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GENERALIZED IMPLICIT HYBRID PROJECTION-PROXIMAL POINT ALGORITHM FOR MAXIMAL MONOTONE OPERATORS IN HILBERT SPACE

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Abstract. In this paper, we introduce a generalized implicit hybrid projection-proximal point algorithm for finding zeros of a maximal monotone operator in a Hilbert space setting. The global convergence of the method for the weak topology under appropriate assumptions on the algorithm parameters is established.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A set-valued operator $A : H \to 2^H$ is called maximal monotone if A is monotone, i.e., $\forall x, y \in H,\ \forall v \in A(x),\ \forall w \in A(y),\ \langle v - w, x - y \rangle \geq 0$, and the graph $\mathrm{Gr}A = \{(x, v) \in H \times H : v \in A(x)\}$ is not properly contained in the graph of any other monotone operators. We deal in this paper with iterative methods for finding zeros of maximal monotone operators in H; i.e., given a maximal monotone operator $A : H \to 2^H$, find $x \in H$ such that

$$0 \in A(x). \tag{1.1}$$

A classical method to solve the problem $0 \in A(x)$ is the proximal point algorithm (PPA) which was proposed and studied in [5, 6]. The PPA generates a sequence $\{x^k\} \subset H$ by the successive approximation scheme

$$x^{k+1} = x^k - \lambda_k v^k, \quad v^k \in A(x^{k+1}), \quad k = 0, 1, ...,$$

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where $\{\lambda_k\}$ is a sequence of positive regularization parameters. It is easy to see that the PPA is equivalent to the following iteration:

$$\text{(PPA)} \qquad x^{k+1} = J^A_{\lambda_k}(x^k),$$

where the single-valued function $J^A_\lambda := (I + \lambda A)^{-1} : H \to H$ is the resolvent of A of parameter λ [7]. We have the following characterization.

$$(1.2) \qquad J^A_\lambda(x) = x \text{ if and only if } 0 \in A(x).$$

See, e.g., [3]. Given variable parameters $\lambda_k > 0$ and $\alpha_k \in [0, 1)$, the following inertial type iteration was proposed in [1]:

$$\text{(IPPA)} \qquad x^{k+1} = J^A_{\lambda_k}(x^k + \alpha_k(x^k - x^{k-1})),$$

under the conditions:

$$(1.3) \qquad \lambda := \inf_{k\geq 0} \lambda_k > 0,$$

$$(1.4) \qquad \forall k \in \mathrm{N},\ \alpha_k \in [0, 1) \quad \text{and} \quad \alpha := \sup_{k\geq 0} \alpha_k < 1,$$

$$(1.5) \qquad \sum \alpha_k \|x^k - x^{k-1}\|^2 < \infty.$$

We also recall the relaxed proximal point algorithm which was proposed in [4]:

$$\text{(RPPA)} \qquad x^{k+1} = [(1 - \rho_k)I + \rho_k J^A_{\lambda_k}](x^k),$$

where $\rho_k \in (0, 2)$ is a relaxation factor satisfying

$$(1.6) \qquad R_1 := \inf_{k\geq 0} \rho_k > 0 \quad \text{and} \quad R_2 := \sup_{k\geq 0} \rho_k < 2.$$

Recently, Alvarez [2] constructed an inexact relaxed and inertial hybrid projection-proximal point algorithm for which weak convergence was proved under conditions (1.3)-(1.6) and additional conditions on α_k are given in order to ensure (1.5) a priori.

Motivated by the work in [2], the first aim in this paper is to introduce a generalized, relaxed and inertial hybrid projection-proximal point algorithm which improves, extends and unifies [2, Algorithm 1.1] and for which weak convergence is proved under the conditions more general than those in [2, Theorem 1.1]. The second goal is to construct a generalized inexact version of the RIPPA which includes the more standard inexact version of the RIPPA in [2, Theorem 1.2] as a special case, and for which weak convergence holds under the conditions more general than those in [2, Theorem 1.2].

This paper is organized as follows. Section 2 introduces a generalized, relaxed and inertial hybrid projection-proximal point algorithm for which weak convergence is proved under very general conditions. Next a generalized inexact version of the RIPPA is considered in Section 3 for which weak convergence holds under appropriate summability conditions on the errors.

2. Generalized Implicit Hybrid Projection-proximal Point Algorithm

We first introduce the following generalized relaxed and inertial hybrid projection-proximal point algorithm.

Algorithm 2.1. Let $\sigma \in [0,1)$ be a fixed relative error tolerance.

