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MAPS PRESERVING LIE PRODUCT ON $B(X)$

Xiuping Yu and Fangyan Lu

Abstract. Let X and Y be complex Banach spaces. Let ϕ be a bijection from $B(X)$ onto $B(Y)$ satisfying $\phi([A, B]) = [\phi(A), \phi(B)]$ for all $A, B \in B(X)$. Then $\phi = \psi + \tau$, where ψ is a ring isomorphism or a negative of a ring anti-isomorphism from $B(X)$ onto $B(Y)$, and τ is a map from $B(X)$ into $\mathbb{C}I$ satisfying $\tau([A, B]) = 0$ for all $A, B \in B(X)$.

1. Introduction and the Main Result

Given an associative ring $\mathcal{R}$, one can render it into a Lie ring by defining, for $a, b \in \mathcal{R}$, the Lie product $[a, b]$ to be $ab - ba$. The study of the Lie structure is an active research area in ring theory and operator theory. See [1-3, 5, 9, 12, 15-17, 23-25] and references therein. In this paper we investigate the relationship between the Lie multiplication structure and the addition structure. This is also motivated by the study, in for example [8, 14, 20, 21], of question of when a multiplicative bijection is additive, and by the study, in for example [6, 10, 11, 18, 19], of question of when a bijection preserving Jordan product is additive.

Let X be a complex Banach space. By $B(X)$ we mean the algebra of all bounded linear operators on X. Our main result reads as follows.

Theorem 1.1. *Let X and Y be complex Banach spaces and suppose that X is of dimension > 1. Let ϕ be a bijection from $B(X)$ onto $B(Y)$ satisfying $\phi([A, B]) = [\phi(A), \phi(B)]$ for all $A, B \in B(X)$. Then one of the following holds.*

(1) *$\phi = \psi + \tau$, where ψ is a ring isomorphism from $B(X)$ onto $B(Y)$, and τ is a map from $B(X)$ into $\mathbb{C}I$ satisfying $\tau([A, B]) = 0$ for all $A, B \in B(X)$.*

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(2) $\phi = \psi + \tau$, *where* ψ *is a negative of a ring anti-isomorphism from* $B(X)$ *onto* $B(Y)$*, and* τ *is a map from* $B(X)$ *into* $\mathbb{C}I$ *satisfying* $\tau([A, B]) = 0$ *for all* $A, B \in B(X)$.

The proof of the theorem will be given in Section 3. It should be mentioned that some parts of the proof what follows are essentially due to Hua [7] and Martindale [13]. For the sake of completeness (and also because of the inaccessibility of their papers), we shall reproduce their proofs in some detail when the occasion demands.

2. Preliminaries

Throughout this section, Z is a complex Banach space with the topological dual X^*. For $z \in Z$ and $f \in Z^*$, the operator $z \otimes f$ is defined by $x \mapsto f(x)z$ for $x \in Z$.

Lemma 2.1. *Let* A, B, E, F *be in* $B(Z)$ *and suppose that* E *and* F *are non-zero idempotents. If* $EAETF = ETFBF$ *for all* $T \in B(Z)$*, then* $EAE = \lambda E$ *and* $FBF = \lambda F$ *for some* $\lambda \in \mathbb{C}$*. In particular, if* $EAETF = 0$ *for all* $T \in B(Z)$ *then* $EAE = 0$*; and if* $ETFBF = 0$ *for all* $T \in B(Z)$ *then* $FBF = 0$.

Proof. Fix a functional f in Z^* such that $f(Fz_0) = 1$ for $z_0 \in Z$. Putting $T = Ez \otimes f$ in $EAETF = ETFBF$ and applying the equation to z_0, we get a scalar λ such that $AEz = \lambda Ez$ for all $z \in Z$. So $EAE = \lambda E$. Hence $ETFBF = \lambda ETF$ for all $T \in B(X)$. That is $ET(FBF - \lambda F) = 0$ for all $T \in B(X)$. Since it is well-known that $B(Z)$ is a prime ring, $FBF = \lambda F$.

Following Hua [7], we define an operator A in $B(Z)$ to be an I-operator if $A = B + \lambda I$, where B is an idempotent in $B(Z)$, I is the identity operator on Z and λ is a (complex) scalar. The following characterizes I-operators in $B(Z)$.

Lemma 2.2. *Let* A *be in* $B(Z)$*. Then* A *is an* I*-operator if and only if* $[A, [A, [A, T]]] = [A, T]$ *for all* $T \in B(Z)$.

Proof. The "only if" part is obvious. Now assume that $[A, [A, [A, T]]] = [A, T]$ for all $T \in B(X)$, i.e.,

$$(A^3 - A)T - 3A^2TA + 3ATA^2 - T(A^3 - A) = 0. \tag{2.1}$$

Suppose that A is not a scalar multiple of I. Then there exist a vector $z_0 \in Z$ and a functional $f \in Z^*$ such that $f(z_0) = 0$ and $f(Az_0) = 1$. Putting $T = z \otimes f$ in Eq. (2.1) and applying this equation to z_0, we get two scalars λ and μ such that $A^2z + \lambda Az + \mu z = 0$ for all $z \in Z$. Namely, $A^2 + \lambda A + \mu I = 0$. Translating A by a scalar multiple of I, we get that $A^2 = \gamma I$ for some $\gamma \in \mathbb{C}$. Since A is not a scalar multiple of I, Eq. (2.1) yields $\gamma = \frac{1}{4}$. Now $A + \frac{1}{2}I$ is an idempotent. This completes the proof.

