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LATTICE OPERATIONS OF POSITIVE BILINEAR MAPPINGS

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Abstract. In this paper we establish extension theorems for additive mappings $\varphi : A^+ \times B^+ \mapsto C^+$, where A, B are Riesz spaces (lattice ordered spaces or vector lattices) and C is an order complete Riesz space, to the whole of $A \times B$, thereby extending well-known results for additive mappings between Riesz spaces. We prove, in particular, that when A, B and C are order complete Riesz spaces, the ordered vector space $\mathcal{B}_b(A \times B, C)$ of all order bounded bilinear mappings has the structure of a lattice space.

1. Introduction

The extension theory of positive operators on a Riesz space has been well-documented; see, for example, the book by Aliprantis and Burkinshaw [1]. It is well-known that the ordered vector space $\mathcal{L}_b(E, F)$ of all order bounded linear mappings of a Riesz space E into an order complete Riesz space F has the structure of a lattice space. This important result was first proved by Riesz [5] for the special case $F = IR$, and later extended to the general setting by Kantorovic [2, 3]. In this paper we consider order bounded bilinear mappings $\varphi : A \times B \mapsto C$, where A, B and C are Riesz spaces. In §§2 and 3 we establish extension theorems for additive mappings $\varphi : A^+ \times B^+ \mapsto C^+$ to the whole of $A \times B$. In particular, we prove in §3 that φ may be extended uniquely to an order bounded bilinear mapping on $A \times B$. This enables us to define lattice operations on the space $\mathcal{B}_b(A \times B, C)$ when A, B and C are order complete Riesz spaces.

For the elementary theory of Riesz space and terminology not explained here we refer to [1, 4].

2. Quasi-bilinear Mappings

Definition 2.1. Let A, B and C be ordered vector spaces.

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(i) A mapping $\varphi : A \times B \mapsto C$ is said to be positive (in notations $\varphi \geq 0$ or $\varphi \leq 0$) whenever $\varphi(x, y) \in C^+$ (i.e., $\varphi(x, y) \geq 0$) holds for all $(x, y) \in A^+ \times B^+$.

(ii) A mapping $\varphi : A^+ \times B^+ \mapsto C^+$ is said to be additive whenever

$$\varphi(x + y, z) = \varphi(x, z) + \varphi(y, z) \quad \text{and} \quad \varphi(x, w + z) = \varphi(x, w) + \varphi(x, z)$$

hold for all $x, y \in A^+$ and $w, z \in B^+$.

(iii) A mapping $\psi : A^+ \times B \mapsto C$ (respectively $\varphi : A \times B^+ \mapsto C$) is said to be a right (respectively left) quasi-bilinear mapping if it is linear in the second variable (first variable) and additive in the first (second) or, equivalently,

$$\psi(x + y, \lambda u + v) = \lambda\psi(x, u) + \psi(x, v) + \lambda\psi(y, u) + \psi(y, v)$$

$$(\varphi(\lambda x + y, u + v) = \lambda\varphi(x, u) + \lambda\varphi(x, v) + \varphi(y, u) + \varphi(y, v))$$

for all $\lambda \in IR$, $x, y \in A^+$ *and* $u, v \in B$ *(*$x, y \in A$ *and* $u, v \in B^+$*).*

The collection of all right (left) quasi-bilinear mappings of $A^+ \times B$ into C (respectively $A \times B^+$ into C) will be denoted by $\mathcal{QB}(A^+ \times B, C)$ (respectively $\mathcal{QB}(A \times B^+, C)$). Evidently, $\mathcal{QB}(A^+ \times B, C)$ (respectively $\mathcal{QB}(A \times B^+, C)$) is an ordered vector space under the ordering, for all $x \in A^+$ and $y \in B^+$, $\varphi_1 \geq \varphi_2$ if and only if $\varphi_1(x, y) \geq \varphi_2(x, y)$.

In this paper we shall concentrate on right quasi-bilinear mappings; similar results hold for left quasi-bilinear mappings ([6]).

The following result follows almost immediately from the definition.

Lemma 2.2. *Let A, B and C be ordered vector spaces. If* $\varphi : A^+ \times B^+ \mapsto C^+$ *is an additive mapping, then* $(x, y) \leq (a, b)$ *in* $A^+ \times B^+$ *implies* $\varphi(x, y) \leq \varphi(a, b)$ *in* C^+.

