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THE EXACT SOLUTIONS OF THE PROBLEMS IN FORCED CAPILLARY-GRAVITY WAVES GENERATED BY A PLANE WAVEMAKER UNDER HOCKING'S EDGE CONDITION

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Abstract. The purpose of this paper is to construct solutions for the problems when capillary-gravity waves are generated by a vertical plane wavemaker in consideration of surface tension and the edge condition proposed by L. M. Hocking [4]. The uniqueness of these solutions will also be proven here.

1. Introduction

The problem of forced capillary-gravity waves was first studied by Havlock [3], who published a paper regarding the problem of forced surface waves under gravity generated by a plane wave maker. Later Evans [1, 2] studied the problem of a heaving cylinder in a fluid with the effect of surface tension included and proposed an edge condition. Hocking [4] proposed another dynamic edge condition that at a contact line the time derivative of the free surface is proportional to the slope of the free surface. Both of these edge conditions have been studied for their contributions to the solution of related problems. In general, Hocking's model [4] is considered more physically plausible than that of Evans's. Rhodes-Robinson [6] studied the problems of forced capillary-gravity waves generated by a plane or cylindrical wavemaker under Evans's edge condition. Mandal and Bandyopadhyay [5] took a different approach to the same problems generated by a plane wavemaker under Evans's edge condition by adopting the method of Fourier transform. In both Rhodes-Robinson's and Mandal et al's approaches, the case of finite depth and the case of infinite depth were treated separately. However, the solutions under Hocking's edge condition were not considered in either of these papers.

Later Shen and Yeh [7] found the unique solution of forced capillary-gravity waves in a circular basin under Hocking's edge condition using Green's function

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method. Here we incorporate the techniques due to Mandal et al. as well as Shen et al. to find the desired Green's function in order that the solution for Hocking's edge condition can be constructed.

In section 2, the solution of finite-depth plane wavemaker problem is constructed, and its uniqueness is proven. The solution of infinite depth case and the proof of its uniqueness are done in section 3.

2. Solution of the Finite Depth Problem

2.1. Formulation

We consider the irrotational motion of an incompressible inviscid fluid due to a harmonically oscillating vertical plane wave maker of infinite length under the action of gravity and surface tension. We use a rectangular coordinate system in which the y-axis is pointing vertically upwards, so that $y = 0$, $x > 0$ is the undisturbed state. The fluid occupies the region $x > 0$ and $-h < y < 0$, and at equilibrium it is of uniform depth h. The motion is two-dimensional and time-harmonic and is described by a velocity potential $\Psi(x, y, t)$. Then the linearized equations governing the liquid motion are the following (Mandal et al. [5]):

$$\nabla_2^2 \Psi = 0 \text{ in the fluid region } V. \tag{2.1}$$

On the free surface S $(y = 0,)$ we find

$$\Psi_y = Z_t, \tag{2.2}$$

and

$$\Psi_t + \mathrm{g} Z = \mathrm{T} \nabla_1^2 Z, \tag{2.3}$$

where $\nabla_2^2, \nabla_1^2 (\nabla_1^2 \equiv \frac{\partial^2}{\partial X^2})$ represent two-dimensional and one-dimensional Laplacian respectively, g is the gravitational constant, $\rho \mathrm{T}$ is the surface tension constant, ρ is the fluid density, and Z denotes free surface. At the wave maker M,

$$\Psi_x = u(y) e^{\mathrm{i}\omega t} \quad \text{on} \quad M, \tag{2.4}$$

where ω is the angular frequency. At the bottom B,

$$\Psi_y = 0 \quad \text{on} \quad y = -h, \tag{2.5}$$

The radiation condition that represents the behavior of outgoing waves at large distance from the wave maker can be expressed as

$$\Psi \to C_0 e^{\mathrm{i}(k_0 x + \omega t)} \cosh k_0 (h + y) \;\text{ as } x \to \infty, \tag{2.6}$$

where $\alpha = k_0$ is the unique positive real root of the equation (Rhodes-Robinson [6]):

$$\Delta_0(\alpha) = \alpha\left(\mathrm{T}\alpha^2 + 1\right)\sinh\alpha h - \omega^2\cosh\alpha h = 0. \tag{2.7}$$

Note that C_0 is unknown and will be dealt with in the following sections.

Finally, Hocking's edge condition [4] prescribing the free surface slope at the wave maker associated with the effect of surface tension on the wave is given by

$$Z_t = \lambda Z_x, \quad (\lambda \equiv \frac{1}{\delta}) \quad \text{at} \quad x = 0 = y \quad (\Gamma) \tag{2.8}$$

Let

$$\Psi(x, y, t) = \psi(x, y)\, e^{\mathrm{i}\omega t}, \tag{2.9}$$

and

$$Z(x, t) = \varsigma(x)\, e^{\mathrm{i}\omega t}, \tag{2.10}$$

and then measure x, y, Z and ζ in units of h, t in units of $(h/\mathrm{g})^{\frac{1}{2}}$, Ψ and ψ in units of $\mathrm{g}h^{\frac{3}{2}}$, ω in units of $(\mathrm{g}/h)^{\frac{1}{2}}$, T in units of $\mathrm{g}h^2$, $u(y)$ in units of $(\mathrm{g}h)^{\frac{1}{2}}$ and δ in units of $(\mathrm{g}/h)^{\frac{1}{2}}$. In terms of ψ and ζ, then we find that (2.1) to (2.8) become

$$\nabla_2^2\psi = 0 \text{ in } V, \tag{2.11}$$

$$\psi_y = \mathrm{i}\omega\varsigma \text{ and } \mathrm{i}\omega\psi + \mathrm{g}\varsigma = \mathrm{T}\nabla_1^2\varsigma \text{ on } S, \tag{2.12}$$

$$\psi_x = u(y) \text{ on } M, \tag{2.13}$$

$$\psi_y = 0 \text{ on } B, \tag{2.14}$$

$$\psi \to C_0 e^{\mathrm{i}k_0 x}\cosh\left(k_0(1 + y)\right) \text{ as } x \to \infty, \tag{2.15}$$

and

$$\psi_{xy} = \mathrm{i}\omega\delta\psi_y \text{ at } \Gamma. \tag{2.16}$$

2.2. Construction of the solution

To find the solution according to Hocking's edge condition, let us consider first a Green's function $G = G(x, y, \xi, \eta)$ satisfying the following equations:

$$\nabla_2'^2 G = -\delta(x - \xi)\delta(y - \eta) \qquad \text{in} \qquad V'; \tag{2.17}$$

$$-\omega^2 G + G_\eta - \mathrm{T}\nabla'^2_1 G_\eta = 0, \eta = 0 \qquad \text{on} \qquad S'; \tag{2.18}$$

$$G_\eta = 0, \eta = -1 \qquad \text{on} \qquad B'; \tag{2.19}$$

$$G_\xi = 0, \xi = 0 \qquad \text{on} \qquad M'; \tag{2.20}$$

$$G \to C_1 e^{\mathrm{i}k_0\xi} \cosh(k_0(1+\eta)) \quad \text{as} \quad \xi \to \infty; \tag{2.21}$$

$$G_{\xi\eta} = 0, \xi = 0 = \eta \qquad \text{at} \qquad \Gamma'; \tag{2.22}$$

where ∇'^2_2 and $\nabla'^2_1 (\nabla'^2_1 \equiv \frac{\partial^2}{\partial \xi^2})$ are the Laplacians with respect to ξ and η, C_1 is a constant. Such a Green's function does exist, according to Rhodes-Robinson [6]. Now, by using Green's identity, we find

$$\begin{aligned} \Phi &= \int\int_{V'} (G\nabla'^2_2\Phi - \Phi\nabla'^2_2 G)\mathrm{d}\xi\mathrm{d}\eta = \int_{\partial V'} (G\Phi_n - \Phi G_n)\mathrm{d}A' \\ &= \int_{S'} (G\Phi_\eta - \Phi G_\eta)\mathrm{d}\xi + \int_{B'} (G\Phi_\eta - \Phi G_\eta)\mathrm{d}\xi + \int_{M'} (G\Phi_\xi - \Phi G_\xi)\mathrm{d}\eta \\ &\quad + \int_\infty (G\Phi_\xi - \Phi G_\xi)\mathrm{d}\eta. \end{aligned} \tag{2.23}$$

Note that Φ_n and G_n are the normal derivatives of Φ and G respectively. Moreover,

$$G = \frac{1}{\omega^2}(G_\eta - \mathrm{T}\nabla'^2_1 G_\eta) \quad \text{and} \quad \Phi = \frac{1}{\omega^2}(\Phi_\eta - \mathrm{T}\nabla'^2_1 \Phi_\eta) \quad \text{in} \quad S'; \tag{2.24}$$

$$\Phi_\eta = 0 = G_\eta \quad \text{on} \quad B'; \tag{2.25}$$

$$\Phi_\xi = u(\eta) \quad \text{and} \quad G_\xi = 0 \quad \text{on} \quad M'; \tag{2.26}$$

$$\begin{aligned} \Phi_\xi &\to C_0(\mathrm{i}k_0)e^{\mathrm{i}k_0\xi}\cosh(k_0(1+\eta)) \text{ and} \\ G_\xi &\to C_1(\mathrm{i}k_0)e^{\mathrm{i}k_0\xi}\cosh(k_0(1+\eta)) \quad \text{as} \quad \xi \to \infty. \end{aligned} \tag{2.27}$$

