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APPROXIMATION OF HARMONIC FUNCTIONS CLASSES WITH SINGULARITIES ON QUASICONFORMAL CURVES

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Abstract. In the present paper the direct theorem on approximation of classes of harmonic functions with singularities on closed quasiconformal curves by the harmonic rational function is obtained.

1. Introduction and Main Results

Let Γ be an arbitrary closed Jordan curve with complements $\Omega = C\Gamma = \Omega_1 \cup \Omega_2$ $(0 \in \Omega_1, \infty \in \Omega_2)$. Consider the function $w = \phi_i(z)$, $(i = 1, 2)$ that conformally and univalently maps Ω_i onto $\Omega_i^{\prime}$ respectively $\Omega_1^{\prime}$ $=(\{w : |w| < 1\}$, $\Omega_2^{\prime} = \{w : |w| > 1\})$ with normalization

$$\phi_1(0) = 0, \quad \phi_1^{\prime}(0) > 0, \qquad \phi_2(\infty) = \infty, \qquad \lim_{z\to\infty} \phi_2(z)/z > 0.$$

The function inverse to $w = \phi_i(z)$ denote by $z = \psi_i(w)$, $(i = 1, 2)$.

Assume at arbitrary natural n

$$\Gamma_{1+\frac{(-1)^i}{n}} := \left\{\zeta : \zeta \in \Omega_i, \quad |\phi_i(\zeta)| = 1 + \frac{(-1)^i}{n}\right\}, \qquad i = 1, 2,$$

$$\rho_{1+\frac{(-1)^i}{n}}(z) := \inf_{\zeta\in\Gamma_{1+\frac{(-1)^i}{n}}} |\zeta - z|, \quad \rho_{1/n}(z) := \min_{i=1,2}\left\{\rho_{1+\frac{(-1)^i}{n}}(z)\right\}, \qquad i = 1, 2$$

For $\delta > 0$ we assume

$$U(z, \delta) := \{\varsigma : |\varsigma - z| < \delta\}, \quad z \in \mathbb{C}$$

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and

$$G_n{:}=\bigcup_{\varsigma\in\Gamma}U\left(\varsigma,\frac{1}{2}\rho_{1/n}(\varsigma)\right).$$

Let C^{∞} be a class of infinitely frequently differentiable in $\mathbb{C}$ functions, $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$ be a Laplace operator.

Let $\omega(\delta)$, $\delta>0$ be a function of continuity module type, i,e, a positive, non - decreasing (where $\omega(+0)=0$) function satisfying at some $C=const>0$ the condition $\omega(t\delta)\leq Ct\omega(\delta)$; $\delta>0$, $t>1$.

By $C^{\omega}_{\Delta}(\Gamma)$ we denote a class of real valued continuous in $\overline{\mathbb{C}}$, harmonic in $\overline{\mathbb{C}}\setminus\Gamma$ functions $u(z)$ satisfying at z and $\varsigma\in\mathbb{C}$ the condition

$$|u(z)-u(\varsigma)|\leq C\omega(|z-\varsigma|),\ C=C(u)=const \tag{1}$$

In sequel, by C, C_1, ... we'll denote constant (in different relations, generally speaking different) depending, unless, specially stipulated, only on Γ. We'll also use the symbol $A\preccurlyeq B$ that means $A\leq CB$, where $C=const>0$ is independent on A and B, and $A\asymp B$ if simultaneously $A\preccurlyeq B$.and $B\preccurlyeq A$.

In the present paper we'll interested in the case when Γ is a quasiconformal curve. The convenient geometrical quasiconformality definition of the curve is the following (see [4, p. 100]):

Let's consider a Jordan curve Γ and two arbitrary points z_1 and z_2 on it. By $\Gamma(z_1,z_2)$ we denote one of the two curves (with less diameter) on which the points z_1 and z_2 divide the curve Γ. The feasibility of the relation

$$diam\,\Gamma(z_1,z_2)\preccurlyeq|z_1-z_2|$$

is the necessary and sufficient condition for the quasiconformality of the curve Γ.

As P. P. Belinskii's example shows (see [5. p. 42]) the quasiconformal curves may be unrectifiable at any of its part.