Lemma 2.3. *Let A, B and C be Riesz spaces, with C order complete. If* $\varphi : A^+ \times B^+ \mapsto C^+$ *is additive, then* φ *is positive homogeneous in both variables; that is,* $\varphi(x, \lambda y) = \lambda\varphi(x, y)$ *and* $\varphi(\lambda x, y) = \lambda\varphi(x, y)$ *for all* $\lambda \geq 0$ *and* $(x, y) \in A^+ \times B^+$.

Proof. The result is trivial for $\lambda = 0$, and so we assume that $\lambda > 0$. We shall only prove that $\varphi(x, \lambda y) = \lambda\varphi(x, y)$ for all $\lambda > 0$, $(x, y) \in A^+ \times B^+$; the second equation can be established similarly. If λ is rational, then $\lambda = \frac{p}{q}$ for some positive integers $p, q > 0$. By the additivity of φ,

$$\varphi(a, pb) = p\varphi(a, b) \quad \text{and} \quad \varphi(a, b) = \varphi(a, q(\frac{b}{q})) = q\varphi(a, \frac{b}{q}),$$

which implies that $\varphi(a, \lambda b) = \lambda\varphi(a, b)$ for all positive rationals λ.

If λ is irrational, then choose two sequences of rational numbers $\{\epsilon_n\}$ and $\{\eta_n\}$ such that $0 \leq \epsilon_n \uparrow \lambda$ and $\eta_n \downarrow \lambda$. Given $a \in A^+$ it follows from $\epsilon_n b \leq \lambda b \leq \eta_n b$ in B^+ that $(a, \epsilon_n b) \leq (a, \lambda b) \leq (a, \eta_n b)$ in $A^+ \times B^+$ for all $a \in A^+$. By Lemma 2.2, $\varphi(a, \epsilon_n b) \leq \varphi(a, \lambda b) \leq \varphi(a, \eta_n b)$, and so

$$\epsilon_n \varphi(a, b) \leq \varphi(a, \lambda b) \leq \eta_n \varphi(a, b) \quad \text{for } n = 1, 2, \ldots.$$

Moreover, $\epsilon_n \varphi(a, b) \uparrow \lambda\varphi(a, b)$ and $\eta_n \varphi(a, b) \downarrow \lambda\varphi(a, b)$ in C^+. It follows that $\lambda\varphi(a, b) \leq \varphi(a, \lambda b)$. Since $\varphi(a, b) \geq 0$ in C and C is Archimedean (note that every order complete Riesz space is Archimedean), it follows from $(\eta_n - \epsilon_n) \downarrow 0$ and the inequalities

$$0 \leq \varphi(a, \lambda b) - \lambda\varphi(a, b) \leq \varphi(a, \lambda b) - \epsilon_n \varphi(a, b) \leq (\eta_n - \epsilon_n)\varphi(a, b)$$

that $\varphi(a, \lambda b) = \lambda\varphi(a, b)$, as required.

Similarly we can show that $\varphi(\lambda a, b) = \lambda\varphi(a, b)$, for all $\lambda \geq 0$ and $(a, b) \in A^+ \times B^+$. This proves that an additive mapping φ is positive homogeneous from $A^+ \times B^+$ into C^+.

Theorem 2.4. *Let A, B and C be Riesz spaces, with C order complete. If $\varphi : A^+ \times B^+ \mapsto C^+$ is an additive mapping, then φ extends uniquely to a positive right quasi-bilinear mapping, for all $x \in A^+$ and $y \in B$,*

$$\widetilde{\varphi} : A^+ \times B \mapsto C \qquad \text{such that} \qquad \widetilde{\varphi}(x, y) = \varphi(x, y^+) - \varphi(x, y^-).$$

Proof. We first observe that if $y = u - v$ with $u, v \in B^+$, then

$$\varphi(x, y^+) - \varphi(x, y^-) = \varphi(x, u) - \varphi(x, v)$$

for $x \in A^+$. Indeed, it follows from $y = y^+ - y^- = u - v$ that $y^+ + v = u + y^-$, and so, by the additivity of φ on $A^+ \times B^+$,

$$\varphi(x, y^+) + \varphi(x, v) = \varphi(x, y^+ + v) = \varphi(x, u + y^-) = \varphi(x, u) + \varphi(x, y^-),$$

from which it follows that $\varphi(x, y^+) - \varphi(x, y^-) = \varphi(x, u) - \varphi(x, v)$. Therefore, since every $y \in A$ has at least one decomposition by the properties of Riesz spaces, if we define

$$\widetilde{\varphi}(x, y) = \varphi(x, u) - \varphi(x, v) \ \ ((x, y) \in A^+ \times B),$$

where $y = u - v$ $(u, v \in B^+)$, then $\widetilde{\varphi}(x, y)$ depends only on (x, y) in $A^+ \times B$ and not on the particular decomposition of (x, y). Thus $\widetilde{\varphi}$ is well-defined on $A^+ \times B$. Moreover, $\widetilde{\varphi}(x, y) = \varphi(x, y)$ holds for every $(x, y) \in A^+ \times B^+$, and so $\widetilde{\varphi} : A^+ \times B \mapsto C$ is a positive mapping.

