TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 3, pp. 719-740, June 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

OPTIMALITY AND DUALITY FOR MULTIOBJECTIVE FRACTIONAL PROBLEMS WITH r-INVEXITY

Jin-Chirng Lee and Shun-Chin Ho

Abstract. We establish necessary and sufficient conditions for efficiency of multiobjective fractional programming problems involving r-invex functions. Using the optimality conditions, we investigate the parametric type dual, Wolfe type dual and Mond-Weir type dual for multiobjective fractional programming problems concerning r-invexity. Some duality theorems are also proved for such problem in the framework of r-invexity.

1. INTRODUCTION

A vector minimization problem involving nonlinear fractional functions is a natural extension of multiobjective linear fractional programming. Linear fractional criteria are frequently encountered in financial problem, game theory, decision theory, and all optimal decision problems with noncomparable criteria, e.g., in corporate planning and bank balance sheet management. Multiobjective (fractional) programming is indeed an interesting topic. Recently, there are many articles that have been of much interest, e.g., see [1, 3, 4, 6, 7, 9, 11, 14-21]. In particular, Antczak [2] introduced the concept of differentiable V-r-invexity which is a generalization of invexity. He got the Kuhn-Tucker type necessary optimality theorem, weak, strong and strictly converse duality for a multiobjective optimization programming involving differentiable V-r-invex functions. The concepts of efficiency and proper efficiency play a key role in fractional vector optimization problems. Several authors including Singh and Hanson [12], Datta [5], Kaul and Lyall [8], Liu [10] have discussed efficiency and proper efficiency to fractional vector minimization problems. In [13] Singh derived the necessary conditions for efficient optimality of differentiable multiobjective programming under a constraint qualification.

Received March 28, 2006, accepted September 14, 2006.
Communicated by Wen-Wei Lin.
2000 *Mathematics Subject Classification*: 26A51, 90C25, 90C32, 90C46.
Key words and phrases: Multiobjective fractional programming, Efficient solution, r-Invexity, Duality.

Using the results in [2] to multiobjective fractional programmings, we will investigate sufficient optimality, weak, strong and strictly converse duality for a multiobjective optimization programming involving differentiable r-invex function in this paper. We organize our paper as follows. Basic definitions and notations are given in Section 2. In Section 3, we show that Singh's necessary conditions can be applied to the multibjective fractional programming and we give a sufficient conditions for the multiobjective fractional programming. Employing these results, we construct three dual problems in Sections 4-6 where the weak, strong and strict converse duality theorems are established in the framework of r-invex functions.

2. Definitions and Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $\mathbb{R}^n_+$ its non-negative orthant.
The following convention for equalities and inequalities will be used throughout the paper.

For any $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$, we define:

(i) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \cdots, n$;

(ii) $x \geqq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2, \cdots, n$;

(iii) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

Throughout the paper, let X be a nonempty open subset of $\mathbb{R}^{\ltimes}$ and we use the same notation for row and column vectors when the interpretation is obvious.

Definition 1. [2] Let r be an arbitrary real number and η a function from $X \times X$ into $\mathbb{R}^{\ltimes}$. A differentiable function $f : X \longrightarrow \mathbb{R}$ is called a *(strictly) r-invex* function with respect to η at $u \in X$ on X if there exists a function $\alpha : X \times X \longrightarrow \mathbb{R}_+ \backslash \{0\}$ such that for each $x \in X$, the relation

$$\begin{aligned} &\frac{1}{r}e^{rf(x)} - \frac{1}{r}e^{rf(u)} \\ &\geqq e^{rf(u)}\alpha(x,u)\nabla f(u)\eta(x,u) \quad (> \text{ with } x \neq u), \quad \text{for} \quad r \neq 0, \end{aligned} \tag{2.1}$$

holds.
If $r \longrightarrow 0$, the expression (2.1) becomes

$$f(x) - f(u) \geqq \alpha(x,u)\nabla f(u)\eta(x,u) \qquad (> \text{ with } x \neq u) \tag{2.2}$$

holds.

In view of the limit, one can regard (2.2) as $r = 0$ in (2.1).

If the above inequalities are satisfied at any point $u \in X$, then f is said to be *(strictly) r-invex* with respect to η on X.

Remark 1. In the case when $\alpha(x,u) = 1$ for all $x, u \in X$ in (2.2), then f is an invex function at u on X with respect to η.

