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WEAKLY STABLE IDEALS OF EXCHANGE RINGS

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Abstract. Let I be an ideal of an exchange ring R. We say that I is weakly stable provided that every regular element in $1+I$ is one-sided unit-regular. If I is weakly stable, then so is $M_n(I)$ as an ideal of $M_n(R)$. Also every square regular matrix over such an ideal admits a diagonal reduction by right or left invertible matrices. These extend the corresponding results of [1],[6-8],[10] and [12-13].

A ring R is an exchange ring if for every right R-module A and any two decompositions $A = M' \oplus N = \bigoplus_{i\in I} A_i$, where $M'_R \cong R_R$ and the index set I is finite, there exist submodules $A'_i \subseteq A_i$ such that $A = M' \oplus \big(\bigoplus_{i\in I} A'_i\big)$. The class of exchange rings is very large, it includes local rings, semiperfect rings, semiregular rings, π-regular rings, strongly π-regular rings, C^*-algebras with real rank one, while there still exist exchange rings which belong to none of the above classes. An element $a \in R$ is regular in case there exists a $x \in R$ such that $a = axa$. If x is right or left invertible, then $a \in R$ is said to be one-sided unit-regular. Following Wei, an exchange ring R is weakly stable if and only if every regular element in R is one-sided unit-regular(see [12, Theorem 3]). Many equivalent characterizations of weakly stable exchange rings have been studied by Wei(cf. [12-13]). A natural problem asks that whether weakly stable property is Morita invariant for exchange rings.

Let I be an ideal of an exchange ring R. We say that I is weakly stable provided that every regular element in $1 + I$ is one-sided unit-regular. Let V be an infinite-dimensional vector space over a division ring D. Let $R = \text{End}_D(V)$ and $S = \text{End}\big(G \oplus H\big)$, where G is the direct sum of infinitely many copies of Z_p and H is the direct sum of infinitely many copies of Z_q with distinct primes p and q. Set $T = R \oplus S$. Then T is a regular ring, while it is not weakly stable. But S is weakly stable as an ideal of T. Thus the concept of weakly stable ideal is a nontrivial generalization of that of weakly stable ring.

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We prove, in this paper, that if an ideal I of an exchange ring R is weakly stable then so is $M_n(I)$ as an ideal of $M_n(R)$. This gives an affirmative answer to the previous problem. Also every square regular matrix over such an ideal admits a diagonal reduction by right or left invertible matrices. These extend the corresponding results of [1, 6-8, 10, 12-13].

Throughout, all rings are associative with identity and all modules are right modules. $\mathrm{U}_<(R)$ stands for the set of all right or left invertible elements in R. $M \lesssim^{\oplus} N$ means that a right R-module M is isomorphic to a direct summand of a right R-module N. The pair (a,b) is called right unimodular in case $aR+bR=R$. The right unimodular (a,b) is called right weakly reducible if there exists $y\in R$ such that $a+by\in \mathrm{U}_<(R)$.

Lemma 1. *Let (a,b) be right unimodular in a ring R. Let $u,v\in U(R)$ and $c\in R$. Then $(vau+vbc, vb)$ is also right unimodular. Furthermore, (a,b) is right weakly reducible if and only if so is $(vau+vbc, vb)$.*

Proof. By [3, Lemma 6.3], $(vau+vbc, vb)$ is right unimodular. Assume that (a,b) is right weakly reducible. Then we have a $y\in R$ such that $a+by\in \mathrm{U}_<(R)$. Choose $z=yu-c$. We have $(vau+vbc)+(vb)z=v(a+by)u\in \mathrm{U}_<(R)$; hence, $(au+vbc, vb)$ is right weakly reducible. Conversely, assume that there exists $z\in R$ such that $vau+vbc+vbz\in \mathrm{U}_<(R)$. Then $v\big(a+b(c+z)u^{-1}\big)u\in \mathrm{U}_<(R)$, so $a+b(c+z)u^{-1}\in \mathrm{U}_<(R)$. Therefore (a,b) is right weakly reducible. ∎

Lemma 2. *Let I be an ideal of an exchange ring R. Then the following are equivalent:*

