TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 1, pp. 89-104, February 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

A WAY OF CONSTRUCTING SPHERICAL ZONAL TRANSLATION NETWORK OPERATORS WITH LINEAR BOUNDED OPERATORS

Sheng Baohuai, Wang Jianli and Zhou Songping

Abstract. A kind of spherical zonal translation network operator is constructed with the help of the de la Vallée poussin means of the spherical harmonic polynomial and the Riesz means of the Jacobi orthogonal polynomials, and, moreover, its degree of approximation in $L^p(S^q)$ spaces is deduced. The method presented in the present paper is actually a way of constructing spherical zonal translation network operators with spherical linear bounded operators.

1. Introduction

In recent years there has been growing interest in the problem of neural network and related approximation, and many important results on the quantitative estimate of degree of approximation have been made. The concept of sigmoidal function of order k is defined in [1] and it was proved by H. N. Mhaskar and C. A. Micchelli (see [1]) that if ϕ is not a polynomial function defined on R and $K \subset R^d$ is a compact set, the function class

$$\triangle_\phi(x) = \{\phi(\lambda(x-t)) : \lambda \in R^d, t \in R^d\} \cup \{1\}, x \in K$$

is dense in $L^p(K)$. Moreover, a kind of neural network operator was constructed by B-spline functions and a Jackson type theorem of approximation in $C([0,1])$ was established. Let $s \geq d \geq 1$ be integers. $\phi^* : R^d \to R$ is a $2\pi-$periodic function and $\phi^* \in L^p([-\pi,\pi]^d), 1 \leq p \leq +\infty, J = J_{d,s}$ is the class of all $d \times s$ matrices of rank d with integer entries, and let

$$\triangle^*_\phi(x) = \{\phi^*(Ax+t) : A \in J, t \in [-\pi,\pi]^d\} \cup \{1\}, x \in [-\pi,\pi]^s.$$

Received September 17, 2004, accepted July 24, 2006.
Communicated by Sen-Yen Shaw.
2000 *Mathematics Subject Classification*: 41A36.
Key words and phrases: Spherical harmonics, Translation network operators, de la Vallée poussin means, Approximation.

Then, H. N. Mhaskar and C. A. Micchelli gave the necessary and sufficient conditions that $\triangle_{\phi}^{*}(x)$ is dense in $L^p([-\pi,\pi]^s)$ and constructed with the de la Vallée Poussin means of Fourier series a very important network operator and obtained estimates on the degree of approximation in $L^p([-\pi,\pi]^s)$ (see [2]). In a recent paper, [3] considered the problem of approximation of non-periodic functions by the function class

$$\triangle_{\phi}^{c}(x)=\{\phi[\cos(A\arccos x+t)]:A\in J,t\in[-1,1]^d\}\cup\{1\},x\in[-1,1]^s,$$

and gave the necessary and sufficient condition such that $\triangle_{\phi}^{c}(x)$ is dense in the weighted space $L_W^p[-1,1]^s$ (where $W(x)=(1-x_1^2)^{-\frac{1}{2}}(1-x_2^2)^{-\frac{1}{2}}\cdots(1-x_s^2)^{-\frac{1}{2}}$) as well, and constructed with the de la Vallée Poussin means of the Chebyshev polynomials of the first kind a kind of network operator and obtained an estimate on the degree of approximation in weighted $L_W^p[-1,1]^s$ space. Let $q\geq 1$ be an integer which will be fixed throughout the rest of this paper, and let S^q be the unit sphere in the Euclidean space R^{q+1}. By $L^p(S^q)$ $(1\leq p\leq+\infty)$ we denote the function space consisting of real or complex functions defined on S^q such that $\|f\|_{p,S^q}<+\infty$, where

$$\|f\|_{p,S^q}=\begin{cases}\left(\int_{S^q}|f(x)|^p d\mu_q(x)\right)^{\frac{1}{p}}, & 1\leq p<+\infty,\\ ess\sup\limits_{x\in S^q}|f(x)|, & p=+\infty,\end{cases}$$

with $d\mu_q(x)$ being the usual volume element on S^q. The volume of S^q is

$$\omega_q=\int_{S^q}d\mu_q=\frac{2\pi^{\frac{q+1}{2}}}{\Gamma(\frac{q+1}{2})}.$$

By $L_{W_q}^p[-1,1]$ we denote the the real or complex function ϕ defined on [-1, 1] such that $\|\phi\|_{p,W_q}<+\infty$, where

$$\|\phi\|_{p,W_q}=\begin{cases}\left(\int_{-1}^{1}|\phi(x)|^p W_q(x)dx\right)^{\frac{1}{p}}<+\infty, & 1\leq p<+\infty,\\ ess\sup\limits_{x\in[-1,1]}|\phi(x)|<+\infty, & p=+\infty,\end{cases}$$

with $W_q(x)=(1-x^2)^{\frac{q}{2}-1}$. Then H. N. Mhaskar, F. J. Narcowich and J. D. Ward showed in [4] that if $\phi\in L_{W_q}^p[-1,1]$ satisfies certain conditions, the zonal translation network class

$$\triangle_{\phi}^{S}(x)=\{\phi(x\cdot y):y\in S^q\}\cup\{1\},x\in S^q,$$

(where $x \cdot y$ denote the inner product of vectors x and y) is dense in $L^p(S^q)$ and gave a Jackson estimate on the degree of approximation in $L^p(S^q)$. The fact that Bernstein operators are useful in constructing neural network operators (see [6]) inspires us to construct specific zonal translation network operators on the unit sphere S^q. We shall constructed, along the lines of [4], a kind of specific spherical zonal translation network operator with the de la Vallée Poussin means of the spherical harmonic polynomials, the Riesz means of the Jacobi orthogonal algebraic polynomials and the Gauss integral formula obtained in [7]-[8], and give its degree of approximation in $L^p(S^q)$. What we shall show in the present paper is in fact a way of constructing spherical zonal translation network linear operator from a specific bounded operator on the unit sphere.