Consider a multiobjective nonlinear fractional programming problem as the following form:

$$(FP) \qquad \text{Minimize} \quad \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_k(x)}{g_k(x)}\right),$$

$$\text{subject to} \quad x \in X \in \mathbb{R}^{\ltimes} \text{ such tha}$$

$$h(x) \leqq 0, \tag{2.3}$$

where $f_i, g_i : X \longrightarrow \mathbb{R}, i = 1, 2, \cdots, k$ and $h : X \longrightarrow \mathbb{R}^{\geqslant}$ are differentiable functions. Without loss of generality, we can assume that $f_i(x) \geqq 0$, $g_i(x) > 0$, $i = 1, 2, \cdots, k$, for all $x \in X$.

Denote by $X^\circ = \{x \in X : h(x) \leqq 0\}$ the feasible solutions to (FP) and let

$$\phi_i(x) = \frac{f_i(x)}{g_i(x)} \quad \text{and} \quad \phi(x) = (\phi_1(x), \phi_2(x), \cdots, \phi_k(x)).$$

A feasible solution x^* to (FP) is said to be an *efficient solution* to (FP) if there does not exist any feasible solution x to (FP) such that $\phi(x) \leq \phi(x^*)$.

We need the following definition:

Definition 2. [13] Let $Y \subseteq \mathbb{R}^{\ltimes}$. The vector $\mu \in \mathbb{R}^{\ltimes}$ is called a *convergence vector* for Y at $\nu^\circ \in Y$ if and only if there exist a sequence $\{\nu_{\widetilde{k}}\}$ in Y and a sequence $\{\alpha_{\widetilde{k}}\}$ of positive real numbers such that

$$\text{if} \lim_{\widetilde{k} \longrightarrow \infty} \nu_{\widetilde{k}} = \nu^\circ \quad \text{and} \lim_{\widetilde{k} \longrightarrow \infty} \alpha_{\widetilde{k}} = 0, \quad \text{then} \lim_{\widetilde{k} \longrightarrow \infty} \frac{(\nu_{\widetilde{k}} - \nu^\circ)}{\alpha_{\widetilde{k}}} = \mu.$$

Denoted by $C(Y, \nu^\circ)$ the set of all convergent vectors for Y at ν°.

We say that the constraint h satisfies the *constraint qualification* at x° (cf. [13]) if

$$D \subseteq C(X^\circ, x^\circ), \tag{2.4}$$

where $C(X^\circ, x^\circ)$ is the set of all convergence vectors for X° at x° and $D = \{d \in \mathbb{R}^{\ltimes} : \nabla h_j(x^\circ) d \leqq 0 \quad \text{for all} \quad j \in J\}$ where $J = \{j : h_j(x^\circ) = 0\}$.

3. Necessary and Sufficient Optimality Conditions

In this section, we establish some necessary and sufficient conditions for efficient solution. The following Kuhn Tucker type necessary optimality conditions refers to Singh [13]

Lemma 1. *If x^* is an efficient solution to (FP) and h satisfies the constraint qualification (2.4) at x^*, then there exist $y^* \in \mathbb{R}^{\urcorner}_{+}$ and $z^* \in \mathbb{R}^{>}$ such that*

$$\begin{aligned}
&\sum_{i=1}^{k} y_i^* \nabla \phi_i(x^*) + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*) = 0,\\
&z_j^* h_j(x^*) = 0 \qquad \textit{for all} \quad j = 1, 2, \cdots, m,\\
&h_j(x^*) \leqq 0 \qquad \textit{for all} \quad j = 1, 2, \cdots, m,\\
&y^* \in I, z^* \in \mathbb{R}^{\geqslant}_{+},
\end{aligned}$$

where $I = \{y \in \mathbb{R}^{\urcorner} : y = (y_1, y_2, \cdots, y_k) > 0$ and $\sum_{i=1}^{k} y_i = 1\}$.