Lemma 2.3. *Suppose that E and F in $B(Z)$ are idempotents and satisfy $EF = FE$. Then the statement that either $EF = 0$ or $(I-E)(I-F) = 0$ is true if and only if $[[E,[E,[T,F]]],F] = [E,[T,F]]$ holds for all $T \in B(X)$.*

Proof. A computation gives

$$\begin{aligned}&[[E,[E,[T,F]]],F] - [E,[T,F]]\\ &= 2(EFT(I-E)(I-F) + (I-E)(I-F)TEF).\end{aligned}$$

Then the necessity is obviously seen, and the fact that $B(Z)$ is a prime ring gives the sufficiency.

3. The Proof of the Theorem

The proof of the theorem will be given in some steps. The main idea is to divide $B(X)$ into the three-by-three block matrix algebra and to identify the behavior of ϕ on each block. We note that this idea is inspired by Martindale [13].

3.1. Elementary Results

We begin with a trivial one.

Lemma 3.1. *We have $\phi(0) = 0$.*

Proof. Indeed, $\phi(0) = \phi([0,0]) = [\phi(0), \phi(0)] = 0$.

We will make a crucial use of the following two results.

Lemma 3.2. *Let $S, A_1, A_2, \ldots, A_n \in B(X)$ and $\lambda \in \mathbb{C}$. Suppose that $\phi(S) = \sum_{i=1}^n \phi(A_i) + \lambda I$. Then for all $T \in B(X)$, we have $\phi([T,S]) = \sum_{i=1}^n \phi([T,A_i])$.*

Proof. Multiplying $\phi(S) = \sum_{i=1}^n \phi(A_i) + \lambda I$ by $\phi(T)$ from both sides separately, we get that

$$\phi(T)\phi(S) = \sum_{i=1}^n \phi(T)\phi(A_i) + \phi(T)\lambda I$$

and

$$\phi(S)\phi(T) = \sum_{i=1}^n \phi(A_i)\phi(T) + \lambda I\phi(T).$$

Then

$$\begin{aligned}\phi([T,S]) &= [\phi(T),\phi(S)] = \phi(T)\phi(S) - \phi(S)\phi(T)\\ &= \sum_{i=1}^{n}\phi(T)\phi(A_i) - \sum_{i=1}^{n}\phi(A_i)\phi(T)\\ &= \sum_{i=1}^{n}(\phi(T)\phi(A_i) - \phi(A_i)\phi(T))\\ &= \sum_{i=1}^{n}([\phi(T),\phi(A_i)]) = \sum_{i=1}^{n}\phi([T,A_i]),\end{aligned}$$

completing the proof.

Lemma 3.3. *Let A be in $B(X)$. Then $\phi(A+\mathbb{C}I) = \phi(A)+\mathbb{C}I$. In particular, $\phi(\mathbb{C}I) = \mathbb{C}I$.*

Proof. Let λ be in $\mathbb{C}$. Since ϕ is surjective, we can choose S from $B(X)$ such that $\phi(S) = \phi(A) + \lambda I$. Then for $T \in B(X)$, making use of Lemma 3.2, we get that

$$\phi([T,S]) = \phi([T,A]).$$

Since ϕ is injective, $[T,S] = [T,A]$. So $T(S-A) = (S-A)T$ for all $T \in B(X)$. By Lemma 2.1, $S - A = \mu I$ for some $\mu \in \mathbb{C}$. Consequently, we have that $\phi^{-1}(\phi(A) + \mathbb{C}I) \subseteq A + \mathbb{C}I$; namely, $\phi(A) + \mathbb{C}I \subseteq \phi(A+\mathbb{C}I)$. Considering ϕ^{-1}, we have that $\phi(A+\mathbb{C}I) \subseteq \phi(A)+\mathbb{C}I$. So $\phi(A+\mathbb{C}I) = \phi(A)+\mathbb{C}I$. In particular, assuming $A = 0$, we get that $\phi(\mathbb{C}I) = \mathbb{C}I$.

3.2. The Assumptions

If X is of dimension 1, then $B(X)$ is commutative and hence so is $B(Y)$. Thus any bijective map from $B(X)$ onto $B(Y)$ can be presented in the form $\psi + \tau$, where ψ is an arbitrary ring isomorphism and τ is a map from $B(X)$ into $B(Y)$. Consequently, the statement (1) is true in this case.