In what follows, we shall write $A = O(B)$ if there exists a constant $C > 0$ such that $A \leq CB$, and we shall write $A \sim B$ if $A = O(B)$ and $B = O(A)$.

2. Some Preliminaries of Spherical Harmonics

For an integer $l \geq 0$, the class of all one variable algebraic polynomials of degree $\leq l$ defined on [-1, 1] is denoted by P_l, the restriction to S^q of a homogeneous harmonic polynomial of degree l is called a spherical harmonic of degree l. The class of all spherical harmonics of degree l will be denoted by H_l^q, and the class of all spherical harmonics of degree $l \leq n$ will be denoted by Π_n^q. Of course, $\Pi_n^q = \bigoplus_{l=0}^{n} H_l^q$, and it comprises the restriction to S^q of all algebraic polynomials in $q+1$ variables of total degree not exceeding n. The dimension of H_l^q is given by (see [9, P.65])

$$d_l^q = \dim H_l^q = \begin{cases} \dfrac{2l+q-1}{l+q-1} \begin{pmatrix} l+q-1 \\ q-1 \end{pmatrix}, & l \geq 1, \\ 1, & l = 0, \end{cases}$$

and that of Π_n^q is $\sum_{l=0}^{n} d_l^q$. By [10] we know

$$L^2(S^q) = \text{closure}\{\bigoplus_l H_l^q\}.$$

Hence,if we choose an orthonormal basis $\{Y_{l,k} : k = 1, 2, \cdots, d_l^q\}$ for each H_l^q, then the set $\{Y_{l,k} : l = 0, 1, 2, \cdots ; k = 1, 2, \cdots, d_l^q\}$ forms an orthonormal basis for $L^2(S^q)$. One has the well-known addition formula(see [11]):

$$\sum_{k=1}^{d_l^q} Y_{l,k}(x)\overline{Y_{l,k}(y)} = \frac{d_l^q}{\omega_q} p_l^{q+1}(x \cdot y), l = 0, 1, \cdots,$$

where $p_l^{q+1}(x)$ is the degree-l Legendre polynomial. The Legendre polynomials are normalized so that $p_l^{q+1}(1)=1$, and satisfy the orthogonality relation

$$\int_{-1}^{1} p_l^{q+1}(x)p_k^{q+1}(x)W_q(x)dx = \frac{\omega_q}{\omega_{q-1}d_l^q}\delta_{l,k}.$$

From the fact that $\Pi_n^q = \bigoplus_{l=0}^{n} H_l^q$ and the addition formula, we know that for any $p \in \Pi_n^q$ and $x \in S^q$

$$p(x) = \sum_{l=0}^{n} \frac{d_l^q}{\omega_q}\int_{S^q} p(y)p_l^{q+1}(x\cdot y)d\mu_q(y).$$

In addition to the inner product and norms defined above on S^q, we shall need the following related norms for $[-1,1]$, with generalized Jacobi weight functions $W_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ $(\alpha > -1, \beta > -1)$

$$\|f\|_{p,W_{\alpha,\beta}} = \begin{cases} \left(\int_{-1}^{1} |f(x)|^p W_{\alpha,\beta}(x)dx\right)^{\frac{1}{p}}, & 1 \le p < +\infty; \\ ess \sup\limits_{x\in[-1,1]} |f(x)|, & p = +\infty. \end{cases}$$

We also note that the Funk-Hecke formula (see [9, Chapter 3]) implies the following useful connection between integrals over S^q and integrals over [-1, 1] with respect to the weight function $W_q(x)$. For any $\phi \in L^1_{W_q}[-1,1], x \in S^q$, and any $Y_l \in H_l^q$ we have

$$\int_{S^q} \phi(x\cdot z)Y_l(z)d\mu_q(z) = \frac{\omega_q}{d_l^q}\hat{\phi}(l)Y_l(x),$$

where

$$\hat{\phi}(l) = \frac{\omega_{q-1}d_l^q}{\omega_q}\int_{-1}^{1} \phi(x)p_l^{q+1}(x)W_q(x)dx.$$

Moreover, we have the following relation

$$\int_{S^q} \phi(x\cdot y)d\mu_q(x) = \omega_{q-1}\int_{-1}^{1} \phi(x)W_q(x)dx.$$

The orthogonal projection $Y_k(f,x)$ of a function $f \in L^1(S^q)$ on H_k^q is defined by (see [10])

$$Y_k(f,x) = \frac{d_k^q}{\omega_q}\int_{S^q} p_k^{q+1}(x\cdot y)f(y)d\mu_q(y).$$

Correspondingly we have the following Fourier-Laplace expansion of f

$$f(x) \sim \sum_{k=0}^{\infty} Y_k(f, x), x \in S^q.$$

Let $\mathcal{C}$ be a finite set of distinct points on S^q. The mesh norm of $\mathcal{C}$ is defined to be

$$\delta_{\mathcal{C}} = \max_{x \in S^q} \text{dist}(x, \mathcal{C}) = \max_{x \in S^q} \min_{y \in \mathcal{C}} \text{dist}(x, y),$$

where $\text{dist}(x, y) = \arccos(x \cdot y)$ is the geodesic distance between x and y.