To construct a harmonic rational function $R_n(z)$ it is used a harmonic rational core, defined by

$$\pi_n(\varsigma,z):=Re\sum_{j=-n}^{n}a_j(\varsigma)z^j, \tag{2}$$

where a_j, $j=-n,....,n$ are complex-valued summable functions.

The main result of the given paper is the following theorem.

Theorem 1. *Let Γ be a closed quasiconformal curve, $u(z)\in C^{\omega}_{\Delta}(\Gamma)$. Then for each natural $n=1,2,...$ there exists a harmonic rational function $R_n(z)$, that for $z\in G_n$ it is fulfilled the inequality*

$$|u(z)-R_n(z)|\leq C\omega\left[\rho_{1/n}(z)\right], \tag{3}$$

where the constant $C > 0$ is independent of z and n.

The harmonic functions play an important role in many areas of applied mathematics and mechanics. It is actually the approximation of these functions by rational functions or some other functions, which can be found easily. Therefore, Theorem 1 is important for the approximate construction of the harmonic functions with singularites on quasiconformal curves. The similar result for the harmonic function with singularites on quasiconformal arcs was obtained in [1].

2. Auxiliary Results

Lemma 1. [6] *Let Γ be a quasiconformal curve, $z \in \Gamma$, $\varsigma \in \Omega_i$, $|\varsigma| \geq \varepsilon > 0$ ($\varepsilon > 0$ is a fixed number), and let*

$$\varsigma_\Gamma := \psi_i \left[\phi_i(\varsigma) \left| \phi_i(\varsigma) \right|^{-1} \right]$$

and

$$\overset{\sim i}{z}_R := \phi_i \left[(1 + (-1)^i R) \phi_i(z) \right], \; \overset{\sim i}{\varsigma}_R := \psi_i \left[(1 + (-1)^i R) \phi_i(\varsigma) \right], \;\; i = 1, \; 2$$

for $R \in (0,1)$. Then

$$\begin{aligned}
d(\varsigma, \Gamma) &\asymp |\varsigma - \varsigma_\Gamma| ; \\
\rho_{1+(-1)^i R}(z) &\asymp \left| z - \overset{\sim i}{z}_R \right| ; \\
\left| \varsigma - \overset{\sim i}{\varsigma}_R \right| &\asymp \rho_{1+(-1)^i R}(z), \;\; \text{if} \;\; |\varsigma - z| \preccurlyeq \rho_{1+(-1)^i R}(z); \\
\left| \tfrac{\varsigma - \overset{\sim i}{\varsigma}_R}{\varsigma - z} \right| &\preccurlyeq \left| \frac{\rho_{1+(-1)^i R}(z)}{\varsigma - z} \right|^\gamma, \;\; \text{if} \;\; |\varsigma - z| \succcurlyeq \rho_{1+(-1)^i R}(z),
\end{aligned}$$

where the constant $\gamma > 0$ is dependent only on Γ.

Corollary 1. (see [2, p. 149]). *Let $0 < R < 1$, $\varsigma \in \Gamma$ and $\rho_R(\varsigma) = \rho_{1+(-1)^i R}(\varsigma)$ for some $i = 1, 2$. Then there exists a constant γ $(0 < \gamma < 1)$, that*

$$\frac{\rho_R(\varsigma)}{\left| \overset{\sim i}{\varsigma}_R - z \right|} \preccurlyeq \left| \frac{\rho_R(z)}{\overset{\sim i}{\varsigma}_R - z} \right|^\gamma, \quad \forall z \in G_R. \tag{4}$$

Remark 1. Lemma 1 and Corollary 1 are cited in slightly modified form.

Lemma 2. *Let* Γ *be an arbitrary, closed Jordan curve and* $u \in C_{\Delta}^{\omega}(\Gamma)$. *Then*

$$|u(z) - u_n(z)| \le C\omega\left[\rho_{1/n}(z)\right], z \in G_n; \tag{5}$$

$$u(z) = u(z), \ z \in \mathbb{C} \backslash G_n \tag{6}$$

$$|\Delta u_n(z)| \le C\omega\left[\rho_{1/n}(z)\right]/\rho_{1/n}^2(z) \tag{7}$$

Proof. Let's consider an arbitrary averaging core $K(z) \in C^{\infty}(\mathbb{C})$, with the properties

$$\int_{\mathbb{C}} K(z)\, d\sigma_z = 1;$$

$$K(|z|) = K(z) \ge 0, z \in \mathbb{C};$$

$$K(z) = 0, |z| \ge 1.$$

By $\delta_n^{(i)}(z)$, $z \in \mathbb{C} \cap \bar{\bar{\Omega}}_i \setminus \Gamma_{1+\frac{(-1)^i}{n}}$ denote a regulated distance from z to the level line $\Gamma_{1+\frac{(-1)^i}{n}}$ $(i = 1, 2)$ (see [7, p. 203]).