If X is of dimension 2, it follows from the following Proposition 3.4 that Y is also of dimension 2. So, in this case, ϕ is a bijection preserving Lie product from $M_2(\mathbb{C})$ onto itself. By Lemma 2.2, $\phi(e_{ii}) = f_{ii} + \lambda_i I$, $i = 1, 2$. Here f_{11} and f_{22} are commuting idempotents. Hence by Lemma 2.3, either $f_{11}f_{22} = 0$ or $(I - f_{11})(I - f_{22}) = 0$ (cf. Proposition 3.4). If $(I - f_{11})(I - f_{22}) = 0$ then $(I - f_{11}) + (I - f_{22}) = I$ (cf. Lemma 3.5). This in turn implies that $f_{11}f_{22} = 0$. Therefore we always have that $f_{11}f_{22} = 0$ and $f_{11} + f_{22} = I$. So there exists an invertible matrix $T \in M_2$ such that $T\phi(e_{ii})T^{-1} = e_{ii} + \lambda_i I$, $i = 1, 2$. Define $\psi = T\phi T^{-1}$. Then $\psi(\mathbb{C}e_{ij}) = \mathbb{C}e_{ij}$, $1 \le i \ne j \le 2$ (cf. Lemma 3.6) and

$\psi(\mathbb{C}e_{ii}) \subseteq \mathbb{C}e_{ii} + \mathbb{C}I$, $i = 1, 2$ (cf. Lemma 3.7). Using those facts we can prove that the statement (1) holds. We omit details.

Assumption 1. X is of dimension> 2.

Now there exist three non-trivial idempotent operators P_1, P_2, P_3 on X such that $P_1 + P_2 + P_3 = I$ and $P_iP_j = 0$ for all $i \neq j$. For each $i \in \{1, 2, 3\}$, by Lemma 2.2, there exists an idempotent operator Q_i in $B(Y)$ such that $\phi(P_i) - Q_i$ is a scalar multiple of I. Since P_i is non-trivial, it follows from Lemma 3.3 that Q_i is also non-trivial. Therefore, such a Q_i is unique. In the forgoing, we shall fix those P_i and Q_i.

Proposition 3.4. *Either* $Q_iQ_j = 0$ *for all* $i \neq j$*, or* $(I - Q_i)(I - Q_j) = 0$ *for all* $i \neq j$.

Proof. Since any pair of $\{P_1, P_2, P_3\}$ commute, it follows that any pair of $\{Q_1, Q_2, Q_3\}$ commute. Making use of the necessity of Lemma 2.3, $[[P_i, [P_i, [T, P_j]]], P_j] = [P_i, [T, P_j]]$ for all $T \in B(X)$, $i \neq j$. Since ϕ is surjective, it follows that $[[Q_i, [Q_i, [S, Q_j]]], Q_j] = [Q_i, [S, Q_j]]$ for all $S \in B(Y)$. Making use of the sufficiency of Lemma 2.3, either $Q_iQ_j = 0$ or $(I - Q_i)(I - Q_j) = 0$. If $(I - Q_1)(I - Q_2) = (I - Q_1)(I - Q_3) = 0$ but $Q_2Q_3 = 0$, then $I - Q_1 = (I - Q_1)Q_2 = (I - Q_1)(I - Q_3) = 0$. This conflicts with the fact that $Q_1 \neq I$, completing the proof.

In the forgoing, we shall prove the theorem only for one of cases.

Assumption 2. $Q_iQ_j = 0$ for all $i \neq j$.

Under this assumption, we shall show that the statement (1) of the theorem holds. If $(I - Q_i)(I - Q_j) = 0$ for all $i \neq j$, a similar argument establishes the statement (2) of the theorem. We note that there is an easy treatment for this case when Y is reflexive. Suppose that $(I - Q_i)(I - Q_j) = 0$ for all $i \neq j$ and that Y is reflexive. Then the equation $\theta(A) = A^*$ for $A \in B(Y)$ defines an algebraic anti-isomorphism from $B(Y)$ onto $B(Y^*)$. It is easily seen that $\Phi = -\theta(\phi)$ is a bijection preserving Lie product from $B(X)$ onto $B(Y^*)$. Moreover, Q_i' is a unique idempotent such that $\Phi(P_i) - Q_i'$ is a scalar multiple of I and $Q_i'Q_j' = 0$ for all $i \neq j$, where $Q_i' = \theta(I - Q_i)$. Thus, applying the argument what follows, the statement (1) of the theorem holds for Φ. Hence the statement (2) of the theorem holds for $\phi = -\theta^{-1}(\Phi)$.

3.3. The Peirce Decompositions

Let $\mathcal{A}_{ij} = P_iB(X)P_j$, $1 \leq i, j \leq 3$. Then $B(X) = \sum_{i,j=1}^{3} \mathcal{A}_{ij}$ since $P_1 + P_2 + P_3 = I$. This is the Peirce decomposition of $B(X)$. We note that this kind of machinery already proved effective in [6, 10, 11, 14, 18, 19] where serval results are obtained on the additivity of maps which preserve certain product. Similarly, let $\mathcal{B}_{ij} = Q_iB(Y)Q_j$. In the sequel, when writing A_{ij} (or B_{ij}), it indicates that $A_{ij} \in \mathcal{A}_{ij}$ ($B_{ij} \in \mathcal{B}_{ij}$, respectively).