Step 1. Given $x^k, x^{k-1} \in H,\ \lambda_k > 0,\ \alpha_k, \beta_k \in [0,1)$, and $\rho_k \in (0,2)$, find $u^k \in H$ such that

$$\tag{2.1} \begin{cases} \tilde{x}^k &= J^A_{\lambda_k}(x^k + e^k), \\ y^k &= \beta_k \tilde{x}^k + (1-\beta_k)[x^k + \alpha_k(x^k - x^{k-1})], \\ \lambda_k v^k &= u^k + \tilde{\eta}^k \text{ for some } v^k \in \rho_k A(y^k - u^k/\rho_k) \end{cases}$$

where the residuals $e^k, \tilde{\eta}^k \in H$ satisfy

$$\tag{2.2} \begin{cases} \|e^k\| \le \mu_k \|\tilde{x}^k - x^k\| \text{ with } \lim_{k\to\infty} \mu_k = 0, \\ \|\tilde{\eta}^k\| \le \sigma \max\{\|u^k\|, \lambda_k \|v^k\|\}. \end{cases}$$

Step 2. If $\beta_k = 0$ and $v^k = 0$, then set $x^n := y^k$ for all $n \ge k+1$ and stop. Otherwise:

(i) Let $P_k : H \to H$ be the orthogonal projection operator onto the hyperplane

$$\tag{2.3} H_k = \{x \in H : \langle v^k, x - y^k \rangle = -\langle v^k, u^k \rangle / \rho_k\}.$$

(ii) Set

$$\tag{2.4} x^{k+1} = y^k + \rho_k(P_k y^k - y^k) = y^k - \frac{\langle v^k, u^k \rangle}{\|v^k\|^2} v^k.$$

(iii) Let $k \leftarrow k+1$ and return to Step 1.

Remark 2.1. (i) If we take $\beta_k = 0$, $u^k = y^k - z^k$ and

$$\tilde{\eta}^k = \lambda_k \eta^k \quad \text{with } \|\eta^k\| \le \sigma \max\{\|z^k - y^k\|/\lambda_k, \|v^k\|\},$$

then Algorithm 2.1 reduces to [2, Algorithm 1.1]. (ii) Observe that (2.1) amounts to

$$u^k = \rho_k(y^k - J^A_{\lambda_k}(y^k + \tilde{\eta}^k/\rho_k)).$$

Indeed the latter is equivalent to $y^k + \tilde{\eta}^k/\rho_k \in (I + \lambda_k A)(y^k - u^k/\rho_k)$ which can be written as $u^k/\rho_k + \tilde{\eta}^k/\rho_k \in \lambda_k A(y^k - u^k/\rho_k)$ which is exactly (2.1). Thus it is clear that the algorithm described above is well defined.

In order to prove weak convergence of Algorithm 2.1, we need the following lemmas.

Lemma 2.1. [2, Lemma 2.1]. *Let* $\sigma \in [0, 1)$. *If* $v = u + \eta$ *with* $\|\eta\| \le \sigma \max\{\|u\|, \|v\|\}$, *then*

(*i*) $\|v\| \le \|u\|/(1 - \sigma)$;

(*ii*) $\langle v, u\rangle \ge (1 - \sigma)\|u\|\|v\|$.

Lemma 2.2. [2, Lemma 2.3]. *Let* $\varphi_k \ge 0$ *and* $\delta_k \ge 0$ *be such that*

$$\varphi_{k+1} \le \varphi_k + \alpha_k(\varphi_k - \varphi_{k-1}) + \delta_k$$

with $\sum \delta_k < \infty$, *and* $0 \le \alpha_k \le \alpha < 1$. *Then the following hold:*

(*i*) $\sum[\varphi_k - \varphi_{k-1}]_+ < \infty$, *where* $[t]_+ := \max\{t, 0\}$;

(*ii*) *There exists* $\varphi^* \ge 0$ *such that* $\lim_{k\to\infty} \varphi_k = \varphi^*$.

Lemma 2.3. [2, Lemma 2.4]. *Let* H *be a Hilbert space and* $\{x^k\}$ *a sequence such that there exists a nonempty set* $S \subset H$ *satisfying the following conditions:*

(*a*) *For every* $\bar{x} \in S$, $\lim_{k\to\infty} \|x^k - \bar{x}\|$ *exists;*

(*b*) *If* $x^{k_j} \rightharpoonup \hat{x}$ *weakly in* H *for a subsequence* $k_j \to \infty$, *then* $\hat{x} \in S$.

Then there exists $x^* \in S$ *such that* $x^k \rightharpoonup x^*$ *weakly in* H *as* $k \to \infty$.

Lemma 2.4. *Let* $\{a_n\}$,$\{b_n\}$ *and* $\{c_n\}$ *be three nonnegative real sequences satisfying the following condition:*

$$(*) \qquad a_{n+1} \le (1 + b_n)a_n + c_n, \quad n \ge n_0$$

for some integer $n_0 \ge 1$, *where* $\sum b_n < \infty$ *and* $\sum c_n < \infty$. *Then the limit* $\lim_{n\to\infty} a_n$ *exists.*

Proof. On account of $(*)$, we derive

$$a_{n+m+1} \le (\prod_{j=n}^{n+m}(1+b_j))(\sum_{i=n}^{n+m} c_i + a_n),$$

for all $m, n \ge n_0$, and hence

$$\limsup_{m\to\infty} a_m \le \exp(\sum_{j=n}^{\infty} b_j)(\sum_{i=n}^{\infty} c_i + a_n).$$

This immediately implies that

$$\limsup_{m\to\infty} a_m \le \liminf_{n\to\infty} a_n;$$

that is, $\lim_{n\to\infty} a_n$ exists. ■

We now state and prove the main result of this section.