Let's fix such sufficient small number $\varepsilon > 0$, that

$$h^{(i)}(z) =: \varepsilon\delta_n^{(i)}(z) \le \frac{1}{4}\rho_{1+\frac{(-1)^i}{n}}(z), \quad z \in \mathbb{C} \cap \bar{\bar{\Omega}}_i \setminus \Gamma_{1+\frac{(-1)^i}{n}}.$$

Then denoting $h(z) := \min\limits_{i=1,2} h^{(i)}(z)$, we consequently have

$$U(z, h(z)) \subset \mathbb{C} \setminus \Gamma, \ \forall z \in \mathbb{C} \setminus G_n.$$

Consider the function

$$u_n(z) : \quad = [h(z)]^{-2} \int_{\mathbb{C}} u(\varsigma) K\left(\frac{\varsigma - z}{h(z)}\right) d\sigma_{\varsigma}$$

$$= \int_{\mathbb{C}} u(z + h(z)\xi) K(\xi)\, d\sigma_{\xi}, \quad z \notin \Gamma_{1+\frac{(-1)^i}{n}};$$

and

$$u_n(z) := u(z), \quad z \in \Gamma_{1+\frac{(-1)^i}{n}}.$$

Let $z \in \mathbb{C} \backslash G_n$. Then by the mean value theorem for a harmonic function we get

$$\begin{aligned} u_n(z) &= \int_0^1 K(\rho)\,\rho d\rho \int_0^{2\pi} u\left(z + h(z)\,\rho e^{i\varphi}\right) d\varphi \\ &= 2\pi u(z) \int_0^1 K(\rho)\,\rho d\rho = u(z) \int_{\mathbb{C}} K(z)\, d\sigma_z = u(z)\,. \end{aligned}$$

Now, let $z \in G_n$. We have

$$\begin{aligned} |u(z) - u_n(z)| &= \left| \int_{\mathbb{C}} \left[u\left(z + h(z)\,\xi\right) - u(z)\right] K(\xi)\, d\sigma_\xi \right| \\ &\preccurlyeq \omega\left[h(z)\right] \int_{\mathbb{C}} K(\xi)\, d\xi \preccurlyeq \omega\left[\rho_{1/n}(z)\right]. \end{aligned}$$

Let $z_0 \in U(z_0, h)$ be an arbitrary fixed point. Assume $h =: h(z)$ and $r =: |\varsigma - z|/h$. Then

$$\begin{aligned} \frac{\partial u_n(z)}{\partial x} &= \frac{\partial}{\partial x}\left[u_n(z) - u(z_0)\right] \\ &= -\frac{2h'_x}{h^3} \int_{\mathbb{C}} \left[u(\varsigma) - u(z_o)\right] K(r)\, d\sigma_\varsigma \\ &+ h^{-2} \int_{\mathbb{C}} \left[u(\varsigma) - u(z_0)\right] K'_r(r) \left(\frac{|\varsigma - z|'_x}{h} - \frac{r h'_x}{h}\right) d\sigma_\varsigma. \end{aligned}$$