Since the function ϕ_i is differentiable for $i = 1, 2, \cdots, k$. From Lemma 1, there exist $y \in \mathbb{R}^{\urcorner}_{+}$ and $z \in \mathbb{R}^{>}$ such that

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{y_i}{g_i(x^*)} [\nabla f_i(x^*) - v_i^* \nabla g_i(x^*)] + \sum_{j=1}^{m} z_j \nabla h_j(x^*) = 0,\\
&z_j h_j(x^*) = 0 \qquad \text{for all} \quad j = 1, 2, \cdots, m,\\
&h_j(x^*) \leqq 0 \qquad \text{for all} \quad j = 1, 2, \cdots, m,\\
&y \in I \text{ and } z \in \mathbb{R}^{\geqslant}_{+},
\end{aligned}$$

where $v_i^* = \dfrac{f_i(x^*)}{g_i(x^*)}$, for all $i = 1, 2, \cdots, k$. Now we let $y_i^* = \dfrac{\frac{y_i}{g_i(x^*)}}{\sum_{i=1}^{k} \frac{y_i}{g_i(x^*)}}$, $i = 1, 2, \cdots, k$, then $y^* \in I$ and let $z_j^* = \dfrac{z_j}{\sum_{i=1}^{k} \frac{y_i}{g_i(x^*)}}$, $j = 1, 2, \cdots, m$, then we obtain

$$\begin{aligned}
&\sum_{i=1}^{k} y_i^* [\nabla f_i(x^*) - v_i^* \nabla g_i(x^*)] + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*) = 0,\\
&f_i(x^*) - v_i^* g_i(x^*) = 0 \qquad \text{for all} \quad i = 1, 2, \cdots, k,\\
&z_j^* h_j(x^*) = 0 \qquad \text{for all} \quad j = 1, 2, \cdots, m,\\
&h_j(x^*) \leqq 0 \qquad \text{for all} \quad j = 1, 2, \cdots, m,\\
&y^* \in I \text{ and } z^* \in \mathbb{R}^{\geqslant}_{+}.
\end{aligned}$$

Hence we can rewrite the result of Lemma 1 as the following theorem.

Theorem 1. (Necessary Optimality Conditions) *Let* x^**be an efficient solution to* (FP). *Assume that* h *satisfies the constraint qualification* (2.4) *at* x^*. *Then there exist* $y^* \in \mathbb{R}^k_+, z^* \in \mathbb{R}^m, v^* \in \mathbb{R}^{\urcorner}$ *such that* (x^*, v^*, y^*, z^*) *satisfies,*

$$\sum_{i=1}^{k} y_i^*[\nabla f_i(x^*) - v_i^* \nabla g_i(x^*)] + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*) = 0, \tag{3.1}$$

$$f_i(x^*) - v_i^* g_i(x^*) = 0 \qquad \textit{for all} \quad i = 1, 2, \cdots, k, \tag{3.2}$$

$$z_j^* h_j(x^*) = 0 \qquad \textit{for all} \quad j = 1, 2, \cdots, m, \tag{3.3}$$

$$h_j(x^*) \leqq 0 \qquad \textit{for all} \quad j = 1, 2, \cdots, m, \tag{3.4}$$

$$y^* \in I \textit{ and } z^* \in \mathbb{R}^{>}_{+}. \tag{3.5}$$

The necessary optimality conditions follows from the inverse of necessary optimality conditions with extra assumptions. We will establish the sufficient conditions under the r-invex function.

Theorem 2. (Sufficient Optimality Conditions) *Let* $x^* \in X^\circ$ *be a feasible solution to* (FP)*, and there exist* $y^* \in I \subset \mathbb{R}^{\urcorner}$, $v^* \in \mathbb{R}^{\urcorner}$, $z^* \in \mathbb{R}^{>}$ *satisfying the conditions* $(3.1) \sim (3.5)$ *at* x^*. *Furthermore suppose that any one of the conditions* (a) *or* (b) *holds:*

- (a) $A(x) = \sum_{i=1}^{k} y_i^*[f_i(x) - v_i^* g_i(x)] + \sum_{j=1}^{m} z_j^* h_j(x)$ *is an* r*-invex function with respect to* η *at* x^* *on* X°*,*
- (b) $B(x) = \sum_{i=1}^{k} y_i^*[f_i(x) - v_i^* g_i(x)]$ *and* $L(x) = \sum_{j=1}^{m} z_j^* h_j(x)$ *are the* r*-invex functions with respect to* η *at* x^* *on* X°*,*

Then x^* *is an efficient solution to* (FP).