Lemma 2.1. *There exist constants $\alpha_q$and N_q with the following property. Let $1 \le p \le +\infty$, $\mathcal{C}$ be a finite set of distinct points on S^q and n be an integer with $N_q \le n \le \alpha_q \delta_{\mathcal{C}}^{-1}$. Then there exist nonnegative weights $\{A_\xi\}_{\xi \in \mathcal{C}}$and $\{a_\xi\}_{\xi \in \mathcal{C}}$ with*

$$\sum_{\xi \in \mathcal{C}} \frac{a_\xi}{A_\xi} \le C,$$

such that for every $p \in \Pi_n^q$,

$$\frac{1}{\omega_q} \int_{S^q} p(x) d\mu_q(x) = \sum_{\xi \in \mathcal{C}} a_\xi p(\xi),$$

and

$$\|p\|_{\mathcal{C},p} \sim \|p\|_{p,S^q},$$

where

$$\|p\|_{\mathcal{C},p} = \begin{cases} \left(\sum\limits_{\xi \in \mathcal{C}} \left|p(\xi)\right|^p A_\xi \right)^{\frac{1}{p}}, & 1 \le p < +\infty, \\ \sup\limits_{\xi \in \mathcal{C}} \{\left|p(\xi)\right|\}, & p = +\infty. \end{cases}$$

Further, $\left|\{\xi : a_\xi \ne 0\}\right| \sim n^q \sim \dim(\Pi_n^q)$.

Proof. See [7-8].

Let $n \ge 1$ be an integer and

$$\omega_{n,k}^{(\lambda)} = \frac{n!}{(n-k)!} \frac{\Gamma(n + 2\lambda + 1)}{\Gamma(n + k + 2\lambda + 1)}, 0 \le k \le n, 2\lambda = q - 1.$$

Then H. Berens and Li Luoqing (see [12]) introduced the following de la Vallée Poussin operator on spherical harmonics

$$V_n(f,\mu)=\sum_{k=0}^{n}\omega_{n,k}^{(\lambda)}Y_k(f,\mu),f\in L^1(S^q),$$

and deduced the following estimate.

Lemma 2.2. *Let* $f\in L^p(S^q)$. *Then*

$$\|V_n(f)-f\|_{p,S^q}\le\frac{C}{n+1}\sum_{k=0}^{[\sqrt{n}]}(k+1)E_k(f)_{p,S^q},$$

where

$$E_n(f)_{p,S^q}=\inf_{p\in\Pi_n^q}\|f-p\|_{p,S^q}.$$

Proof. See [12].

3. The Riesz means on Jacobi Orthogonal Polynomials

In this section, we shall give the estimate of the degree of approximation of Riesz means of Jacobi polynomials. To define the $K-$functional, we first define the Jacobi differential operator

$$P_{\alpha,\beta}(D)=W_{\alpha,\beta}(x)^{-1}\frac{d}{dx}W_{\alpha,\beta}(x)(1-x^2)\frac{d}{dx}.$$

Its eigenfunctions are the Jacobi polynomials $p_k^{(\alpha,\beta)}(x)$ and

$$P_{\alpha,\beta}(D)p_k^{(\alpha,\beta)}(x)=-k(k+\alpha+\beta+1)p_k^{(\alpha,\beta)}(x),$$

where the Jacobi polynomials $p_k^{(\alpha,\beta)}(x)$ are normalized by

$$\int_{-1}^{1}p_k^{(\alpha,\beta)}(x)p_m^{(\alpha,\beta)}(x)W_{\alpha,\beta}(x)dx=\delta_{k,m}.$$

The formal expansion of $f\in L^1_{W_{\alpha,\beta}}[-1,1]$ corresponding to $p_k^{(\alpha,\beta)}(x)$ is

$$f(x)\sim\sum_{k=0}^{+\infty}a_k(f)p_k^{(\alpha,\beta)}(x),x\in[-1,1],$$

where

$$a_k(f) = \int_{-1}^{1} f(x) p_k^{(\alpha,\beta)}(x) W_{\alpha,\beta}(x) dx.$$

The partial sum $S_n(f,x)$ for a given n is

$$S_n(f,x) = \sum_{k=0}^{n} a_k(f) p_k^{(\alpha,\beta)}(x), x \in [-1,1],$$

and for an integer $b > 0$ the Riesz means $R_n^{(\alpha,\beta),b}(f)$ are

$$R_n^{(\alpha,\beta),b}(f,x) = \sum_{0 \le k < n} \Big(1 - \frac{k(k+\alpha+\beta+1)}{n(n+\alpha+\beta+1)}\Big)^b a_k(f) p_k^{(\alpha,\beta)}(x), x \in [-1,1].$$

A K-functional $K^{(\alpha,\beta)}(f,t)_p$(see [13]) corresponding to the differential operator $P_{\alpha,\beta}(D)$ is defined by

$$K^{(\alpha,\beta)}(f,t)_p = \inf_{g \in C^2[-1,1],\, P_{\alpha,\beta}(D) g \in L^p_{W_{\alpha,\beta}}[-1,1]} (\|f-g\|_{p,W_{\alpha,\beta}} + t\|P_{\alpha,\beta}(D)g\|_{p,W_{\alpha,\beta}}), t > 0,$$

where $P_{\alpha,\beta}(D)^j g(x)$ is given in such a way that $P_{\alpha,\beta}(D)^0 = P_{\alpha,\beta}(D), P_{\alpha,\beta}(D)^j = P_{\alpha,\beta}(D)(P_{\alpha,\beta}(D)^{j-1})$, and

$$\int_{-1}^{1} P_{\alpha,\beta}(D)^j g(x) p_k^{(\alpha,\beta)}(x) W_{\alpha,\beta}(x) dx = \int_{-1}^{1} g(x) P_{\alpha,\beta}(D)^j p_k^{(\alpha,\beta)}(x) W_{\alpha,\beta}(x) dx$$

hold for any $p_k^{(\alpha,\beta)}(x), k = 0,1,2,\cdots$. Then, Z. Ditzian showed that (see [14])

$$\|R_n^{(0,0),1}(f) - f\|_{p,W_{0,0}} \sim K^{(0,0)}(f,\ n^{-2})_p, f \in L^p[-1,1].$$

For the needs of constructing zonal translation operators in the next paragraph we give here an upper estimate of convergence rate of $R_n^{(\alpha,\beta),b}(f)$ in case of $(\alpha,\beta) \neq (0,0)$.