Theorem 2.1. Let $\{x^k\} \subset H$ be a bounded sequence generated by (2.1)-(2.4) where $A : H \to 2^H$ is a maximal monotone operator with $S := A^{-1}(\{0\}) \ne \emptyset$, $\sigma \in [0, 1)$ and the parameters α_k and ρ_k satisfy (1.4) and (1.6), respectively. Under (1.5) and $\sum \beta_k \mu_k^2 < \infty$, we have

(i) For all $\bar{x} \in S$, $\|x^k - \bar{x}\|$ is convergent, and

$$\lim_{k\to\infty} \|x^{k+1} - y^k + u^k/\rho_k\| = 0; \tag{2.5}$$

(ii) If λ_k additionally satisfies (1.3), then $\lim_{k\to\infty} \|v^k\| = 0$ and there exists $x^* \in S$ such that $x^k \rightharpoonup x^*$ weakly in H as $k \to \infty$.

Proof. We divide the proof into several steps.

(i) Let x^* be any element of S. Then it follows from (2.1) that

$$\frac{1}{\lambda_k}(x^k - \tilde{x}^k + e^k) \in A(\tilde{x}^k).$$

Since x^* is a root of A, so, $0 \in A(x^*)$. Thus from the monotonicity of A we obtain

$$\langle \frac{1}{\lambda_k}(x^k - \tilde{x}^k + e^k) - 0, \tilde{x}^k - x^* \rangle \ge 0.$$

By the assumption that $\lambda_k > 0$, we immediately have

$$\langle x^k - \tilde{x}^k + e^k, \tilde{x}^k - x^* \rangle \ge 0. \tag{2.6}$$

Note that

$$\|u+v\|^2 = \|u\|^2 - \|v\|^2 + 2\langle v, u+v\rangle, \quad \forall u, v \in H.$$

From (2.6) we get

$$\begin{aligned} \|\tilde{x}^k - x^*\|^2 &= \|x^k - x^*\|^2 - \|\tilde{x}^k - x^k\|^2 + 2\langle \tilde{x}^k - x^k, \tilde{x}^k - x^*\rangle \\ &= \|x^k - x^*\|^2 - \|\tilde{x}^k - x^k\|^2 + 2\langle e^k, \tilde{x}^k - x^*\rangle \\ &\quad -2\langle x^k - \tilde{x}^k + e^k, \tilde{x}^k - x^*\rangle \\ &\le \|x^k - x^*\|^2 - \|\tilde{x}^k - x^k\|^2 + 2\langle e^k, \tilde{x}^k - x^*\rangle. \end{aligned} \tag{2.7}$$

(ii) For $\mu_k > 0$, using the Cauchy-Schwartz inequality we have

$$2\langle e^k, \tilde{x}^k - x^*\rangle \le \frac{1}{2\mu_k^2}\|e^k\|^2 + 2\mu_k^2\|\tilde{x}^k - x^*\|^2. \tag{2.8}$$

Since $\lim_{k\to\infty}\mu_k = 0$, there exists an integer $N_0 \ge 0$ such that for all $n \ge N_0$, $1 - 2\mu_k^2 > 0$. Substituting (2.8) in (2.7) and utilizing (2.2), we obtain

$$\begin{aligned} \|\tilde{x}^k - x^*\|^2 &\le \|x^k - x^*\|^2 - \|\tilde{x}^k - x^k\|^2 + \tfrac{1}{2\mu_k^2}\|e^k\|^2 + 2\mu_k^2\|\tilde{x}^k - x^*\|^2 \\ &\le \|x^k - x^*\|^2 - \|\tilde{x}^k - x^k\|^2 + \tfrac{1}{2}\|\tilde{x}^k - x^k\|^2 + 2\mu_k^2\|\tilde{x}^k - x^*\|^2 \\ &= \|x^k - x^*\|^2 - \tfrac{1}{2}\|\tilde{x}^k - x^k\|^2 + 2\mu_k^2\|\tilde{x}^k - x^*\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} \|\tilde{x}^k - x^*\|^2 &\le \left(1 + \frac{2\mu_k^2}{1-2\mu_k^2}\right)\|x^k - x^*\|^2 - \frac{1}{2(1-2\mu_k^2)}\|\tilde{x}^k - x^k\|^2 \\ &\le \left(1 + \frac{2\mu_k^2}{1-2\mu_k^2}\right)\|x^k - x^*\|^2 - \frac{1}{2}\|\tilde{x}^k - x^k\|^2. \end{aligned} \tag{2.10}$$