We now show that $\widetilde{\varphi}$ is additive on $A^+ \times B$. To see this, let $x_1, x_2 \in A^+$ and $y \in B$. Then $(x_1 + x_2, y) = (x_1 + x_2, y^+ - y^-)$ in $A^+ \times B$, and so, it follows from the additivity of φ on $A^+ \times B^+$ in the first variable that

$$\widetilde{\varphi}(x_1 + x_2, y) = \widetilde{\varphi}(x_1, y) + \widetilde{\varphi}(x_2, y).$$

Similarly we see that $\widetilde{\varphi}(x, y + z) = \widetilde{\varphi}(x, y) + \widetilde{\varphi}(x, z)$ as $(x, y + z) = (x, (y^+ + z^+) - (y^- + z^-))$ in $A^+ \times B$ for all $x \in A^+$ and $y, z \in B$.

For the homogeneity of $\widetilde{\varphi}$ on $A^+ \times B$, let $\lambda \in I\!R$, $x \in A^+$ and $y \in B$. Then $(x, \lambda y) = (x, (\lambda y)^+ - (\lambda y)^-)$ holds in $A^+ \times B$. If $\lambda \geq 0$, then we have $(x, \lambda y) = (x, \lambda y^+ - \lambda y^-)$ in $A^+ \times B$. Since φ is positive homogeneous by Lemma 2.3,

$$\widetilde{\varphi}(x, \lambda y) = \varphi(x, \lambda y^+) - \varphi(x, \lambda y^-) = \lambda\varphi(x, y^+) - \lambda\varphi(x, y^-) = \lambda\widetilde{\varphi}(x, y).$$

If $\lambda \leq 0$, then $-\lambda \geq 0$, and so $(\lambda y)^+ = (-\lambda)y^-$ and $(\lambda y)^- = (-\lambda)y^+$. Hence $(x, \lambda y) = (x, (-\lambda)y^- - (-\lambda)y^+)$, and so since φ is positive homogeneous by Lemma 2.3,

$$\widetilde{\varphi}(x, \lambda y) = \varphi(x, (-\lambda)y^-) - \varphi(x, (-\lambda)y^+) = \lambda(\varphi(x, y^+) - \varphi(x, y^-)) = \lambda\widetilde{\varphi}(x, y).$$

This proves that $\widetilde{\varphi}$ is homogeneous on $A^+ \times B$ in the second variable.

Similarly it can be seen that $\widetilde{\varphi}$ is positive homogeneous on $A^+ \times B$ in the first variable.

So far, we have proved that $\widetilde{\varphi}$ is a positive right quasi-bilinear mapping from $A^+ \times B$ into C. Finally, it remains to show that $\widetilde{\varphi}$ is unique. Assume that ψ is another right quasi-bilinear mapping from $A^+ \times B$ into C which extends φ; that is, $\psi(x, y) = \varphi(x, y)$ for all $(x, y) \in A^+ \times B^+$. By the decomposition property of Riesz spaces, given $y \in B$, there exist u and v in B^+ such that $y = u - v$. Hence $\psi(x, y) = \psi(x, u) - \psi(x, v) = \varphi(x, u) - \varphi(x, v) = \widetilde{\varphi}(x, y)$ for all $(x, y) \in A^+ \times B$, as required.

Remark 2.5. Let A, B and C be Riesz spaces, with C order complete. If $\varphi : A^+ \times B^+ \mapsto C^+$ is an additive mapping in both variables, then the left quasi-bilinear mapping $\widetilde{\varphi} : A \times B^+ \mapsto C$ defined by $\widetilde{\varphi}(x, y) = \varphi(x^+, y) - \varphi(x^-, y)$ for all $x \in A$ and $y \in B^+$ is the unique extension of φ.