Using this relation we can write

$$\begin{aligned} \frac{\partial^2 u_n(z)}{\partial x^2} &= 2\left(\frac{3h'^2_x}{h^4} - \frac{h''_{xx}}{h^3}\right) \int_{\mathbb{C}} \left[u(\varsigma) - u(z_o)\right] K(r)\, d\sigma_\varsigma \\ &- \frac{4h'_x}{h^3} \int_{\mathbb{C}} \left[u(\varsigma) - u(z_o)\right] K'_r(r) \left(\frac{|\varsigma - z|'_x}{h} - \frac{r h'_x}{h}\right) d\sigma_\varsigma \\ &+ h^{-2} \int_{\mathbb{C}} \left[u(\varsigma) - u(z_o)\right] K''_{rr}(r) \left(\frac{|\varsigma - z|'_x}{h} - \frac{r h'_x}{h}\right)^2 d\sigma_\varsigma \\ &+ h^{-2} \int_{\mathbb{C}} \left[u(\varsigma) - u(z_o)\right] K'_r(r) \\ &\left(\frac{|\varsigma - z|''_{xx}}{h} - \frac{2\,|\varsigma - x|'_x\, h'_x + |\varsigma - z|''_{xx} - 2r h'_x}{h^2}\right)^2 d\sigma_\varsigma. \end{aligned}$$

By the properties of the function $\delta_n^{(k)}(z)$ $(k = 1,\ 2)$ we get

$$\begin{aligned} \frac{\partial^2 u_n(z)}{\partial x^2} &\preccurlyeq h^{-4}\omega(h) \int_{\mathbb{C}} K(r)\, d\sigma_\varsigma + h^{-4}\omega(h) \int_{U(z,h)} d\sigma_\varsigma + h^{-2}\omega(h) \\ &\int_{U(z,h)} \left(\frac{1}{|\varsigma - z|\, h} + \frac{1}{h^2}\right) d\sigma_\varsigma \preccurlyeq \frac{\omega(h)}{h^2}. \end{aligned} \tag{8}$$

The inequality

$$\left|\frac{\partial^2 u_n(z)}{\partial y^2}\right| \preccurlyeq \frac{\omega(h)}{h^2} \tag{9}$$

is proved quite similar to the previous case.

Consequently by (8) and (9) we get

$$|\Delta u_n(z)| \preccurlyeq \omega\left[\rho_{1/n}(z)\right]\left[\rho_{1/n}(z)\right]^{-2}, \qquad z \in G_n.$$

Lemma 3 *For any fixed* $p > 0$ *and* $n = 1, 2...$ *there exists a rational core of the form (2) satisfying for* $z, \varsigma \in G_n$ *the inequality*

$$\left|\ell n\left|\frac{\varsigma - z}{\varsigma_o - z}\right| - \pi_n(\varsigma, z)\right| \le \left\{\begin{array}{ll} c_1\rho^p_{1/n} \big/ |\varsigma - z|^p, & |\varsigma - z| \ge \rho_{1/n}(z); \\ c_2 \ell n 2\rho_{1/n}(z) / |\varsigma - z|, & |\varsigma - z| < \rho_{1/n}(z). \end{array}\right\} \tag{10}$$

Proof. Let $z \in G_n$and $\varsigma \in \Omega_2 \cap \text{int } \Gamma_{2.}$. Let's consider V.K. Dzjadyk's polynomial core $K_{1,s,k,n}(\varsigma, z)$ (see for instance [8, p. 429]). By [2, p. 161] for $k \ge 9$ and $p = 0, 1$ it is valid the following

$$\left|\frac{\partial^p}{\partial z^p}\left[\frac{1}{\varsigma - z} - K_{1,s,k,n}(\varsigma, z)\right]\right| \preccurlyeq \frac{\left|\widetilde{\varsigma} - \varsigma\right|^{ks}}{|\varsigma - z|^{p+1}\left|\widetilde{\varsigma} - z\right|^{ks}}; \tag{11}$$

$$\left|\frac{\partial^p}{\partial z^p} K_{1,s,k,n}(\varsigma, z)\right| \preccurlyeq \left|\widetilde{\varsigma} - z\right|^{-p-1}, \tag{12}$$

here $\widetilde{\varsigma} = \psi_2\left[(1 + \frac{1}{n})\phi_2(\varsigma)\right]$.

