TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 3, pp. 741-751, June 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

CHARACTERIZATION OF THE WEIGHTED LIPSCHITZ FUNCTION BY THE GARSIA-TYPE NORM ON THE UNIT BALL

Hong Rae Cho[1], Su-Mi Kwon and Jinkee Lee[2]

Abstract. The space $BMOA$ can be characterized by the boundedness of Garsia norm. In this paper like as the $BMOA$ function we characterize the holomorphic weighted Lipschitz function by the boundedness of the Grasia-type norm on the unit ball in $\mathbb{C}^n$.

1. Introduction and Statement of Results

Let E be a bounded subset of $\mathbb{R}^N$. Let ω be a *majorant*, i.e., a non-negative continuous increasing function on $(0, t_0)$, where t_0 is large enough, such that $\limsup_{t\to 0^+} \omega(t) = 0$ and that $\omega(t)/t$ non-increasing $(t > 0)$. Then the weighted Lipschitz space $\Lambda_\omega(E)$ is defined by the requirement

$$(1.1) \qquad |f(x) - f(y)| \le M\omega(|x-y|), \ x, y \in E.$$

The norm $\|f\|_{\Lambda_{\omega(E)}}$ is defined as the smallest M in (1.1).

Weighted Lipschitz spaces have been studied by many authors (see [2-6, 8, 9], and references in their papers).

In this paper, we deal with the case where E is the unit ball $\mathbf{B} = \{z \in \mathbb{C}^n; |z| < 1\}$, f is holomorphic in $\mathbf{B}$ and continuous up to the boundary $\mathbf{S} = \partial\mathbf{B}$, and ω is a majorant satisfying

$$(1.2) \qquad \int_0^t \frac{\omega(s)}{s} ds \le C(\omega)\omega(t) \quad (0 < t < 1),$$

Received April 13, 2006, accepted May 10, 2006.
Communicated by Der-Chen Chang.
2000 *Mathematics Subject Classification*: 32A37.
Key words and phrases: Hardy-Littlewood type characterization, Weighted Lipschitz spaces, A regular majorant, Poisson integral, Garsia-type norm.
[1]This work was supported by Korea Research Foundation Grant funded by Korea Government (MOEHRD, Basic Reasearch Promotion Fund) (R05-2004-000-10469-0).
[2]This work was supported by Korea Research Foundation Grant (KRF-2004-002-C00013).

and

$$t\int_t^\infty \frac{\omega(s)}{s^2}ds \le C(\omega)\omega(t) \quad (0<t<1). \tag{1.3}$$

A majorant is henceforth called *fast* (resp., *slow*) if condition (1.2) (resp., (1.3)) is fulfilled. Following Dyakonov a majorant is called *regular* if it is both fast and slow.

A real function ω, defined on some interval, is called almost increasing if there exists a constant C such that $x<y$ implies $\omega(x)<C\omega(y)$. (An almost decreasing function is defined similarly.)

Remark 1.1. (i) We assume that ω is a majorant satisfying the conditions:

$$\frac{\omega(t)}{t^\alpha} \text{ is almost increasing for some } \alpha>0 \tag{1.4}$$

and

$$\frac{\omega(t)}{t^\beta} \text{ is almost decreasing for some } 0<\beta<1. \tag{1.5}$$

We know that for a majorant conditions (1.2) (resp., (1.3)) and (1.4) (resp., (1.5)) are equivalent.

(ii) A non-trivial example of the majorant satisfying conditions (1.4) and (1.5) is of the form $\omega(t)=t^\alpha(1+|\log t|)^\beta$ where $0<\alpha<1$ and $\beta\in\mathbb{R}$.