3. Order Bounded Bilinear Mappings

Definition 3.1. Let A, B and C be ordered vector spaces. A subset D of $A \times B$ is called order bounded if there exist (a, b) and $(\tilde{a}, \tilde{b})$ in $A \times B$ such that $(a, b) \leq (x, y) \leq (\tilde{a}, \tilde{b})$ for all $(x, y) \in D$. A bilinear mapping $\varphi : A \times B \mapsto C$ is said to be order bounded if φ maps order bounded subsets of $A \times B$ onto order bounded subsets of C. In other words, $\varphi : A \times B \mapsto C$ is order bounded if there exist $u, v \in C$ such that $u \leq \varphi(x, y) \leq v$ for all $(x, y) \in A \times B$ satisfying $(a, b) \leq (x, y) \leq (\tilde{a}, \tilde{b})$ for some $(a, b), (\tilde{a}, \tilde{b}) \in A \times B$.

The set of all order bounded bilinear mappings of $\mathcal{B}(A \times B, C)$ will be denoted by $\mathcal{B}_b(A \times B, C)$. It is not difficult to see that $\mathcal{B}_b(A \times B, C)$ is an ordered linear subspace of $B(A \times B, C)$. In this section we show that, for an order complete Riesz space C, $\mathcal{B}_b(A \times B, C)$ is an order complete Riesz space. We start with the following lemma.

Lemma 3.2. *If A, B and C are Riesz spaces, then every positive bilinear mapping $\varphi : A \times B \mapsto C$ is order bounded.*

Proof. Let D an order bounded subset of $A \times B$; that is, there exists (u, v) in $A^+ \times B^+$ such that $(-u, -v) \leq (x, y) \leq (u, v)$ in $A \times B$ for all $(x, y) \in D$. It follows from $(x + u, y + v) \geq (0, 0)$ and $(u - x, v - y) \geq (0, 0)$ that $\varphi(x, y) + \varphi(x, v) + \varphi(u, y) + \varphi(u, v) \geq 0$ and $\varphi(u, v) - \varphi(u, y) - \varphi(x, v) + \varphi(x, y) \geq 0$, and so $-\varphi(u, v) \leq \varphi(x, y)$ in C. Similarly, from $(u - x, y + v) \geq (0, 0)$ and $(x + u, v - y) \geq (0, 0)$, we see that $\varphi(x, y) \leq \varphi(u, v)$ in C. Hence $-\varphi(u, v) \leq \varphi(x, y) \leq \varphi(u, v)$ in C, or, equivalently, since C is a Riesz space, $|\varphi(x, y)| \leq \varphi(u, v)$ in C, as required.

Theorem 3.3. [Extension Theorem] *Let A, B and C be Riesz spaces, with C order complete. If $\varphi : A^+ \times B^+ \mapsto C^+$ is an additive mapping, then φ extends uniquely to a positive bilinear mapping $\varphi : A \times B \mapsto C$ such that $\varphi(x, y) = \widetilde{\varphi}_l(x, y^+) - \widetilde{\varphi}_l(x, y^-)$ for all $x \in A$ and $y \in B$, where $\widetilde{\varphi}_l$ is the unique positive left quasi-bilinear mapping from $A \times B^+$ into C, as given in Remark 2.5.*

Proof. We first show that φ is unambiguously defined on $A \times B$. For this reason, suppose that $y = u - v$ with $u, v \in B^+$. It follows from $(x, y^+ + v) = (x, u + y^-)$ in $A \times B^+$ that $\widetilde{\varphi}_l(x, y^+) - \widetilde{\varphi}_l(x, y^-) = \widetilde{\varphi}_l(x, u) - \widetilde{\varphi}_l(x, v)$. Hence, since every $y \in B$ has at least one decomposition $y = u - v$ with $u, v \in B^+$, if we define $\varphi(x, y) = \widetilde{\varphi}_l(x, u) - \widetilde{\varphi}_l(x, v)$, then $\varphi(x, y)$ depends only on (x, y) in $A \times B$; not on the particular decomposition of (x, y).

By repeating the same arguments as the ones used to prove the extension theorem (Theorem 2.4), it follows from the left quasi-bilinearity of $\widetilde{\varphi}_l$ on $A^+ \times B$ that φ is bilinear. Moreover, φ is positive since $\widetilde{\varphi}_l$ on $A^+ \times B^+$ is positive.