Lemma 3.1. *Let $f \in L^p_{W_{\alpha,\beta}}, 1 \le p \le +\infty$, $b > max(\alpha + \frac{1}{2}, \beta + \frac{1}{2}), \alpha, \beta > -1$,and $\alpha + \beta \ge -1$. Then,there is a constant $C(b,\alpha,\beta) > 0$, which depends only on b, α, and β, such that*

$$\|R_n^{(\alpha,\beta),b}(f)\|_{p,W_{\alpha,\beta}} \le C(b,\alpha,\beta)\|f\|_{p,W_{\alpha,\beta}}.$$

Proof. See [13, Theorem A].

Lemma 3.2. *Suppose* $g \in C^2[-1,1], 1 \le p \le +\infty$*,* $b > max(\alpha + \frac{1}{2}, \beta + \frac{1}{2}), \alpha, \beta > -1$*, and* $\alpha + \beta \ge -1$*. Then we have*

$$\|R_n^{(\alpha,\beta),b}(g) - g\|_{p,W_{\alpha,\beta}} = O\Big(\frac{1}{n^2}\Big)\|P_{\alpha,\beta}(D)g\|_{p,W_{\alpha,\beta}}.$$

Proof.

$$\begin{aligned} R_n^{(\alpha,\beta),b}(R_n^{(\alpha,\beta),b}(g) - g)(x) &= \sum_{0\le k<n} \Big[1 - \frac{k(k+\alpha+\beta+1)}{n(n+\alpha+\beta+1)}\Big]^b a_k \\ &\quad \times (R_n^{(\alpha,\beta),b}(g) - g) p_k^{(\alpha,\beta)}(x) \\ &= \sum_{0\le k<n} \Big[1 - \frac{k(k+\alpha+\beta+1)}{n(n+\alpha+\beta+1)}\Big]^b \\ &\quad \times \Big(\Big[1 - \frac{k(k+\alpha+\beta+1)}{n(n+\alpha+\beta+1)}\Big]^b - 1\Big) \\ &\quad \times a_k(g) p_k^{(\alpha,\beta)}(x) \\ &= \sum_{0\le k<n} \Big[1 - \frac{k(k+\alpha+\beta+1)}{n(n+\alpha+\beta+1)}\Big]^b \\ &\quad \times \sum_{i=1}^{b} (-1)^i \binom{b}{i} \Big(\frac{k(k+\alpha+\beta+1)}{n(n+\alpha+\beta+1)}\Big)^i \\ &\quad \times a_k(g) p_k^{(\alpha,\beta)}(x) \\ &= \sum_{i=1}^{b} \binom{b}{i} \Big(\frac{1}{n(n+\alpha+\beta+1)}\Big)^i \\ &\quad \times \sum_{0\le k<n} \Big[1 - \frac{k(k+\alpha+\beta+1)}{n(n+\alpha+\beta+1)}\Big]^b \\ &\quad \times a_k(g) \Big(-k(k+\alpha+\beta+1)\Big)^i p_k^{(\alpha,\beta)}(x). \end{aligned}$$

On the other hand, by

$$P_{\alpha,\beta}(D)^r p_k^{(\alpha,\beta)}(x) = \big(-k(k+\alpha+\beta+1)\big)^r p_k^{(\alpha,\beta)}(x),$$

we have

$$\begin{aligned} R_n^{(\alpha,\beta),b}(R_n^{(\alpha,\beta),b}(g) - g)(x) &= \sum_{i=1}^{b} \binom{b}{i} \Big(\frac{1}{n(n+\alpha+\beta+1)}\Big)^i \\ &\quad \times P_{\alpha,\beta}(D)^i R_n^{(\alpha,\beta),b}(g,x). \end{aligned}$$

By the definition of $P_{\alpha,\beta}(D)^i$ one has

$$P_{\alpha,\beta}(D)^i R_n^{(\alpha,\beta),b}(g,x) = R_n^{(\alpha,\beta),b}(P_{\alpha,\beta}(D)^i g, x).$$

Since

$$R_m^{(\alpha,\beta),b}(R_n^{(\alpha,\beta),b}(g),x) = R_n^{(\alpha,\beta),b}(R_m^{(\alpha,\beta),b}(g),x),$$

we have

$$\begin{aligned}
& R_m^{(\alpha,\beta),b}(R_m^{(\alpha,\beta),b}(g),x) - R_{m+1}^{(\alpha,\beta),b}(R_m^{(\alpha,\beta),b}(g),x) \\
= & \Big(R_m^{(\alpha,\beta),b}(R_m^{(\alpha,\beta),b}(g),x) - R_m^{(\alpha,\beta),b}(g,x)\Big) \\
& + \Big(R_m^{(\alpha,\beta),b}(g,x) - R_{m+1}^{(\alpha,\beta),b}(R_m^{(\alpha,\beta),b}(g),x)\Big) \\
= & \sum_{i=1}^{b} \binom{b}{i} P_{\alpha,\beta}(D)^i R_m^{(\alpha,\beta),b}(g,x)\Big(\frac{1}{\big(m(m+\alpha+\beta+1)\big)^i} \\
& - \frac{1}{\big((m+1)(m+\alpha+\beta+2)\big)^i}\Big) \\
= & \sum_{i=1}^{b} \binom{b}{i} P_{\alpha,\beta}(D)^i R_m^{(\alpha,\beta),b}(g,x) O\Big(\frac{1}{m^{2i+1}}\Big).
\end{aligned}$$

