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ON STRONGLY REVERSIBLE RINGS

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Abstract. A ring R is called strongly reversible, if whenever polynomials $f(x)$, $g(x)$ in $R[x]$ satisfy $f(x)g(x) = 0$, then $g(x)f(x) = 0$. It is proved that a ring R is strongly reversible if and only if its polynomial ring $R[x]$ is strongly reversible if and only if its Laurent polynomial ring $R[x, x^{-1}]$ is strongly reversible. We also show that for a right Ore ring R with Q its classical right quotient ring, R is strongly reversible if and only if Q is strongly reversible .

1. Introduction

Throughout this paper, unless stated, any ring is associative and has an identity. In [1], Cohn introduced the notion of a reversible ring. A ring R is said to be reversible, if whenever $a, b \in R$ satisfy $ab = 0$, then $ba = 0$. Anderson-Camillo [2] used the term ZC_2 for what is called reversible. While Krempa-Niewieczerzal [3] took the term C_0 for it. In [4], Lambek called R be symmetric, if $rst = 0$ implies $rts = 0$ for all $r, s, t \in R$, while Anderson-Camillo [2] took the term ZC_3 for this notion. A ring R is called semicommutative, if whenever $ab = 0$, then $aRb = 0$ for all $a, b \in R$. Reduced rings (i.e., rings with no nonzero nilpotent elements in R) are symmetric by [4, P. 361], symmetric rings are clearly reversible, and reversible rings are semicommutative by [4, Prop. 1.3], but the converses are not true. Kim and Lee showed that polynomial rings over reversible rings need not be reversible [5, Example 2.1]. In the paper, we consider these reversible rings over which polynomial rings are reversible and call them be strongly reversible, i.e., a ring R is called strongly reversible, if whenever polynomials $f(x)$, $g(x)$ in $R[x]$

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satisfy $f(x)g(x) = 0$, then $g(x)f(x) = 0$. Reversible Armendariz rings are such rings [5, Prop. 2.4], so reduced rings are strongly reversible, but the converse is not true by Proposition 3.5. We will show that strongly reversible rings are not necessarily symmetric and symmetric rings are not strongly reversible in general, though they both two are generalizations of reduced rings. It is proved that a ring R is strongly reversible if and only if its polynomial ring $R[x]$ is strongly reversible if and only if its Laurent polynomial ring $R[x, x^{-1}]$ is strongly reversible. At last, we also show that for a right Ore ring R with Q its classical right quotient ring, R is strongly reversible if and only if Q is strongly reversible.

2. Strongly Reversible Rings and Symmetric Rings

Definition 2.1. A ring R is called strongly reversible, if whenever polynomials $f(x)$, $g(x)$ in $R[x]$ satisfy $f(x)g(x) = 0$, then $g(x)f(x) = 0$.

Clearly, any strongly reversible ring is reversible, but the converse is not true [5, Example 2.1], also, the class of strongly reversible rings is closed under subrings and direct products. It is obvious that any reduced rings are both strongly reversible and symmetric, strongly reversible rings and symmetric rings are all reversible. In this part, we show that strongly reversible rings are not necessarily symmetric and symmetric rings are not strongly reversible in general.

Example 2.1. See [6, Example 5] for detail. Let D be a commutative domain, define the free algebra $F = D < a, b, c >$, and let

$$I = (FaF)^2 + (FbF)^2 + (FcF)^2 + FabcF + FbcaF + FcabF \subset F.$$

Put $R = F/I$. Then R is a local ring generated as a D-module by the following elements:

$$w_0 = 1, w_1 = a, w_2 = b, w_3 = c, w_4 = ab, w_5 = ba, w_6 = ac,$$
$$w_7 = ca, w_8 = bc, w_9 = cb, w_{10} = acb, w_{11} = cba, w_{12} = bac.$$

Obviously, R is not symmetric since $acb \notin I$ and $abc \in I$. Note that $R[x] \simeq F[x]/I[x]$, where $F[x] = D[x] < a, b, c >$ is the free algebra, and $D[x]$ is also a commutative domain, so $R[x]$ is reversible, hence R is strongly reversible.