(iii) Note that for all $x, y \in H$ and $0 \le \lambda \le 1$, there holds the following well-known identity:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Thus from (2.1) and (2.10) we conclude that

$$\begin{aligned} \|y^k - x^*\|^2 &= \|\beta_k(\tilde{x}^k - x^*) + (1-\beta_k)(x^k - x^* + \alpha_k(x^k - x^{k-1}))\|^2 \\ &\le \beta_k\|\tilde{x}^k - x^*\|^2 + (1-\beta_k)\|x^k - x^* + \alpha_k(x^k - x^{k-1})\|^2 \\ &\le \beta_k[(1 + \tfrac{2\mu_k^2}{1-2\mu_k^2})\|x^k - x^*\|^2 - \tfrac{1}{2}\|\tilde{x}^k - x^k\|^2] \\ &\quad +(1-\beta_k)[\|x^k - x^*\|^2 + 2\alpha_k\langle x^k - x^*, x^k - x^{k-1}\rangle \\ &\quad +\alpha_k^2\|x^k - x^{k-1}\|^2]. \end{aligned} \tag{2.11}$$

Observe that

$$\|x^k - x^*\|^2 = \|x^{k-1} - x^*\|^2 + 2\langle x^k - x^*, x^k - x^{k-1}\rangle - \|x^k - x^{k-1}\|^2$$

and hence

$$\|x^k - x^*\|^2 - \|x^{k-1} - x^*\|^2 + \|x^k - x^{k-1}\|^2 = 2\langle x^k - x^*, x^k - x^{k-1}\rangle.$$

This together with (2.11) implies that for all $k \geq N_0$

$$\begin{aligned}
&\|y^k - x^*\|^2 \\
&\leq \beta_k[(1 + \frac{2\mu_k^2}{1-2\mu_k^2})\|x^k - x^*\|^2 - \frac{1}{2}\|\tilde{x}^k - x^k\|^2] \\
&\quad +(1-\beta_k)[\|x^k - x^*\|^2 + \alpha_k(\|x^k - x^*\|^2 - \|x^{k-1} - x^*\|^2 \\
&\quad +\|x^k - x^{k-1}\|^2) + \alpha_k^2\|x^k - x^{k-1}\|^2] \\
&= (1+\frac{2\beta_k\mu_k^2}{1-2\mu_k^2})\|x^k - x^*\|^2 + \alpha_k(1-\beta_k)(\|x^k - x^*\|^2 - \|x^{k-1} - x^*\|^2) \\
&\quad +\alpha_k(1-\beta_k)\|x^k - x^{k-1}\|^2 + \alpha_k^2(1-\beta_k)\|x^k - x^{k-1}\|^2 \\
&\leq (1+\frac{2\beta_k\mu_k^2}{1-2\mu_k^2})\|x^k - x^*\|^2 + \alpha_k(1-\beta_k)(\|x^k - x^*\|^2 - \|x^{k-1} - x^*\|^2) \\
&\quad +2\alpha_k\|x^k - x^{k-1}\|^2.
\end{aligned} \tag{2.12}$$

(iv) From now on assume that $v^k \neq 0$ for all $k \geq 1$; otherwise, the algorithm finishes in a finite number of iterations providing a solution to (1.1).

As in the proof of [2, Theorem 2.2], we define $\varphi_k := \frac{1}{2}\|x^k - x^*\|^2$. It follows from (2.4) that

$$\begin{aligned}
\varphi_{k+1} &= \frac{1}{2}\|y^k - x^*\|^2 + \rho_k\langle P_k y^k - y^k, y^k - x^*\rangle + \frac{\rho_k^2}{2}\|P_k y^k - y^k\|^2 \\
&= \frac{1}{2}\|y^k - x^*\|^2 - \rho_k(1-\rho_k/2)\|P_k y^k - y^k\|^2 \\
&\quad +\rho_k\langle P_k y^k - y^k, P_k y^k - x^*\rangle.
\end{aligned} \tag{2.13}$$

Next notice that by Lemma 2.1 (i), $v^k \neq 0$ implies $u_k \neq 0$ due to (2.1) and (2.2). Then by virtue of Lemma 2.1 (ii),

$$\langle v^k, u^k\rangle \geq (1-\sigma)\|v^k\|\|u^k\| > 0. \tag{2.14}$$

Now we define $l_k(x) := \langle v^k, x - y^k\rangle$. As $v^k/\rho_k \in A(y^k - u^k/\rho_k)$. The monotonicity of A leads to $\langle v^k/\rho_k, x^* - (y^k - u^k/\rho_k)\rangle \leq 0$ and hence $\langle v^k, x^* - y^k\rangle \leq$

$-\langle v^k, u^k\rangle/\rho_k$. This shows that x^* lies in the half-space $H_k^{\leq} = \{x \in H : l_k(x) \leq -\langle v^k, u^k\rangle/\rho_k\}$. Therefore from $\rho_k > 0$ and (2.14) we know that the hyperplane H_k given by (2.3) strictly separates y^k from x^*. Moreover since the orthogonal projection of y^k onto H_k is also the orthogonal projection onto the half-space $H_k^{\leq}$, one gets $\langle P_k y^k - y^k, P^k y^k - x^*\rangle \leq 0$. It follows from (2.13) that

$$\varphi_{k+1} \leq \frac{1}{2}\|y^k - x^*\|^2 - \rho_k(1-\rho_k/2)\|P_k y^k - y^k\|^2. \tag{2.15}$$