From now on we assume that ω is a regular majorant. By the same argument as the proofs of Lemma 10.1.6 in [9] or Lemma 6.4.8 in [10], we can prove that for a harmonic function f on $\mathbf{B}$ it follows that

$$\|f\|_{\Lambda_\omega(\mathbf{B})} \sim \sup_{z\in\mathbf{B}}\left[\frac{1-|z|}{\omega(1-|z|)}|\nabla f(z)|\right]. \tag{1.6}$$

Let $\mathcal{R}f$ be the radial derivative of a holomorphic function f in $\mathbf{B}$ defined by

$$\mathcal{R}f(z)=\sum_{j=1}^n z_j\frac{\partial f}{\partial z_j}(z).$$

It is obvious that $|\mathcal{R}f(z)|\le|\nabla f(z)|$ on $\mathbf{B}$. Even though these two gradients have no the same size, we can see that (see Proposition 3.1)

$$\sup_{z\in\mathbf{B}}\left[\frac{1-|z|}{\omega(1-|z|)}|\nabla f(z)|\right] \sim \sup_{z\in\mathbf{B}}\left[\frac{1-|z|}{\omega(1-|z|)}|\mathcal{R}f(z)|\right]. \tag{1.7}$$

By (1.6) and (1.7), we obtain the following.

Theorem 1.2. *If f is a holomorphic function on $\mathbf{B}$, we have*

$$\|f\|_{\Lambda_\omega(\mathbf{B})} \sim \sup_{z\in\mathbf{B}} \left[\frac{1-|z|}{\omega(1-|z|)}|\mathcal{R}f(z)|\right].$$

We recall that

$$\|f\|_{\Lambda_\omega(\mathbf{S})} = \inf\{M; |f(\zeta)-f(\xi)| \le M\omega(|\zeta-\xi|),\ \zeta,\xi\in\mathbf{S}\}.$$

Theorem 1.3. *Let $C[f]$ be the Cauchy transform of f on $\mathbf{S}$. Then*

$$\sup_{z\in\mathbf{B}} \left[\frac{1-|z|}{\omega(1-|z|)}|\mathcal{R}C[f](z)|\right] \lesssim \|f\|_{\Lambda_\omega(\mathbf{S})}. \tag{1.8}$$

By (1.8), we see that $\|f\|_{\Lambda_\omega(\mathbf{B})}$ and $\|f\|_{\Lambda_\omega(\mathbf{S})}$ are equivalent for holomorphic function $f\in C(\bar{\mathbf{B}})$.

For $0<p<\infty$ the Hardy space H^p consists of holomorphic functions f in $\mathbf{B}$ such that

$$\sup_{0<r<1}\int_{\mathbf{S}} |f(r\zeta)|^p d\sigma(\zeta) < \infty,$$

where $d\sigma$ is the surface measure on $\mathbf{S}$, normalized so that $\sigma(\mathbf{S})=1$.

For any $\zeta\in\mathbf{S}$ and $r>0$, the non-isotropic ball in $\mathbf{S}$ with center ζ and radius $r>0$ is the set

$$Q(\zeta,r) = \{\xi\in\mathbf{S} : |1-<\zeta,\xi>|^{1/2} < r\}.$$

Let $BMOA$ denote the space of functions f in H^2 such that

$$\sup_Q \frac{1}{\sigma(Q)}\int_Q |f-f_Q|^2 d\sigma < \infty,$$

where

$$f_Q = \frac{1}{\sigma(Q)}\int_Q f d\sigma$$

is the average of f over Q and the supremum is taken over $Q=Q(\zeta,r)$ for all $\zeta\in\mathbf{S}$ and all $r>0$.

With each point $z\in\mathbf{B}$ we associate the Poisson kernel

$$P(z,\zeta) = \frac{1-|z|^2}{|\zeta-z|^{2n}},\ \zeta\in\mathbf{S}. \tag{1.9}$$

The Poisson integral of a function $f \in L^1(\mathbf{S})$ is defined, as usual, by

$$\mathcal{P}f(z) = \int_{\mathbf{S}} f(\zeta)P(z,\zeta)\,d\sigma(\zeta),\ z \in \mathbf{B}.$$

Further, with $f \in H^1$ we associate the nonnegative function

$$\mathcal{G}_f(z) = \mathcal{P}|f|(z) - |f(z)|,\ z \in \mathbf{B}.$$

The Garsia norm is defined by

$$\|f\|_G = \sup\{\mathcal{G}_{f^2}(z)^{1/2} : z \in \mathbf{B}\},\ f \in H^2.$$

It is proved that $f \in BMOA$ if and only if the Garsia norm of f is finite. One-dimensional case was proved by Garsia (see [7]) and the n-dimensional case by Sh. Axler and J. Shapiro (see [1]).