Hence

$$\int_{S'} (G\Phi_\eta - \Phi G_\eta)\mathrm{d}\xi = \frac{\mathrm{T}}{\omega^2}\int_0^\infty (G_\eta\nabla'^2_1\Phi_\eta - \Phi_\eta\nabla'^2_1 G_\eta)\mathrm{d}\xi, \tag{2.28}$$

$$\int_{B'} (G\Phi_\eta - \Phi G_\eta)\mathrm{d}\xi = 0, \tag{2.29}$$

$$\int_{M'} (G\Phi_\xi - \Phi G_\xi)\mathrm{d}\eta = \int_{-1}^0 G|_{\xi=0} u(\eta)\mathrm{d}\eta, \tag{2.30}$$

$$\int_{\infty}(G\Phi_\xi - \Phi G_\xi)\mathrm{d}\eta = \int_{-1}^{0}[C_1C_0(\mathrm{i}k_0)e^{2\mathrm{i}k_0\xi}\cosh^2(k_0(1+\eta)) - C_0C_1(\mathrm{i}k_0)e^{2\mathrm{i}k_0\xi}\cosh^2(k_0(1+\eta))]\mathrm{d}\eta = 0, \tag{2.31}$$

and

$$\begin{aligned}\Phi &= \frac{\mathrm{T}}{\omega^2}\int_0^{\infty}(G_\eta\nabla'^2_1\Phi_\eta - \Phi_\eta\nabla'^2_1 G_\eta)\mathrm{d}\xi + \int_{-1}^{0}G|_{\xi=0}u(\eta)\mathrm{d}\eta \\ &= \frac{\mathrm{T}}{\omega^2}(G_\eta\Phi_{\xi\eta} - \Phi_\eta G_{\xi\eta})_{\eta=0}|_0^\infty + \int_{-1}^{0}G|_{\xi=0}u(\eta)\mathrm{d}\eta,\end{aligned} \tag{2.32}$$

where

$$\Phi_{\xi\eta}|_{\xi\to\infty} = C_0(\mathrm{i}k_0^2)e^{\mathrm{i}\xi k_0}\sinh(k_0(1+\eta)), \tag{2.33}$$

and

$$G_{\xi\eta}|_{\xi\to\infty} = C_1(\mathrm{i}k_0^2)e^{\mathrm{i}k_0\xi}\sinh(k_0(1+\eta)), \tag{2.34}$$

$$\begin{aligned}&\lim_{\xi\to\infty}\frac{\mathrm{T}}{\omega^2}(G_\eta\Phi_{\xi\eta} - \Phi_\eta G_{\xi\eta})_{\eta=0} \\ = &\lim_{\xi\to\infty}\frac{\mathrm{T}}{\omega^2}[(-k_0^3)C_1C_0e^{2\mathrm{i}k_0\xi}(\cosh k_0)(\sinh k_0) \\ &-(-k_0^3)C_0C_1e^{2\mathrm{i}k_0\xi}(\sinh k_0)(\cosh k_0)] \\ = &\ 0.\end{aligned} \tag{2.35}$$

Then we have

$$\Phi = \int_{-1}^{0}G|_{\xi=0}u(\eta)\mathrm{d}\eta - \frac{\mathrm{T}}{\omega^2}G_\eta\Phi_{\xi\eta}|_{\Gamma'}. \tag{2.36}$$

Evans [1, 2] suggested that at the contact line, the edge condition could be

$$Z_x = \mathrm{i}\omega\lambda_0 e^{\mathrm{i}\omega t} \quad \text{at} \quad \Gamma, \tag{2.37}$$

which can be reduced to

$$\varphi^0_{xy} = \lambda_0 \quad \text{at} \quad \Gamma, \tag{2.38}$$

where $\varphi^0 e^{\mathrm{i}\omega t}$ is the solution of (2.1) to (2.6) and (2.37). We note that, according to (2.37), (2.36) becomes

$$\varphi^0 = \int_{-1}^{0}G|_{\xi=0}u\mathrm{d}\eta - \frac{\mathrm{T}\lambda_0}{\omega^2}G_\eta|_{\Gamma'}, \tag{2.39}$$

while Hocking's edge condition implies that (2.36) should be

$$\Phi = \int_{-1}^{0} G|_{\xi=0} u \mathrm{d}\eta - \frac{\mathrm{i}\mathrm{T}\delta}{\omega} G_\eta \Phi_\eta|_{\Gamma'}. \tag{2.40}$$

It is obvious that by finding $G_\eta|_{\Gamma'}$, we may find the desired solution under Hocking's edge condition, and instead of finding G we will solve the equations for φ^0 to determine $G_\eta|_{\Gamma'}$. Now let

$$\varphi^0 = C_0 e^{\mathrm{i}k_0 x} \cosh(k_0(1+y)) + \varphi(x,y), \tag{2.41}$$

then

$$\nabla_2^2 \varphi = 0 \qquad \text{in} \qquad V, \tag{2.42}$$

$$-\omega^2 \varphi + \varphi_y - \mathrm{T}\nabla_1^2 \varphi_y = 0 \qquad \text{on} \qquad S, \tag{2.43}$$

$$\varphi_y = 0 \qquad \text{on} \qquad B, \tag{2.44}$$

$$\varphi_x = v(y) = u(y) - \mathrm{i}C_0 k_0 \cosh(k_0(1+y)) \quad \text{on} \quad M, \tag{2.45}$$

$$\varphi \to 0 \qquad \text{as} \qquad x \to \infty, \tag{2.46}$$

$$\varphi_{xy} = \lambda_0 - \mathrm{i}C_0 k_0^2 \sinh k_0 \qquad \text{at} \qquad \Gamma. \tag{2.47}$$

By Fourier transforming φ with respect to x, we let

$$\chi(y,\xi) = \int_0^\infty \varphi \cos \xi x \mathrm{d}x, \tag{2.48}$$

and χ satisfies

$$\chi_{yy} - \xi^2 \chi = v(y), \quad -1 < y < 0; \tag{2.49}$$

$$-\omega^2 \chi + (1 + \mathrm{T}\xi^2)\chi_y + \mathrm{T}(\lambda_0 - \mathrm{i}C_0 k_0^2 \sinh k_0) = 0, \ \ y = 0; \tag{2.50}$$

$$\chi_y = 0, \ \ y = -1. \tag{2.51}$$

To solve χ, let us construct a Green's function $\hat{g} = \hat{g}(y,\eta)$ satisfying the following equations:

$$\hat{g}_{\eta\eta} - \xi^2 \hat{g} = \delta(\eta - y), \ \ -1 < \eta < 0; \tag{2.52}$$

$$-\omega^2 \hat{g} + (1 + \mathrm{T}\xi^2)\hat{g}_\eta = 0, \ \ \eta = 0; \tag{2.53}$$

$$\hat{g}_\eta = 0, \quad \eta = -1. \tag{2.54}$$

From (2.52), we can write $\hat{g}$ as

$$\begin{aligned} \hat{g} &= \mathrm{C}_1 e^{\xi\eta} + \mathrm{D}_1 e^{-\xi\eta}, -1 \le \eta < y \le 0 \\ &= \mathrm{C}_2 e^{\xi\eta} + \mathrm{D}_2 e^{-\xi\eta}, -1 \le y < \eta \le 0. \end{aligned} \tag{2.55}$$

By continuity of $\hat{g}$ at $y = \eta$,

$$(\mathrm{C}_2 - \mathrm{C}_1)e^{\xi y} + (\mathrm{D}_2 - \mathrm{D}_1)e^{-\xi y} = 0, \tag{2.56}$$

and the jump condition at $y = \eta$ suggests

$$(\mathrm{C}_2 - \mathrm{C}_1)e^{\xi y} - (\mathrm{D}_2 - \mathrm{D}_1)e^{-\xi y} = \frac{1}{\xi}. \tag{2.57}$$

From (2.56) and (2.57), (2.55) can be rewritten as

$$\hat{g} = (\mathrm{C}_1 e^{\xi\eta} + \mathrm{D}_1 e^{-\xi\eta}) + \frac{H(\eta - y)}{\xi} \sinh(\xi(\eta - y)), \tag{2.58}$$

where H is the Heaviside function. Using from (2.53) and (2.54), we find from (2.58) that

$$\mathrm{C}_1 = -\frac{e^{\xi}}{2\xi} \times \frac{\xi(\xi^2\mathrm{T} + 1)\cosh\xi y + \omega^2 \sinh\xi y}{\Delta(\xi)}, \tag{2.59}$$

$$\mathrm{D}_1 = -\frac{e^{-\xi}}{2\xi} \times \frac{\xi(\mathrm{T}\xi^2 + 1)\cosh\xi y + \omega^2 \sinh\xi y}{\Delta(\xi)}, \tag{2.60}$$

and

$$\Delta(\xi) \equiv \xi(\xi^2 + 1)\sinh\xi - \omega^2\cosh\xi. \tag{2.61}$$

Finally, we obtain

$$\begin{aligned} \hat{g} = &-\frac{\cosh(\xi(1+\eta))}{\xi\Delta(\xi)}\left[\xi(\mathrm{T}\xi^2 + 1)\cosh\xi y + \omega^2\sinh\xi y\right] \\ &+\frac{H(\eta - y)}{\xi}\sinh(\xi(\eta - y)). \end{aligned} \tag{2.62}$$