To get $B(Y) = \sum_{i,j=1}^{3} \mathcal{B}_{ij}$, we need the following lemma.

Lemma 3.5. $Q_1 + Q_2 + Q_3 = I$.

Proof. Choose $S = \sum_{i,j=1}^{3} S_{ij} \in B(X)$ such that $\phi(S) = Q_1+Q_2+Q_3$. Then $\phi(S) = \phi(P_1) + \phi(P_2) + \phi(P_3) + \mu I$ for some $\mu \in \mathbb{C}$. Fix an index $i \in \{1, 2, 3\}$. For all $T_{ii} \in \mathcal{A}_{ii}$, by Lemma 3.2,

$$\phi([T_{ii}, S]) = \sum_{j=1}^{3} \phi([T_{ii}, P_j]) = 0.$$

So we have that

$$T_{ii}S - ST_{ii} = 0 \tag{3.2}$$

for all $T_{ii} \in \mathcal{A}_{ii}$. In particular, $P_iS - SP_i = 0$ and therefore $S_{ij} = P_iSP_j = 0$ for each $j \in \{1, 2, 3\}$ with $j \neq i$. Now (3.2) becomes $T_{ii}S_{ii} - S_{ii}T_{ii} = 0$ for all $T_{ii} \in \mathcal{A}_{ii}$. So $S_{ii} = \lambda_iP_i$ for some $\lambda_i \in \mathbb{C}$ by Lemma 2.1. Thus the arbitrariness of $i \in \{1, 2, 3\}$ gives that $S = \lambda_1P_1 + \lambda_2P_2 + \lambda_3P_3$. Since $Q_1 + Q_2 + Q_3$ is idempotent, it follows from Lemma 2.2 that S is an I-operator. So the spectrum of S is contained in $\{\lambda, \lambda + 1\}$ for some $\lambda \in \mathbb{C}$. Namely, $\{\lambda_1, \lambda_2, \lambda_3\} \subseteq \{\lambda, \lambda + 1\}$. We now show that $\lambda_1 = \lambda_2 = \lambda_3$. Otherwise, without loss of generality, we may suppose that $\lambda_1 = \lambda_2 + 1$. Let $T_{12} \in \mathcal{A}_{12}$ be non-zero. Then

$$\phi(T_{12}) = \phi([S, T_{12}]) = \sum_{k=1}^{3} \phi([P_k, T_{12}]) = \phi(T_{12}) + \phi(-T_{12}). \tag{3.3}$$

It follows from the injectivity of ϕ that $-T_{12} = 0$. This contradiction shows that $\lambda_1 = \lambda_2 = \lambda_3$. So S is a scalar multiple of I. Hence by Lemma 3.3 $Q_1+Q_2+Q_3$ is also a scalar multiple of I. Consequently, $Q_1+Q_2+Q_3 = I$ since it is idempotent.

We note that if $(I - Q_i)(I - Q_j) = 0$ for all $i \neq j$ and $\phi(S) = (I - Q_1) + (I - Q_2) + (I - Q_3)$ then Eq. (3.3) becomes

$$\phi(T_{12}) = \phi([S, T_{12}]) = -\sum_{k=1}^{3} \phi([P_k, T_{12}]) = -\phi(T_{12}) - \phi(-T_{12}).$$

Therefore $2\phi(T_{12}) = -\phi(-T_{12})$. Hence $2\phi(-T_{12}) = -\phi(T_{12}) = \frac{1}{2}\phi(-T_{12})$. So $\frac{3}{2}\phi(-T_{12}) = 0$, which is also a contradiction. This is the main difference between the proof for the case $Q_iQ_j = 0$ for all $i \neq j$ and the one for the case $(I - Q_i)(I - Q_j) = 0$ for all $i \neq j$.

3.4. The Behavior of ϕ on $\mathcal{A}_{ij}$

Making use of Lemma 3.3, we see that P_i is a unique idempotent such that $\phi^{-1}(Q_i) - P_i$ is a scalar multiple of I. So the behavior of ϕ acting on $\mathcal{A}_{ij}$ and the behavior of ϕ^{-1} acting on $\mathcal{B}_{ij}$ are same.

Lemma 3.6. $\phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}, i \neq j$.

Proof. Let $A \in \mathcal{A}_{ij}$. Then

$$\phi(A) = \phi([A, P_j]) = [\phi(A), \phi(P_j)] = [\phi(A), Q_j].$$

So $Q_j\phi(A)Q_i = 0$. Hence

$$\begin{aligned}\phi(A) &= \phi([P_i, [A, P_j]]) = [Q_i, [\phi(A), Q_j]] \\ &= Q_i\phi(A)Q_j + Q_j\phi(A)Q_i = Q_i\phi(A)Q_j \in \mathcal{B}_{ij}.\end{aligned}$$

Therefore, $\phi(\mathcal{A}_{ij}) \subseteq \mathcal{B}_{ij}$. Considering ϕ^{-1}, we get $\phi(\mathcal{A}_{ij}) \supseteq \mathcal{B}_{ij}$, completing the proof.