Finally, for the uniqueness of φ, assume that $\widetilde{\varphi}$ is another bilinear mapping from $A \times B$ into C which extends φ; that is, $\widetilde{\varphi}(x, y) = \varphi(x, y)$ for all $(x, y) \in A^+ \times B^+$. Given $x \in A$ and $y \in B^+$, there exist $u, v \in A^+$ and $w, z \in B^+$ such that $x = u - v$ and $y = w - z$. Hence

$$\begin{aligned}\widetilde{\varphi}(x,y) &= \widetilde{\varphi}(u,w)+\widetilde{\varphi}(v,z)-\widetilde{\varphi}(v,w)-\widetilde{\varphi}(u,z)\\ &= ((\widetilde{\varphi}_l(u,w)-\widetilde{\varphi}_l(v,w))-(\widetilde{\varphi}_l(u,z)-\widetilde{\varphi}_l(v,z))\\ &= \varphi(x,w)-\varphi(x,z)=\varphi(x,y)\end{aligned}$$

for all $(x,y)\in A\times B$, which shows that $\widetilde{\varphi}=\varphi$, as required.

Theorem 3.4. *Let A, B and C be Riesz spaces, with C order complete. A bilinear mapping $\varphi : A\times B \mapsto C$ is order bounded if and only if there exist positive bilinear mappings $\varphi_1,\varphi_2 : A\times B\mapsto C$ such that $\varphi=\varphi_1-\varphi_2$.*

Proof. Suppose first that $\varphi=\varphi_1-\varphi_2$, where φ_1 and φ_2 are positive bilinear mappings. Since φ_1 and φ_2 are order bounded by Lemma 3.2, φ is order bounded.

Conversely, suppose that φ is an order bounded bilinear mapping from $A\times B$ into C. Then, for $x\in A$ and $y\in B$, the set

$$\{|\varphi(u,v)| : -x\le u\le x, -y\le v\le y\}$$

is an order bounded subset of C^+; in particular, the set $\{\varphi(a,b) : 0\le a\le x, 0\le b\le y\}$ is an order bounded subset of C^+. Hence $\bigvee_{\substack{0\le a\le x\\ 0\le b\le y}}\varphi(a,b)$ exists in C since C is order complete. If we set

$$\psi(x,y)=\bigvee_{\substack{0\le a\le x\\ 0\le b\le y}}\varphi(a,b),$$

then it is clear that $\psi(x,y)\ge 0$ in C for all $(x,y)\ge(0,0)$ in $A\times B$.

We show that ψ is additive on $A^+\times B^+$. If $a,b\in A$ and $c\in B$ satisfy $0\le a\le x,\ 0\le b\le y$ in A and $0\le c\le z$ in B, then $0\le a+b\le x+y$ in A, and so it follows from

$$\varphi(a,c)+\varphi(b,c)=\varphi(a+b,c)\le\bigvee_{\substack{0\le u\le x+y\\ 0\le v\le z}}\varphi(u,v)=\psi(x+y,z)$$

that $\psi(x,z)+\psi(y,z)\le\psi(x+y,z)$.

On the other hand, if $a\in A^+$ and $b\in B^+$ satisfy $0\le a\le x+y$ in A and $0\le b\le z$ in B, then there exist a_1 and a_2 in A such that $0\le a_1\le x,\ 0\le a_2\le y$ and $a_1+a_2=a$, by the decomposition property of Riesz spaces (see, e.g., [1, Theorem 1.9]). Hence

$$\begin{aligned}\varphi(a,b) &= \varphi(a_1,b)+\varphi(a_2,b)\le\bigvee_{\substack{0\le u_1\le x\\ 0\le v\le z}}\varphi(u_1,v)+\bigvee_{\substack{0\le u_2\le y\\ 0\le v\le z}}\varphi(u_2,v)\\ &= \psi(x,z)+\psi(y,z),\end{aligned}$$

from which it follows that $\bigvee_{\substack{0\le a\le x+y\\0\le b\le z}}\varphi(x+y,z)\le\psi(x,z)+\psi(y,z)$; that is, $\psi(x+y,z)\le\psi(x,z)+\psi(y,z)$. Therefore $\psi(x+y,z)=\psi(x,z)+\psi(y,z)$.