By Green's identity,

$$\begin{aligned} &\int_{-1}^{0} \chi(\hat{g}_{\eta\eta} - \xi^2\hat{g}) - \hat{g}(\chi_{\eta\eta} - \xi^2\chi)\mathrm{d}\eta \\ &= -\int_{-1}^{0} \hat{g}v\mathrm{d}\eta + \chi = (\chi\hat{g}_\eta - \hat{g}\chi_\eta)_{-1}^{0} \\ &= \frac{1}{\omega^2}\left[(1+\mathrm{T}\xi^2)\hat{g}_\eta\chi\eta + \mathrm{T}\hat{g}_\eta(\lambda_0 - \mathrm{i}C_0k_0^2\sinh k_0) - (1+\mathrm{T}\xi^2)\chi_{\eta\hat{g}\eta}\right]|_{\eta=0} \\ &= \frac{\mathrm{T}}{\omega^2}(\lambda_0 - \mathrm{i}C_0k_0^2\sinh k_0)\hat{g}_\eta|_{\eta=0}, \end{aligned} \tag{2.63}$$

and we have

$$
\begin{aligned}
\chi &= \int_{-1}^{0} \hat{g}v(\eta)\mathrm{d}\eta + \frac{\mathrm{T}}{\omega^2}(\lambda_0 - \mathrm{i}C_0k_0^2\sinh k_0)\hat{g}_\eta|_{\eta=0} \\
&= -\frac{\xi(\mathrm{T}\xi^2+1)\cosh\xi y + \omega^2\sinh\xi y}{\Delta(\xi)}\big[\frac{1}{\xi}\int_{-1}^{0} v\cosh(\xi(1+\eta))\mathrm{d}\eta \\
&\quad + \frac{\mathrm{T}}{\omega^2}(\lambda_0 - \mathrm{i}C_0k_0^2\sinh k_0)\sinh\xi\big] + \frac{1}{\xi}\int_{y}^{0}\sinh(\xi(\eta-y))v\mathrm{d}\eta \\
&\quad + \frac{\mathrm{T}}{\omega^2}(\lambda_0 - \mathrm{i}C_0k_0^2\sinh k_0)\cosh\xi y.
\end{aligned}
\tag{2.64}
$$

Let

$$
a(\xi) = \int_{-1}^{0} u\cosh(\xi(1+\eta))\mathrm{d}\eta, \tag{2.65}
$$

$$
b(\xi, y) = \int_{y}^{0} u\sinh(\xi(\eta-y))\mathrm{d}\eta, \tag{2.66}
$$

then

$$
\begin{aligned}
\chi &= -\frac{\xi(\mathrm{T}\xi^2+1)\cosh\xi y + \omega^2\sinh\xi y}{\Delta(\xi)} \\
&\quad \big[\frac{a(\xi)}{\xi} - \frac{\mathrm{i}C_0k_0}{\xi(k_0^2-\xi^2)}((k_0-\xi)\sinh(k_0+\xi) \\
&\quad + (k_0+\xi)\sinh(k_0-\xi)) + \frac{\mathrm{T}}{\omega^2}(\lambda_0 - \mathrm{i}C_0k_0^2\sinh k_0)\sinh\xi\big] \\
&\quad + \frac{b(\xi,y)}{\xi} - \frac{\mathrm{i}C_0k_0}{\xi(k_0^2-\xi^2)}\ \big[k_0(\sinh k_0)(\sinh\xi y) \\
&\quad - \xi(\cosh(k_0(1+y))\ (-\cosh k_0)(\cosh\xi y)\big] \\
&\quad + \frac{\mathrm{T}}{\omega^2}(\lambda_0 - \mathrm{i}C_0k_0^2\sinh k_0)(\cosh\xi y).
\end{aligned}
\tag{2.67}
$$

From the identity in (2.7), it follows that

$$
\omega^2 = k_0(\mathrm{T}k_0^2+1)\tanh k_0. \tag{2.68}
$$

By rearranging (2.67), we have

$$\chi = -\frac{\xi(\mathrm{T}\xi^2+1)\cosh\xi y + \omega^2\sinh\xi y}{\xi\Delta(\xi)}\big[a(\xi) - \frac{\mathrm{i}C_0k_0(\cosh k_0)}{(\xi^2-k_0^2)(\mathrm{T}k_0^2+1)}$$
$$\times\Big(\xi(\mathrm{T}\xi^2+1)\sinh\xi - \omega^2\cosh\xi\Big) + \frac{\mathrm{T}\lambda_0}{\omega^2}\xi\sinh\xi\big] + \frac{b(\xi,y)}{\xi}$$
$$-\frac{\mathrm{i}C_0k_0}{\xi(\xi^2-k_0^2)}\big[\xi\big(\cosh(k_0(1+y)) - (\cosh k_0)(\cosh\xi y)\big)$$
$$-k_0(\sinh k_0)(\sinh\xi y)\big] + \frac{\mathrm{T}}{\omega^2}(\lambda_0\mathrm{i}C_0k_0^2\sinh k_0)\cosh\xi y \tag{2.69}$$
$$= -\frac{\xi(\mathrm{T}\xi^2+1)\cosh\xi y + \omega^2\sinh\xi y}{\xi\Delta(\xi)}\big[a(\xi) - \frac{\mathrm{i}C_0k_0\cosh k_0}{(\xi^2-k_0^2)(\mathrm{T}k_0^2+1)}\Delta(\xi)$$
$$+\frac{\mathrm{T}\lambda_0}{\omega^2}\xi\sinh\xi\big] + \frac{b(\xi,y)}{\xi} - \frac{\mathrm{i}C_0k_0}{\xi(\xi^2-k_0^2)}\big[\xi(\cosh(k_0(1+y))$$
$$-(\cosh k_0)(\cosh\xi y)) - k_0(\sinh k_0)(\sinh\xi y))\big]$$
$$+\tfrac{\mathrm{T}}{\omega^2}(\lambda_0 - \mathrm{i}C_0k_0^2\sinh k_0)\cosh\xi y.$$

Since χ is obtained from Fourier Transform, we should expect χ without singularity for $\xi \in [0,\infty]$. As seen from (2.69), χ can be expressed as

$$\chi = \mathfrak{K}_1(\xi,y) + \frac{b(\xi,y)}{\xi} + \mathfrak{K}_2(\xi,y) + \frac{\mathrm{T}}{\omega^2}\left(\lambda_0 - \mathrm{i}C_0k_0^2\sinh k_0\right)\cosh\xi y, \tag{2.70}$$

$$\mathfrak{K}_1(\xi,y) \equiv -\frac{\xi(\mathrm{T}\xi^2+1)\cosh\xi y + \omega^2\sinh\xi y}{\xi\Delta(\xi)}\big[a(\xi)$$
$$-\frac{\mathrm{i}C_0k_0\cosh k_0}{(\xi^2-k_0^2)(\mathrm{T}k_0^2+1)}\Delta(\xi) + \frac{\mathrm{T}\lambda_0}{\omega^2}\xi\sinh\xi\big], \tag{2.71}$$

$$\mathfrak{K}_2(\xi,y) \equiv -\frac{\mathrm{i}C_0k_0}{\xi(\xi^2-K_0^2)}\big[\xi\big(\cosh(k_0(1+y))$$
$$-(\cosh k_0)(\cosh\xi y)\big) - k_0(\sinh k_0)(\sinh\xi y)\big], \tag{2.72}$$

where $\mathfrak{K}_1$ and $\mathfrak{K}_2$ may have singularities at $\xi = k_0$ and $\xi = 0$. Let us check the following limits:

(2.73)

$$\lim_{\xi\to 0}\mathfrak{K}_1(\xi,y) = \lim_{\xi\to 0} -\frac{1}{\Delta(\xi)}\left[(\mathrm{T}\xi^2+1)\cosh\xi y + \omega^2\frac{\sinh\xi y}{\xi}\right]$$
$$\times\left[a(\xi) - \frac{\mathrm{i}C_0k_0\cosh k_0}{(\xi^2-k_0^2)(\mathrm{T}k_0^2+1)}\Delta(\xi) + \frac{\mathrm{T}\lambda_0}{\omega^2}\xi\sinh\xi\right]$$
$$= \lim_{\xi\to 0} -\frac{1}{\Delta(\xi)}\left[(\mathrm{T}\xi^2+1)\cosh\xi y + \omega^2\frac{\sinh\xi y}{\xi}\right]$$

$$\times\left[a(\xi)-\frac{\mathrm{i}C_0k_0\cosh k_0}{(\xi+k_0)(\mathrm{T}k_0^2+1)}\times\frac{\Delta(\xi)}{\xi-k_0}+\frac{\mathrm{T}\lambda_0}{\omega^2}\xi\sinh\xi\right]$$

$$=-\frac{1+\omega^2y}{\Delta(0)}\left[a(0)+\frac{\mathrm{i}C_0\cosh k_0}{k_0(\mathrm{T}k_0^2+1)}\Delta(0)\right],\quad exist.$$