By lemma 1 we have

$$\left|\widetilde{\varsigma} - z\right| \asymp |\varsigma - z| + \rho_{1+\frac{1}{n}}(z);$$

$$\left|\tfrac{\widetilde{\varsigma} - \varsigma}{\varsigma - z}\right| = \left|\frac{\widetilde{\varsigma} - \varsigma}{\varsigma - \varsigma_\Gamma}\right|\left|\frac{\widetilde{\varsigma} - \varsigma_\Gamma}{\widetilde{\varsigma} - z}\right| \preccurlyeq \left|\frac{\rho_{1+\frac{1}{n}}(\varsigma_\Gamma)}{\widetilde{\varsigma} - \varsigma_\Gamma}\right|^\gamma \left|\frac{\widetilde{\varsigma} - \varsigma_\Gamma}{\widetilde{\varsigma} - z}\right|^\gamma = \left|\frac{\rho_{1+\frac{1}{n}}(\varsigma_\Gamma)}{\widetilde{\varsigma} - z}\right|^\gamma,$$

here γ $(0 < \gamma < 1)$ is a constant from Lemma 1.

Then for sufficiently large fixed number $m > 0$ (choice of the number m will be mentioned below) the parameters s and κ, $z \in G_n^{(1)}$ and $\varsigma \in \Omega_2 \cap \; int\Gamma_2$ are chosen and the inequalities

$$\left|\frac{1}{\varsigma - z} - K_{1,s,k,n}(\varsigma, z)\right| \preccurlyeq \left[\frac{\rho_{1+\frac{1}{n}}(\varsigma_\Gamma)}{|\varsigma - z| + \rho_{1+\frac{1}{n}}(\varsigma_\Gamma)}\right]^m |\varsigma - z|^{-1}; \tag{13}$$

$$|K_{1,s,k,n}(\varsigma, z)| \preccurlyeq \left[|\varsigma - z| + \rho_{1+\frac{1}{n}}(\varsigma_\Gamma)\right]^{-1} \tag{14}$$

are fulfilled.

Assume $K_n(\varsigma, z) = K_{1,s,k,[\varepsilon n]}(\varsigma, z)$, here a sufficiently small constant $\varepsilon = \varepsilon(k, s, m)$ is chosen so that $\deg K_n \leq n$. From the open covering $\left\{U\left(\varsigma, 2\rho_{1+\frac{1}{n}}(\varsigma)\right)\right\}_{\varsigma\in\Gamma} \cap \Omega_2$ of the set $G_n^{(1)}$ we select the final subcovering $\{U\left(\varsigma_k, 2\rho_{1+\frac{1}{n}}(\varsigma_k)\right)\}_{k=1}^r \cap \Omega_2, \varsigma_k \in \Gamma$, $k = 1, 2, ..., r$. Let the point ξ_k, $k = 1, 2, ..., r$ be chosen so that $|\xi_k - \varsigma_k| = 2\rho_{1+\frac{1}{n}}(\varsigma_k), |\varphi_2(\xi_k)| \geq \frac{1}{n}$.

Let's fix the index $k = 1, 2, ..., r$. Connect the points $\varsigma_0 \in \Gamma$ and ξ_k by the arc

$$\ell_k = \{\xi : |\phi_2(\xi)| = 2, \;\; \arg\phi_2(\zeta_0) < \arg\phi_2(\xi) < \arg\phi_2(\xi_k)\}$$
$$\cup\{\xi : |\phi_2(\xi_k)| \leq |\phi_2(\xi)| \leq 2, \arg\phi_2(\xi) = \arg\phi_2(\xi_k)\}.$$

By lemma 4 of the paper [9] the *arc* ℓ_k possesses the following properties

$$d(z, \ell_k) \asymp |z - \xi_k|, \forall z \in G_n^{(1)};$$
$$mes\,\ell_k(\xi_k, \xi) \asymp |\xi - \xi_k|, \forall \xi \in \ell_k,$$

where the symbol mes denotes lebesgue's linear measure.

Then we assume

$$\pi_n(\xi_k, z) = Re\int_{\ell_k} K_n(\xi, z)\, d\xi.$$

On the other hand

$$Re\int_{\ell_k}\frac{d\xi}{\xi - z} = \ell n\left|\frac{\xi - z}{\varsigma_0 - z}\right|. \tag{15}$$