Similarly we can show that $\psi(x,y+z)=\psi(x,y)+\psi(x,z)$ for all $x\in A^+$ and $y,z\in B^+$. This proves that ψ is an additive mapping from $A^+\times B^+$ into C^+. By Theorem 3.3, there exists a unique positive bilinear mapping; say φ_1, from $A\times B$ into C which extends φ, i.e., $\varphi_1(x,y)=\varphi(x,y)$ for all $(x,y)\in A^+\times B^+$. Write $\varphi_2=\varphi_1-\varphi$. Clearly φ_2 defines a bilinear mapping on $A\times B$, φ_1 and φ are in the space $\mathcal{B}_b(A\times B,C)$ of all order bounded bilinear mappings of $\varphi:A\times B\mapsto C$. Thus, by the definition of ψ,

$$\varphi_1(x,y)=\psi(x,y)\ge\varphi(x,y),\quad\text{and so}\quad\varphi_2(x,y)=\varphi_1(x,y)-\varphi(x,y)\ge 0$$

for all $(x,y)\in A^+\times B^+$. Hence $\varphi_2\ge 0$; that is, φ_2 is positive, and so is order bounded by Lemma 3.2. Therefore we have $\varphi=\varphi_1-\varphi_2$ with $\varphi_1,\varphi_2\ge 0$ in $\mathcal{B}_b(A\times B,C)$, as required.

Theorem 3.5. *Let A, B and C be Riesz spaces, with C order complete, and let $\mathcal{B}_b(A\times B,C)$ be the space of all order bounded bilinear mappings of $\varphi:A\times B$ into C. If order in $\mathcal{B}_b(A\times B,C)$ is defined by*

$$\varphi_1\ge\varphi_2\quad\textit{if and only if}\quad\varphi_1(x,y)\ge\varphi_2(x,y)$$

for all $(x,y)\in A^+\times B^+$, then $\mathcal{B}_b(A\times B,C)$ becomes an order complete Riesz space.

Proof. We first prove that $\mathcal{B}_b(A\times B,C)$ is a Riesz space. In order to do this, in the view of the identities in Riesz spaces

$$\varphi\vee\psi=(\varphi-\psi)^++\psi\quad\text{and}\quad\varphi\wedge\psi=-((-\varphi)\vee(-\psi)),$$

it is enough to show that φ^+ exists and belongs to $\mathcal{B}_b(A\times B,C)$ for every $\varphi\in\mathcal{B}_b(A\times B,C)$. To this end, let $\varphi\in\mathcal{B}_b(A\times B,C)$. As in the proof of the preceding theorem, if we define

$$\psi:A^+\times B^+\mapsto C\quad\text{by}\quad\psi(x,y)=\bigvee_{\substack{0\le a\le x\\0\le b\le y}}\varphi(a,b)$$

for all $(x,y)\in A^+\times B^+$, then we see that ψ is an additive mapping. By the extension theorem (Theorem 3.3), ψ defines a positive bilinear mapping from $A\times B$ into C (more precisely, ψ extends uniquely to a positive bilinear mapping, again denoted by ψ, from $A\times B$ into C).

We have to show that ψ is the least upper bound of φ and 0. Clearly $\psi\ge 0$ and $\psi\ge\varphi$ since $\psi(x,y)\ge\varphi(x,y)$ for all $(x,y)\in A^+\times B^+$. Hence $\psi\ge\varphi\vee 0$;

that is, ψ is any other upper bound of φ and 0 in $\mathcal{B}_b(A \times B, C)$. Suppose that ψ' is an upper bound of φ and 0 in $\mathcal{B}_b(A \times B, C)$. Then $\psi'(x, y) \geq \psi'(a, b) \geq \varphi(a, b)$ for all $(0, 0) \leq (x, y) \leq (a, b)$ in $A \times B$. It follows that

$$\psi'(x, y) \geq \bigvee_{\substack{0 \leq a \leq x \\ 0 \leq b \leq y}} \varphi(a, b) = \psi(x, y)$$

for all $(x, y) \in A^+ \times B^+$, and so $\psi' \geq \psi$. Therefore ψ is the least upper bound of φ and 0; that is, $\psi = \varphi \vee 0$ in $\mathcal{B}_b(A \times B, C)$. In the usual notation, $\psi = \varphi^+$ holds in $\mathcal{B}_b(A \times B, C)$. This shows that $\varphi^+ \in \mathcal{B}_b(A \times B, C)$ for each $\varphi \in \mathcal{B}_b(A \times B, C)$ and satisfies $\varphi^+(x, y) = \bigvee_{\substack{0 \leq a \leq x \\ 0 \leq b \leq y}} \varphi(a, b)$ for all $(x, y) \in A^+ \times B^+$, as required.

Finally we establish that $\mathcal{B}_b(A \times B, C)$ is order complete, as follows.