(2.74)

$$\lim_{\xi\to k_0}\mathfrak{K}_1(\xi,y)=\lim_{\xi\to k_0}-\frac{\xi(\mathrm{T}\xi^2+1)\cosh\xi y+\omega^2\sinh\xi y}{\xi\Delta(\xi)}\times\left[a(\xi)-\frac{\mathrm{i}C_0k_0\cosh k_0}{(\xi+k_0)(\mathrm{T}k_0^2+1)}\times\frac{\Delta(\xi)}{\xi-k_0}+\frac{\mathrm{T}\lambda_0}{\omega^2}\xi\sinh\xi\right].$$

For $\mathfrak{K}_1(\xi,y)$ to have limit at $\xi=k_0$, we must have

$$\lim_{\xi\to k_0}\left[a(\xi)-\frac{\mathrm{i}C_0k_0\cosh k_0}{(\xi+k_0)(\mathrm{T}k_0^2+1)}\times\frac{\Delta(\xi)}{\xi-k_0}+\frac{\mathrm{T}\lambda_0}{\omega^2}\xi\sinh\xi\right]=0,\tag{2.75}$$

since

$$\Delta(\xi)=(\xi-k_0)f(\xi)\qquad\text{and}\quad f(k_0)\neq 0.\tag{2.76}$$

Using (2.75), and applying (2.68) again, we obtain

$$C_0=-\frac{4\mathrm{i}(\mathrm{T}k_0^2+1)a(k_0)+4\mathrm{i}\mathrm{T}\lambda_0\cosh k_0}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0}.\tag{2.77}$$

Then look at

$$\begin{aligned}\lim_{\xi\to k_0}\mathfrak{K}_2(\xi,y)&=\lim_{\xi\to k_0}-\frac{\mathrm{i}C_0k_0}{\xi(\xi+k_0)}\times\\&\left\{\frac{\xi\left[(\cosh k_0(1+y))-(\cosh k_0)(\cosh\xi y)\right]}{\xi-k_0}-\frac{k_0(\sinh k_0)(\sinh\xi y)}{\xi-k_0}\right\}\\&=-\frac{\mathrm{i}C_0}{2k_0}\times\lim_{\xi\to k_0}\left\{\xi\left[(\cosh k_0(1+y))-(\cosh k_0)y(\sinh\xi y)\right]\right.\\&+\left[(\cosh k_0(1+y))-(\cosh k_0)(\cosh\xi y)\right]\\&\left.-k_0(\sinh k_0)(y\cosh\xi y)\right\}\\&=-\frac{\mathrm{i}C_0}{2k_0}\left[(k_0+1)(\cosh k_0(1+y))-k_0y(\sinh k_0(1+y))\right.\\&\left.-(\cosh k_0)(\cosh k_0y)\right],\end{aligned}\tag{2.78}$$

which exists. And

$$\lim_{\xi\to 0}\mathfrak{K}_2(\xi,y)=\lim_{\xi\to 0}-\frac{\mathrm{i}C_0k_0}{(\xi^2-k_0^2)}$$

(2.79)
$$\left\{\left[(\cosh k_0(1+y))-(\cosh k_0)(\cosh\xi y)\right]-k_0(\sinh k_0)\frac{\sinh\xi y}{\xi}\right\}$$

$$=\frac{\mathrm{i}C_0}{k_0}\left\{(\cosh k_0(1+y))-(\cosh k_0)(\cosh k_0 y)-k_0 y(\sinh k_0)\right\}.$$

Subsequently, φ can be determined as

$$\varphi=\frac{2}{\pi}\int_0^\infty\chi\cos\xi x\mathrm{d}\xi=\frac{1}{\pi}\int_0^\infty\chi(e^{\mathrm{i}\xi x}+e^{-\mathrm{i}\xi x})\mathrm{d}\xi.$$

(2.80)
$$=\frac{1}{\pi}\left[\int_{-\infty}^{0}\chi e^{\mathrm{i}|\xi|x}\mathrm{d}\xi+\int_0^{-\infty}\chi e^{\mathrm{i}|\xi|x}\mathrm{d}\xi\right]+\frac{2\pi\mathrm{i}}{\pi}\sum_{\xi=a\mathrm{i},a>0}\mathrm{Res}(\chi e^{\mathrm{i}\xi x})$$

$$=2\mathrm{i}\sum_{n=1}^{\infty}\underset{\xi=k_n\mathrm{i}}{\mathrm{Res}}(\chi e^{\mathrm{i}\xi x}),$$

where $k_1,k_2,k_3\cdots k_n,\cdots>0$, $k_n<k_{n+1}$, $\forall n\in\mathbb{N}$; χ has only singularities $\{k_n\mathrm{i}\}_{n=1}^{\infty}$ on the upper-half complex plane and $\{\mp k_n\mathrm{i}\}_{n=1}^{\infty}$ are the pure imaginary roots for equation (2.7). Also note that χ is considered as an even function when extended to $(-\infty,\infty)$. It follows from (2.80) and (2.70) to (2.72) that

$$\varphi=2\mathrm{i}\sum_{n=1}^{\infty}\underset{\xi=k_n\mathrm{i}}{\mathrm{Res}}\chi e^{\mathrm{i}\xi x}$$

$$=2\mathrm{i}\sum_{n=1}^{\infty}\lim_{\xi\to\mathrm{i}k_n}\mathrm{Res}\left[\chi e^{\mathrm{i\xi x}}\cdot(\xi-\mathrm{ik_n})\right]$$

(2.81)
$$=2\mathrm{i}\sum_{n=1}^{\infty}\lim_{\xi\to\mathrm{i}k_n}\left\{\left[\mathfrak{K}_1(\xi,y)+\mathfrak{K}_2(\xi,y)\frac{b(\xi,y)}{\xi}\right.\right.$$

$$\left.\left.+\frac{\mathrm{T}}{\omega^2}(\lambda_0-\mathrm{i}C_0k_0^2\sinh k_0)\cosh\xi y\right]\cdot e^{\mathrm{i}\xi x}(\xi-\mathrm{i}k_n)\right\}.$$

Note that, aside from $\mathfrak{K}_1(\xi,y)$, none of the remaining terms in χ has singularities at $\mathrm{i}k_n$; hence for each n,

(2.82)
$$\lim_{\xi\to\mathrm{i}k_n}\left[\mathfrak{K}_2(\xi,y)+\frac{b(\xi,y)}{\xi}+\frac{\mathrm{T}}{\omega^2}(\lambda_0-\mathrm{i}C_0k_0^2\sinh k_0)\cosh\xi y\right]$$