Lemma 3.7. $\phi(\mathcal{A}_{ii}) \subseteq \mathcal{B}_{ii} + \mathbb{C}I$ *for each* $i \in \{1, 2, 3\}$. *Moreover, for each* $B_{ii} \in \mathcal{B}_{ii}$ *there is* $A_{ii} \in \mathcal{A}_{ii}$ *such that* $\phi(A_{ii}) = B_{ii} + \lambda I$ *for some* $\lambda \in \mathbb{C}$.

Proof. We only consider the case $i = 1$. The proof for the other cases is similar.

Let A be in $\mathcal{A}_{11}$ and write $\phi(A) = \sum_{i,j=1}^{3} B_{ij}$ corresponding to the decomposition of $B(Y)$. Then for all $j \in \{1, 2, 3\}$, we have that

$$0 = \phi([A, P_j]) = [\phi(A), Q_j] = \sum_{i \neq j}(B_{ij} - B_{ji}).$$

From this, we get that $B_{ij} = 0$ for all $i \neq j$. Thus $\phi(A) = B_{11} + B_{22} + B_{33}$. For $R_{23} \in \mathcal{B}_{23}$, by Lemma 3.6 there exists $T_{23} \in \mathcal{A}_{23}$ such that $\phi(T_{23}) = R_{23}$. Then

$$B_{22}R_{23} - R_{23}B_{33} = [\sum_{i=1}^{3} B_{ii}, R_{23}] = [\phi(A), \phi(T_{23})] = \phi([A, T_{23}]) = 0.$$

So, by Lemma 2.1, $B_{22} = \lambda Q_2$ and $B_{33} = \lambda Q_3$ for some $\lambda \in \mathbb{C}$. Thus

$$\phi(A) = B_{11} + \lambda(Q_2 + Q_3) = B_{11} - \lambda Q_1 + \lambda I.$$

Therefore $\phi(\mathcal{A}_{11}) \subseteq \mathcal{B}_{11} + \mathbb{C}I$.

Now let $B_{ii} \in \mathcal{B}_{ii}$. Applying the preceding result to ϕ^{-1}, there exist an $A_{ii} \in \mathcal{A}_{ii}$ and a scalar $\lambda \in \mathbb{C}$ such that $\phi(A_{ii} + \lambda I) = B_{ii}$. By Lemma 3.3, we can suppose that $\phi(A_{ii} + \lambda I) = \phi(A_{ii}) + \mu I$ for some $\mu \in \mathbb{C}$. Then $\phi(A_{ii}) = B_{ii} - \mu I$, completing the proof.

3.5. The Definition of ψ

By Lemma 3.7, for $A_{ii} \in \mathcal{A}_{ii}$ with $i \in \{1, 2, 3\}$, there exists a unique scalar $f_i(A_{ii})$ such that $\phi(A_{ii}) - f_i(A_{ii})I \in \mathcal{B}_{ii}$. Now for $\sum_{i,j=1}^{3} A_{ij} \in \sum_{i,j=1}^{3} \mathcal{A}_{ij}$, we define

$$\psi(\sum_{i,j=1}^{3} A_{ij}) = \sum_{i,j=1}^{3} \phi(A_{ij}) - \sum_{k=1}^{3} f_k(A_{kk})I.$$

Lemma 3.8. *We have that*

(*i*) $\psi(A_{ij}) = \phi(A_{ij})$, $i \neq j$*;*

(*ii*) $\psi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}$ *for all* $i, j \in \{1, 2, 3\}$*;*

(*iii*) $\psi(\sum_{i,j=1}^{3} A_{ij}) = \sum_{i,j=1}^{3} \psi(A_{ij})$*;*

(*iv*) ψ *is surjective.*

Proof. If $i \neq j$, $\psi(A_{ij}) = \phi(A_{ij})$ by the definition, and hence $\psi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}$ by Lemma 3.6. By the definition again, $\psi(A_{ii}) = \phi(A_{ii}) - f_i(A_{ii})$. So $\psi(\mathcal{A}_{ii}) = \mathcal{B}_{ii}$ by Lemma 3.7 and

$$\psi(\sum_{i,j=1}^{3} A_{ij}) = \sum_{i=1}^{3}(\phi(A_{ii}) - f_i(A_{ii})I) + \sum_{i \neq j}^{3} \phi(A_{ij}) = \sum_{i,j=1}^{3} \psi(A_{ij}).$$

So far, we have proved the former three parts. Now the last part is an easy consequence of parts (ii) and (iii).

3.6. The Additivity of ψ

We begin with the " weak additivity " of ϕ on each row.

Lemma 3.9. *Let* $k \in \{1, 2, 3\}$ *and* $A_{kj} \in \mathcal{A}_{kj}$ *for* $j = 1, 2, 3$. *Then* $\phi^{-1}(\sum_{j=1}^{3} \phi(A_{kj})) \in \sum_{j=1}^{3} A_{kj} + \mathbb{C}I$.