Suppose that $0 \leq \varphi_\tau \uparrow \leq \varphi_0$ holds in $\mathcal{B}_b(A \times B, C)$. We have to show that $\bigvee_\tau \varphi_\tau$ exists in $\mathcal{B}_b(A \times B, C)$. To this end, let $\varphi(x, y) = \bigvee_\tau \varphi_\tau(x, y)$ for all $(x, y) \in A^+ \times B^+$. Clearly $\varphi(x, y)$ exists as an element of C^+ since C is order complete. For $x, y \in A^+$ and $z \in B^+$, the nets $\{\varphi_\tau(x, z)\}$ and $\{\varphi_\tau(y, z)\}$ are upwards directed in C^+ and it follows from the bilinearity of φ_τ for all τ and the properties of Riesz spaces (see, e.g., [4, Theorem 15.8(iii)]) that

$$\varphi(x + y, z) = \bigvee_\tau \varphi_\tau(x, z) + \bigvee_\tau \varphi_\tau(y, z) = \varphi(x, z) + \varphi(y, z).$$

Similarly we can show that $\varphi(x, y + z) = \varphi(x, y) + \varphi(x, z)$ for all $x \in A^+$ and $y, z \in B^+$. This shows that φ is an additive mapping from $A^+ \times B^+$ into C^+ in both variables. Hence, by the extension theorem (Theorem 3.3), there exists a unique positive bilinear mapping ψ from $A \times B$ into C which extends φ. It follows that $\psi(x, y) = \bigvee_\tau \varphi_\tau(x, y)$ for all $(x, y) \in A^+ \times B^+$ since $\psi(x, y) = \varphi(x, y)$ for all $(x, y) \in A^+ \times B^+$. Therefore $\varphi_\tau \uparrow \psi$ holds in $\mathcal{B}_b(A \times B, C)$; that is, ψ is the desired supremum of the net $\{\varphi_\tau\}$ satisfying $0 \leq \varphi_\tau \uparrow \leq \varphi_0$ in $\mathcal{B}_b(A \times B, C)$. This proves that $\mathcal{B}_b(A \times B, C)$ is an order complete Riesz space.

Considering Theorem 2.4 and following the proofs of Theorem 2.5, the following can be established.

Remark 3.6. If A, B and C be Riesz spaces, with C order complete, then the space of all right quasi-bilinear mappings $(\mathcal{QB})_b(A^+ \times B, C)$ and the space of all left quasi-bilinear mappings $(\mathcal{QB})_b(A \times B^+, C)$ are both order complete Riesz spaces.

We observe that $\mathcal{B}_b(A \times B, C) \subseteq (\mathcal{QB})_b(A^+ \times B, C)$ and $\mathcal{B}_b(A \times B, C) \subseteq (\mathcal{QB})_b(A \times B^+, C)$, and so $\mathcal{B}_b(A \times B, C) \subseteq (\mathcal{QB})_b(A^+ \times B, C) \cap (\mathcal{QB})_b(A \times B^+, C)$. Hence $\mathcal{B}_b(A \times B, C)$ is an order complete Riesz subspace of both $(\mathcal{QB})_b(A^+ \times B, C)$ and $(\mathcal{QB})_b(A \times B^+, C)$.

We are now in a position to express the lattice operations of the space $\mathcal{B}_b(A \times B, C)$.

Theorem 3.7. *Let A, B and C be Riesz spaces, with C order complete. For every $\varphi \in \mathcal{B}_b(A \times B, C)$ and $(x, y) \in A^+ \times B^+$, the following statements hold.*

(1) $\varphi^+(x, y) = \bigvee_{\substack{0 \le a \le x \\ 0 \le b \le y}} \varphi(a, b)$.

(2) $\varphi^-(x, y) = \bigvee_{\substack{0 \le a \le x \\ 0 \le b \le y}} -\varphi(a, b)$.

(3) $|\varphi(x, y)| \le |\varphi|(x, y)$.

(4) $|\varphi(x, y)| \le |\varphi|(|x|, |y|)$ *for all* $(x, y) \in A \times B$.

(5) $|\varphi|(x, y) = \bigvee_{\substack{|a| \le x \\ |b| \le y}} \varphi(a, b) = \bigvee_{\substack{|a| \le x \\ |b| \le y}} |\varphi(a, b)|$.

Proof. We first note that φ^+ (and hence φ^- and $|\varphi|$) is well-defined in $\mathcal{B}_b(A \times B, C)$ since $\mathcal{B}_b(A \times B, C)$ is an order complete Riesz space by Theorem 3.5.

(1) This is obvious since φ^+ is the mapping ψ in the proof of Theorem 3.5.