(1) *I is weakly stable.*

(2) *Whenever $ax+b=1$ with $a\in 1+I, b\in I$, there exists a $y\in R$ such that $a+by\in U_<(R)$*

Proof. (1)⇒(2) Suppose that $ax+b=1$ with $a\in 1+I, b\in I$. Since R is an exchange ring, by [11, Proposition 29.1], we can find an idempotent $e\in R$ such that $e\in bR$ and $1-e\in(1-b)R$. So $e=bs$ and $1-e=(1-b)t=axt$ for some $s,t\in R$. Hence $(1-e)axt(1-e)+e=1$, and then $(1-e)a=(1-e)axt(1-e)a$, i.e., $(1-e)a\in 1+I$ is regular. So we can find a $u\in U_<(R)$ such that $(1-e)a=(1-e)au(1-e)a$. Let $f=u(1-e)a$. Then

$$\begin{aligned} & f\big(xt(1-e)+ue\big)+(1-f)ue \\ = {} & fxt(1-e)+ue \\ = {} & u\big((1-e)axt(1-e)+e\big) \\ = {} & u. \end{aligned}$$

If $vu = 1$ for some $v \in R$, then

$$\begin{aligned} & (1-f)uev(1-f)ue \\ = \ & (1-f)ue(vu - vfu)e \\ = \ & (1-f)ue\big(1 - vu(1-e)au\big)e \\ = \ & (1-f)ue. \end{aligned}$$

If $uv = 1$ for some $v \in R$, then

$$\begin{aligned} & (1-f)uev(1-f)ue \\ = \ & (1-f)(1-f)uev(1-f)ue \\ = \ & (1-f)\big(uv - f(xt(1-e) + ue)v\big)(1-f)ue \\ = \ & (1-f)uv(1-f)ue \\ = \ & (1-f)ue. \end{aligned}$$

In any case, we get $(1-f)ue = (1-f)uev(1-f)ue$. Let $g = (1-f)uev(1-f)$. Then $f\big(xt(1-e)+ue\big) + gue = u$. Since $f = f^2, g = g^2$ and $fg = gf = 0$, we see that $f\big(xt(1-e)+ue\big) = fu$ and $gue = gu$. Thus

$$\begin{aligned} & u\big(a + bs(v(1-f)(1+fuev(1-f)) - a)\big)\big(1 - fuev(1-f)\big)u \\ = \ & u\big((1-e)a + ev(1-f)(1+fuev(1-f))\big)\big(1 - fuev(1-f)\big)u \\ = \ & \big(f + uev(1-f)(1+fuev(1-f))\big)\big(1 - fuev(1-f)\big)u \\ = \ & \big(f(1-fuev(1-f)) + uev(1-f)\big)u \\ = \ & \big(f + (1-f)uev(1-f)\big)u \\ = \ & (f+g)u \\ = \ & f\big(xt(1-e)+ue\big) + gue \\ = \ & u. \end{aligned}$$

It follows from $u \in U_<(R)$ that $a + bs\big(v(1-f)(1+fuev(1-f)) - a\big) \in U_<(R)$, as required.

(2)⇒(1) For any regular $x \in 1 + I$, we have a $y \in R$ such that $x = xyx$ and $y = yxy$. Clearly, $yx + (1-yx) = 1$ with $y \in 1 + I, 1 - yx \in I$. Hence, we have a $z \in R$ such that $y + (1-yx)z = u \in U_<(R)$. Therefore $x = x\big(y + (1-yx)z\big)x = xux$, as asserted. ■

In [5, Corollary 7], the author proved that if R is one-sided unit-regular then so is $M_n(R)$ by virtue of self-cancellations of modules. But we can not apply that method to weakly stable ideals of exchange rings. Now we extend this fact to ideals of an exchange ring by a new route.