Proof. If hypothesis (a) holds for $r \neq 0$, then from the r-invexity of A, there exists $a : X^\circ \times X^\circ \longrightarrow \mathbb{R}_+ \backslash \{0\}$ such that

$$\tfrac{1}{r} e^{rA(x)} - \tfrac{1}{r} e^{rA(x^*)} \geqq e^{rA(x^*)} a(x, x^*) \nabla A(x^*) \eta(x, x^*). \tag{3.6}$$

From (3.1), we know

$$\left[\sum_{i=1}^{k} y_i^*[\nabla f_i(x^*) - v_i^* \nabla g_i(x^*)] + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*)\right] \eta(x, x^*) = 0. \tag{3.7}$$

From (3.6) and (3.7), we have

$$A(x) \geqq A(x^*). \tag{3.8}$$

This is also valid for $r = 0$. If x^* were not an efficient solution to problem (FP), then there exists $x \in X^\circ$ such that

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(x^*)}{g_i(x^*)} = v_i^* \quad \text{for} \quad i = 1, 2, \cdots, k,$$
$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(x^*)}{g_t(x^*)} = v_t^* \quad \text{for some} \quad t \in \underline{k} = \{1, 2, \cdots, k\},$$

that is,

$$f_i(x) - v_i^* g_i(x) \leqq f_i(x^*) - v_i^* g_i(x^*) \text{ for } i = 1, 2, \cdots, k,$$
$$f_t(x) - v_t^* g_t(x) < f_t(x^*) - v_t^* g_t(x^*) \qquad \text{for some} \quad t \in \underline{k}.$$

The above relations together with the relation (3.5) imply that

$$\sum_{i=1}^{k} y_i^* [f_i(x) - v_i^* g_i(x)] < \sum_{i=1}^{k} y_i^* [f_i(x^*) - v_i^* g_i(x^*)]. \tag{3.9}$$

From the relations (2.3), (3.3) and (3.5), we obtain

$$\sum_{j=1}^{m} z_j^* h_j(x) \leqq \sum_{j=1}^{m} z_j^* h_j(x^*). \tag{3.10}$$

Consequently, (3.9) and (3.10) yield

$$\sum_{i=1}^{k} y_i^* [f_i(x) - v_i^* g_i(x)] + \sum_{j=1}^{m} z_j^* h_j(x) < \sum_{i=1}^{k} y_i^* [f_i(x^*) - v_i^* g_i(x^*)] + \sum_{j=1}^{m} z_j^* h_j(x^*),$$

which contradicts (3.8). Hence x^* is an efficient solution to (FP).

If hypothesis (b) holds for $r \neq 0$. Since $L(x)$ is an r-invex function, then there exists $l : X^\circ \times X^\circ \longrightarrow \mathbb{R}_+ \backslash \{0\}$ such that

$$\frac{1}{r} e^{rL(x)} - \frac{1}{r} e^{rL(x^*)} \geqq e^{rL(x^*)} l(x, x^*) \nabla L(x^*) \eta(x, x^*). \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\nabla L(x^*) \eta(x, x^*) \leqq 0. \tag{3.12}$$

By (3.7) and (3.12), we get

$$\left[\sum_{i=1}^{k} y_i^* [\nabla f_i(x^*) - v_i^* \nabla g_i(x^*)] \right] \eta(x, x^*) \geqq 0. \tag{3.13}$$

Since $B(x)$ is an r-invex function, then there exists $b : X^\circ \times X^\circ \longrightarrow \mathbb{R}_+\backslash\{0\}$ such that

$$\frac{1}{r}e^{rB(x)} - \frac{1}{r}e^{rB(x^*)} \geqq e^{rB(x^*)}b(x,x^*)\nabla B(x^*)\eta(x,x^*). \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$B(x) \geqq B(x^*). \tag{3.15}$$

If x^* were not an efficient solution to the problem (FP), then we get (3.9) in the same way. But (3.9) contradicts (3.15). Therefore, x^* is an efficient solution to the problem (FP) and the proof is completed. ■

4. Parametric Duality Model

We consider a parametric type dual problem as follows:

(DFP_v1) Maximize $v = (v_1, v_2, \cdots, v_k)$,

subject to

$$\sum_{i=1}^{k} y_i[\nabla f_i(u) - v_i \nabla g_i(u)] + \sum_{j=1}^{m} z_j \nabla h_j(u) = 0, \tag{4.1}$$

$$f_i(u) - v_i g_i(u) \geqq 0 \quad \text{for all} \quad i = 1, 2, \cdots, k, \tag{4.2}$$

$$\sum_{j=1}^{m} z_j h_j(u) \geqq 0, \tag{4.3}$$

$$u \in X, y \in I \subset \mathbb{R}^{\urcorner}, z \in \mathbb{R}^{\gtrdot}_{+}, v \geqq 0. \tag{4.4}$$

Let Γ denote the set of all feasible points of (DFP_v1). Moreover, we denote by $pr_X\Gamma$ the projection of the set Γ on X.