(2) Follows from the fact that $\varphi^- = (-\varphi)^+$ in $\mathcal{B}_b(A \times B, C)$ since $\mathcal{B}_b(A \times B, C)$ is a Riesz space.

(3) By the properties of Riesz spaces again, for all $(x, y) \in A^+ \times B^+$, $\varphi(x, y) \le \varphi^+(x, y) \le |\varphi|(x, y)$ and $-\varphi(x, y) \le \varphi^-(x, y) \le |\varphi|(x, y)$ hold in C. Hence $|\varphi(x, y)| = (\varphi(x, y)) \vee (-\varphi(x, y)) \le |\varphi|(x, y)$ for all $(x, y) \in A^+ \times B^+$.

(4) Using the decomposition property of Riesz spaces, bilinearity of φ and (3), for all $x \in A$ and $y \in B$, we have

$$|\varphi(x, y)| \le |\varphi|(x^+, y^+) + |\varphi|(x^+, y^-) + |\varphi|(x^-, y^+) + |\varphi|(x^-, y^-)| = |\varphi|(|x|, |y|).$$

(5) If $|a| \le x$ in A and $|b| \le y$ hold in B, then $\varphi(a, b) \le |\varphi(a, b)| \le |\varphi|(|a|, |b|) \le |\varphi|(x, y)$ by (4) and the positivity of $|\varphi|$ in the Riesz space $\mathcal{B}_b(A \times B, C)$. It follows that

$$\bigvee_{\substack{|a| \le x \\ |b| \le y}} \varphi(a, b) \le |\varphi|(x, y) \quad \text{and} \quad \bigvee_{\substack{|a| \le x \\ |b| \le y}} |\varphi(a, b)| \le |\varphi|(x, y).$$

For the converse direction, we first observe that $0 \le a_1 \le x$ and $0 \le a_2 \le x$ imply $a_1 - a_2 \le x$ and $a_2 - a_1 \le x$, and so $|a_1 - a_2| \le x$. Similarly $0 \le b_1 \le y$ and $0 \le b_2 \le y$ imply $|b_1 - b_2| \le y$. It now follows from $|\varphi| = \varphi^+ + \varphi^-$ in

$\mathcal{B}_b(A \times B, C)$ that

$$\begin{aligned}
|\varphi|(x,y) &= \varphi^+(x,y) + \varphi^-(x,y) \\
&= \bigvee_{\substack{0\le u\le x\\ 0\le v\le y}} \varphi(u,v) + \bigvee_{\substack{0\le w\le x\\ 0\le z\le y}} -\varphi(w,z) \quad \text{(by (1) and (2))} \\
&= \bigvee_{\substack{0\le u\le x\\ 0\le v\le y}} \varphi(u,v) + \bigvee_{\substack{0\le w\le x\\ 0\le z\le y}} \varphi(-w,z) \\
&\le \Big(\bigvee_{\substack{0\le u\le x\\ 0\le v\le y}} \varphi(u,v) + \bigvee_{\substack{0\le w\le x\\ 0\le z\le y}} \varphi(-w,z) + \bigvee_{\substack{0\le u\le x\\ 0\le v\le y}} \varphi(u,v) + \bigvee_{\substack{0\le w\le x\\ 0\le v\le y}} \varphi(-w,v) \Big) \\
&\quad + \bigvee_{\substack{0\le u\le x,\, 0\le w\le x\\ 0\le z\le y}} \varphi(u-w,-2z) \\
&= \bigvee_{\substack{0\le u\le x,\, 0\le w\le x\\ 0\le v\le y,\, 0\le z\le y}} \varphi(u-w,v+z) + \bigvee_{\substack{0\le u\le x,\, 0\le w\le x\\ 0\le z\le y}} \varphi(u-w,-2z) \\
&= \bigvee_{\substack{0\le u\le x,\, 0\le w\le x\\ 0\le v\le y,\, 0\le z\le y}} \varphi(u-w,v-z) \le \bigvee_{\substack{|a|\le x\\ |b|\le y}} \varphi(a,b) \le \bigvee_{\substack{|a|\le x\\ |b|\le y}} |\varphi(a,b)|.
\end{aligned}$$

Combining the above and preceding gives

$$|\varphi|(x,y) = \bigvee_{\substack{|a|\le x\\ |b|\le y}} \varphi(a,b) = \bigvee_{\substack{|a|\le x\\ |b|\le y}} |\varphi(a,b)|,$$

as required.

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