$$=(\xi-\mathrm{i}k_n)e^{\mathrm{i}\xi x}0,$$

and then

$$\begin{aligned}
\varphi &= 2\mathrm{i}\sum_{n=1}^{\infty}\lim_{\xi\to \mathrm{i}k_n} \mathfrak{K}_1(\xi,y)e^{\mathrm{i}\xi x}(\xi-\mathrm{i}k_n)\\
&= 2\mathrm{i}\sum_{n=1}^{\infty}\lim_{\xi\to \mathrm{i}k_n} -\frac{\xi(\mathrm{T}\xi^2+1)\cosh\xi y+\omega^2\sinh\xi y}{\xi\Delta(\xi)}\\
&\quad\times\left[a(\xi)-\frac{\mathrm{i}C_0k_0\cosh k_0}{(\xi^2-k_0^2)(\mathrm{T}k_0^2+1)}\Delta(\xi)+\frac{\mathrm{T}}{\omega^2}(\lambda_0\xi\sinh\xi)\right](\xi-\mathrm{i}k_n)e^{\mathrm{i}\xi x}\\
&= -2\mathrm{i}\sum_{n=1}^{\infty}\frac{\mathrm{i}k_n(1-\mathrm{T}k_n^2)\cos k_n y+\mathrm{i}\omega^2\sin k_n y}{\mathrm{i}k_n}e^{-k_n x}\times\\
&\qquad\lim_{\xi\to \mathrm{i}k_n}\left[\frac{a(\xi)}{\Delta(\xi)}-\frac{\mathrm{i}C_0k_0\cosh k_0}{(\xi^2-k_0^2)(\mathrm{T}k_0^2+1)}+\frac{\mathrm{T}\lambda_0}{\omega^2}\xi\cdot\frac{\sinh\xi}{\Delta(\xi)}\right](\xi-\mathrm{i}k_n)\\
&= -2\mathrm{i}\sum_{n=1}^{\infty}\frac{k_n(1-\mathrm{T}k_n^2)\cos k_n y+\omega^2\sin k_n y}{k_n}e^{-k_n x}\\
&\quad\times\left[a(\mathrm{i}k_n)+\frac{\mathrm{T}\lambda_0\cos k_n}{-\mathrm{T}k_n^2+1}\right]\times\lim_{\xi\to \mathrm{i}k_n}\frac{\xi-\mathrm{i}k_n}{\Delta(\xi)} \qquad (2.83)\\
&= -2\mathrm{i}\sum_{n=1}^{\infty}\frac{1}{k_n}\left[k_n(1-\mathrm{T}k_n^2)\cos k_n y+k_n(\mathrm{T}k_n^2-1)(\tan k_n)(\sin k_n y)\right]e^{-k_n x}\\
&\quad\times\left[a(\mathrm{i}k_n)+\frac{\mathrm{T}\lambda_0\cos k_n}{-\mathrm{T}k_n^2+1}\right]\times\frac{-2\cos k_n}{\mathrm{i}[(1-3\mathrm{T}k_n^2)\sin 2k_n+2k_n(1-\mathrm{T}k_n^2)]}\\
&= 4\sum_{n=1}^{\infty}(1-\mathrm{T}k_n^2)\left[(\cos k_n)(\cos k_n y)-(\sin k_n)(\sin k_n y)\right]e^{-k_n x}\times\\
&\qquad\left[a(\mathrm{i}k)+\frac{\mathrm{T}\lambda_0\cos k_n}{-\mathrm{T}k_n^2+1}\right]\times\left[(1-3\mathrm{T}k_n^2)\sin 2k_n+2k_n(1-\mathrm{T}k_n^2)\right]^{-1}\\
&= 4\sum_{n=1}^{\infty}\frac{(1-\mathrm{T}k_n^2)e^{-k_n x}\cos k_n(1+y)}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\left[\int_{-1}^{0}u(\eta)\cos k_n(1+\eta)\mathrm{d}\eta\right.\\
&\qquad\left.+\frac{\mathrm{T}\lambda_0\cos k_n}{-\mathrm{T}k_n^2+1}\right]
\end{aligned}$$

where

$$a(\mathrm{i}k_n)=\int_{-1}^{0}u(\eta)\cos k_n(1+\eta)\mathrm{d}\eta. \qquad (2.84)$$

We also note that in the derivations given above, ω^2 has been replaced by the identity in (2.68) as well as

$$\omega^2 = k_n(\mathrm{T}k_n^2 - 1)\tan k_n, \tag{2.85}$$

since $\{\mathrm{i}k_n\}_{n=1}^{\infty}$ are also the roots of the equation (2.7).

Eventually, we find

$$\begin{aligned}
\varphi^0 &= \varphi + C_0 e^{\mathrm{i}k_0 x}\cosh(k_0(1+y)) \\
&= -4\mathrm{i}\frac{(1+\mathrm{T}k_0^2)e^{\mathrm{i}k_0x}\cosh(k_0(1+y))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0} \\
&\quad\times\left[\int_{-1}^{0} u\cosh(k_0(1+\eta))\mathrm{d}\eta + \frac{\mathrm{T}\lambda_0(\cosh k_0)}{1+\mathrm{T}k_0^2}\right] \\
&\quad -4\sum_{n=1}^{\infty}\frac{(1-\mathrm{T}k_n^2)e^{-k_nx}\cos(k_n(1+y))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n} \\
&\quad\times\left[\int_{-1}^{0} u\cos(k_n(1+\eta))\mathrm{d}\eta + \frac{\mathrm{T}\lambda_0(\cos k_n)}{1-\mathrm{T}k_n^2}\right],
\end{aligned}\tag{2.86}$$

which is equivalent to the solution obtained by Rhodes-Robinson [6] and Mandal et al. [5] using left-handed coordinate, while our result here is based on right-handed coordinated system. Also by (2.39), we conclude that

$$\begin{aligned}
\int_{-1}^{0} G|_{\xi=0}u\mathrm{d}\eta &= -4\Bigg\{\mathrm{i}\frac{(1+\mathrm{T}k_0^2)e^{\mathrm{i}k_0x}\cosh(k_0(1+y))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0} \\
&\quad\times\int_{-1}^{0} u\cosh(k_0(1+\eta))\mathrm{d}\eta \\
&\quad+\sum_{n=1}^{\infty}\frac{(1-\mathrm{T}k_n^2)e^{-k_nx}\cos(k_n(1+y))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n} \\
&\quad\times\int_{-1}^{0} u\cos(k_n(1+\eta))\mathrm{d}\eta\Bigg\}
\end{aligned}\tag{2.87}$$

and

$$\begin{aligned}
\tfrac{\mathrm{T}\lambda_0}{\omega^2}G_\eta|_{\Gamma'} &= 4\mathrm{T}\lambda_0\Bigg\{\mathrm{i}\frac{e^{\mathrm{i}k_0x}(\cosh k_0)(\cosh(k_0(1+y)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0} \\
&\quad+\sum_{n=1}^{\infty}\frac{e^{-k_nx}(\cos k_n)(\cos(k_n(1+y)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\Bigg\},
\end{aligned}\tag{2.88}$$

since (2.87) is the term associated with integration of $G|_{\xi=0}u$ and independent of

λ_0, while (2.88) is the term associated with λ_0 and independent of u. So

$$\begin{aligned} G_\eta|_{\Gamma'} \;=\; 4\omega^2\Bigg\{ &\mathrm{i}\frac{e^{\mathrm{i}k_0x}(\cosh k_0)(\cosh(k_0(1+y)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0} \\ &+\sum_{n=1}^{\infty}\frac{e^{-k_nx}(\cos k_n)(\cos(k_n(1+y)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\Bigg\}. \end{aligned} \tag{2.89}$$

Now by (2.40), let us find the expression for $\Phi_\eta|_{\Gamma'}$. First we differentiate both sides of (2.40) by y, and set $x=y=0$, to find

$$\Phi_y|_\Gamma = \frac{\partial}{\partial y}\left(\int_{-1}^0 G|_{\xi=0}u\mathrm{d}\eta\right)\Big|_\Gamma - \frac{\mathrm{i}\mathrm{T}\delta}{\omega}G_{y\eta}|_{\Gamma\Gamma'}\Phi_\eta|_{\Gamma'}. \tag{2.90}$$

It follows that

$$\left(1+\frac{\mathrm{i}\mathrm{T}\delta}{\omega}G_{y\eta}|_{\Gamma\Gamma'}\right)\Phi_y|_\Gamma = \frac{\partial}{\partial y}\left(\int_{-1}^0 G|_{\xi=0}u\mathrm{d}\eta\right)\Big|_\Gamma, \tag{2.91}$$

and we obtain

$$\Phi_y|_\Gamma = \Phi_\eta|_{\Gamma'} = \left[\frac{\partial}{\partial y}\left(\int_{-1}^0 G|_{\xi=0}u\mathrm{d}\eta\right)\Big|_\Gamma\right] \Big/ \left(1+\frac{\mathrm{i}\mathrm{T}\delta}{\omega}G_{y\eta}|_{\Gamma\Gamma'}\right), \tag{2.92}$$

where

$$\begin{aligned} &\frac{\partial}{\partial y}\left(\int_{-1}^0 G|_{\xi=0}u\mathrm{d}\eta\right)\Big|_\Gamma \\ &= -4\Bigg\{\mathrm{i}\frac{k_0(1+\mathrm{T}k_0^2)e^{\mathrm{i}k_0x}\sinh(k_0(1+y))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0}\times\int_{-1}^0 u\cosh(k_0(1+\eta))\mathrm{d}\eta \\ &\quad+\sum_{n=1}^{\infty}\frac{k_n(1-\mathrm{T}k_n^2)e^{-k_nx}\sin(k_n(1+y))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\int_{-1}^0 u\cos(k_n(1+\eta))\mathrm{d}\eta\Bigg\}\Big|_\Gamma \\ &\quad-4\omega^2\Bigg\{\mathrm{i}\frac{\cosh k_0}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0}\int_{-1}^0 u\cosh(k_0(1+\eta))\mathrm{d}\eta \\ &\quad+\sum_{n=1}^{\infty}\frac{\cos k_n}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\int_{-1}^0 u\cos(k_n(1+\eta))\mathrm{d}\eta\Bigg\}, \end{aligned} \tag{2.93}$$

and

$$
\begin{aligned}
&\left(1+\frac{\mathrm{iT}\delta}{\omega}G_{y\eta}|_{\Gamma\Gamma'}\right)\\
&= 1+\frac{\mathrm{iT}\delta}{\omega}(-4\omega^2)\Bigg\{\mathrm{i}\frac{k_0e^{\mathrm{i}k_0x}(\cosh k_0)(\sinh(k_0(1+y)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0}\\
&\quad-\sum_{n=1}^{\infty}\frac{k_ne^{-k_nx}(\cos k_n)(\sin(k_n(1+y)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\Bigg\}\Big|_{\Gamma}\\
&= 1-4\mathrm{iT}\delta\omega\Bigg\{\mathrm{i}\frac{k_0(\cosh k_0)(\sinh k_0)}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0}\\
&\quad-\sum_{n=1}^{\infty}\frac{k_n(\cos k_n)(\sin k_n)}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\Bigg\}.
\end{aligned}
\tag{2.94}
$$