Proof. Choose $S \in B(X)$ such that

$$\phi(S) = \phi(A_{11}) + \phi(A_{12}) + \phi(A_{13}). \tag{3.4}$$

Then for $T_{22} \in \mathcal{A}_{22}$, we have that $\phi([T_{22}, S]) = \phi(-A_{12}T_{22})$. So $S_{12}T_{22} = A_{12}T_{22}$ and $T_{22}S_{22} - S_{22}T_{22} = T_{22}S_{21} = T_{22}S_{23} = S_{32}T_{22} = 0$. Therefore $S_{12} = A_{12}$, $S_{21} = S_{32} = S_{23} = 0$ and $S_{22} = \lambda_2 P_2$ for some $\lambda_2 \in \mathbb{C}$.

For $T_{33} \in \mathcal{A}_{33}$, by Eq. (3.4) and Lemma 3.2, we have that $\phi([T_{33}, S]) = \phi(-A_{13}T_{33})$. So $S_{13}T_{33} = A_{13}T_{33}$ and $T_{33}S_{31} = T_{33}S_{33} - S_{33}T_{33} = 0$. It follows that $S_{13} = A_{13}$, $S_{31} = 0$ and $S_{33} = \lambda_3 P_3$ for some $\lambda_3 \in \mathbb{C}$.

For $T_{12} \in \mathcal{A}_{12}$, $\phi([T_{12}, S]) = \phi(-A_{11}T_{12})$. So $T_{12}S_{22} - S_{11}T_{12} = -A_{11}T_{12}$. Hence $S_{11}T_{12} = (A_{11} + \lambda_2 I)T_{12}$. From this we see that $S_{11} = A_{11} + \lambda_2 P_1$. Similarly, $S_{11} = A_{11} + \lambda_3 P_1$. Consequently, $\lambda_2 = \lambda_3$ and $S = A_{11} + A_{12} + A_{13} + \lambda_2 I$, completing the proof.

Lemma 3.10. *ψ is additive on $\mathcal{A}_{ij}$ for $1 \le i \ne j \le 3$.*

Proof. Let A_{12} and B_{12} be in $\mathcal{A}_{12}$. Making use of the above lemma, we see that the following equalities

$$\begin{aligned}
&\psi(A_{12}) + \psi(B_{12}) = \phi(A_{12}) + \phi(B_{12})\\
&= [Q_1 + \phi(A_{12}), Q_2 + \phi(B_{12})]\\
&= [\phi(P_1) + \phi(A_{12}), \phi(P_2) + \phi(B_{12})]\\
&= [\phi(P_1 + A_{12} + \mu_1 I), \phi(P_2 + B_{12} + \mu_2 I)]\\
&= \phi([P_1 + A_{12} + \mu_1 I, P_2 + B_{12} + \mu_2 I)])\\
&= \phi(A_{12} + B_{12}) = \psi(A_{12} + B_{12}).
\end{aligned}$$

hold true.

Lemma 3.11. *ψ is additive on $\mathcal{A}_{ii}$, $i = 1, 2, 3$.*

Proof. For clarify of exposition, we assume that $i = 1$. Let A_{11} and B_{11} be in $\mathcal{A}_{11}$. Choose $S \in B(X)$ such that $\phi(S) = \psi(A_{11}) + \psi(B_{11})$. Then

$$\phi(S) = \phi(A_{11}) + \phi(B_{11}) + \lambda I, \tag{3.5}$$

where $\lambda = f_1(A_{11}) + f_1(B_{11})$.

For $T_{kk} \in \mathcal{A}_{kk}$ with $k \in \{2, 3\}$, by Lemma 3.2, we have that $\phi([T_{kk}, S]) = 0$ and then $[T_{kk}, S] = 0$. Therefore, $S_{ij} = 0$ for all $1 \le i \ne j \le 3$, $S_{22} = \lambda_2 P_2$ and $S_{33} = \lambda_3 P_3$ for some $\lambda_2, \lambda_3 \in \mathbb{C}$.

For $T_{1k} \in \mathcal{A}_{1k}$ with $k \in \{2, 3\}$, by Eq. (3.5) and Lemmas 3.2 and 3.10, we have that

$$\phi([S, T_{1k}]) = \phi(A_{11}T_{1k}) + \phi(B_{11}T_{1k}) = \phi((A_{11} + B_{11})T_{1k}).$$

So $[S, T_{1k}] = (A_{11} + B_{11})T_{1k}$. In particular, $S_{11}T_{1k} - T_{1k}S_{kk} = (A_{11} + B_{11})T_{1k}$. Since $S_{kk} = \lambda_k P_k$, it follows that $S_{11}T_{1k} = (A_{11} + B_{11} + \lambda_k I)T_{1k}$. Hence $S_{11} = A_{11} + B_{11} + \lambda_k P_1$ for each $k \in \{2, 3\}$. So $\lambda_2 = \lambda_3$ and then $S = A_{11} + B_{11} + \lambda_2 I$. Now

$$\begin{aligned} &\psi(A_{11}) + \psi(B_{11}) = \phi(A_{11} + B_{11} + \lambda I) \\ &= \phi(A_{11} + B_{11}) + \mu I = \psi(A_{11} + B_{11}) + f_1(A_{11} + B_{11})I + \mu I. \end{aligned}$$

Since $\psi(A_{11}) + \psi(B_{11}) - \psi(A_{11} + B_{11})$ is in $\mathcal{A}_{11}$, it follows that $f_1(A_{11} + B_{11})I + \mu I = 0$. Consequently, $\psi(A_{11} + B_{11}) = \psi(A_{11}) + \psi(B_{11})$, completing the proof.

