta[\hat{x}]$ is a nonempty, bounded, closed and convex subset of X. Now, putting $K = X$ and $C = B_\delta[\hat{x}]$, by Theorem 3.1 we have $T^{-1}0 \neq \emptyset$. The proof is complete. ∎

Remark 3.1. Motivated by Browder [12] and Minty [13], Matsushita and Takahashi [20] considered the coercivity condition similar to (8) under the assumption that $T : X \to 2^{X^*}$ is a pseudomonotone operator such that each Tx is a nonempty weakly compact convex subset of X^* as follows: There exists $\hat{x} \in X$ such that

$$\lim_{\|y\|\to\infty} \Big(\inf_{y^* \in T(y)} \frac{\langle y^*, y-\hat{x}\rangle}{\|y\|} \Big) = \infty.$$

It is well known that Asplund [18] has shown that a reflexive Banach space X has an equivalent norm such that X is a strictly convex and smooth Banach space. A Banach space X is said to be smooth provided the limit

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t} \tag{10}$$

exists for each $x, y \in X$ with $\|x\| = \|y\| = 1$. In this case, the norm of X is said to be Gâteaux differentiable. X is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|\|x\|,\ \|x^*\| = \|x\|\}.$$

Without confusion, one understands that $\|x^*\|$ is the X^*-norm and $\|x\|$ is the X-norm. Many properties of the normalized duality mapping J have been studied. For the details, one may see Takahashi [17]. We list some properties below for easy reference:

(P_1) for any $x \in X$, $J(x)$ is nonempty, bounded, closed and convex;

(P_2) for any $x \in X$ and a real number α, $J(\alpha x) = \alpha J(x)$;

(P_3) if X is reflexive, J is a mapping of X onto X^*;

(P_4) if X is smooth, J is a single-valued continuous mapping on X;

(P_5) if X is strictly convex, J is one-to-one.

Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space. For any $x \in X$, there exists a unique point $x_0 \in C$ such that

$$\|x_0 - x\| = \min_{y \in C} \|y - x\|.$$

The mapping $P_C : X \to C$ defined by $P_C x = x_0$ is called the metric projection from X onto C. For each $x \in X$, $P_C x$ satisfies

$$\langle J(x - P_C x), P_C x - y \rangle \geq 0, \quad \forall y \in C. \tag{11}$$

(see [17, 19] for more details).

Further we characterize the existence of zero points of a pseudomonotone multifunction under the assumptions that $K = X$ and X is reflexive, strictly convex and smooth.

Theorem 3.4. *Let X be a real reflexive, strictly convex and smooth Banach space. Let $A : X^* \to X^*$ be such that the map $u \mapsto \langle Au, y \rangle$ is weak* sequentially continuous for each $y \in X$, and let $T : X \to 2^{X^*}$ be a nonempty weak* compact valued multifunction which is H^*-hemicontinuous and pseudomonotone with respect to A. Then the following statements are equivalent:*

(i) $(AT)^{-1}0 \neq \emptyset$;

(ii) *there exists a nonempty, bounded, closed and convex subset C of X such that*

$$-AT(x) \subset (N_C(x) \setminus \{0\})^c, \quad \forall x \in C$$

where $N_C(x) = \{x^* \in X^* : \langle x^*, x - y \rangle \geq 0, \ \forall y \in C\}$ for all $x \in C$.

Proof. First, let us show that (i) $\Rightarrow$ (ii). Indeed, take a fixed $x_0 \in (AT)^{-1}0$ and $r > 0$. Assume that there exists $z \in B_r[x_0]$ such that

$$-AT(z) \cap (N_{B_r[x_0]}(z) \setminus \{0\}) \neq \emptyset,$$

that is, there exists $\bar{u} \in T(z)$ such that

$$-A\bar{u} \in N_{B_r[x_0]}(z) \setminus \{0\}, \tag{12}$$

Since $N_{B_r[x_0]}(x) \setminus \{0\} = \emptyset$ whenever $\|x\| < r$, we may assume $z \in \partial B_r[x_0]$ without loss of generality.