Finally, the solution Φ is obtained as

$$
\begin{aligned}
\Phi &= -4\Bigg\{\mathrm{i}\frac{(1+\mathrm{T}k_0^2)e^{\mathrm{i}k_0x}\cosh(k_0(1+y))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0}\int_{-1}^{0}u\cosh(k_0(1+\eta))\mathrm{d}\eta\\
&\quad+\sum_{n=1}^{\infty}\frac{(1-\mathrm{T}k_n^2)e^{-k_nx}\cos(k_n(1+y))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\int_{-1}^{0}u\cos(k_n(1+\eta))\mathrm{d}\eta\Bigg\}\\
&\quad-\frac{\mathrm{iT}\delta}{\omega}(-4\omega^2)\Bigg\{\mathrm{i}\frac{e^{\mathrm{i}k_0x}(\cosh k_0)(\cosh(k_0(1+y)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0}\\
&\quad+\sum_{n=1}^{\infty}\frac{e^{-k_nx}(\cos k_n)(\cos(k_n(1+y)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\Bigg\}\Phi_\eta|_{\Gamma'}\\
&= 4\Bigg\{\mathrm{i}\frac{e^{\mathrm{i}k_0x}\cosh(k_0(1+y))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0}\\
&\quad\left[-(1+\mathrm{T}k_0^2)\int_{-1}^{0}u\cosh(k_0(1+\eta))\mathrm{d}\eta+\mathrm{iT}\delta\omega(\cosh k_0)\Phi_\eta|_{\Gamma'}\right]\\
&\quad+\sum_{n=1}^{\infty}\frac{e^{-k_nx}\cos(k_n(1+y))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\\
&\quad\left[-(1-\mathrm{T}k_n^2)\int_{-1}^{0}u\cos(k_n(1+\eta))\mathrm{d}\eta+\mathrm{iT}\delta\omega(\cos k_n)\Phi_\eta|_{\Gamma'}\right]\Bigg\},
\end{aligned}
\tag{2.95}
$$

where $\Phi_\eta|_{\Gamma'}$ is expressed by (2.92), (2.93) and (2.94).

2.3. Green's function and the uniqueness

As described before, the Green's function G satisfying (2.17) to (2.22) does exist. Now we use this result to show the uniqueness of the solution under Hocking's edge condition.

Consider Φ_0 being the solution of the following equations:

$$\nabla_2^2\Phi_0 = 0 \quad \text{in} \quad V; \tag{2.96}$$

$$-\omega^3\Phi_0 + \Phi_{0y} - \mathrm{T}\nabla_1^2\Phi_{0y} = 0 \quad \text{on} \quad S; \tag{2.97}$$

$$\Phi_{0y} = 0 \quad \text{on} \quad B; \tag{2.98}$$

$$\Phi_{0x} = 0 \quad \text{on} \quad M; \tag{2.99}$$

$$\Phi_0 \ \to\ 0 \quad \text{as} \quad x \to \infty; \tag{2.100}$$

$$\Phi_{0xy} = \mathrm{i}\omega\delta\Phi_{0x} \quad \text{at} \quad \Gamma. \tag{2.101}$$

Then, by (2.40) and (2.89), we obtain

$$\begin{aligned}\Phi_0 = {} & \left(-\frac{\mathrm{iT}\delta}{\omega}\right)(-4\omega^2)\Bigg[\mathrm{i}\frac{e^{\mathrm{i}k_0x}(\cosh k_0)(\cosh(k_0(1+y)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0} \\ & +\sum_{n=1}^{\infty}\frac{e^{-k_nx}(\cos k_n)(\cos(k_n(1+y)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\Bigg]\Phi_{0\eta}\Big|_{\Gamma},\end{aligned} \tag{2.102}$$

We differentiate both sides with respect to y, let $x = y = 0$, and obtain that

$$\begin{aligned}\Phi_{0y}\Big|_{\Gamma} = {} & -4\mathrm{T}\omega\delta\Bigg[\frac{k_0(\cosh k_0)(\sinh k_0)}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0} \\ & +\mathrm{i}\sum_{n=1}^{\infty}\frac{k_n(\cos k_n)(\sin k_n)}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\Bigg]\Phi_{0y}\Big|_{\Gamma}.\end{aligned} \tag{2.103}$$

It follows that

$$\begin{aligned}\Phi_{0y}\Big|_{\Gamma}\Bigg\{1+4\mathrm{T}\omega\delta\Bigg[& \frac{k_0(\cosh k_0)(\sinh k_0)}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0} \\ & +\mathrm{i}\sum_{n=1}^{\infty}\frac{k_n(\cos k_n)(\sin k_n)}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\Bigg]\Bigg\} = 0\end{aligned} \tag{2.104}$$

Since $k_0, k_1, k_2, \cdots k_n, \cdots > 0$, and ω, δ are considered as real numbers, it is obvious that for $\delta \geq 0$,

$$1 + 4\mathrm{T}\omega\delta\left[\frac{k_0(\cosh k_0)(\sinh k_0)}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0}\right.$$
$$\left.+\mathrm{i}\sum_{n=1}^{\infty}\frac{k_n(\cos k_n)(\sin k_n)}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\right] \neq 0. \tag{2.105}$$

By (2.104), we have

$$\Phi_{0y}\Big|_{\Gamma} = 0, \tag{2.106}$$

and hence by (2.102),

$$\Phi_0 \equiv 0 \quad \text{in} \quad V. \tag{2.107}$$

Thus the solution of homogeneous equations is zero, and we conclude that the solution of (2.11) to (2.16) is unique.

3. Solution of the Infinite Depth Problem

3.1. Formulation

Now let us consider the case of infinite depth. For the velocity potential $\Psi_\infty = \Psi_\infty(x,y,t)$, the radiation condition representing the behavior of outgoing waves at large distance from the wave maker is expressed as

$$\Psi_\infty \to \hat{C}_0 e^{k_0 y} e^{\mathrm{i}(k_0 x+\omega t)} \quad \text{as} \quad x \to \infty, \tag{3.1}$$

where $\hat{C}_0$ is a constant to be determined and k_0 is the positive root of the following equation

$$\alpha(\mathrm{T}\alpha^2+1) - \omega^2 = 0. \tag{3.2}$$

As for the remaining linearized governing equations for Ψ_∞, we find

$$\nabla_2^2\Psi_\infty = 0 \quad \text{in} \quad V = \{(x,y)|x\in(0,\infty), y\in(-\infty,0)\}, \tag{3.3}$$

$$\Psi_{\infty y} = Z_{\infty t} \text{ and} \tag{3.4}$$

$$\Psi_{\infty t} + \mathrm{g}Z_\infty = \mathrm{T}\nabla_1^2 Z_\infty \quad \text{on} \quad S, \tag{3.5}$$

$$\Psi_{\infty x} = u(y)e^{\mathrm{i}\omega t} \quad \text{on} \quad M, \tag{3.6}$$

$$\Psi_{\infty y} \to 0 \quad \text{as} \quad y \to -\infty, \tag{3.7}$$

$$Z_{\infty t} = \lambda Z_{\infty x} \quad \text{at} \quad \Gamma, \tag{3.8}$$

where we use Z_∞ as the free surface. Then again to time-reduce the problem, we set Φ_∞ as the time–reduced potential, and ζ_∞ as the time–reduced free surface. Nondimensionalizing the variables as before, we obtain

$$\nabla_2^2\Phi_\infty = 0 \quad \text{in} \quad V, \tag{3.9}$$

$$\omega^2\Phi_\infty + \Phi_{\infty y} - \mathrm{T}\nabla_1^2\Phi_{\infty y} = 0 \quad \text{on} \quad S, \tag{3.10}$$

$$\Phi_{\infty x} = u(y) \quad \text{on} \quad M, \tag{3.11}$$

$$\Phi_{\infty y} \to 0 \quad \text{as} \quad y \to -\infty, \tag{3.12}$$

$$\Phi_\infty \to \hat{C}_0 e^{k_0(y+\mathrm{i}x)} \quad \text{as} \quad x \to \infty, \tag{3.13}$$

$$\Phi_{\infty xy} = \mathrm{i}\omega\delta\Phi_{\infty y} \quad \text{at} \quad \Gamma. \tag{3.14}$$

3.2. Construction of the solution

Similarly, we construct a Green's function G_∞ satisfying the homogeneous boundary conditions corresponding to (3.10), (3.11), and (3.12), together with

$$G_{\infty\eta} \to 0 \quad \text{as} \quad \eta \to -\infty, \tag{3.15}$$

$$G_\infty \to \hat{C}_1 e^{(\mathrm{i}\xi+\eta)k_0} \quad \text{as} \quad \xi \to \infty, \tag{3.16}$$

and Φ_∞ in terms of G_∞ is given by

$$\Phi_\infty = \int_{-\infty}^{0} G_\infty\big|_{\xi=0} u(\eta)\mathrm{d}\eta + \frac{\mathrm{T}}{\omega^2}\left(G_{\infty_\eta}\Phi_{\infty_{\xi\eta}} - \Phi_{\infty_\eta}G_{\infty_{\xi\eta}}\right)_{\eta=0}\Big|_{\xi=0}^{\xi=\infty}; \tag{3.17}$$

furthermore, we use the solution φ_∞^0 for Evans's edge condition to obtain $G_{\infty\eta}|_{\Gamma'}$ in order to obtain Φ_∞. Since the construction is similar to that of $G_\eta|_\Gamma$, we may assume that

$$\varphi_\infty^0 = \varphi_\infty + \hat{C}_0 e^{k_0(y+\mathrm{i}x)}, \tag{3.18}$$

$$\varphi_\infty = \frac{2}{\pi}\int_0^\infty \chi_\infty \cos\xi x \mathrm{d}\xi, \tag{3.19}$$

where φ_∞ and χ_∞ are the functions similar to φ and χ as in (2.41) and (2.70), respectively.