We define the Garsia-type norm by

$$\|f\|_{G,\omega} = \sup\left\{\frac{1}{\omega(1-|z|)}\mathcal{G}_{f^2}(z)^{1/2} : z \in \mathbf{B}\right\},\ f \in H^2. \tag{1.10}$$

Theorem 1.4. *Let $f \in H^2$. If ω and ω^2 are regular, then $\|f\|_{\Lambda_\omega(\mathbf{B})} \sim \|f\|_{G,\omega}$.*

Like as estimates for the classical Lipschitz case we need some key integral estimates proved in Section 2. They are non-trivial and not proved by the same methods as the classical Lipschitz case.

2. Integral Estimates

In this section we prove some basic integral estimates that will be used in the proofs of main results.

Lemma 2.1. *We have*

$$\int_{\mathbf{B}} \frac{\omega(1-|w|)}{(1-|w|)|1-\langle z,w\rangle|^{n+1}}dV(w) \lesssim \frac{\omega(1-|z|)}{1-|z|},$$

where dV is the volume measure on $\mathbf{B}$, normalized so that $V(\mathbf{B}) = 1$.

Proof. By the polar coordinate and the inequality in [10, Proposition 1.4.10], we have

$$\begin{aligned}\int_{\mathbf{B}} \frac{\omega(1-|w|)}{(1-|w|)|1-\langle z,w\rangle|^{n+1}}dV(w) &= \int_0^1 \int_{\mathbf{S}} \frac{\omega(1-r)}{(1-r)|1-\langle rz,\zeta\rangle|^{n+1}}d\sigma(\zeta)r^{2n-1}dr \\ &\lesssim \int_0^1 \frac{\omega(1-r)}{(1-r)(1-r|z|)}dr.\end{aligned}$$

Note that $1-r|z|=(1-r)+(1-|z|)-(1-|z|)(1-r)$. Thus we have

$$\int_0^1 \frac{\omega(1-r)}{(1-r)(1-r|z|)}dr=\int_0^1 \frac{\omega(s)}{s[s+(1-|z|)-(1-|z|)s]}ds,$$

by putting $1-r=s$. We decompose the integral by two parts as following

$$\begin{aligned}\int_0^1 \frac{\omega(s)}{s[s+(1-|z|)-(1-|z|)s]}ds &= \int_0^{1-|z|}+\int_{1-|z|}^1 \frac{\omega(s)}{s[s+(1-|z|)-(1-|z|)s]}ds\\ &=(\mathrm{I})+(\mathrm{II}).\end{aligned}$$

For the first part we have

$$\begin{aligned}(\mathrm{I})=\int_0^{1-|z|}\frac{\omega(s)}{s[s+(1-|z|)-(1-|z|)s]}ds &= \int_0^{1-|z|}\frac{\omega(s)}{s[s|z|+(1-|z|)]}ds\\ &\lesssim \frac{1}{1-|z|}\int_0^{1-|z|}\frac{\omega(s)}{s}ds\\ &\lesssim \frac{\omega(1-|z|)}{1-|z|}.\end{aligned}$$

Now for the second part we have

$$(\mathrm{II})=\int_{1-|z|}^1 \frac{\omega(s)}{s[s+(1-|z|)-(1-|z|)s]}ds\lesssim\int_{1-|z|}^1\frac{\omega(s)}{s^2}ds\lesssim\frac{\omega(1-|z|)}{1-|z|}.$$

We use the regular condition of ω at the last step of estimates for two parts of the integral. Thus we complete the proof. ■