Example 2.2. We refer to the argument [7, Example 2] and [5, Example 2.1]. Let Z_2 be the field of integers modulo 2 and $A = Z_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra of polynomials with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over Z_2. Note that A is a ring without identity and consider an ideal of the ring $Z_2 + A$, say I, generated by $a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,\ b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2$,$(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2)$, and $r_1r_2r_3r_4$. Where $r, r_1, r_2, r_3, r_4 \in A$. Then clearly $A^4 \in I$. Next let $R = (Z_2+A)/I$ and consider $R[x] \cong ((Z_2+A)[x])/(I[x])$. R is not strongly reversible by [5, Example 2.1]. Next we show that R is symmetric.

Proof. We call each product of the indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ a monomial and say that α is a monomial of degree n if it is a product of exactly n number of indeterminates. Let H_n be the set of all linear combinations of monomials of degree n over Z_2. Notice that H_n is finite for any n and that the ideal I of R is homogeneous (i.e., if $\sum_{i=1}^{s} r_i \in I$ with $r_i \in H_i$ then every r_i is in I).

Suppose $f, g, h \in Z_2 + A$ satisfy $fgh \in I$. We want to show $fhg \in I$. Since R is a reversible local ring, we can assume without loss of generality that $f + I$, $g + I$, and $h + I$ are non-units and hence belong to the maximal ideal A/I of R. write $f = f_1 + f_2 + f_3 + f_4, g = g_1 + g_2 + g_3 + g_4$, and $h = h_1 + h_2 + h_3 + h_4$, where $f_i, g_i, h_i \in H_i$ for $i = 1, 2, 3, 4$. Then

$$\begin{aligned} fgh \in I \quad &\Leftrightarrow f_1g_1h_1 \in I \\ &\Leftrightarrow \{a_i, b_j\} \subseteq \{f_1, g_1, h_1\}\ for\ i, j = 0, 2\ or \{a_0{+}a_1{+}a_2, b_0{+}b_1{+}b_2\} \\ &\qquad \subseteq \{f_1, g_1, h_1\} \\ &\Leftrightarrow f_1h_1g_1 \in I \\ &\Leftrightarrow fhg \in I. \end{aligned}$$

Thus we obtain that R is symmetric but not strongly reversible.

3. Strongly Reversible Rings

Proposition 3.1. *Let R be a ring, e a central idempotent of R, Δ be a multiplicative closed subset consisting central regular elements of R. Then the following statements are equivalent:*

(*1*) *R is strongly reversible.*

(*2*) *eR and $(1 - e)R$ are strongly reversible.*

(*3*) *$\Delta^{-1}R$ is strongly reversible.*

Proof. (1)⇔(2) is straightforward since subrings and direct products of strongly reversible rings are strongly reversible.
(3)⇒(1) is obvious.

(1)⇒(3). Let $f(x)=\sum_{i=0}^{m}u_i^{-1}a_ix^i$, $g(x)=\sum_{j=0}^{n}v_j^{-1}b_jx^j\in\Delta^{-1}R[x]$ satisfy $f(x)g(x)=0$. Then $F(x)=(u_mu_{m-1}\cdots u_0)f(x)$, $G(x)=(v_nv_{n-1}\cdots v_0)g(x)\in R[x]$ and $F(x)G(x)=0$, so $G(x)F(x)=0$ since R is strongly reversible. Thus we have $g(x)f(x)=0$ since all $u_i,v_j,i=0,1,\ldots,m,j=0,1,\ldots,n$ are regular and central.