Let us consider χ_∞ first. We find

$$\chi_\infty = K_{\infty_1}(\xi, y) + \frac{b(\xi, y)}{\xi} + K_{\infty_2}(\xi, y) + \frac{\mathrm{T}}{\omega^2}(\lambda_0 - \mathrm{i}\hat{C}_0 k_0^2)\cosh \xi y, \tag{3.19}$$

where

$$\begin{aligned} K_{\infty_1}(\xi, y) = & -\frac{\xi(\mathrm{T}\xi^2+1)\cosh\xi y + \omega^2 \sinh \xi y}{\xi \Delta_\infty(\xi)} \\ & \times \left[a_\infty(\xi) - \frac{\mathrm{i}\hat{C}_0 k_0 \Delta_\infty(\xi)}{(\xi^2 - k_0^2)(\mathrm{T}k_0^2+1)} + \frac{\mathrm{T}\lambda_0}{\omega^2}\xi \right], \end{aligned} \tag{3.21}$$

$$\Delta_\infty(\xi) \equiv \xi(\mathrm{T}\xi^2+1) - \omega^2, \tag{3.22}$$

$$a_\infty(\xi) = \int_{-\infty}^{0} u e^{\xi\eta} \mathrm{d}\eta, \tag{3.23}$$

$$K_{\infty_2}(\xi, y) = -\frac{\mathrm{i}\hat{C}_0 k_0}{\xi(\xi^2 - k_0^2)} [\xi(e^{k_0 y} - \cosh \xi y) - k_0 \sinh \xi y], \tag{3.24}$$

and $b(\xi, y)$ is specified as in (2.66). Since ξ_∞ should not have any singularity over $[0, \infty]$, we find

$$\lim_{\xi \to k_0} \left[a_\infty(\xi) - \frac{\mathrm{i}\hat{C}_0 k_0 \Delta_\infty(\xi)}{(\xi^2 - k_0^2)(\mathrm{T}k_0^2+1)} + \frac{\mathrm{T}\lambda_0}{\omega^2}\xi \right] = 0, \tag{3.25}$$

then

$$\hat{C}_0 = -\frac{2\mathrm{i}}{k_0(3\mathrm{T}k_0^2+1)}[\omega^2 a_\infty(k_0) + \mathrm{T}k_0\lambda_0]. \tag{3.26}$$

Next, from (3.19),

$$\begin{aligned} \varphi_\infty &= \frac{1}{\pi}\int_{-\infty}^{\infty} \chi_\infty e^{\mathrm{i}|\xi|x} \mathrm{d}\xi \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \left[K_{\infty_1}(\xi, y) + \frac{b(\xi, y)}{\xi} \right. \\ & \qquad \left. + K_{\infty_2}(\xi, y) + \tfrac{\mathrm{T}}{\omega^2}(\lambda_0 - \mathrm{i}\hat{C}_0 k_0^2)\cosh \xi y \right] \times e^{\mathrm{i}|\xi|x} \mathrm{d}\xi. \end{aligned} \tag{3.27}$$

Now we rotate the integration contour along the positive imaginary axis for the integral involving $e^{\mathrm{i}\xi x}$, and along the negative imaginary axis for the integral involving $e^{-\mathrm{i}\xi x}$, and obtain

$$
\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{\infty} K_{\infty_1}(\xi,y)e^{\mathrm{i}|\xi|x}\mathrm{d}\xi \\
&= \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{\xi(\mathrm{T}\xi^2+1)\cosh\xi y+\omega^2\sinh\xi y}{\xi\Delta_\infty(\xi)} \\
&\quad\times\left[a_\infty(\xi)\frac{\mathrm{i}\hat{C}_0k_0\Delta_\infty(\xi)}{(\xi^2-k_0^2)(\mathrm{T}k_0^2+1)}+\frac{\mathrm{T}\lambda_0}{\omega^2}\xi\right]e^{\mathrm{i}|\xi|x}\mathrm{d}\xi \\
&= \frac{-1}{\pi}\int_{\infty}^{0}\frac{(-\mathrm{i}\alpha)(1-\mathrm{T}\alpha^2)\cos\alpha y+(-\mathrm{i})\omega^2\sin\alpha y}{(-\mathrm{i}\alpha)[(-\mathrm{i}\alpha)(1-\mathrm{T}\alpha^2)-\omega^2]}\Bigg\{a_\infty(-\mathrm{i}\alpha) \\
&\quad-\frac{\mathrm{i}\hat{C}_0k_0[(-\mathrm{i}\alpha)(1-\mathrm{T}\alpha^2)-\omega^2]}{(-\alpha^2-k_0^2)(1+\mathrm{T}k_0^2)}+\frac{\mathrm{T}\lambda_0}{\omega^2}(-\mathrm{i}\alpha)\Bigg\}e^{-\alpha x}\mathrm{d}(\mathrm{i}\alpha) \\
&\quad+\frac{-1}{\pi}\int_{\infty}^{0}\frac{(\mathrm{i}\alpha)(1-\mathrm{T}\alpha^2)\cos\alpha y+(\mathrm{i})\omega^2\sin\alpha y}{(\mathrm{i}\alpha)[(\mathrm{i}\alpha)(1-\mathrm{T}\alpha^2)-\omega^2]}\Bigg\{a_\infty(\mathrm{i}\alpha) \\
&\quad-\frac{\mathrm{i}\hat{C}_0k_0[(\mathrm{i}\alpha)(1-\mathrm{T}\alpha^2)-\omega^2]}{(-\alpha^2-k_0^2)(1+\mathrm{T}k_0^2)}+\frac{\mathrm{T}\lambda_0}{\omega^2}(\mathrm{i}\alpha)\Bigg\}e^{-\alpha x}\mathrm{d}(\mathrm{i}\alpha) \\
&= -\frac{2}{\pi}\int_{\infty}^{0}\frac{\alpha(1-\mathrm{T}\alpha^2)\cos\alpha y+\omega^2\sin\alpha y}{\alpha^2(1-\mathrm{T}\alpha^2)^2+\omega^4} \\
&\quad\times\left[\int_{-\infty}^{0}u\frac{\alpha(1-\mathrm{T}\alpha^2)\cos\alpha\eta+\omega^2\sin\alpha\eta}{\alpha}\mathrm{d}\eta+\mathrm{T}\lambda_0\right]e^{-\alpha x}\mathrm{d}\alpha,
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{0}\frac{b(\xi,y)}{\xi}e^{\mathrm{i}|\xi|x}\mathrm{d}\xi \\
&= \frac{1}{\pi}\Bigg[\int_{\infty}^{0}e^{-\alpha x}\int_{y}^{0}\frac{(-\mathrm{i})u(\eta)\sin(\alpha(\eta-y))}{(-\mathrm{i}\alpha)}\mathrm{d}\eta\mathrm{d}(\mathrm{i}\alpha) \\
&\quad+\int_{0}^{\infty}e^{-\alpha x}\int_{y}^{0}\frac{(\mathrm{i})u(\eta)\sin(\alpha(\eta-y))}{(\mathrm{i}\alpha)}\mathrm{d}\eta\mathrm{d}(\mathrm{i}\alpha)\Bigg] \\
&= \frac{\mathrm{i}}{\pi}\int_{y}^{0}u(\eta)\Bigg[\int_{\infty}^{0}\frac{e^{-\alpha x}\sin(\alpha(\eta-y))}{\alpha}\mathrm{d}\alpha \\
&\quad+\int_{0}^{\infty}\frac{e^{-\alpha x}\sin(\alpha(\eta-y))}{\alpha}\mathrm{d}\alpha\Bigg]\mathrm{d}\eta=0,
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{\infty}K_{\infty_2}(\xi,y)e^{\mathrm{i}|\xi|x}\mathrm{d}\xi \\
&= \frac{-\mathrm{i}\hat{C}_0k_0}{\pi}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}|\xi|x}}{\xi(\xi^2-k_0^2)}\left[\xi(e^{k_0y}-\cosh\xi y)-k_0\sinh\xi y\right]\mathrm{d}\xi
\end{aligned}
\tag{3.30}
$$