Theorem 3. *Let I be a weakly stable ideal of an exchange ring R. Then $M_n(I)$ is a weakly stable ideal of $M_n(R)(n \geq 1)$.*

Proof. Clearly, the result holds for $n = 1$. Assume inductively that the result holds for n. It will suffice to show that the result also holds for $n + 1$. Suppose that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n+1)} \\ a_{21} & a_{22} & \cdots & a_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & \cdots & a_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1(n+1)} \\ b_{21} & b_{22} & \cdots & b_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(n+1)1} & b_{(n+1)2} & \cdots & b_{(n+1)(n+1)} \end{pmatrix}$$

$$+ \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1(n+1)} \\ c_{21} & c_{22} & \cdots & c_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & \cdots & c_{(n+1)(n+1)} \end{pmatrix} = diag(1, 1, \cdots, 1) \ (*)$$

in $M_{n+1}(R)$, where

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n+1)} \\ a_{21} & a_{22} & \cdots & a_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & \cdots & a_{(n+1)(n+1)} \end{pmatrix} \in diag(1, 1, \cdots, 1) + M_{n+1}(I),$$

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1(n+1)} \\ c_{21} & c_{22} & \cdots & c_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & \cdots & c_{(n+1)(n+1)} \end{pmatrix} \in M_{n+1}(I).$$

Clearly, $a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1(n+1)}b_{(n+1)1} + c_{11} = 1$ with $a_{11} \in 1 + I$. Since I is weakly stable, by virtue of Lemma 2, we can find $z_1 \in R$ such that $a_{11} + (a_{12}b_{21} + \cdots + a_{1(n+1)}b_{(n+1)1} + c_{11})z_1 \in \mathrm{U}_<(R)$. According to Lemma 1, $(*)$ is right weakly reducible if and only if this is so for the row with elements

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(n+1)} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2(n+1)} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & a_{(n+1)3} & \cdots & a_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ b_{21}z_1 & 1 & 0 & \cdots & 0 \\ b_{31}z_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{(n+1)1}z_1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} z_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)}. \end{pmatrix}$$

Thus we may assume that the element $a_{11} \in \mathrm{U}_<(R)$ in $(*)$, so there are $s, t \in R$ such that $sa_{11}t = 1$, where $s = 1$ or $t = 1$. Clearly, we may assume that

$$\begin{pmatrix} a_{33} & \cdots & a_{3(n+1)} \\ a_{43} & \cdots & a_{4(n+1)} \\ \vdots & \ddots & \vdots \\ a_{(n+1)3} & \cdots & a_{(n+1)(n+1)} \end{pmatrix} \in diag(1, 1, \cdots, 1) + M_{n-1}(I),$$

$$\begin{pmatrix} c_{33} & \cdots & c_{3(n+1)} \\ c_{43} & \cdots & c_{4(n+1)} \\ \vdots & \ddots & \vdots \\ c_{(n+1)3} & \cdots & c_{(n+1)(n+1)} \end{pmatrix} \in M_{n-1}(I).$$

One easily verifies that

$$\begin{pmatrix} s & 0 & 0 & \cdots & 0 \\ 1 - ats & at & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(n+1)} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2(n+1)} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & a_{(n+1)3} & \cdots & a_{(n+1)(n+1)} \end{pmatrix}$$

$$\times \begin{pmatrix} t & 1-tsa & 0 & \cdots & 0 \\ 0 & sa & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & b_{12} & b_{13} & \cdots & b_{1(n+1)} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2(n+1)} \\ b_{31} & b_{32} & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{(n+1)1} & b_{(n+1)2} & * & \cdots & * \end{pmatrix},$$

and that

$$\begin{pmatrix} s & 0 & 0 & \cdots & 0 \\ 1-ats & at & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} at & 1-ats & 0 & \cdots & 0 \\ 0 & s & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}^{-1},$$

$$\begin{pmatrix} t & 1-tsa & 0 & \cdots & 0 \\ 0 & sa & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} sa & 0 & 0 & \cdots & 0 \\ 1-tsa & t & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}^{-1} \in \mathrm{GL}_{n+1}(R).$$

Thus $(*)$ is right weakly reducible if and only if this is so for the row with elements

$$\begin{pmatrix} 1 & b_{12} & b_{13} & \cdots & b_{1(n+1)} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2(n+1)} \\ b_{31} & b_{32} & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{(n+1)1} & b_{3(n+1)} & * & \cdots & * \end{pmatrix}, \begin{pmatrix} s & 0 & 0 & \cdots 0 & \\ 1-ats & at & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)} \end{pmatrix}.$$