(v) Substituting (2.12) in (2.15), we deduce that for all $k \geq N_0$

$$\varphi_{k+1} \leq \varphi_k + \alpha_k(1-\beta_k)(\varphi_k - \varphi_{k-1}) + \delta_k - \rho_k(1-\rho_k/2)\|P_k y^k - y^k\|^2 \tag{2.16}$$

where $\delta_k := \alpha_k\|x^k - x^{k-1}\|^2 + \frac{\beta_k\mu_k^2}{1-2\mu_k^2}\|x^k - x^*\|^2$. Since $\sum \beta_k\mu_k^2 < \infty$, it follows that $\sum_{k=N_0}^{\infty} \frac{\beta_k\mu_k^2}{1-2\mu_k^2} < \infty$. Note that $\{x^k\}$ is bounded. Thus from (1.5) we get $\sum \delta_k < \infty$. Observe that $0 \leq \alpha_k(1-\beta_k) \leq \alpha < 1$ due to (1.4). By virtue of Lemma 2.2 applied to (2.16), the sequence $\{\varphi_k\}$ is convergent. Furthermore on account of (1.6), (2.4), (2.14) and (2.16), we derive for all $k \geq N_0$

$$\begin{aligned} &(1/R_2 - 1/2)(1-\sigma)^2\|u^k\|^2 \\ &\leq (1/R_2 - 1/2)(\langle v^k, u^k\rangle/\|v^k\|)^2 \\ &\leq ((1-\rho_k/2)/\rho_k)[\rho_k\|P_k y^k - y^k\|]^2 \\ &= \rho_k(1-\rho_k/2)\|P_k y^k - y^k\|^2 \\ &\leq \varphi_k - \varphi_{k+1} + \alpha_k(1-\beta_k)(\varphi_k - \varphi_{k-1}) + \delta_k \\ &\leq |\varphi_k - \varphi_{k+1}| + |\varphi_k - \varphi_{k-1}| + \delta_k \end{aligned} \tag{2.17}$$

where $R_1 := \inf_{k\geq 0}\rho_k > 0$ and $R_2 := \sup_{k\geq 0}\rho_k < 2$. Since $\lim_{k\to\infty}\delta_k = 0$ and $\lim_{k\to\infty}\varphi_k$ exists, from (2.17) we immediately obtain

$$\lim_{k\to\infty}\|u^k\| = \lim_{k\to\infty}\langle v^k, u^k\rangle/\|v^k\| = 0. \tag{2.18}$$

Consequently from (2.2) and Lemma 2.1 (i) we conclude that

$$\lambda_k\|v^k\| \leq \|u^k\|/(1-\sigma) \to 0 \quad \text{as } k \to \infty. \tag{2.19}$$

Moreover in view of (2.4) we know that the second limit in (2.18) yields $\lim_{k\to\infty}\|x^{k+1} - y^k\| = 0$. From this fact, together with the first limit in (2.18), it follows that (2.5) holds because $R_1 = \inf_{k\geq 0}\rho_k > 0$ due to (1.6). This completes the proof of Theorem 2.1 (i).

(vi) In order to prove Theorem 2.2 (ii), it suffices to show the uniqueness of the weak cluster point. Indeed by Theorem 2.2 (i), condition (a) of Lemma 2.3 holds with $S = A^{-1}(\{0\})$. Next suppose (1.3) holds and let $\hat{x}$ be a weak cluster point of $\{x^k\}$. By (2.5), $\hat{x}$ is also a weak cluster point of $\{y^k - u^k/\rho_k\}$. But

$$v^k/\rho_k \in A(y^k - u^k/\rho_k) \tag{2.20}$$

with $v^k/\rho_k \to 0$ strongly in H by (2.19) together with (1.3) and (1.6). Since the graph of the maximal monotone operator A is closed in $H \times H$ for the weak-strong topology (see [8]), it is possible to pass to the limit in (2.20) to deduce that $0 \in A(\hat{x})$, i.e., $\hat{x} \in S$. Thus condition (b) of Lemma 2.3 is also satisfied. Therefore by Lemma 2.3 we infer that $\{x^k\}$ is weakly convergent to an element in S. ∎

Remark 2.2. According to Remark 2.1, we know that [2, Algorithm 1.1] is a special case of Algorithm 2.1. Now we claim that [2, Theorem 1.1] is exactly a corollary of Theorem 2.2. Indeed it suffices to shows that the sequence $\{x^k\}$ generated by [2, Algorithm 1.1] is bounded. Let $x^* \in S := A^{-1}(\{0\})$ and define $\varphi_k := \frac{1}{2}\|x^k - x^*\|^2$. Since we have $\beta_k = 0$ for the case of [2, Algorithm 1.1], (2.16) reduces to

$$\varphi_{k+1} \le \varphi_k + \alpha_k(\varphi_k - \varphi_{k-1}) + \delta_k - \rho_k(1-\rho_k/2)\|P_k y^k - y^k\|^2$$

where $\delta_k := \alpha_k\|x^k - x^{k-1}\|^2$. According to Lemma 2.2 (ii), the limit $\lim_{k\to\infty}\varphi_k$ exists. This implies that $\{x^k\}$ is bounded.