Proposition 3.2. *Let R be a subdirect sum of strongly reversible rings. Then R is strongly reversible.*

Proof. Let $I_\lambda(\lambda\in\Lambda)$ be ideals of R such that R/I_λ is strongly reversible and $\cap_{\lambda\in\Lambda}I_\lambda=0$. Suppose that $f(x)=\sum_{i=0}^{m}a_ix^i$, $g(x)=\sum_{j=0}^{n}b_jx^j\in R[x]$ satisfy $f(x)g(x)=0$. Then $\overline{g}(x)\overline{f}(x)=0$ in $(R/I_\lambda)[x]$ for each $\lambda\in\Lambda$ since R/I_λ is strongly reversible. So $\sum_{i+j=k}b_ja_i\in I_\lambda$ for $k=0,1,\ldots,m+n$ and any $\lambda\in\Lambda$, which implies that $\sum_{i+j=k}b_ja_i=0$ for $k=0,1,\ldots,m+n$ since $\cap_{\lambda\in\Lambda}I_\lambda=0$, and we obtain $g(x)f(x)=0$.

A ring R is called Armendariz if whenever polynomials $f(x)=\sum_{i=0}^{m}a_ix^m$, $g(x)=\sum_{j=0}^{n}b_jx_n\in R[x]$ satisfy $f(x)g(x)=0$, then $a_ib_j=0$ for each i,j (see [7,8,9] for detail). D.D.Anderson [8, Theorem 2] showed that a ring R is Armendariz if and only if $R[x]$ is Armendariz. It is obvious that R is reduced if and only if $R[x]$ is reduced. We have known that $R[x]$ may not be reversible when R is reversible, but we are able to prove that $R[x]$ is strongly reversible if R is strongly reversible as following results.

Theorem 3.3. *Let R be a ring, then the following statements are equivalent:*

(1) *R is strongly reversible.*

(2) *$R[x]$ is strongly reversible.*

(3) *$R[x,x^{-1}]$ is strongly reversible.*

Proof. (1)⇒(2) Let $f(y)=f_0+f_1y+\cdots+f_py^p, g(y)=g_0+g_1y+\cdots+g_qy^q\in R[x][y]$ satisfy $f(y)g(y)=0$, where $f_i=\sum_{s=0}^{m_i}a_s^{(i)}x^s, g_j=\sum_{t=0}^{n_j}b_t^{(j)}x^t\in R[x]$ for $i=0,1,\ldots,p,j=0,1,\ldots,q$. Let $k=deg(f_0)+deg(f_1)+\cdots+deg(f_p)+deg(g_0)+deg(g_1)+\cdots+deg(g_q)$, where degree is as polynomials in x and the degree of the zero polynomial is taken to be 0. Then $f(x^k)=f_0+f_1x^k+\cdots+f_px^{pk}, g(x^k)=g_0+g_1x^k+\cdots+g_qx^{qk}\in R[x]$ and the set of coefficients of f_i's (resp. g_j's) equals the set of coefficients of $f(x^k)$ (resp. $g(x^k)$). Since $f(y)g(y)=0$ and x commutes with elements of R, we have that $f(x^k)g(x^k)=0$, thus $g(x^k)f(x^k)=0=g(y)f(y)$ since R is strongly reversible, which implies $R[x]$ is strongly reversible.

(2)⇒(3) Follows from Proposition 3.1.

(3)⇒(1) It is clear.

Corollary 3.4. *Let R be a strongly reversible ring and $\{x_\alpha\}$ any set of commuting indeterminates over R. Then any subring of $R[\{x_\alpha\}]$ is strongly reversible.*

Proof. Let $f(y), g(y) \in R[\{x_\alpha\}]$ with $f(y)g(y) = 0$. Then

$$f(y), g(y) \in R[\{x_{\alpha_1}, x_{\alpha_2}, \cdots, x_{\alpha_n}\}][y]$$

for some finite subset $\{x_{\alpha_1}, x_{\alpha_2}, \cdots, x_{\alpha_n}\} \subseteq \{x_\alpha\}$. The ring $R[\{x_{\alpha_1}, x_{\alpha_2}, \cdots, x_{\alpha_n}\}][y]$, by induction, is strongly reversible, so we have that $g(y)f(y) = 0$. Hence $R[\{x_\alpha\}]$ is strongly reversible and thus so is any subring of $R[\{x_\alpha\}]$.