Recalling that for $p_n \in P_n$ there exists the Bernstein type inequality (see [15])

$$\|P_{\alpha,\beta}(D)p_n\|_{p,W_{\alpha,\beta}} \le C(\alpha,\beta) n^2 \|p_n\|_{p,W_{\alpha,\beta}},$$

where $C(\alpha,\beta) > 0$ is a constant depending only on α and β. Hence, by Lemma 3.1 we have

$$\begin{aligned}
\|P_{\alpha,\beta}(D)^i R_m^{(\alpha,\beta),b}(g)\|_{p,W_{\alpha,\beta}} &= \|P_{\alpha,\beta}(D) R_m^{(\alpha,\beta),b}(P_{\alpha,\beta}(D)^{i-1} g)\|_{p,W_{\alpha,\beta}} \\
&= O(m^2)\|R_m^{(\alpha,\beta),b}(P_{\alpha,\beta}(D)^{i-1} g)\|_{p,W_{\alpha,\beta}} \\
&= O(m^{2(i-1)})\|R_m^{(\alpha,\beta),b}(P_{\alpha,\beta}(D) g)\|_{p,W_{\alpha,\beta}} \\
&= O(m^{2(i-1)})\|P_{\alpha,\beta}(D) g\|_{p,W_{\alpha,\beta}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \|R_m^{(\alpha,\beta),b}(R_m^{(\alpha,\beta),b}(g)) - R_{m+1}^{(\alpha,\beta),b}(R_m^{(\alpha,\beta),b}(g))\|_{p,W_{\alpha,\beta}} \\
= & O\Big(\frac{1}{m^3}\Big)\|P_{\alpha,\beta}(D) g\|_{p,W_{\alpha,\beta}}.
\end{aligned}$$

In a similar way, one has

$$\begin{aligned}&\|R_{m+1}^{(\alpha,\beta),b}(R_{m+1}^{(\alpha,\beta),b}(g))-R_{m}^{(\alpha,\beta),b}(R_{m+1}^{(\alpha,\beta),b}(g))\|_{p,W_{\alpha,\beta}}\\ =\ &O\Big(\frac{1}{m^3}\Big)\|P_{\alpha,\beta}(D)g\|_{p,W_{\alpha,\beta}}.\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{m=n}^{\infty}\|R_{m}^{(\alpha,\beta),b}(R_{m}^{(\alpha,\beta),b}(g))-R_{m+1}^{(\alpha,\beta),b}(R_{m+1}^{(\alpha,\beta),b}(g))\|_{p,W_{\alpha,\beta}}\\ =\ &\lim_{N\to+\infty}\sum_{m=n}^{N}\|R_{m}^{(\alpha,\beta),b}(R_{m}^{(\alpha,\beta),b}(g))-R_{m+1}^{(\alpha,\beta),b}(R_{m+1}^{(\alpha,\beta),b}(g))\|_{p,W_{\alpha,\beta}}\\ =\ &O\Big(\lim_{N\to+\infty}\sum_{m=n}^{N}\frac{1}{m^3}\Big)\|P_{\alpha,\beta}(D)g\|_{p,W_{\alpha,\beta}}\\ =\ &O\Big(\frac{1}{n^2}\Big)\|P_{\alpha,\beta}(D)g\|_{p,W_{\alpha,\beta}}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|R_{n}^{(\alpha,\beta),b}(g)-g\|_{p,W_{\alpha,\beta}}\le\ &\|R_{n}^{(\alpha,\beta),b}(R_{n}^{(\alpha,\beta),b}(g))-R_{n}^{(\alpha,\beta),b}(g)\|_{p,W_{\alpha,\beta}}\\ &+\|R_{n}^{(\alpha,\beta),b}(R_{n}^{(\alpha,\beta),b}(g))-g\|_{p,W_{\alpha,\beta}}\\ \le\ &\|R_{n}^{(\alpha,\beta),b}(R_{n}^{(\alpha,\beta),b}(g))-R_{n}^{(\alpha,\beta),b}(g)\|_{p,W_{\alpha,\beta}}\\ &+\|\sum_{m=n}^{N}\big(R_{m}^{(\alpha,\beta),b}(R_{m}^{(\alpha,\beta),b}(g))-R_{m+1}^{(\alpha,\beta),b}(R_{m+1}^{(\alpha,\beta),b}(g))\big)\\ &+\big(R_{N+1}^{(\alpha,\beta),b}(R_{N+1}^{(\alpha,\beta),b}(g))-g\big)\|_{p,W_{\alpha,\beta}}\\ \le\ &\|R_{n}^{(\alpha,\beta),b}(R_{n}^{(\alpha,\beta),b}(g))-R_{n}^{(\alpha,\beta),b}(g)\|_{p,W_{\alpha,\beta}}\\ &+\sum_{m=n}^{N}\|R_{m}^{(\alpha,\beta),b}(R_{m}^{(\alpha,\beta),b}(g))-R_{m+1}^{(\alpha,\beta),b}(R_{m+1}^{(\alpha,\beta),b}(g))\|_{p,W_{\alpha,\beta}}\\ &+\|R_{N+1}^{(\alpha,\beta),b}(R_{N+1}^{(\alpha,\beta),b}(g))-g\|_{p,W_{\alpha,\beta}}.\end{aligned}$$

Lemma 3.1 and the density of polynomials in $L^p_{W_{\alpha,\beta}}$ immediately yield

$$\begin{aligned}\|R_{N+1}^{(\alpha,\beta),b}(R_{N+1}^{(\alpha,\beta),b}(g))-g\|_{p,W_{\alpha,\beta}}\le\ &\|R_{N+1}^{(\alpha,\beta),b}(R_{N+1}^{(\alpha,\beta),b}(g))-R_{N+1}^{(\alpha,\beta),b}(g)\|_{p,W_{\alpha,\beta}}\\ &+\|R_{N+1}^{(\alpha,\beta),b}(g)-g\|_{p,W_{\alpha,\beta}}\to 0\ (N\to+\infty).\end{aligned}$$