Clearly, all $b_{ij} \in I$ for $i \neq j$. If $s = 1$, we see that $b_{22} = a_{21}(1 - ta_{11}) + a_{22}a_{11} \in 1 + I$ because $a_{21} \in I$ and $a_{11}, a_{22} \in 1 + I$. If $t = 1$, then $b_{22} = (1 - a_{11})a_{12} + a_{11}a_{22} \in 1 + I$ because $a_{12} \in I$ and $a_{11}, a_{22} \in 1 + I$. In any case, we have $b_{22} \in 1 + I$. By Lemma 1 again, $(*)$ is right weakly reducible if and only if this is so for the row with elements

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ ** & 1 & 0 & \cdots & 0 \\ ** & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ** & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} s & 0 & 0 & \cdots & 0 \\ 1-ats & at & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)} \end{pmatrix}.$$

Thus, we may assume that $a_{11} = 1, a_{1i} = 0 = a_{i1}$ for $i = 2, \cdots, n+1$ in $(*)$. Furthermore, we may assume that $(*)$ is in the following form:

$$\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & D \end{pmatrix} \begin{pmatrix} b_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_n \end{pmatrix},$$

and that we may assume that $D \in diag(1, 1, \cdots, 1) + M_n(I)$. Clearly, we have $DB_{22} + C_{22} = I_n$. By the induction hypothesis, $M_n(I)$ is weakly stable. In view of Lemma 2, there exists a $Z_2 \in M_n(R)$ such that $D + C_{22}Z_2 \in \mathrm{U}_<\big(M_n(R)\big)$. Thus, we pass to the right unimodular row with elements

$$\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & D \end{pmatrix} + \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} 0 & 0_{1\times n} \\ 0_{n\times 1} & Z_2 \end{pmatrix}, \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

So it suffices to show that the right unimodular with elements

$$\begin{pmatrix} 1 & C_{12}Z_2 \\ 0_{n\times 1} & D + C_{22}Z_2 \end{pmatrix} \text{ and } \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

is weakly right reducible. Since $D + C_{22}Z_2 \in \mathrm{U}_<(M_n(R))$, we conclude that $\begin{pmatrix} 1 & C_{12}Z_2 \\ 0 & D + C_{22}Z_2 \end{pmatrix} \in \mathrm{U}_<(M_{n+1}(R))$. By induction, we complete the proof. ■

Lemma 4. *Let I be a weakly stable ideal of an exchange ring R. Then for any idempotent $e \in I$, eRe is a weakly stable ring.*

Proof. Let$e \in I$ be an idempotent. Given $ax + b = e$ with $a, x, b \in eRe$, we have $(a + 1 - e)(x + 1 - e) + b = 1$ in R. From $a, e \in I$, we know that $a + 1 - e \in 1 + I$. So there is a $y \in R$ such that $a + 1 - e + by \in U_<(R)$.

First, assume $(a+1-e+by)u=1$ for some $u\in R$. As $a,b\in eRe$, we get $(1-e)u=1-e$; hence $eue=ue$. Thus $\big(a+b(eye)\big)(eue)=e$. Now assume $u(a+1-e+by)=1$ for some $u\in R$. It is easy to check that $(eue)\big(a+b(eye)\big)=e$. In any case, we have $a+b(eye)\in U_{<}(eRe)$. Therefore eRe is a weakly stable ring. ■

Let $FP(I)$ denotes the set of all finitely generated projective right R-modules P with $P=PI$. Recall that an ideal I of an exchange ring R is separative provided that for any $A,B\in FP(I)$,

$$A\oplus A\cong A\oplus B\cong B\oplus B\Longrightarrow A\cong B.$$

It is well known that an ideal I of an exchange ring R is separative if and only if eRe is separative for any idempotents $e\in I$. By [6, Theorem 3], every weakly stable exchange ring is separative. In view of Lemma 4, we conclude that every weakly stable ideal I of an exchange ring R is separative; hence, the natural map $GL_1(R,I)\to K_1(R,I)$ is surjective.