Let R be a ring. Suppose that $Z(R)$ contains an infinite subring whose nonzero elements are regular in R, where $Z(R)$ denotes the set of all central elements of R, if R is reversible, then R is strongly reversible by [5, Prop. 2.3]. Another example of a strongly reversible ring is given in the following which also shows that strongly reversible rings are not reduced in general.

Proposition 3.5. *Let R be a ring and n any positive integer. if R is reduced, then $R[x]/(x^n)$ is strongly reversible, where (x^n) is the ideal generated by x^n.*

Proof. It is obvious that $R[x]/(x^n)$ is strongly reversible since $R[x]/(x^n)$ is both reversible [5, Prop. 2.5] and Armendariz [8, Theorem 5].

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$, $m \in M$ and the usual matrix operations are used.

Corollary 3.6. *Let R be a ring and $T = R \bigoplus R$ be the trivial extension of R by R. If R is reduced, then T is strongly reversible.*

Proof. $T \cong R[x]/(x^2)$ is strongly reversible by Proposition 3.5.

Considering corollary 3.6, we may conjecture that if a ring R is strongly reversible, then $T(R, R)$ is strongly reversible. However, this is not true from [5, Example 1.7] and easy check. One may still conjecture that a ring R is strongly reversible if for any strongly reversible nonzero proper ideal I of R, R/I is strongly reversible, I is considered as a ring without the identity, however the following example erases the possibility even if R is semicommutative.

Example 3.7. Let S be a division ring and

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in S \right\}.$$

Then R is not strongly reversible since it is not reversible [5, Example 1.5].

First notice that R has only the following nonzero proper ideals.

$$I_1 = \begin{pmatrix} 0 & S & S \\ 0 & 0 & S \\ 0 & 0 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & S & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, I_3 = \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & S \\ 0 & 0 & 0 \end{pmatrix}, I_4 = \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

I_1 is not strongly reversible by [5, Example 1.5] and $I_j's$ with $j = 2, 3, 4$ are strongly reversible since they are nilpotent of index 2. The following computations are based on [2, I.3] and the condition that S is a division ring. Let φ! :!$R/I_2 \longrightarrow T(S, S)$ by

$$\varphi \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}.$$

It is easy to check that φ is a ring isomorphism, then $R/I_2 \cong T(S, S)$ is strongly reversible by Corollary 3.6. The case of R/I_3 is similar to the preceding one. Next let

$$f(x) = \sum_{i=0}^{m} \begin{pmatrix} a_i & b_i & 0 \\ 0 & a_i & c_i \\ 0 & 0 & a_i \end{pmatrix} x^i, f(x) = \sum_{j=0}^{n} \begin{pmatrix} u_j & v_j & 0 \\ 0 & u_j & w_j \\ 0 & 0 & u_j \end{pmatrix} x^j \in R/I_4[x]$$

satisfy $f(x)g(x) = 0$,then we have that

$$\begin{pmatrix} \sum_{i=0}^{m} a_i x^i & \sum_{i=0}^{m} b_i x^i & 0 \\ 0 & \sum_{i=0}^{m} a_i x^i & \sum_{i=0}^{m} c_i x^i \\ 0 & 0 & \sum_{i=0}^{m} a_i x^i \end{pmatrix} \begin{pmatrix} \sum_{j=0}^{n} u_j x^j & \sum_{j=0}^{n} v_j x^j & 0 \\ 0 & \sum_{j=0}^{n} u_j x^j & \sum_{j=0}^{n} w_j x^j \\ 0 & 0 & \sum_{j=0}^{n} u_j x^j \end{pmatrix} = 0.$$

Which implies that $(\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} u_j x^j) = 0$, hence $\sum_{i=0}^{m} a_i x^i = 0$ or $\sum_{j=0}^{n} u_j x^j = 0$ since S is a division ring, and it is easy to prove that $g(x)f(x) = 0$. Thereby we get that for any strongly reversible nonzero proper ideal I of R , R/I is strongly reversible.