Remark 2.3. If in Theorem 2.1, the assumption of the boundedness of $\{x^k\}$ is removed, and the condition $\sum \alpha_k\|x^k - x^{k-1}\|^2 < \infty$ is replaced by the one $\sum \alpha_k\|x^k - x^{k-1}\| < \infty$ while other assumptions remain the same, then the conclusions (i) and (ii) in Theorem 2.1 remain valid. As a matter of fact, let $x^* \in S := A^{-1}(\{0\})$. Then it follows from (2.10) and (2.15) that for all $k \ge N_0$

$$\begin{aligned}
&\|x^{k+1}-x^*\| \\
&\le \|y^k-x^*\| = \|\beta_k(\tilde{x}^k-x^*) + (1-\beta_k)[x^k-x^* + \alpha_k(x^k-x^{k-1})]\| \\
&\le \beta_k\|\tilde{x}^k-x^*\| + (1-\beta_k)[\|x^k-x^*\| + \alpha_k\|x^k-x^{k-1}\|] \\
&\le \beta_k\Big(1+\frac{2\mu_k^2}{1-2\mu_k^2}\Big)^{1/2}\|x^k-x^*\|+(1-\beta_k)[\|x^k-x^*\|+\alpha_k\|x^k-x^{k-1}\|] \qquad (2.21)\\
&\le \beta_k\Big(1+\frac{\mu_k^2}{1-2\mu_k^2}\Big)\|x^k-x^*\| + (1-\beta_k)[\|x^k-x^*\|+\alpha_k\|x^k-x^{k-1}\|] \\
&\le \Big(1+\frac{\beta_k\mu_k^2}{1-2\mu_k^2}\Big)\|x^k-x^*\|+\alpha_k\|x^k - x^{k-1}\|.
\end{aligned}$$

Hence the boundedness of $\{x^k\}$ immediately follows from Lemma 2.4 applied to (2.21). Also it is easy to see that $\sum \alpha_k\|x^k - x^{k-1}\|^2 < \infty$. Consequently all assumptions in Theorem 2.1 are all satisfied and so the desired conclusions follows.

3. An Alternative Inexact Hybrid Scheme without the Projection Step

The following result which extends [2, Theorem. 3.1] shows the weak convergence of an alternative inexact hybrid scheme without the projection step for solving problem (1.1).

Theorem 3.1. *Let $A : H \to 2^H$ be a maximal monotone operator with $S := A^{-1}(\{0\}) \neq \emptyset$ and $\{x^k\} \subset H$ a bounded sequence satisfying*

$$\left\{\begin{aligned} \tilde{x}^k &= J^A_{\lambda_k}(x^k + e^k), \\ y^k &= \beta_k \tilde{x}^k + (1-\beta_k)[x^k + \alpha_k(x^k - x^{k-1})] \text{ with } \beta_k \in [0,1), \\ \lambda_k v^k &= u^k + \tilde{\eta}^k \text{ for some } v^k \in \rho_k A(y^k - u^k/\rho_k), \\ x^{k+1} &= y^k - (1-\beta_k)u^k, \end{aligned}\right. \tag{3.1}$$

where the residuals $e^k, \tilde{\eta}^k \in H$ satisfy

$$\left\{\begin{array}{l} \|e^k\| \leq \mu_k \|\tilde{x}^k - x^k\| \text{ with } \lim\limits_{k\to\infty} \mu_k = 0, \\ \sum \|\tilde{\eta}^k\| < \infty, \quad \sum \|\tilde{\eta}^k\|\|y^k\| < \infty, \end{array}\right. \tag{3.2}$$

and the parameters λ_k, α_k, and ρ_k satisfy (1.3), (1.4) and (1.6), respectively. Suppose (1.5), $\beta := \sup_{k\geq 0} \beta_k < 1$ and $\sum \beta_k \mu_k^2 < \infty$. Then $v^k \to 0$ strongly in H and there exists $\bar{x} \in S$ such that $x^k \rightharpoonup \bar{x}$ weakly in H.

Proof. We divide the proof into several steps.

(i) Let x^* be any element of S. Then it follows from (2.12) that there exists an integer $N_0 \geq 1$ such that for all $n \geq N_0$

$$\begin{aligned} \|\tilde{y}^k - x^*\|^2 \leq (1 + \frac{2\beta_k \mu_k^2}{1 - 2\mu_k^2})\|x^k - x^*\|^2 + \alpha_k(1-\beta_k)(\|x^k - x^*\|^2 \\ -\|x^{k-1} - x^*\|^2) + 2\alpha_k \|x^k - x^{k-1}\|^2. \end{aligned} \tag{3.3}$$