Which allows

$$\begin{aligned}\|R_n^{(\alpha,\beta),b}(g)-g\|_{p,W_{\alpha,\beta}} &\le \|R_n^{(\alpha,\beta),b}(R_n^{(\alpha,\beta),b}(g))-R_n^{(\alpha,\beta),b}(g)\|_{p,W_{\alpha,\beta}}\\ &\quad+\sum_{m=n}^{+\infty}\|R_m^{(\alpha,\beta),b}(R_m^{(\alpha,\beta),b}(g))\\ &\quad-R_{m+1}^{(\alpha,\beta),b}(R_{m+1}^{(\alpha,\beta),b}(g))\|_{p,W_{\alpha,\beta}}\\ &= O\Big(\frac{\|P_{\alpha,\beta}(D)g\|_{p,W_{\alpha,\beta}}}{n^2}\Big).\end{aligned}$$

Lemma 3.3. *Let* $f\in L^p_{W_{\alpha,\beta}}, 1\le p\le+\infty$*,* $b>max(\alpha+\frac{1}{2},\beta+\frac{1}{2}),\alpha,\beta>-1$*,and* $\alpha+\beta\ge-1$*. Then there exists a constant* $C>0$ *such that*

$$\|R_n^{(\alpha,\beta),b}(f)-f\|_{p,W_{\alpha,\beta}}\le\frac{C}{n^2}\sum_{0<k\le n}kE_k(f)_{p,W_{\alpha,\beta}},$$

where $E_n(f)_{p,W_{\alpha,\beta}}=\inf_{p_n\in P_n}\|f-p_n\|_{p,W_{\alpha,\beta}}$.

Proof. By Lemma 3.1 and Lemma 3.2 we know that for any $P_{\alpha,\beta}(D)g\in L^p_{W_{\alpha,\beta}}$

$$\begin{aligned}\|R_n^{(\alpha,\beta),b}(f)-f\|_{p,W_{\alpha,\beta}} &\le \|R_n^{(\alpha,\beta),b}(f-g)\|_{p,W_{\alpha,\beta}}\\ &\quad+\|R_n^{(\alpha,\beta),b}(g)-g\|_{p,W_{\alpha,\beta}}+\|f-g\|_{p,W_{\alpha,\beta}}\\ &= O\Big(\|f-g\|_{p,W_{\alpha,\beta}}+\frac{\|P_{\alpha,\beta}(D)g\|_{p,W_{\alpha,\beta}}}{n^2}\Big).\end{aligned}$$

Which allows

$$\|R_n^{(\alpha,\beta),b}(f)-f\|_{p,W_{\alpha,\beta}}=O\big(K^{(\alpha,\beta)}(f,n^{-2})_p\big).$$

Since (see [13, Theorem 5.4])

$$K^{(\alpha,\beta)}(f,n^{-2})_p\le Cn^{-2}\sum_{0<k\le n}kE_k(f)_{p,W_{\alpha,\beta}},$$

Lemma 3.3 is therefore deduced.

4. Constructing of Zonal Translation Operators

Choosing an orthonormal basis $\{Y_{l,k}:k=1,2,\cdots,d_l^q\}$ for each H_l^q, then we know that for any $x\in[-1,1],\phi\in L^1_{W_q[-1,1]}$, and any given integer $N>0$

$$Y_{l,k}(x)=\frac{d_l^q}{\omega_q R_{N+1}^{(\frac{q}{2}-1,\frac{q}{2}-1),b}(\phi)(l)}\int_{S^q}R_{N+1}^{(\frac{q}{2}-1,\frac{q}{2}-1),b}(\phi,x\cdot\xi)Y_{l,k}(\xi)d\mu_q(\xi),$$
$$k=1,2,\cdots,d_l^q;\ l=0,1,2,\cdots.$$

Taking $R_N(\phi,x)=R_{N+1}^{(\frac{q}{2}-1,\frac{q}{2}-1),b}(\phi,x)$ and $\hat{f}(l,k)=\int_{S^q}f(u)\overline{Y_{l,k}(u)}d\mu_q(u)$, then for $x\in S^q$, $f\in L^1(S^q)$, one has

$$\begin{aligned}
V_N(f,x) &= \sum_{l=0}^{N}\sum_{k=1}^{d_l^q}\widehat{V_N(f)}(l,k)Y_{l,k}(x)\\
&= \sum_{l=0}^{N}\sum_{k=1}^{d_l^q}\widehat{V_N(f)}(l,k)\frac{d_l^q}{\omega_q\widehat{R_N(\phi)}(l)}\int_{S^q}R_N(\phi,x\cdot\xi)Y_{l,k}(\xi)d\mu_q(\xi)\\
&= \sum_{l=0}^{N}\frac{d_l^q}{\omega_q\widehat{R_N(\phi)}(l)}\sum_{k=1}^{d_l^q}\int_{S^q}V_N(f,u)\overline{Y_{l,k}(u)}d\mu_q(u)\\
&\quad\times\int_{S^q}R_N(\phi,x\cdot\xi)Y_{l,k}(\xi)d\mu_q(\xi)\\
&= \sum_{l=0}^{N}\frac{d_l^q}{\omega_q\widehat{R_N(\phi)}(l)}\sum_{k=1}^{d_l^q}\int_{S^q}R_N(\phi,x\cdot\xi)\\
&\quad\times\Big(\int_{S^q}V_N(f,u)Y_{l,k}(\xi)\overline{Y_{l,k}(u)}d\mu_q(u)\Big)d\mu_q(\xi)\\
&= \sum_{l=0}^{N}\frac{d_l^q}{\omega_q\widehat{R_N(\phi)}(l)}\int_{S^q}R_N(\phi,x\cdot\xi)\\
&\quad\times\Big(\int_{S^q}V_N(f,u)\frac{d_l^q}{\omega_q}p_l^{q+1}(\xi\cdot u)d\mu_q(u)\Big)d\mu_q(\xi)\\
&= \sum_{l=0}^{N}\frac{d_l^q}{\omega_q\widehat{R_N(\phi)}(l)}\int_{S^q}R_N(\phi,x\cdot\xi)Y_l\big(V_N(f),\xi\big)d\mu_q(\xi).
\end{aligned}$$