Allowing for relations (13) and (15) for $z \in G_n^{(2)}$ and sufficiently large $m = m(p, \Gamma)$ we have

$$\begin{aligned}\left|\ell n\left|\tfrac{\xi_k - z}{\varsigma_0 - z}\right| - \pi_n(\xi_k, z)\right| &\preccurlyeq \int_{\ell_k}\frac{\left[\rho_{1+\frac{1}{n}}(\varsigma_k)\right]^m}{|\xi - z|^{m+1}}|d\xi| \preccurlyeq \left(\frac{\rho_{1+\frac{1}{n}}(\varsigma_k)}{|z - \xi_k|}\right)^m \\ &\preccurlyeq \left[\frac{\rho_{1+\frac{1}{n}}(z)}{|z - \varsigma_k| + \rho_{1+\frac{1}{n}}(z)}\right]^p.\end{aligned} \tag{16}$$

Let's construct a system of sets $\{V_k\}_{k=1}^{r}$by the following recurrent principle

$$\begin{aligned} V_1 &= U\left(\varsigma_1, 2\rho_{1+\frac{1}{n}}\left(\varsigma_1\right)\right) \cap \Omega_2; \\ V_k &= U\left(\varsigma_k, 2\rho_{1+\frac{1}{n}}\left(\varsigma_k\right)\right) \cap \Omega_2 \diagdown \cup_{j=1}^{k-1} V_j, \quad k = 2, ..., r. \end{aligned}$$

Let $\varsigma \in V_k, \ \ k = 1, ..., r$. By constructing the core $\pi_n\left(\varsigma, z\right)$ we'll proceed from the identity

$$\frac{1}{\xi - z} = \sum_{j=0}^{s-1} \frac{\left(\xi_k - \xi\right)^j}{\left(\xi_k - z\right)^{j+1}} + \left(\frac{\xi_k - \xi}{\xi_k - z}\right)^s \frac{1}{\xi - z}.$$

Further let $z \in G_n^{(1)}$. We'll give the desired polynomial core by the formula

$$\pi_n\left(\varsigma, z\right) = \pi_n\left(\xi_k, z\right) - \sum_{j=1}^{s} \frac{1}{j} Re\left\{\left(\xi_k - \varsigma\right)^j \left[K_{[n/j]}\left(\xi_k, z\right)\right]^j\right\}.$$

By $\ell\left(\varsigma\right)$ we denote on arc of the circle $\{\xi : |2\xi - (\varsigma + \xi_k)| = |\varsigma - \xi_k|\}$ the most distant from the point z connecting the points ξ_k and ς. It is valid the identity

$$\begin{aligned} Re \textstyle\int_{\ell(\varsigma)} \frac{d\xi}{\varsigma - z} &= \ell n \left|\frac{\varsigma - z}{\xi_k - z}\right| = Re \int_{\ell(\varsigma)} \sum_{j=0}^{s-1} \frac{\left(\xi_k - \xi\right)^j}{\left(\xi_k - z\right)^{j+1}} d\xi \\ & +Re \int_{\ell(\varsigma)} \left(\frac{\xi_k - \xi}{\xi_k - z}\right)^s \frac{d\xi}{\xi - z} = -\sum_{j=1}^{s} \frac{1}{j} Re \left(\frac{\xi_k - \zeta}{\xi_k - z}\right)^j \\ & +Re \int_{\ell(\varsigma)} \left(\frac{\xi_k - \xi}{\xi_k - z}\right)^s \frac{d\xi}{\xi - z}. \end{aligned} \tag{17}$$

Using $(13), (14), (16)$ and (17)

$$\begin{aligned} &\left| Re\{\left(\frac{\xi_k - \zeta}{\xi_k - z}\right)^j - \left(\xi_k - \varsigma\right)^j \left(K_{[n/j]}\left(\xi_k, z\right)\right)^j\}\right| \\ &\leq |\xi_k - \varsigma|^j \left|\frac{1}{\xi_k - \varsigma} - K_{[n/j]}\left(\xi_k, z\right)\right| \sum_{p=0}^{j-1} \frac{1}{|\xi_k - z|^p} \left|K_{[n/j]}\left(\xi_k, z\right)\right|^{j-p-1} \\ &\preccurlyeq \left|\frac{\rho_{1+\frac{1}{n}}\left(\varsigma_k\right)}{\xi_k - z}\right|^m \preccurlyeq \left|\frac{\rho_{1+\frac{1}{n}}\left(z\right)}{\xi_k - z}\right|^p, \end{aligned} \tag{18}$$

where $m = m\left(p, \Gamma\right)$ is a sufficiently large number. Moreover, at sufficiently large $s = s\left(p, \Gamma\right)$ and using (4) we have