Lemma 5. *Let A be a right R-module such that End$_R(A)$ is a weakly stable ring. Then for any right R-module B and C, $A\oplus B\cong A\oplus C$ implies that either $B\lesssim^{\oplus} C$ or $C\lesssim^{\oplus} B$.*

Proof. Suppose that $M=A_1\oplus B_1=A_2\oplus B_2$ with $A_1\cong A\cong A_2, B_1\cong B$ and $B_2\cong C$. By virtue of [10, Proposition 2.5.4], we can find some $C,D\leq M$ such that $M=C\oplus D\oplus B_1=C\oplus B_2$ or $M=C\oplus B_1=C\oplus D\oplus B_2$. If $M=C\oplus D\oplus B_1=C\oplus B_2$, then $B_2\cong D\oplus B_1$; hence, $B_1\lesssim^{\oplus} B_2$. If $M=C\oplus B_1=C\oplus D\oplus B_2$, then $B_1\cong D\oplus B_2$, and so $B_2\lesssim^{\oplus} B_1$. Therefore we complete the proof. ■

Let R be an associative ring with identity. The notatin $R^{n\times 1}$ stands for the set $\left\{\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix} \mid x_1,\cdots,x_n\in R\right\}$, which is a $M_n(R)$-R-bimodule. The notation $R^{1\times n}$ stands for the set $\{(x_1,\cdots,x_n)\mid x_1,\cdots,x_n\in R\}$, which is a R-$M_n(R)$-bimodule. We now observe the following comparability of modules related to weakly stable ideals of exchange rings.

Theorem 6. *Let I be a weakly stable ideal of an exchange ring R, and let $A\in FP(I)$. If B and C are any right R-modules such that $A\oplus B\cong A\oplus C$, then $B\lesssim^{\oplus} C$ or $C\lesssim^{\oplus} B$.*

Proof. Suppose that $A\oplus B\cong A\oplus C$ for right R-modules B and C. Since A is a finitely generated projective right R-module, we have idempotents $e_1,\cdots,e_n\in$

R such that $A \cong e_1R \oplus \cdots \oplus e_nR$ (cf. [11, Exercise 29.9]). Hence, $A \cong diag(e_1, \cdots, e_n)R^{n\times 1}$. Thus, we see that

$$diag(e_1, \cdots, e_n)R^{n\times 1} \oplus B \cong diag(e_1, \cdots, e_n)R^{n\times 1} \oplus C,$$

and then

$$\begin{aligned} & diag(e_1, \cdots, e_n)R^{n\times 1} \bigotimes_R R^{1\times n} \oplus B \bigotimes_R R^{1\times n} \\ \cong\ & diag(e_1, \cdots, e_n)R^{n\times 1} \bigotimes_R R^{1\times n} \oplus C \bigotimes_R R^{1\times n}. \end{aligned}$$

From $A = AI$, we deduce that $e_1R \bigotimes_R (R/I) \oplus \cdots \oplus e_nR \bigotimes_R (R/I) \cong A \bigotimes_R (R/I) = 0$. This means that all $e_iR/(e_iI) \cong e_iR \bigotimes_R (R/I) = 0$, hence $e_i \in e_iR = e_iI \in I$ for all i. Therefore $A \cong e_1R \oplus \cdots e_nR$ with all $e_i \in I$. Thus, $diag(e_1, \cdots, e_n)$ is an idempotent matrix over I. By Theorem 3, $M_n(I)$ is weakly stable as an ideal of $M_n(R)$. Since

$$\begin{aligned} & \text{End}_{M_n(R)}\big(diag(e_1, \cdots, e_n)R^{n\times 1} \bigotimes_R R^{1\times n}\big) \\ \cong\ & diag(e_1, \cdots, e_n)M_n(R)diag(e_1, \cdots, e_n), \end{aligned}$$

it follows from Lemma 4 that $\text{End}_{M_n(R)}\big(diag(e_1, \cdots, e_n)R^{n\times 1} \bigotimes_R R^{1\times n}\big)$ is a weakly stable ring, hence we get $B \bigotimes_R R^{1\times n} \lesssim^{\oplus} C \bigotimes_R R^{1\times n}$ or $C \bigotimes_R R^{1\times n} \lesssim^{\oplus} B \bigotimes_R R^{1\times n}$ by Lemma 5. Clearly, $R^{1\times n} \bigotimes_{M_n(R)} R^{n\times 1} \cong R$ as right R-modules. Therefore either $B \cong B \bigotimes_R R^{1\times n} \bigotimes_{M_n(R)} R^{n\times 1} \lesssim^{\oplus} C \bigotimes_R R^{1\times n} \bigotimes_{M_n(R)} R^{n\times 1} \cong C$ or $C \cong C \bigotimes_R R^{1\times n} \bigotimes_{M_n(R)} R^{n\times 1} \lesssim^{\oplus} B \bigotimes_R R^{1\times n} \bigotimes_{M_n(R)} R^{n\times 1} \cong B$, as required. ■