Since $R_N(\phi,x\cdot\xi)Y_l(V_N(f),\xi)$ is in Π_{2N}^q for a given $\xi\in S^q$, Lemma 2.1 makes

$$V_N(f,x)=\sum_{l=0}^{N}\frac{d_l^q}{\widehat{R_N(\phi)}(l)}\sum_{\xi\in\mathcal{C}}a_\xi R_N(\phi,x\cdot\xi)Y_l\big(V_N(f),\xi\big).$$

The above equation reminds us to define the following zonal translation operators

$$M_{N,\phi}(f,x)=\sum_{l=0}^{N}\frac{d_l^q}{\widehat{R_N(\phi)}(l)}\sum_{\xi\in\mathcal{C}}a_\xi\phi(x\cdot\xi)Y_l\big(V_N(f),\xi\big),\ f\in L^p(S^q),x\in S^q.$$

Theorem 2.2. *Let $f \in L^p(S^q), \phi \in L^p_{W_q}, 1 \le p \le +\infty$. If $\hat{\phi}(l) \ne 0$ holds for any nonnegative integer l, and there exist constant number N_q, α_q, such that $N_q \le 2N \le \alpha_q \delta_{\mathcal{C}}^{-1}$. Then there exists a constant $C > 0$ such that*

$$\|M_{N,\phi}(f) - f\|_{p,S^q} \le C\Big(\frac{1}{N}\sum_{k=0}^{[\sqrt{N}]}(k+1)E_k(f)_{p,S^q} + \frac{\alpha_N^{\phi,q}\omega_{q-1}\|f\|_{p,S^q}}{N^2}\sum_{0<k\le N+1} kE_k(\phi)_{p,W_q}\Big),$$

where $\alpha_N^{\phi,q} = \sum\limits_{l=0}^{N} \frac{(d_l^q)^2\omega_q^{-\frac{1}{p}}\|p_l^{q+1}\|_{p',W_q}}{\left(1-\frac{l(l+q-1)}{(N+1)(N+q)}\right)^b|\hat{\phi}(l)|}, \frac{1}{p}+\frac{1}{p'}=1.$

Proof. By Hölder inequality we have

$$\begin{aligned}\Big|M_{N,\phi}(f,x) - V_N(f,x)\Big| &\le \sum_{\xi\in\mathcal{C}} a_\xi \Big|\sum_{l=0}^{N}\frac{d_l^q}{\widehat{R_N(\phi)}(l)}Y_l(V_N(f),\xi)\Big| \times\big|R_N(\phi,x\cdot\xi)-\phi(x\cdot\xi)\big| \\ &\le \left(\sum_{\xi\in\mathcal{C}} a_\xi\big|R_N(\phi,x\cdot\xi)-\phi(x\cdot\xi)\big|^p\right)^{\frac{1}{p}} \\ &\quad\times\left(\sum_{\xi\in\mathcal{C}} a_\xi\left|\sum_{l=0}^{N}\frac{d_l^q}{\widehat{R_N(\phi)}(l)}Y_l(V_N(f),\xi)\right|^{p'}\right)^{\frac{1}{p'}}.\end{aligned}$$

By Lemma 2.1 we know $\frac{a_\xi}{A_\xi} \le C$. Therefore, using Lemma 2.1 again, one has

$$\begin{aligned}A_N &= \left(\sum_{\xi\in\mathcal{C}} a_\xi\left|\sum_{l=0}^{N}\frac{d_l^q}{\widehat{R_N(\phi)}(l)}Y_l(V_N(f),\xi)\right|^{p'}\right)^{\frac{1}{p'}} \\ &\le C\left(\sum_{\xi\in\mathcal{C}} A_\xi\left|\sum_{l=0}^{N}\frac{d_l^q}{\widehat{R_N(\phi)}(l)}Y_l(V_N(f),\xi)\right|^{p'}\right)^{\frac{1}{p'}} \\ &\le C\|\sum_{l=0}^{N}\frac{d_l^q}{\widehat{R_N(\phi)}(l)}Y_l(V_N(f),x)\|_{p',S^q}.\end{aligned}$$

Since $p_l^{(\frac{q}{2}-1,\frac{q}{2}-1)}(x) = \left(\frac{\omega_{q-1}d_l^q}{\omega_q}\right)^{\frac{1}{2}}p_l^{q+1}(x)$, we have

$$R_N(\phi,x) = \sum_{0<l\le N+1}\Big(1-\frac{l(l+q-1)}{(N+1)(N+q)}\Big)^b\hat{\phi}(l)p_l^{q+1}(x).$$

Which allows

$$\widehat{R_N(\phi)}(l) = \Big(1 - \frac{l(l+q-1)}{(N+1)(N+q)}\Big)^b \hat{\phi}(l).$$

Thus, there exists constant number $C > 0$ such that

$$A_N \le C \sum_{l=0}^{N} \frac{d_l^q}{\big(1 - \frac{l(l+q-1)}{(N+1)(N+q)}\big)^b |\hat{\phi}(l)|} \|Y_l(V_N(f), x)\|_{p', S^q}.$$

Furthermore,

$$\begin{aligned} Y_l(V_N(f), x) &= \frac{d_l^q}{\omega_q}\Big(V_N(f) * p_l^{q+1}\Big)(x) \\ &= \frac{d_l^q}{\omega_q} \int_{S^q} V_N(f, \eta) p_l^{q+1}(x \cdot \eta) d\mu_q(\eta). \end{aligned}$$

The Hölder inequality yields

$$\begin{aligned} \Big|Y_l(V_N(f), x)\Big| &\le \frac{d_l^q}{\omega_q} \|V_N(f)\|_{p,S^q} \left(\int_{S^q} \big|p_l^{q+1}(x\cdot\eta)\big|^{p'} d\mu_q(\eta)\right)^{\frac{1}{p'}} \\ &\le C \frac{d_l^q}{\omega_q} \|f\|_{p,S^q} \left(\omega_{q-1} \int_{-1}^{1} \big|p_l^{q+1}(x)\big|^{p'} W_q(x) dx\right)^{\frac{1}{p'}} \\ &\le C \frac{d_l^q}{\omega_q} \big(\omega_{q-1}\big)^{\frac{1}{p'}} \|p_l^{q+1}\|_{p', W_q} \|f\|_{p,S^q}. \end{aligned}$$