(ii). From (3.1) we can readily see that $u^k = \rho_k(y^k - J^A_{\lambda_k}(y^k + \tilde{\eta}^k/\rho_k))$. Let $\{w^k\}$ be the auxiliary sequence defined by

$$w^k := \rho_k(y^k - J^A_{\lambda_k}(y^k)) \tag{3.4}$$

Since $J^A_{\lambda_k}$ is nonexpansive [3],

$$\|w^k - u^k\| \leq \|\tilde{\eta}^k\|. \tag{3.5}$$

On the other hand, (3.4) may be written as $w^k = \lambda_k \rho_k A_{\lambda_k}(y^k)$ where $A_{\lambda_k} : H \to H$ is given by $A_{\lambda_k} = \frac{1}{\lambda_k}(I - J^A_{\lambda_k})$. By (1.2),

$$0 \in A(x) \quad \Leftrightarrow \quad A_{\lambda_k}(x) = 0. \tag{3.6}$$

Since A_{λ_k} is a cocoercive maximal monotone operator of parameter λ_k,

$$\langle y^k - x^*, A_{\lambda_k}(y^k)\rangle \geq \lambda_k \|A_{\lambda_k}(y^k)\|^2. \tag{3.7}$$

Hence from (3.7) we obtain

$$\begin{aligned}
&\frac{1}{2}\|y^k - (1-\beta_k)w^k - x^*\|^2 \\
&= \frac{1}{2}\|y^k - x^*\|^2 - \rho_k\lambda_k(1-\beta_k)\langle y^k - x^*, A_{\lambda_k}(y^k)\rangle \\
&\quad + \frac{(\rho_k\lambda_k)^2}{2}(1-\beta_k)^2\|A_{\lambda_k}(y^k)\|^2 \\
&\leq \frac{1}{2}\|y^k - x^*\|^2 - \lambda_k^2\rho_k(1-\rho_k/2)(1-\beta_k)\|A_{\lambda_k}(y^k)\|^2 \\
&\leq \frac{1}{2}\|y^k - x^*\|^2 - \lambda_k^2\rho_k(1-\rho_k/2)(1-\beta)\|A_{\lambda_k}(y^k)\|^2.
\end{aligned} \tag{3.8}$$

(iii) Define $\varphi_k := \frac{1}{2}\|x^k - x^*\|^2$. Then from (3.3), (3.5) and (3.8) we get

$$\begin{aligned}
\varphi_{k+1} &= \frac{1}{2}\|y^k - x^* - (1-\beta_k)w^k + (1-\beta_k)w^k - (1-\beta_k)u^k\|^2 \\
&\leq \frac{1}{2}\|y^k - x^* - (1-\beta_k)w^k\|^2 + (1-\beta_k)\|y^k - x^* \\
&\quad - (1-\beta_k)w^k\|\|w^k - u^k\| + \frac{1}{2}(1-\beta_k)^2\|w^k - u^k\|^2 \\
&\leq \frac{1}{2}\|y^k - x^*\|^2 - \lambda_k^2\rho_k(1-\rho_k/2)(1-\beta)\|A_{\lambda_k}(y^k)\|^2 \\
&\quad + \|\tilde{\eta}^k\|\|y^k - x^*\| + \tfrac{1}{2}\|\tilde{\eta}^k\|^2 \\
&\leq \varphi_k + \alpha_k(1-\beta_k)(\varphi_k - \varphi_{k-1}) + \delta_k \\
&\quad - \lambda_k^2\rho_k(1-\rho_k/2)(1-\beta)\|A_{\lambda_k}(y^k)\|^2
\end{aligned} \tag{3.9}$$

where $\delta_k := \frac{\beta_k\mu_k^2}{1-2\mu_k^2}\|x^k - x^*\|^2 + \alpha_k\|x^k - x^{k-1}\|^2 + \|\tilde{\eta}^k\|\|y^k - x^*\| + \frac{1}{2}\|\tilde{\eta}^k\|^2$. Since (1.5) and (3.2) hold and $\{x^k\}$ is bounded, according to $\sum \beta_k\mu_k^2 < \infty$ we deduce that $\sum \delta_k < \infty$. Thus by virtue of Lemma 2.2 (ii), $\{\varphi_k\}$ is convergent.

Furthermore this together with (1.3), (1.6) and (3.9) implies that $\lambda_k A_{\lambda_k}(y^k) \to 0$ strongly in H. Set $\xi^k := y^k - J^A_{\lambda_k}(y^k)$ which amounts to

$$\xi^k/\lambda_k \in A(y^k - \xi^k). \tag{3.10}$$

Since $A_{\lambda_k} = \frac{1}{\lambda_k}(I - J^A_{\lambda_k})$, $\xi^k = \lambda_k A_{\lambda_k}(y_k) \to 0$ strongly in H. Hence from (3.4) and (3.5) we get

$$\begin{aligned} \|u^k\| &\leq \|w^k\| + \|w^k - u^k\| \\ &\leq \rho_k \|y^k - J^A_{\lambda_k}(y^k)\| + \|\tilde{\eta}^k\| \\ &\leq R_2\|\xi^k\| + \|\tilde{\eta}^k\| \to 0 \quad \text{as } k \to \infty. \end{aligned} \tag{3.11}$$