Proposition 3.12. *ψ is additive .*

Proof. Let $A = \sum_{i,j=1}^{3} A_{ij}$ and $B = \sum_{i,j=1}^{3} B_{ij}$ be in $B(X)$. Then Lemmas 3.8, 3.10 and 3.11 are all used in seeing

$$\begin{aligned} &\psi(A + B) = \psi(\sum_{i,j=1}^{3}(A_{ij} + B_{ij})) = \sum_{i,j=1}^{3}\psi(A_{ij} + B_{ij}) \\ &= \sum_{i,j=1}^{3}(\psi(A_{ij}) + \psi(B_{ij})) = \psi(\sum_{i,j=1}^{3} A_{ij}) + \psi(\sum_{i,j=1}^{3} B_{ij}) = \psi(A) + \psi(B) \end{aligned}$$

hold true.

3.7. The Definition of τ

For $A \in B(X)$, we (have to) define $\tau(A) = \phi(A) - \psi(A)$. Then $\tau(A_{ij}) = f_i(A_{ii})I$ if $i = j$ and 0 otherwise. However, to see that $\tau(A)$ lies in $\mathbb{C}I$ for all $A \in B(X)$, we need the following lemma.

Lemma 3.13. *The difference of $\phi(\sum_{i,j=1}^{3} A_{ij})$ and $\sum_{i,j=1}^{3}\phi(A_{ij})$ is a scalar multiple of I for each $\sum_{i,j=1}^{3} A_{ij} \in B(X)$.*

Proof. Let $\sum_{i,j=1}^{3} A_{ij} \in B(X)$ and choose $S = \sum_{i,j=1}^{3} S_{ij}$ from $B(X)$ such that $\phi(S) = \sum_{i,j=1}^{3}\phi(A_{ij})$. Then by Lemma 3.2 and Proposition 3.12, we have that

$$\begin{aligned} \phi([P_1, S]) &= \phi(A_{12}) + \phi(A_{13}) + \phi(-A_{21}) + \phi(-A_{31}) \\ &= \phi(A_{12} + A_{13} - A_{21} - A_{31}). \end{aligned} \tag{3.6}$$

So $[P_1, S] = A_{12} + A_{13} - A_{21} - A_{31}$. From this we see that $S_{12} = A_{12}, S_{13} = A_{13}$, $S_{21} = A_{21}$ and $S_{31} = A_{31}$. Symmetrically, we have that $S_{23} = A_{23}$ and $S_{32} = A_{32}$.

For $T_{12} \in \mathcal{A}_{12}$, we have that

$$\phi([T_{11}, [T_{12}, S]]) = \sum_{i,j=1} \phi([T_{11}, [T_{12}, A_{ij}]])$$
$$= \phi([T_{11}, [T_{12}, A_{21}]]) + \phi(T_{11}T_{12}A_{22}) + \phi(-T_{11}A_{11}T_{12}).$$

Making use of Lemma 3.9, we see that $[T_{11}, [T_{12}, S]] = [T_{11}, [T_{12}, A_{21}]] + T_{11}T_{12}A_{22} - T_{11}A_{11}T_{12} + \lambda I$ for some $\lambda \in \mathbb{C}$. Hence $T_{11}T_{12}S_{22} - T_{11}S_{11}T_{12} = T_{11}T_{12}A_{22} - T_{11}A_{11}T_{12}$, and hence $T_{12}S_{22} - S_{11}T_{12} = T_{12}A_{22} - A_{11}T_{12}$. Namely, $T_{12}(S_{22} - A_{22}) = (S_{11} - A_{11})T_{12}$. By Lemma 2.1, $S_{11} = A_{11} + \mu P_1$ and $S_{22} = A_{22} + \mu P_2$ for some $\mu \in \mathbb{C}$. Symmetrically, there exists a scalar γ such that $S_{11} = A_{11} + \gamma P_1$ and $S_{33} = A_{33} + \gamma P_3$. Consequently, $\mu = \gamma$ and $S = \sum_{i,j=1}^{3} A_{ij} + \mu I$. Lemma 3.3 applies, completing the proof.