Lemma 2.2. *We have*

$$\int_{\zeta\in\mathbf{S}}\frac{\omega(|1-\langle z,\zeta\rangle|)}{|1-\langle z,\zeta\rangle|^{n+1}}\,d\sigma(\zeta)\lesssim\frac{\omega(1-|z|)}{1-|z|}.$$

Proof. For $\tilde{z}=z/|z|$ we let

$$\mathbf{S}_0=\{\zeta\in\mathbf{S}:|1-\langle\tilde{z},\zeta\rangle|<1-|z|\}$$

and

$$\mathbf{S}_j=\{\zeta\in\mathbf{S}:2^{j-1}(1-|z|)\le|1-\langle\tilde{z},\zeta\rangle|<2^j(1-|z|)\},\ j=1,2,\cdots.$$

Then

$$\mathbf{S}=\bigcup_{j=0}^{\infty}\mathbf{S}_j.$$

We decompose

$$\int_{\zeta\in\mathbf{S}} \frac{\omega(|1-\langle z,\zeta\rangle|)}{|1-\langle z,\zeta\rangle|^{n+1}}\, d\sigma(\zeta) \lesssim \sum_{j=0}^{\infty}\int_{\zeta\in\mathbf{S_j}} \frac{\omega(|1-\langle z,\zeta\rangle|)}{|1-\langle z,\zeta\rangle|^{n+1}}\, d\sigma(\zeta).$$

Note that $1-|z| \le |1-\langle z,\zeta\rangle| \le |1-\langle z,\tilde{z}\rangle| + |1-\langle \tilde{z},\zeta\rangle| = 1-|z| + |1-\langle \tilde{z},\zeta\rangle| \le 2(1-|z|)$ for $\zeta\in\mathbf{S}_0$. Thus $|1-\langle z,\zeta\rangle| \sim 1-|z|$ on $\mathbf{S}_0$. Therefore we obtain

$$\begin{aligned}\int_{\zeta\in\mathbf{S}_0} \frac{\omega(|1-\langle z,\zeta\rangle|)}{|1-\langle z,\zeta\rangle|^{n+1}}\, d\sigma(\zeta) &\sim \frac{\omega(1-|z|)}{(1-|z|)^{n+1}}\sigma(\mathbf{S}_0)\\ &\lesssim \frac{\omega(1-|z|)}{(1-|z|)^{n+1}}(1-|z|)^n\\ &= \frac{\omega(1-|z|)}{1-|z|}.\end{aligned}$$

For $j=1,2,\cdots$, we have on $\mathbf{S}_j$

$$\begin{aligned}|1-\langle z,\zeta\rangle| &\le |1-\langle z,\tilde{z}\rangle| + |1-\langle \tilde{z},\zeta\rangle|\\ &= 1-|z| + |1-\langle \tilde{z},\zeta\rangle|\\ &< 1-|z| + 2^j(1-|z|)\\ &< 2\cdot 2^j(1-|z|)\end{aligned}$$

and

$$\begin{aligned}|1-\langle \tilde{z},\zeta\rangle| &\le |1-\langle \tilde{z},z\rangle| + |1-\langle z,\zeta\rangle|\\ &= 1-|z| + |1-\langle z,\zeta\rangle|\\ &\le 2|1-\langle z,\zeta\rangle|.\end{aligned}$$

We note that

$$2^{j-1}(1-|z|) \le |1-\langle \tilde{z},\zeta\rangle| \lesssim |1-\langle z,\zeta\rangle| < 2^j(1-|z|) \quad \text{on} \quad \mathbf{S}_j$$

for $j=1,2,\cdots$. Thus it follows that

$$|1-\langle z,\zeta\rangle| \sim 2^j(1-|z|) \quad \text{on} \quad \mathbf{S}_j, j=1,2,\cdots.$$