In [13, Theorem 6], Wei obtained the internal weak cancellation property of weakly stable ring. He proved that $End_R(A)$ is a weakly stable ring if and only if $A = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$ implies that $B \lesssim^{\oplus} C$ or $C \lesssim^{\oplus} B$. Now we get an external weak cancellation property of weakly stable exchange rings.

Corollary 7. *An exchange ring R is a weakly stable ring if and only if for all finitely generated projective right R-modules A, B and C, $A \oplus B \cong A \oplus C \Longrightarrow B \lesssim^{\oplus} C$ or $C \lesssim^{\oplus} B$.*

Proof. It is obvious from Theorem 6 and [13, Theorem 3.1]. ■

Lemma 8. *Let I be a weakly stable ideal of an exchange ring R. Then $ax + b = 1$ with $a \in I$ implies there exists $y \in R$ such that $a + by \in U_<(R)$.*

Proof. Suppose that $ax+b=1$ with $a\in I, x,b\in R$. Then $(a+b)x+b(1-x)=1$. Furthermore, we have $(a+b)\big(x+b(1-x)\big)+\big(1-(a+b)\big)b(1-x)=1$. Clearly, $a+b\in 1+I$ and $\big(1-(a+b)\big)b(1-x)\in I$. By Lemma 2, we have $z\in R$ such that $(a+b)+\big(1-(a+b)\big)b(1-x)z\in \mathrm{U}_<(R)$. So we have some $s\in R$ such that

$$\begin{aligned} & x+\big(1+s(1-(a+b))\big)b(1-x) \\ = & \ x+b(1-x)+s\big(1-(a+b)\big)b(1-x) \\ & \in U_<(R). \end{aligned}$$

Consequently, we have $t\in R$ such that $a+b\big(1+(1-x)t\big)=a+b+b(1-x)t\in U_<(R)$, as asserted. ∎

Theorem 9. *Let I be a weakly stable ideal of an exchange ring R. Then for any regular $A\in M_n(I)$, there exist an idempotent matrix E and a right or left invertible matrix U such that $A=EU$.*

Proof. Because I is weakly stable, so is $M_n(I)$ as an ideal of $M_n(R)$ by Theorem 3. Let $A\in M_n(I)$ be regular. We have $B\in M_n(R)$ such that $A=ABA$. Applying Lemma 8 to $AB+(I_n-AB)=I_n$, there is a $Y\in M_n(R)$ such that $A+(I_n-AB)Y=U\in \mathrm{U}_<\big(M_n(R)\big)$. Therefore $A=AB\big(A+(I_n-AB)Y\big)=ABU$. Set $E=AB$. Then $E\in M_n(R)$ is an idempotent, as desired. ∎

Corollary 10. *Let I be a weakly stable ideal of an exchange ring R. If $\frac{1}{2}\in R$, then for any regular $A\in M_n(I)(n\geq 2)$, there exist right or left invertible matrices U,V such that $A=U+V$.*

Proof. Let $A\in M_n(I)$ be regular and $n\geq 2$. In view of Theorem 9, there exist an idempotent matrix E and a right or left invertible matrix W such that $A=EW$. Since $2\in \mathrm{U}(R)$, we see that $E=2^{-1}diag(1,\cdots,1)+2^{-1}\big(2E-diag(1,\cdots,1)\big)$ is the sum of two invertible matrices. Set $U=2^{-1}diag(1,\cdots,1)W$ and $V=2^{-1}\big(2E-diag(1,\cdots,1)\big)$. Then $A=U+V$, as asserted . ∎

Let I be a weakly stable ideal of an exchange ring R, and let $A\in M_n(I)$ be regular. Analogously to [13, Theorem 3.6], it follows by Theorem 3 and Lemma 8 that there exists a right or left $U\in M_n(R)$ such that $A=ABA=ABU=UBA$ for some $B\in M_n(R)$.