$$\begin{aligned} \left|Re \textstyle\int_{\ell(\varsigma)} \left(\frac{\xi_k - \xi}{\xi_k - z}\right)^s \frac{d\xi}{\xi - z}\right| &\preccurlyeq \left|\frac{\xi_k - \varsigma}{\xi_k - z}\right|^s \left(1 + \left|\ell n \left|\frac{\xi_k - z}{\varsigma - z}\right|\right|\right) \\ &\preccurlyeq \left|\frac{\rho_{1+\frac{1}{n}}\left(z\right)}{\xi_k - z}\right|^p \left(1 + \left|\ell n \left|\frac{\rho_{1+\frac{1}{n}}\left(z\right)}{\varsigma - z}\right|\right|\right). \end{aligned} \tag{19}$$

By inequalities $(16),(18)$ and (19)we'll get (10). Let $z\in G_n$ and $\varsigma\in\Omega_1\cap ext\Gamma_{1/2}$. Then estimate (10) is proved analogously the previous case. It remains only to note that $\pi_n(\varsigma,z)$ for $\varsigma\in\Omega_1\cap ext\ \Gamma_{1/2}$ is a rational function (see, [6]).

3. The Proof of the Main Result

By Green's formula (see for instance [10, p. 363-367]) for $z\in G_n$ we have

$$\begin{aligned}u_n(z)=&\frac{1}{2\pi}\int_{\Gamma_2}\left[u_n(\varsigma)\frac{\partial}{\partial n_\varsigma^{(1)}}\ell n\left|\varsigma-z\right|-\ell n\left|\varsigma-z\right|\frac{\partial u_n(\varsigma)}{\partial n_\varsigma^{(1)}}\right]|d\varsigma|\\&-\frac{1}{2\pi}\int_{\Gamma_{1/2}}\left[u_n(\varsigma)\frac{\partial}{\partial n_\varsigma^{(1/2)}}\ell n\left|\varsigma-z\right|-\ell n\left|\varsigma-z\right|\frac{\partial u_n(\varsigma)}{\partial n_\varsigma^{(1/2)}}\right]|d\varsigma|\\&+\frac{1}{2\pi}\iint\limits_{\tilde{G_n}}\Delta u_n(\varsigma)\,\ell n\left|\varsigma-z\right|d\sigma_\varsigma,\end{aligned}$$

where $\partial/\partial n_\varsigma^{(i)}$, $i=1,1/2$ is a differentiation operator on exterior normal respectively to the curves Γ_2 and $\Gamma_{1/2}$.

Let's fix arbitrary points $\varsigma_1\in\Gamma_2$ and $\varsigma_2\in\Gamma_{1/2}$ and consider the function

$$\begin{aligned}g_n(z)=&\frac{1}{2\pi}\iint\limits_{G_n^{(2)}}\Delta u_n(\varsigma)\,\ell n\left|\frac{\varsigma-z}{\varsigma_1-z}\right|d\sigma_\varsigma\\&+\frac{1}{2\pi}\iint\limits_{G_n^{(1)}}\Delta u_n(\varsigma)\,\ell n\left|\frac{\varsigma-z}{\varsigma_2-z}\right|d\sigma_\varsigma.\end{aligned}\tag{20}$$

At some fixed vicinity of the curve Γ, $u_n(z)-g_n(z)$ is a harmonic function. By lemma 2 and the known properties of the distance between the continuum and its level lines (see, for instance, [11, p. 181]) this difference is bounded above by cn^4.

Then the function $u_n(z)-g_n(z)$ by Bernstein - Walsh theorem for $z\in G_n$ may be approximated by the sequence of harmonic functions with geometric progression velocity.

Consequently, for $z\in G_n$ we have

$$\left|u_n(z)-g_n(z)-\pi_n(z)\right|\preccurlyeq n^4q^4\preccurlyeq\omega\left[\rho_{1/n}(z)\right].$$

We'll give the desired rational function by the formula

$$\begin{aligned}T_n(z)=&\frac{1}{2\pi}\iint\limits_{G_n^{(2)}}\Delta u_n(\varsigma)\,\pi_n(\varsigma,z)\,d\sigma_\varsigma\\&+\frac{1}{2\pi}\iint\limits_{G_n^{(1)}}\Delta u_n(\varsigma)\,\pi_n(\varsigma,z)\,d\sigma_\varsigma,\end{aligned}\tag{21}$$

here $\pi_n(\varsigma, z)$ is a rational harmonic core from lemma 3.