Hence we obtain

$$\begin{aligned}\int_{\zeta\in\mathbf{S}_j} \frac{\omega(|1-\langle z,\zeta\rangle|)}{|1-\langle z,\zeta\rangle|^{n+1}}\, d\sigma(\zeta) &\lesssim \omega(2^j(1-|z|))\int_{\zeta\in\mathbf{S}_j} \frac{d\sigma(\zeta)}{|1-\langle \tilde{z},\zeta\rangle|^{n+1}}\\ &\lesssim \omega(2^j(1-|z|))\frac{\sigma(\mathbf{S}_j)}{(2^{j-1}(1-|z|))^{n+1}}\\ &\lesssim \omega(2^j(1-|z|))\frac{(2^j(1-|z|))^n}{(2^{j-1}(1-|z|))^{n+1}}\\ &\sim \frac{\omega(2^j(1-|z|))}{2^j(1-|z|)}.\end{aligned}$$

Note that

$$\sum_{j=1}^{\infty} \frac{\omega(2^j(1-|z|))}{2^j(1-|z|)} \lesssim \int_0^{\infty} \frac{\omega(2^t(1-|z|))}{2^t(1-|z|)} dt.$$

By putting $s = 2^t$, we have

$$\begin{aligned}
\int_0^{\infty} \frac{\omega(2^t(1-|z|))}{2^t(1-|z|)}\, dt &= \int_1^{\infty} \frac{\omega(s(1-|z|))}{s(1-|z|)} \frac{1}{s}\, ds \\
&\lesssim \int_{1-|z|}^{\infty} \frac{\omega(x)}{x^2}\, dx \\
&\lesssim \frac{\omega(1-|z|)}{1-|z|}.
\end{aligned}$$

Therefore the proof is complete. ■

If we use Lemma 2.2, by the same argument as the proof of Theorem 6.4.9 in [10], we get Theorem 1.3.

We recall that $\tilde{z} = z/|z| \in \mathbf{S}$ for $z \in \mathbf{B}$.

Lemma 2.3. *Let $z \in \mathbf{B}$, then*

$$\int_{\zeta \in \mathbf{S}} \frac{\omega(|\zeta - \tilde{z}|)}{|\zeta - z|^{2n}}\, d\sigma(\zeta) \lesssim \frac{\omega(1-|z|)}{1-|z|}.$$

Proof. Let g be a function of one complex variable. Then for $\eta \in \mathbf{S}$, we have

$$\int_{\mathbf{S}} g(\langle \zeta, \eta \rangle)\, d\sigma(\zeta) = \frac{n-1}{\pi} \int\int_{\mathbf{D}} (1-|w|^2)^{n-2} g(w)\, dA(w).$$

By the unitary invariance of $d\sigma$ and $|\zeta - \vec{e}_1|^2 = 2\ \mathrm{Re}\ (1-\zeta_1)$ for $\zeta = (\zeta_1, \cdots, \zeta_n)$, $\zeta_1 = x + iy$, $r = |z|$ and $\vec{e}_1 = (1, 0, \cdots, 0)$, we have

$$\begin{aligned}
\int_{\mathbf{S}} \frac{\omega(|\zeta - z|)}{|\zeta - z|^{2n}}\, d\sigma(\zeta) &= \int_{\mathbf{S}} \frac{\omega(|\zeta - \vec{e}_1|)}{|\zeta - r\vec{e}_1|^{2n}}\, d\sigma(\zeta) \\
&= \frac{n-1}{\pi} \int\int_{\mathbf{D}} \frac{\omega(\sqrt{2}\sqrt{1-x})}{(1-2rx+r^2)^n} (1-x^2-y^2)^{n-2}\, dx\, dy \\
&\lesssim \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{\omega(\sqrt{1-x})}{(1-2rx+r^2)^n} (1-x^2-y^2)^{n-2}\, dy\, dx \\
&\lesssim \int_{-1}^{1} \frac{\omega(\sqrt{1-x})}{(1-2rx+r^2)^n} (1-x^2)^{n-3/2}\, dx \\
&\lesssim \frac{1}{1-r} \int_0^{\infty} \frac{\omega((1-r)\sqrt{t})}{(t+1)^n} t^{n-3/2}\, dt.
\end{aligned}$$