But we have an affirmative answer if we take a stronger condition as in the following.

Proposition 3.8. *Suppose that R/I is strongly reversible for some ideal I of a ring R. If I is reduced, then R is strongly reversible.*

Proof. Let $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then we have $g(x)f(x) \in I[x]$. Hence $(g(x)f(x))^2 = 0$ implies $g(x)f(x) = 0$ since polynomial rings over reduced rings are reduced, therefore R is strongly reversible.

A ring R is called right Ore, if given $a, b \in R$ with b regular, there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is a well-known fact that R is right Ore if and only if the classical right quotient ring Q of R exists. It was shown in [10, Theorem 16] and [5, Theorem 2.6] that R is reduced (resp. reversible) if and only if Q is reduced (resp. reversible). In the following argument, we extend the result to strongly reversible rings .

Theorem 3.9. *Suppose that there exists the classical right quotient ring Q of a ring R. Then R is strongly reversible if and only if Q is strongly reversible.*

Proof. It is enough to show that if R is strongly reversible, then Q is strongly reversible. Consider $f(x) = \sum_{i=0}^{m} \alpha_i x^i, g(x) = \sum_{j=0}^{n} \beta_j x^j \in Q[x]$ such that $f(x)g(x) = 0$. By [11, Prop.2.1.16], we may assume that $\alpha_i = a_i u^{-1}, \beta_j = b_j v^{-1}$ with $a_i, b_j \in R$ for $i = 0, 1, \ldots, m, j = 0, 1, \ldots, n$ and regular $u, v \in R$. Also by [11, Prop.2.1.16], for each j, there exists $c_j \in R$ and regular $s \in R$ such that $u^{-1}b_j = c_j s^{-1}$. Put $f_1(x) = \sum_{i=0}^{m} a_i x^i, g_1(x) = \sum_{j=0}^{n} b_j x^j, g_2(x) = \sum_{j=0}^{n} c_j x^j \in R[x]$, then we have that $0 = f(x)g(x) = \sum_{i=0}^{m}\sum_{j=0}^{n} \alpha_i\beta_j x^{i+j} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_i(u^{-1}b_j)v^{-1}x^{i+j} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_i c_j (vs)^{-1} x^{i+j} = f_1(x)g_2(x)(vs)^{-1}$, hence $f_1(x)g_2(x) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_i c_j x^{i+j} = 0$ in $R[x]$. $R[x]$ is semicommutative since reversible rings are semicommutative, so $0 = f_1(x)ug_2(x) = \sum_{i+j=k} a_i u c_j x^{i+j}$ $= \sum_{i+j=k} a_i b_j s x^{i+j} = f_1(x)g_1(x)s$, hence $f_1(x)g_1(x) = 0$ in $R[x]$. Use [11, Prop.2.1.16] again, for each i there exist $d_i \in R$ and regular element $t \in R$ such that $v^{-1}a_i = d_i t^{-1}$. Put $f_2(x) = \sum_{i=0}^{m} d_i x^i$, then we have that $0 = f_1(x)tg_1(x) = \sum_{i+j=k} a_i t b_j x^{i+j} = \sum_{i+j=k} v d_i b_j x^{i+j} = vf_2(x)g_1(x)$, thus $f_2(x)g_1(x) = 0$ in $R[x]$, so $g_1(x)f_2(x) = 0$ since R is strongly reversible. Now we have that $g(x)f(x) = (\sum_{j=0}^{n} b_j v^{-1} x^j)(\sum_{i=0}^{m} a_i u^{-1} x^i) = \sum_{i+j=k} b_j (v^{-1}a_i) u^{-1} x^{i+j} = \sum_{i+j=k} b_j d_i (ut)^{-1} x^{i+j} = g_1(x)f_2(x)(ut)^{-1} = 0$. We prove that Q is strongly reversible.

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