By (20) and (21) we have

$$|g_n(z) - T_n(z)| = \Big| \frac{1}{2\pi} \iint\limits_{G_n^{(2)}} \Delta u_n(\varsigma) \left[\ell n \left| \frac{\varsigma - z}{\varsigma_1 - z} \right| - \pi_n(\varsigma, z) \right] d\sigma_\varsigma + \frac{1}{2\pi} \iint\limits_{G_n^{(1)}} \Delta u_n(\varsigma) \left[\ell n \left| \frac{\varsigma - z}{\varsigma_2 - z} \right| - \pi_n(\varsigma, z) \right] d\sigma_\varsigma \Big| \tag{22}$$

To prove theorem 1, obviously, it suffices to be convinced in the validity of the inequality

$$|g_n(z) - T_n(z)| \preccurlyeq C\omega\left[\rho_{1/n}(z)\right], z \in G_n \tag{23}$$

Denote

$$\rho = \rho_{1/n}(z), \ A = U(z, \rho/2), B = G_n \setminus A.$$

If $\varsigma \in A$, then by lemma 2 $\Delta u_n(\varsigma) \preccurlyeq \omega(\rho)/\rho^2$. Allowing for this and lemma 3 we have

$$\left| \iint\limits_{A} \Delta u_n(\varsigma) \left[\ell n \left| \frac{\varsigma - z}{\varsigma_1 - z} \right| - \pi_n(\varsigma, z) \right] d\sigma_\varsigma \right| \preccurlyeq \frac{\omega(\rho)}{\rho^2} \left(1 + \ell n \frac{\rho}{|\varsigma - z|} \right) d\sigma_\varsigma \preccurlyeq \omega[\rho] \tag{24}$$

Above we used the easily verifiable inequality

$$\int_0^\rho x \ell n \frac{\rho}{x} dx \le \rho^2/e.$$

Let $\varsigma \in B$. Then by corollary 1 and lemma 2 we get

$$|\Delta u_n(\varsigma)| \preccurlyeq \frac{\omega\left[\rho_{1/n}(z)\right]}{\left[\rho_{1/n}(z)\right]^2} \preccurlyeq \frac{\omega(|\varsigma - z|)}{|\varsigma - z|} \left| \frac{\varsigma - z}{\rho_{1/n}(z)} \right|^2 \frac{1}{|\varsigma - z|} \preccurlyeq \frac{\omega(\rho)|\varsigma - z|^{2/c - 1}}{\rho^{2/c + 1}}. \tag{25}$$

Assume $2/c - 1 = \alpha > 0$. Then, using (25) and lemma 3 we have

$$\left| \iint\limits_{B} \Delta u_n(\varsigma) \left[\ell n \left| \frac{\varsigma - z}{\varsigma_1 - z} \right| - \pi_n(\varsigma, z) \right] d\sigma_\varsigma \right| \preccurlyeq \iint\limits_{B} \frac{\omega(\rho)}{\rho^{\alpha - p + 2}} |\varsigma - z|^\alpha \frac{d\sigma_\varsigma}{|\varsigma - z|^p} \preccurlyeq \omega(\rho), \tag{26}$$

here $p > 2 + \alpha$.

The validity of the inequalities

$$\left| \int\int_{A} \Delta u_n (\varsigma) \left[\ell n \left| \frac{\varsigma - z}{\varsigma_2 - z} \right| - \pi_n (\varsigma, z) \right] d\sigma_\varsigma \right| \preccurlyeq \omega (\rho) ; \tag{27}$$

$$\left| \int\int_{B} \Delta u_n (\varsigma) \left[\ell n \left| \frac{\varsigma - z}{\varsigma_2 - z} \right| - \pi_n (\varsigma, z) \right] d\sigma_\varsigma \right| \preccurlyeq \omega (\rho) \tag{28}$$

is established similarly.

Using equality (22) and estimates $(24), (26), (27)$ and (28) we arrive at inequality (22).

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