To obtain the last inequality, use $1-x^2 \le 2(1-x)$,

$$3(1-2rx+r^2) \ge 1-x+(1-r)^2,$$

and put $1-x=(1-r)^2 t$. We have

$$\begin{aligned}\int_0^\infty \frac{\omega((1-r)\sqrt{t})}{(t+1)^n} t^{n-3/2}\,dt &= \int_0^1 \frac{\omega((1-r)\sqrt{t})}{(t+1)^n} t^{n-3/2}\,dt \\ &\quad + \int_1^\infty \frac{\omega((1-r)\sqrt{t})}{(t+1)^n} t^{n-3/2}\,dt \\ &= \text{(I)} + \text{(II)}.\end{aligned}$$

Estimating the two integrals, we see that the first right-hand side is

$$\begin{aligned}\text{(I)} = \int_0^1 \frac{\omega((1-r)\sqrt{t})}{(t+1)^n} t^{n-\frac{3}{2}}\,dt &\lesssim \omega(1-r)\int_0^1 \frac{t^{n-\frac{3}{2}}}{(t+1)^n}\,dt \\ &\lesssim \omega(1-r).\end{aligned}$$

By putting $(1-r)\sqrt{t}=s$, the second term is

$$\begin{aligned}\text{(II)} = \int_1^\infty \frac{\omega((1-r)\sqrt{t})}{(t+1)^n} t^{n-\frac{3}{2}}\,dt &\lesssim (1-r)\int_{1-r}^\infty \frac{\omega(s)s}{((1-r)^2+s^2)^{\frac{3}{2}}}\,ds \\ &\lesssim (1-r)\int_{1-r}^\infty \frac{\omega(s)}{s^2}\,ds \\ &\lesssim \omega(1-r).\end{aligned}$$

Thus we get the result. ∎

3. Hardy-littlewood Type Characterization

Proposition 3.1. *Let f be holomorphic in $\mathbf{B}$ with $\mathcal{R}f \in L^1(\mathbf{B})$. We have*

$$\sup_{z\in\mathbf{B}}\left[\frac{1-|z|}{\omega(1-|z|)}|\nabla f(z)|\right] \sim \sup_{z\in\mathbf{B}}\left[\frac{1-|z|}{\omega(1-|z|)}|\mathcal{R}f(z)|\right].$$

Proof. By the reproducing property of the Bergman projection, it follows that

$$\mathcal{R}f(z) = \int_{\mathbf{B}} \frac{\mathcal{R}f(w)}{(1-\langle z,w\rangle)^{n+1}} dV(w).$$

Thus we have

$$\begin{aligned} f(z)-f(0) &= \int_0^1 \mathcal{R}f(tz)\frac{dt}{t} \\ &= \int_{\mathbf{B}} \mathcal{R}f(w)dV(w)\int_0^1 \frac{1}{(1-t\langle z,w\rangle)^{n+1}}\frac{dt}{t} \end{aligned}$$

and so that

$$\begin{aligned} \frac{\partial f}{\partial z_j}(z) &= \int_{\mathbf{B}} \mathcal{R}f(w)dV(w)\int_0^1 \frac{\partial}{\partial z_j}\left[\frac{1}{(1-t\langle z,w\rangle)^{n+1}}\right]\frac{dt}{t} \\ &= \int_{\mathbf{B}} \mathcal{R}f(w)dV(w)\int_0^1 \frac{(n+1)\bar{w}_j}{(1-t\langle z,w\rangle)^{n+2}}dt. \end{aligned}$$

We know that

$$\begin{aligned} \int_0^1 \frac{(n+1)\bar{w}_j}{(1-t\langle z,w\rangle)^{n+2}}dt &= \frac{\bar{w}_j}{\langle z,w\rangle}\left[\frac{1}{(1-\langle z,w\rangle)^{n+1}}-1\right] \\ &= \frac{e(z,w)}{(1-\langle z,w\rangle)^{n+1}}, \end{aligned}$$

where $e \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$. It follows that