From (12),

$$-A\bar{u} \neq 0 \text{ and } \langle -A\bar{u}, z - y \rangle \geq 0, \ \forall y \in B_r[x_0],$$

and hence

$$-A\bar{u} \neq 0 \text{ and } \langle J(z - J^{-1}A\bar{u} - z), z - y\rangle \geq 0, \ \forall y \in B_r[x_0], .$$

It follows from (11) that

$$z = P_{B_r[x_0]}(z - J^{-1}A\bar{u}).$$

In particular, $z - J^{-1}A\bar{u} \notin B_r[x_0]$. Indeed, if $z - J^{-1}A\bar{u} \in B_r[x_0]$, then

$$z = P_{B_r[x_0]}(z - J^{-1}A\bar{u}) = z - J^{-1}A\bar{u},$$

and hence $-A\bar{u} = 0$, which leads to a contradiction.

Let $t = \frac{r}{\|z - J^{-1}A\bar{u} - x_0\|}$ and $w_0 = t(z - J^{-1}A\bar{u} - x_0) + x_0 \in \partial B_r[x_0]$. We next show that

$$w_0 = P_{B_r[x_0]}(z - J^{-1}A\bar{u}).$$

For each $y \in B_r[x_0]$, we have

$$\begin{aligned}
\langle J(z - J^{-1}A\bar{u} - w_0), w_0 - y\rangle &= \langle J(z - J^{-1}A\bar{u} - x_0 + x_0 - w_0), w_0 - y\rangle \\
&= \langle J(\frac{1}{t}(w_0 - x_0) + x_0 - w_0), w_0 - y\rangle \\
&= \frac{1-t}{t}\langle J(w_0 - x_0), w_0 - x_0 + x_0 - y\rangle \\
&= \frac{1-t}{t}(\|w_0 - x_0\|^2 + \langle J(w_0 - x_0), x_0 - y\rangle) \\
&\geq \frac{1-t}{t}(r^2 - \|w_0 - x_0\|\|x_0 - y\|) \\
&= \frac{1-t}{t}r(r - \|y - x_0\|) \geq 0.
\end{aligned}$$

This together with (11) implies that $w_0 = P_{B_r[x_0]}(z - J^{-1}A\bar{u})$. Consequently,

$$t(z - J^{-1}A\bar{u} - x_0) + x_0 = w_0 = P_{B_r[x_0]}(z - J^{-1}A\bar{u}) = z,$$

and hence

$$A\bar{u} = \frac{1-t}{t}J(x_0 - z).$$

Since T is pseudomonotone with respect to A, from $\bar{u} \in Tz$ and $x_0 \in (AT)^{-1}0$, we have

$$0 \leq \langle A\bar{u}, z - x_0\rangle = \frac{1-t}{t}\langle J(x_0 - z), z - x_0\rangle = -\frac{1-t}{t}\|z - x_0\|^2 < 0,$$

which is a contradiction. Therefore, we deduce that

$$-AT(x) \subset (N_{B_r[x_0]}(x) \setminus \{0\})^c, \quad \forall x \in B_r[x_0].$$

Second, we claim that (ii) $\Rightarrow$ (i). Indeed, it follows from Theorem 3.1 that (ii) implies (i). ■

Corollary 3.3. *Let X be a real reflexive, strictly convex and smooth Banach space and let $T : X \to 2^{X^*}$ be a nonempty weak* compact valued multifunction which is H^*-hemicontinuous and pseudomonotone. Then the following are equivalent:*

(i) $T^{-1}0 \neq \emptyset$;

(ii) *there exists a nonempty, bounded, closed and convex subset C of X such that*

$$-T(x) \subset (N_C(x) \setminus \{0\})^c, \quad \forall x \in C$$

where $N_C(x) = \{x^* \in X^* : \langle x^*, x - y\rangle \geq 0, \ \forall y \in C\}$ for all $x \in C$.

Proof. Put $A = I$ the identity mapping of X^*. Then it is obvious that $I : X^* \to X^*$ is such that the map $u \mapsto \langle Au, y\rangle$ is weak* sequentially continuous for each $y \in X$. By Theorem 3.4 we obtain the desired result. ■

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