3.8. The Multiplicativity of ψ

Lemma 3.14. *Let $A_{ik} \in \mathcal{A}_{ik}$ and $B_{kj} \in \mathcal{A}_{kj}$, $i \neq j$. Then $\psi(A_{ik}B_{kj}) = \psi(A_{ik})\psi(B_{kj})$.*

Proof. Since $\psi(A_{pq}) \in \mathcal{B}_{pq}$, we see the following equalities

$$\begin{aligned} \psi(A_{ik}B_{kj}) &= \phi(A_{ik}B_{kj}) = \phi([A_{ik}, B_{kj}]) \\ &= [\phi(A_{ik}), \phi(B_{kj})] = [\psi(A_{ik}), \psi(B_{kj})] \\ &= \psi(A_{ik})\psi(B_{kj}) \end{aligned}$$

hold true.

Lemma 3.15. *Let A_{ii}, B_{ii} be in $\mathcal{A}_{ii}$, $i \in \{1, 2, 3\}$. Then $\psi(A_{ii}B_{ii}) = \psi(A_{ii})\psi(B_{ii})$.*

Proof. Let j be in $\{1, 2, 3\}$ such that $i \neq j$. Making use of the above lemma, we have, for $T_{ij} \in \mathcal{A}_{ij}$, that

$$\psi(A_{ii}B_{ii}T_{ij}) = \psi(A_{ii}B_{ii})\psi(T_{ij}),$$

which is also equal to

$$\psi(A_{ii}B_{ii}T_{ij}) = \psi(A_{ii})\psi(B_{ii}T_{ij}) = \psi(A_{ii})\psi(B_{ii})\psi(T_{ij}).$$

So $\psi(A_{ii}B_{ii})\psi(T_{ij}) = \psi(A_{ii})\psi(B_{ii})\psi(T_{ij})$. Since $\psi(\mathcal{A}_{ij}) = \phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}$, it follows from Lemma 2.1 that $\psi(A_{ii}B_{ii}) = \psi(A_{ii})\psi(B_{ii})$.

Lemma 3.16. *Let $A_{ij} \in \mathcal{A}_{ij}$ and $B_{ji} \in \mathcal{A}_{ji}$, $i \neq j$. Then $\psi(A_{ij}B_{ji}) = \psi(A_{ij})\psi(B_{ji})$.*

Proof. According to the definition of τ and ψ and making use of the additivity of ψ,

$$\begin{aligned}\tau([A_{ij}, B_{ji}]) &= \phi([A_{ij}, B_{ji}]) - \psi([A_{ij}, B_{ji}]) \\ &= [\phi(A_{ij}), \phi(B_{ji})] - \psi([A_{ij}, B_{ji}]) \\ &= [\psi(A_{ij}), \psi(B_{ji})] - \psi([A_{ij}, B_{ji}]) \\ &= \psi(A_{ij})\psi(B_{ji}) - \psi(B_{ji})\psi(A_{ij}) - \psi(A_{ij}B_{ji}) + \psi(B_{ji}A_{ij}) \\ &\in \mathcal{B}_{ii} + \mathcal{B}_{jj}.\end{aligned}$$

So $\tau([A_{ij}, B_{ji}]) = 0$ and hence $\psi(A_{ij}B_{ji}) = \psi(A_{ij})\psi(B_{ji})$.

Proposition 3.16. *ψ is multiplicative.*

Proof. Let A and B be in $B(X)$. Write $A = \sum_{i,j=1}^{3} A_{ij}$ and $B = \sum_{i,j=1}^{3} B_{ij}$ corresponding to the Peirce decomposition of $B(X)$. Since ψ is additive and $\psi(\mathcal{A}_{ij}) \subseteq \mathcal{B}_{ij}$, we have that

$$\psi(AB) = \sum_{i,k,j=1}^{3} \psi(A_{ik}B_{kj})$$

and

$$\psi(A)\psi(B) = \sum_{i,k,j=1}^{3} \psi(A_{ik})\psi(B_{kj}).$$

So in order to prove $\psi(AB) = \psi(A)\psi(B)$, it suffices to show $\psi(A_{ik}B_{kj}) = \psi(A_{ik})\psi(B_{kj})$ for all $1 \le i, j, k \le 3$. But those equalities are assured by Lemmas 3.14-3.16. The proof is complete.

3.9. The Remaining Proof

We will complete our proof by showing that ψ is injective and that τ vanishes on commutators.

Suppose that $\psi(A) = 0$ for $A \in B(Y)$. Then for $i \in \{1, 2, 3\}$, $\psi(AB) = \psi(A)\psi(B) = 0$ and so $\phi(AP_i) = \tau(AP_i) \in \mathbb{C}I$. It follows from Lemma 3.3 that $AP_i \in \mathbb{C}I$. This implies that $AP_i = 0$ for $i \in \{1, 2, 3\}$. Hence $A = 0$ since $P_1 + P_2 + P_3 = I$.

Let A and B be in $B(X)$. By the definition,

$$\tau([A, B]) = \phi([A, B]) - \psi([A, B]) = [\psi(A), \psi(B)] - \psi([A, B]).$$

Note that ψ is actually a Lie isomorphism. It follows that $\tau([A, B]) = 0$.

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Xiuping Yu and Fangyan Lu
Department of Mathematics,
Suzhou University,
Suzhou 215006,
P. R. China
E-mail: xpyu80@126.com
fylu@suda.edu.cn