Lemma 11. *For any regular $a,b\in R$, if $\psi: aR\cong bR$, then $Ra=R\psi(a)$ and $\psi(a)R=bR$.*

Proof. It is easy to see that $\psi(a)R=bR$. By the regularity of b, we see that $\psi(a)\in R$ is regular, and thus we have $c\in R$ such that $\psi(a)=\psi(a)c\psi(a)=$

$\psi\big(ac\psi(a)\big)$. This implies that $a = ac\psi(a) \in R\psi(a)$; hence, $Ra \subseteq R\psi(a)$. In addition, we have $a = ada$ for a $d \in R$. So $\psi(a) = \psi(a)da \in Ra$; hence, $R\psi(a) \subseteq Ra$. Therefore $Ra = R\psi(a)$, as asserted. ■

Lemma 12. *Let I be a weakly stable ideal of an exchange ring R. Then $aR \cong bR$ with regular $a \in I$ implies $b = uav$ for some right or left invertible $u, v \in R$.*

Proof. Given $\psi : aR \cong bR$ with regular $a \in I$, by Lemma 11, we have $Ra = R\psi(a)$ and $\psi(a)R = bR$. As $a \in I$, we get $\psi(a) \in I$; hence, $b \in I$. Clearly, there is some $s \in R$ such that $sa = \psi(a)$. Because of regularity of a, we can find $y \in R$ such that $a = aya$. Hence $(say)a = \psi(a)$. Set $c = say$. Then $ca = \psi(a)$ and $c \in I$. Likewise, we have a $d \in I$ such that $a = d\psi(a)$. Inasmuch as $dc + (1 - dc) = 1$ and $d \in I$, from Lemma 8, there exists a $z \in R$ such that $d + (1 - dc)z \in \mathrm{U}_<(R)$. Using [4, Lemma 1], $c + t(1 - dc) = u \in \mathrm{U}_<(R)$ for some $t \in R$. Hence, $ua = \big(c + t(1 - dc)\big)a = ca = \psi(a)$. Since $\psi(a)R = bR$, we also have $p \in I$ and $q \in I$ such that $\psi(a)p = b$ and $bq = \psi(a)$. From $pq + (1 - pq) = 1$ and $p \in I$, there exists a $k \in R$ such that $p + (1 - pq)k = v \in \mathrm{U}_<(R)$. Thus, $b = \psi(a)p = \psi(a)\big(p + (1 - pq)k\big) = \psi(a)v$, and therefore $b = \psi(a)v = uav$. ■

By [6, Theorem 3], every weakly stable exchange ring is separative. Thus every square regular matrix over weakly stable exchange rings admits a diagonal reduction(cf. [1]). A natural problem is how to extend this fact to matrices over ideals of exchange rings. Now we observe the following result, which also gives a nontrivial generalization of [7, Theorem 15] and [8, Theorem 3].

Theorem 13. *Let I be a weakly stable ideal of an exchange ring R. Then every square regular matrix over I admits a diagonal reduction by right or left invertible matrices.*

Proof. Given any regular $A \in M_n(R)$, there is a split right R-modules exact sequence $0 \longrightarrow \mathrm{Ker}E \longrightarrow R^n \xrightarrow{E} ER^n \longrightarrow 0$. Hence ER^n is a generated projective right R-module. Clearly, there are idempotents $e_1, \cdots, e_n \in R$ such that $ER^n \cong e_1R \oplus \cdots \oplus e_nR \cong diag(e_1, \cdots, e_n)$ as right R-modules, so $ER^{n\times 1} \cong diag(e_1, \cdots, e_n)\ R^{n\times 1}$. Therefore $(ER^{n\times 1}) \bigotimes_R R^{1\times n} \cong diag(e_1, \cdots, e_n)R^{n\times 1} \bigotimes_R R^{1\times n}$. It follows by $R^{n\times 1} \bigotimes R^{1\times n} \cong M_n(R)$ that $AM_n(R) = EM_n(R) \cong diag(e_1, \cdots, e_n)M_n(R)$. As $A \in M_n(I)$, by Lemma 12, there are $U, V \in \mathrm{U}_<\big(M_n(R)\big)$ such that $UAV = diag(e_1, \cdots, e_n)$. ■