Thus,

$$\begin{aligned} A_N &\le C \sum_{l=0}^{N} \frac{(d_l^q)^2 \omega_{q-1}^{\frac{1}{p'}} \omega_q^{-\frac{1}{p}} \|p_l^{q+1}\|_{p', W_q}}{\big(1 - \frac{l(l+q-1)}{(N+1)(N+q)}\big)^b |\hat{\phi}(l)|} \|f\|_{p,S^q} \\ &= C \omega_{q-1}^{\frac{1}{p'}} \alpha_N^{\phi;q} \|f\|_{p,S^q}, \end{aligned}$$

and

$$\begin{aligned} &\|M_{N,\phi}(f) - V_N(f)\|_{p,S^q} \\ \le\ & C \sum_{\xi \in \mathcal{C}} a_\xi \left(\int_{S^q} \Big| R_N(\phi, x\cdot\xi) - \phi(x \cdot \xi)\Big|^p d\mu_q(x)\right)^{\frac{1}{p}} \omega_{q-1}^{\frac{1}{p'}} \alpha_N^{\phi;q} \|f\|_{p,S^q} \\ \le\ & C \sum_{\xi \in \mathcal{C}} a_\xi \left(\int_{-1}^{1} \Big| R_N(\phi, x) - \phi(x)\Big|^p W_q(x) dx\right)^{\frac{1}{p}} \omega_{q-1} \alpha_N^{\phi;q} \|f\|_{p,S^q}. \end{aligned}$$

By Lemma 2.1 we know $\sum_{\xi\in\mathcal{C}} a_\xi = 1$. Hence, Lemma 3.3 makes

$$\|M_{N,\phi}(f) - V_N(f)\|_{p,S^q} \le C\frac{\alpha_N^{\phi,q}\omega_{q-1}}{N^2}\sum_{0<k\le N+1} kE_k(\phi)_{p,W_q}\|f\|_{p,S^q}.$$

It follows that

$$\begin{aligned}\|M_{N,\phi}(f) - f\|_{p,S^q} &\le \|V_N(f) - f\|_{p,S^q} + \|M_{N,\phi}(f) - V_N(f)\|_{p,S^q} \\ &\le C\Big(\frac{1}{N+1}\sum_{k=0}^{[\sqrt{N}]}(k+1)E_k(f)_{p,S^q} \\ &\quad + \frac{\alpha_N^{\phi,q}\omega_{q-1}}{N^2}\sum_{0<k\le N+1} kE_k(\phi)_{p,W_q}\|f\|_{p,S^q}\Big).\end{aligned}$$

Acknowledgment

The authors thank the referee for valuable comments and references.

References

1. H. N. Mhaskar and C. A. Micchelli, Approximation by superposition of sigmoidal and radial basis functions, *Advanced in Applied Mathematics,* **13** (1992), 350-373.
2. H. N. Mhaskar and C. A. Micchelli, Degree of approximation by neural and translation networks with single hidden layer, *Advanced in Applied Math.,* **16** (1995), 151-183.
3. J. L. Wang, B. H. Sheng and S. P. Zhou, On approximation by non-periodic neural and translation networks in L_w^p spaces, *Acta Mathematica Sinica* [*in Chinese*], **46** (2003), 65-74.
4. H. N. Mhaskar, F. J. Narcowich and J. D. Ward, Approximation properties of zonal function networks using scattered data on the sphere, *Advances in Computational Mathematics,* **11** (1999), 121-127.
5. H. N. Mhaskar, F. J. Narcowich and D. Ward, Zonal function network frames on the sphere, *Neural Networks,* **16** (2003), 183-203.
6. C. K. Chui and X. Li, Realization of neural networks with one hidden layer, in: *Multivariate approximation: From CAGD to wavelets,* (K. Jetter and F. L. Utreras Ed.), Proceeding of the International Workshop, Santiago, (1992), 77-89.
7. H. N. Mhaskar, F. J. Narcowich and J. D. Ward, Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature, *Mathematics of Computation*, **70** (2000), 1113-1130.

8. H. N. Mhaskar, F. J. Narcowich, J. and D. Ward, Corrigendum to "Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature ". *Mathematics of Computation,* **71** (2001), 453-454.

9. H. Groemer, *Geometric applications of Fourier series and spherical harmonics*, Cambridge University Press, (1996).

10. K. Y. Wang and L. Q. Li, *Harmonic analysis and approximation on the unit sphere*, Science Press, Beijing/New York, (2000).

11. C. Muller, *Spherical harmonic,* Lecture Notes in Mathematics, Vol. 17, Springer-Verlag, Berlin, (1966).

12. H. Berens and L. Q. Li, On the de la Vallée Poussin means on the sphere. *Results in Math.,* **24** (1993), 14-26.

13. W. Chen and Z. Ditzian, Best approximation and K-functionals. *Acta Math. Hungar,* **75** (1997), 165-208.

14 Z. Ditzian, A K-functional and the rate of convergence of some linear polynomial operators, *Proc. Amer. Math. Soc.,* **124** (1996), 1773-1781.

15 AR. S. Dzafarov, Bernstein inequality for differential operators, *Analysis Math.,* **12** (1986), 251-268.

Sheng Baohuai and Wang Jianli
Department of Mathematics,
Shaoxing College of Arts and Sciences,
Shaoxing, Zhejiang 312000,
P. R. China
E-mail: bhsheng@zscas.edu.cn

Zhou Songping
Institute of Mathematics,
Zhejiang University of Science and Technology,
Hangzhou, Zhejiang 310018,
P. R. China