Together with this (3.1) implies that $\|v^k\| = \frac{1}{\lambda_k}\|u^k + \tilde{\eta}^k\| \to 0$ and $\|x^{k+1} - y^k\| = (1-\beta_k)\|u^k\| \to 0$ as $k \to \infty$. Let $\hat{x}$ be a weak cluster point of $\{x^k\}$. Then $\hat{x}$ is also a weak cluster point of $\{y^k\}$ and consequently $\hat{x}$ is a weak cluster point of $\{y^k - \xi^k\}$. By the weak-strong closedness of the graph of A, letting $k \to \infty$ in (3.10) yields $0 \in A(\hat{x})$. Therefore condition (b) of Lemma 2.3 holds. This completes the proof. ■

Remark 3.1. (i) If in Theorem 3.1, we take $\beta_k = 0$ and $\tilde{\eta}^k = \lambda_k \eta^k$, then (3.1)-(3.2) reduce to (1.12)-(1.13) in [2, Theorem 1.2]. (ii) [2, Theorem 1.2] is exactly a corollary of Theorem 3.1. Indeed it suffices to shows that $\{x^k\}$ is bounded under the conditions of [2, Theorem 1.2]. As a matter of fact, combining (3.9) with $\beta_k = 0$ implies that

$$\varphi_{k+1} \leq \varphi_k + \alpha_k(\varphi_k - \varphi_{k-1}) + \delta_k$$

where $\delta_k := \alpha_k\|x^k - x^{k-1}\|^2 + \lambda_k\|\eta^k\|\|y^k - x^*\| + \frac{\lambda_k^2}{2}\|\eta^k\|^2$. From (1.5) and (1.13), we obtain $\sum \delta_k < \infty$. Thus in view of Lemma 2.2 (ii), $\{\varphi_k\}$ is convergent and hence $\{x^k\}$ is bounded.

Remark 3.2. If in Theorem 3.1, the assumption of the boundedness of $\{x^k\}$ is removed, and the condition $\sum \alpha_k\|x^k - x^{k-1}\|^2 < \infty$ is replaced by $\sum \alpha_k \|x^k - x^{k-1}\| < \infty$ while other assumptions remain the same, then the conclusion in Theorem 3.1 remains valid. As a matter of fact, let $x^* \in S := A^{-1}(\{0\})$. Then it follows from (2.10), (3.1), (3.5) and (3.8) that for all $k \geq N_0$

$$\begin{aligned} &\|x^{k+1} - x^*\| \\ &= \|y^k - x^* - (1-\beta_k)w^k + (1-\beta_k)(w^k - u^k)\| \\ &\leq \|y^k - x^* - (1-\beta_k)w^k\| + (1-\beta_k)\|w^k - u^k\| \\ &\leq \|y^k - x^*\| + (1-\beta_k)\|\tilde{\eta}^k\| \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&\leq \beta_k\|\tilde{x}^k - x^*\| + (1-\beta_k)[\|x^k - x^*\| + \alpha^k\|x^k - x^{k-1}\|] + \|\tilde{\eta}^k\| \\
&\leq \beta_k(1+\frac{2\mu_k^2}{1-2\mu_k^2})^{1/2}\|x^k - x^*\| + (1-\beta_k)[\|x^k - x^*\| \\
&\quad +\alpha_k\|x^k - x^{k-1}\|] + \|\tilde{\eta}^k\| \\
&\leq \beta_k(1+\frac{\mu_k^2}{1-2\mu_k^2})\|x^k - x^*\| + (1-\beta_k)[\|x^k - x^*\| \\
&\quad +\alpha_k\|x^k - x^{k-1}\|] + \|\tilde{\eta}^k\| \\
&\leq (1+\frac{\beta_k\mu_k^2}{1-2\mu_k^2})\|x^k - x^*\| + \alpha_k\|x^k - x^{k-1}\| + \|\tilde{\eta}^k\|.
\end{aligned}
$$

Hence the boundedness of $\{x^k\}$ immediately follows from Lemma 2.4 applied to (3.12). Also it is easy to see that $\sum \alpha_k\|x^k - x^{k-1}\|^2 < \infty$. Consequently all assumptions in Theorem 3.1 are actually satisfied and so the desired conclusion is attained.

REFERENCES

1. F. Alvarez and H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, in Wellposedness in Optimization and Related Topics (Gargnano, 1999), *Set-Valued Anal.*, **9** (2001), 3-11.

2. F. Alvarez, Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space, *SIAM J. Optim.*, **14** (2004), 773-782.

3. H. Brezis, *Operateurs maximaux monotones et semi-groups de contractions dans les espaces de Hilbert*, Mathematics Studies 5, North-Holland, Amsterdam, 1973.

4. J. Eckstein and D. P. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, *Math. Program.*, **55** (1992), 293-318.

5. B. Martinet, Regularisation d'inequations variationnelles par approximations successives, Rev. *Francaise Informat. et Recherche Operationnelle*, **4** (1970), 154-158.

6. B. Martinet, Determination approachee d'un point fixe d'une application pseudo-contractante, C. R. *Acad. Sci. Paris Ser. A-B,* **274** (1972), 163-165.

7. G. Minty, Monotone (nonlinear) operators in Hilbert space, *Duke Math. J.*, **29** (1962), 341-346.

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