Corollary 14. *Let R be an exchange ring R, and let $A \in M_n(R)$ be regular. If $M_n(R)AM_n(R)$ is weakly stable, then A admits a diagonal reduction by right or left invertible matrices.*

Proof. Since $M_n(R)AM_n(R)$ is an ideal of $M_n(R)$, there exists an ideal I of R such that $M_n(I) = M_n(R)AM_n(R)$. In view of [10, Proposition 2.3.14], I is weakly stable. Clearly, $A \in M_n(I)$, and therefore we complete the proof by Theorem 13. ■

Recall that a ring R is regular in case every element in R is regular.

Lemma 15. *Let I be an ideal of a regular ring R. Then the following are equivalent*:

(1) *I is weakly stable.*

(2) *For all idempotents $e \in I$, eRe is one-sided unit-regular.*

Proof. $(1) \Rightarrow (2)$ is obvious from Lemma 4.

$(2) \Rightarrow (1)$ Suppose that $ax + b = 1$ with $a \in 1 + I$ and $x, b \in R$. Since R is regular, there exists an idempotent $e \in I$ such that $1 - a = (1 - a)e$; hence, $a(1-e) = 1-e$. Clearly, $aR + bR = R$. So $ar + bs = (1-a)e$ for some $r, s \in R$. Thus we have $eae(e + ere) + ebse = e - ea(1-e)re = e$. Inasmuch as eRe is one-sided unit-regular, we can find a $z \in eRe$ such that $eae + ebsez = u \in U_<(eRe)$. Set $w = (1-e)ae + (1-e)bsez$.

First, assume $uv = e$ for some $v \in eRe$. Then $(ae + bsez)(v - wv + 1 - e) = (ae + bsez)v = av + bsezv = wv + e$ and $a(1-e)(v - wv + 1 - e) = (1-e)(v - wv + 1 - e) = -wv + 1 - e$. Combining these two equalities, we get $(a + bsez)(v - wv + 1 - e) = 1$.

Now assume $vu = e$ for some $v \in eRe$. Then $(v - wv + 1 - e)(ae + bsez) = e - w + (ae + bsez) - u = e$ and $(v - wv + 1 - e)a(1-e) = (v - wv + 1 - e)(1-e) = 1 - e$. Hence, $(v1 - wv + 1 - e)(a + bsez) = 1$. In any case, we have $a + b(sez) \in U_<(R)$. ■

Theorem 16. *Let I be a weakly stable ideal of a regular ring R. Then every square matrix over I admits a diagonal reduction by invertible matrices.*

Proof. Given any $A \in M_n(I)$, we have the entries $a_{11}, \cdots, a_{ij}, \cdots, a_{nn} \in I$. Since R is a regular ring, by [7, Lemma 6], we have an idempotent $e \in I$ such that $a_{11}, \cdots, a_{ij}, \cdots, a_{nn} \in eRe$. Thus we see that $A \in M_n(eRe)$. Inasmuch as I is weakly stable, from Lemma 15, eRe is an one-sided unit-regular ring; hence, it is separative. According to [1, Theorem 2.4], there exist some invertible matrices $U', V' \in M_n(eRe)$ such that $U'AV'$ is a diagonal matrix. Choose $U = U' + diag(1-e, \cdots, 1-e)$ and $V = V' + diag(1-e, \cdots, 1-e)$. Then $U, V \in M_n(R)$ are invertible. Furthermore, we see that $UAV = U'AV'$ is a diagonal matrix, and we are through. ■

An element $e \in R$ is infinite if there exist orthogonal idempotents $e, f \in R$ such that $e = f + g$ while $eR \cong fR$ and $g \neq 0$. A simple ideal$I$ of a ring R is said to be purely infinite if every nonzero right ideal of I(as a ring without units) contains an infinite idempotent. Let I be a purely infinite, simple and essential ideal of a regular ring R. Then every $A \in M_n(I)$ admits a diagonal reduction by invertible matrices. In view of [7, Lemma 10], I is weakly stable. Therefore we are done by Theorem 16.

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