$$= -\frac{\mathrm{i}\hat{C}_0 k_0}{\pi}\Bigg\{\int_\infty^0 \frac{\mathrm{i}e^{-\alpha x}}{(-\mathrm{i}\alpha)(-\alpha^2-k_0^2)} \times\Bigg[(-\mathrm{i}\alpha)(e^{k_0 y}-\cos\alpha y)-(-\mathrm{i})k_0\sin\alpha y\Bigg]\mathrm{d}\alpha + \int_0^\infty \frac{\mathrm{i}e^{-\alpha x}}{(\mathrm{i}\alpha)(-\alpha^2-k_0^2)}\Bigg[(\mathrm{i}\alpha)(e^{k_0 y}-\cos\alpha y)-(-\mathrm{i})k_0\sin\alpha y\Bigg]\mathrm{d}\alpha\Bigg\} = 0,$$

and

$$\begin{aligned}(3.31)\quad &\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{T}}{\omega^2}(\lambda_0-\mathrm{i}\hat{C}_0k_0)(\cosh\xi y)e^{\mathrm{i}|\xi|x}\mathrm{d}\xi \\ &= \frac{\mathrm{T}(\lambda_0-\mathrm{i}\hat{C}_0k_0)}{\pi\omega^2}\left[\int_\infty^0 e^{-\alpha x}(\cos\alpha y)\mathrm{d}(\mathrm{i}\alpha)+\int_0^\infty e^{-\alpha x}(\cos\alpha y)\mathrm{d}(\mathrm{i}\alpha)\right] \\ &= 0.\end{aligned}$$

That is,

$$(3.32)\quad \varphi_\infty = -\frac{2}{\pi}\int_0^\infty \frac{\alpha(1-\mathrm{T}\alpha^2)\cos\alpha y+\omega^2\sin\alpha y}{\alpha^2(1-\mathrm{T}\alpha^2)^2+\omega^4}\times \left[\int_{-\infty}^0 u\frac{\alpha(1-\mathrm{T}\alpha^2)\cos\alpha\eta+\omega^2\sin\alpha\eta}{\alpha}\mathrm{d}\eta+\mathrm{T}\lambda_0\right]e^{-\alpha x}\mathrm{d}\alpha.$$

Finally,

$$\begin{aligned}(3.33)\quad \varphi_\infty^0 &= \varphi_\infty+\hat{C}_0e^{k_0(\mathrm{i}x+y)} \\ &= -\frac{2}{\pi}\int_0^\infty \frac{\alpha(1-\mathrm{T}\alpha^2)\cos\alpha y+\omega^2\sin\alpha y}{\alpha^2(1-\mathrm{T}\alpha^2)^2+\omega^4}\times \\ &\quad\left[\int_{-\infty}^0 u\frac{\alpha(1-\mathrm{T}\alpha^2)\cos\alpha\eta+\omega^2\sin\alpha\eta}{\alpha}\mathrm{d}\eta+\mathrm{T}\lambda_0\right]e^{-\alpha x}\mathrm{d}\alpha \\ &\quad -\frac{2\mathrm{i}}{k_0(3\mathrm{T}k_0^2+1)}[\omega^2a_\infty(k_0)+\mathrm{T}k_0\lambda_0]e^{k_0(\mathrm{i}x+y)},\end{aligned}$$

which agrees with the solution of Rhodes-Robinson [6] and of Mandal et al. [5].

By applying the same method, we find the solution $\Phi_\infty(x,y)$ for Hocking's edge condition as

$$
\begin{aligned}
&\Phi_\infty(x,y)\\
&= -2\Bigg\{\frac{1}{\pi}\int_0^\infty \frac{\alpha(1-\mathrm{T}\alpha^2)\cos\alpha y+\omega^2\sin\alpha y}{\alpha^2(1-\mathrm{T}\alpha^2)^2+\omega^4}\\
&\times e^{-\alpha x}\left[\int_{-\infty}^0 u\frac{\alpha(1-\mathrm{T}\alpha^2)\cos\alpha\eta+\omega^2\sin\alpha\eta}{\alpha}\mathrm{d}\eta-(\mathrm{i}\mathrm{T}\omega\delta)\Phi_{\infty_\eta}|_{\Gamma'}\right]\mathrm{d}\alpha\\
&+\frac{\mathrm{i}e^{k_0(\mathrm{i}x+y)}}{3\mathrm{T}k_0^2+1}\left[(1+\mathrm{T}k_0^2)\int_{-\infty}^0 ue^{k_0\eta}\mathrm{d}\eta-(\mathrm{i}\mathrm{T}\omega\delta)\Phi_{\infty_\eta}|_{\Gamma'}\right]\Bigg\},
\end{aligned}
\tag{3.34}
$$

where

$$
\Phi_{\infty_\eta}|_{\Gamma'} = \frac{\partial}{\partial y}\left(\int_{-\infty}^0 G_\infty|_{\xi=0}u\mathrm{d}\eta\right)\Big|_{\Gamma'}\Big/\left(1+\frac{\mathrm{i}\mathrm{T}\delta}{\omega}G_{\eta y}|_{\Gamma'\Gamma}\right),
\tag{3.35}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial y}\left(\int_{-\infty}^0 G_\infty|_{\xi=0}u\mathrm{d}\eta\right)\Big|_{\Gamma}\\
&= -2\omega^2\Bigg\{\frac{1}{\pi}\int_0^\infty\frac{e^{-\alpha x}}{\alpha^2(1-\mathrm{T}\alpha^2)^2+\omega^4}\\
&\times\left(\int_{-\infty}^0 u(\alpha(1-\mathrm{T}\alpha^2)\cos\alpha\eta+\omega^2\sin\alpha\eta)\mathrm{d}\eta\right)\mathrm{d}\alpha\Big|_{x=0}\\
&+\frac{\mathrm{i}}{3\mathrm{T}k_0^2+1}\left(\int_{-\infty}^0 ue^{k_0\eta}\mathrm{d}\eta\right)\Bigg\},
\end{aligned}
\tag{3.36}
$$

and

$$
\begin{aligned}
&1+\tfrac{\mathrm{i}\mathrm{T}\delta}{\omega}G_{\eta y}|_{\Gamma'\Gamma} = 1-2\mathrm{i}\mathrm{T}\omega\delta\\
&\qquad\times\left\{\frac{\omega^2}{\pi}\int_0^\infty\frac{\alpha e^{-\alpha x}}{\alpha^2(1-\mathrm{T}\alpha^2)^2+\omega^4}\mathrm{d}\alpha|_{x=0}+\frac{k_0\mathrm{i}}{3\mathrm{T}k_0^2+1}\right\}.
\end{aligned}
\tag{3.37}
$$

3.3. Uniqueness of the infinite depth problem

The uniqueness of the solution follows from the fact that when $u=0$, we obtain the solution $\Phi_\infty^0(x,y)$ satisfying

$$
\begin{aligned}
&\Phi_{\infty y}^0|_\Gamma = (2\mathrm{i}\mathrm{T}\omega\delta)\\
&\qquad\times\left[\frac{\omega^2}{\pi}\int_0^\infty\frac{\alpha e^{-\alpha x}}{\alpha^2(1-\mathrm{T}\alpha^2)^2+\omega^4}\mathrm{d}\alpha|_{x=0}+\frac{\mathrm{i}k_0}{3\mathrm{T}k_0^2+1}\right]\Phi_{\infty\eta}^0|_{\Gamma'},
\end{aligned}
\tag{3.38}
$$

which implies that

$$\Phi^0_{\infty y}|_\Gamma\left\{\left(1+\frac{2\mathrm{T}\omega\delta k_0}{3\mathrm{T}k_0^2+1}\right)-\frac{2\mathrm{i}\mathrm{T}\omega^3\delta}{\pi}\int_0^\infty\frac{\alpha e^{-\alpha x}}{\alpha^2(1-\mathrm{T}\alpha^2)^2+\omega^4}\mathrm{d}\alpha\Big|_{x=0}\right\}=0, \tag{3.39}$$

then

$$\Phi^0_{\infty y}|_\Gamma = 0 \quad \text{and so} \quad \Phi^0_\infty(x,y)=0; \tag{3.40}$$

since for all $k_0 \geq 0$ and $\delta \geq 0$,

$$\left(1+\frac{2\mathrm{T}\omega\delta k_0}{3\mathrm{T}k_0^2+1}\right)-\frac{2\mathrm{i}\mathrm{T}\omega^3\delta}{\pi}\int_0^\infty\frac{\alpha e^{-\alpha x}}{\alpha^2(1-\mathrm{T}\alpha^2)^2+\omega^4}\mathrm{d}\alpha\Big|_{x=0}\neq 0. \tag{3.41}$$

Thus we conclude that the solution obtained from (3.9) to (3.14) is unique.

Rhodes–Robinson [6] suggested that there exist certain forms of expansion for the prescribed normal velocity at the wave maker. The results are stated as "expansion theorems," yet no proof is given and the motivation for the expansions by themselves is unclear. Yeh [8] gave a rigorous proof and presented a way of using the theorems to construct exact solutions to the same problems. Their solving processes will be made available upon request to save space.

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