$$\begin{aligned} \left|\frac{\partial f}{\partial z_j}(z)\right| &\lesssim \int_{\mathbf{B}} |\mathcal{R}f(w)|\frac{1}{|1-\langle z,w\rangle|^{n+1}}dV(w) \\ &\lesssim \sup_{\mathbf{B}}\left[\frac{1-|w|}{\omega(1-|w|)}|\mathcal{R}f(w)|\right]\int_{\mathbf{B}} \frac{\omega(1-|w|)}{(1-|w|)|1-\langle z,w\rangle|^{n+1}}dV(w) \\ &\lesssim \frac{\omega(1-|z|)}{1-|z|}\sup_{\mathbf{B}}\left[\frac{1-|w|}{\omega(1-|w|)}|\mathcal{R}f(w)|\right], \end{aligned}$$

by using Lemma 2.1. Thus we get

$$\sup_{z\in\mathbf{B}}\left[\frac{1-|z|}{\omega(1-|z|)}|\nabla f(z)|\right] \lesssim \sup_{z\in\mathbf{B}}\left[\frac{1-|z|}{\omega(1-|z|)}|\mathcal{R}f(z)|\right]$$

and the converse inequality is obvious. ∎

By (1.6) and Proposition 3.1, we complete the proof of Theorem 1.2.

4. Characterization of the Weighted Lipschitz Norm by Using the Garsia Norm

We recall that $\tilde{z} = z/|z| \in \mathbf{S}$ for $z \in \mathbf{B}$.

Lemma 4.1. *Let $f \in \Lambda_\omega(\mathbf{S})$ and $z \in \mathbf{B}$, then*

$$|\mathcal{P}f(z) - f(\tilde{z})| \lesssim \|f\|_{\Lambda_\omega(\mathbf{S})}\ \omega(1-|z|).$$

Proof. We have

$$\begin{aligned}|\mathcal{P}f(z)-f(\tilde{z})| &= \left|\int_{\mathbf{S}} f(\zeta)\frac{1-|z|^2}{|\zeta-z|^{2n}}\,d\sigma(\zeta)-\int_{\mathbf{S}} f(\tilde{z})\frac{1-|z|^2}{|\zeta-z|^{2n}}\,d\sigma(\zeta)\right| \\ &\lesssim \int_{\mathbf{S}}\frac{1-|z|^2}{|\zeta-z|^{2n}}|f(\zeta)-f(\tilde{z})|\,d\sigma(\zeta) \\ &\lesssim \|f\|_{\Lambda_\omega(\mathbf{S})}(1-|z|^2)\int_{\mathbf{S}}\frac{\omega(|\zeta-\tilde{z}|)}{|\zeta-z|^{2n}}\,d\sigma(\zeta) \\ &\lesssim \|f\|_{\Lambda_\omega(\mathbf{S})}\omega(1-|z|)\end{aligned}$$

where the last inequality relies on Lemma 2.3. ■

Proof of Theorem 1.4. (i) For $\eta,\zeta\in\mathbf{S}$, a straight calculation in the Poisson kernel (1.9) shows that

$$\begin{aligned}\left|\frac{\partial}{\partial r}P(r\eta,\zeta)\right| &= \left|\frac{\partial}{\partial r}\left(\frac{1-r^2}{|r\eta-\zeta|^{2n}}\right)\right| \\ &\lesssim \frac{1}{|r\eta-\zeta|^{2n}}.\end{aligned}$$

Then we have the radial derivative of the Poisson integral formula

$$\begin{aligned}|\mathcal{R}f(r\eta)| &\le \left|\int_{\mathbf{S}} r\frac{\partial}{\partial r}P(r\eta\zeta)[f(\zeta)-f(r\eta)]\,d\sigma(\zeta)\right| \\ &\lesssim \int_{\mathbf{S}}\frac{|f(\zeta)-f(r\eta)|}{|r\eta-\zeta|^{2n}}\,d\sigma(\zeta).\end{aligned}$$

Thus we have

$$(1-|z|)^2|\mathcal{R}f(z)|\lesssim\int_{\mathbf{S}}\frac{(1-|z|^2)|f(\zeta)-f(z)|}{|z-\zeta|^{2n}}\,d\sigma(\zeta)$$

which implies that $\|f\|_{\Lambda_\omega(\mathbf{B})}\lesssim\|f\|_{G,\omega}$.

For the converse we let $\zeta\in\mathbf{S}$ and $z\in\mathbf{B}$. Then

$$\begin{aligned}|f(\zeta)-f(z)| &\le |f(\zeta)-f(\tilde{z})|+|f(\tilde{z})-f(z)| \\ &\lesssim \|f\|_{\Lambda_\omega(\mathbf{S})}\omega(|\zeta-\tilde{z}|)+\|f\|_{\Lambda_\omega(\mathbf{S})}\omega(1-|z|)\end{aligned}\tag{4.1}$$

where we use Lemma 4.1 to estimate the second inequality.

By (4.1), we still have

$$|f(\zeta)-f(z)|^2\lesssim\|f\|^2_{\Lambda_\omega(\mathbf{S})}\omega(|\zeta-\tilde{z}|)^2+\|f\|^2_{\Lambda_\omega(\mathbf{S})}\omega(1-|z|)^2.\tag{4.2}$$

Integrating (4.2) against $P(z,\zeta)d\sigma$, we arrive at

$$\begin{aligned}\frac{1}{1-|z|^2}\int_{\mathbf{S}}|f(\zeta)-f(z)|^2P(z,\zeta)\ d\sigma(\zeta) &\lesssim \|f\|^2_{\Lambda_\omega(\mathbf{S})}\int_{\mathbf{S}}\frac{\omega(|\zeta-\tilde{z}|)^2}{|\zeta-z|^{2n}}\ d\sigma(\zeta)\\ &\quad+\|f\|^2_{\Lambda_\omega(\mathbf{S})}\omega(1-|z|)^2\int_{\mathbf{S}}\frac{d\sigma(\zeta)}{|\zeta-z|^{2n}}\\ &\lesssim \|f\|^2_{\Lambda_\omega(\mathbf{S})}\frac{\omega(1-|z|)^2}{1-|z|}.\end{aligned}$$

Therefore we have $\|f\|_{G,\omega}\lesssim\|f\|_{\Lambda_\omega(\mathbf{B})}$. ∎

References

1. Sh. Axler and J. Shapiro, Putman's theorem, Alexander's spectral area estimate and VMO, *Math. Ann.*, **271** (1985), 161-183.
2. S. Bloom and G. S. De Souza, Atomic decomposition of generalized Lipschitz spaces, *Illinois J. Math.*, **33(2)** (1989), 181-209.
3. S. Bloom and G. S. De Souza, Weighted Lipschitz spaces and their analytic characterizations, *Constr. Approx.*, **10** (1994), 339-376.
4. K. M. Dyakonov, Equivalent norms on Lipschitz-type spaces of holomorphic functions, *Acta. Math.*, **178** (1997), 143-167.
5. K. M. Dyakonov, Holomorphic functions and quasiconformal mappings with smooth moduli, *Adv. Math.*, **187** (2004), 146-172.
6. K. M. Dyakonov, Strong Hardy-Littlewood theorems for analytic functions and mappings of finite distortion, *Math. Z.*, **249(3)** (2005), 597-611.
7. J. B. Garnett, *Boundede Analytic Functions*, Academic Press, Orlando, Fla., 1981.
8. M. Pavlović, On Dyakonov's paper "Equivalent norms on Lipschitz-type spaces of holomorphic functions", *Acta. Math.*, **183** (1999), 141-143.
9. M. Pavlović, *Introduction to Function Spaces on the Disk*, Posebna izdanja 20, Mathematički Institut SANU, Beograd 2004.
10. W. Rudin, *Function Theory in the Unit Ball of* $\mathbf{C}^n$, Springer-Verlag Press, New York, 1980.

Hong Rae Cho, Su-Mi Kwon and Jinkee Lee
Department of Mathematics,
Pusan National University,
Pusan 609-735,
South Korea
E-mail: chohr@pusan.ac.kr
jklee235@pusan.ar.kr