Theorem 3. (Weak Duality) *Let x be (FP)-feasible and let (u, y, z, v) be (DFP_v1)-feasible. Suppose that any one of the following conditions (a) or (b) holds:*

(a) $O(\cdot) = \sum_{i=1}^{k} y_i[f_i(\cdot) - v_i g_i(\cdot)] + \sum_{j=1}^{m} z_j h_j(\cdot)$ *is an r-invex function with respect to η on $X^\circ \cup pr_X\Gamma$,*

(b) $P(\cdot) = \sum_{i=1}^{k} y_i[f_i(\cdot) - v_i g_i(\cdot)]$ *and* $Q(\cdot) = \sum_{j=1}^{m} z_j h_j(\cdot)$ *are the r-invex functions with respect to η on $X^\circ \cup pr_X\Gamma$.*

Then $\phi(x) \nleq v$.

Proof. Let x be (FP)-feasible and let (u, y, z, v) be (DFP_v1)-feasible. By (4.1), we have

$$\left[\sum_{i=1}^{k} y_i[\nabla f_i(u) - v_i \nabla g_i(u)] + \sum_{j=1}^{m} z_j \nabla h_j(u)\right] \eta(x,u) = 0. \tag{4.5}$$

If hypothesis (a) holds for $r \neq 0$, then from the r-invexity of O, there exists $o : (X^\circ \cup pr_X\Gamma) \times (X^\circ \cup pr_X\Gamma) \longrightarrow \mathbb{R}_+ \backslash \{0\}$ such that

$$\frac{1}{r} e^{rO(x)} - \frac{1}{r} e^{rO(u)} \geqq e^{rO(u)} o(x,u) \nabla O(u) \eta(x,u). \tag{4.6}$$

From (4.5) and (4.6), we can get

$$O(x) \geqq O(u). \tag{4.7}$$

This is valid for $r = 0$.
Assume on the contrary that $\phi(x) \leq v$. Then

$$\frac{f_i(x)}{g_i(x)} \leqq v_i \quad \text{for} \quad i = 1, 2, \cdots, k,$$

and

$$\frac{f_t(x)}{g_t(x)} < v_t \quad \text{for some} \quad t \in \underline{k}.$$

Hence we have

$$f_i(x) - v_i g_i(x) \leqq 0 \leqq f_i(u) - v_i g_i(u) \quad \text{for} \quad i = 1, 2, \cdots, k,$$

and

$$f_t(x) - v_t g_t(x) < 0 \leqq f_t(u) - v_t g_t(u) \quad \text{for some} \quad t \in \underline{k}.$$

The above relations together with the relation (4.4) imply that

$$\sum_{i=1}^{k} y_i[f_i(x) - v_i g_i(x)] < \sum_{i=1}^{k} y_i[f_i(u) - v_i g_i(u)]. \tag{4.8}$$

From (2.3), (4.3) and (4.4), we obtain

$$\sum_{j=1}^{m} z_j h_j(x) \leqq \sum_{j=1}^{m} z_j h_j(u). \tag{4.9}$$

Consequently, (4.8) and (4.9) yield

$$\sum_{i=1}^{k} f_i(x) - v_i g_i(x) + \sum_{j=1}^{m} z_j h_j(x) < \sum_{i=1}^{k} f_i(u) - v_i g_i(u) + \sum_{j=1}^{m} z_j h_j(u)$$

which contradicts (4.7). Therefore, $\phi(x) \nleq v$.

If hypothesis (b) holds for $r \neq 0$, then from the r-invexity of Q, there exists $q : (X^\circ \cup pr_X\Gamma) \times (X^\circ \cup pr_X\Gamma) \longrightarrow \mathbb{R}_+\backslash\{0\}$ such that

$$\frac{1}{r}e^{rQ(x)} - \frac{1}{r}e^{rQ(u)} \geqq e^{rQ(u)} q(x,u)\nabla Q(u)\eta(x,u). \tag{4.10}$$

From (4.9) and (4.10), we obtain

$$\nabla Q(u)\eta(x,u) \leqq 0. \tag{4.11}$$

From (4.5) and (4.11), we have

$$\left[\sum_{i=1}^{k} y_i[\nabla f_i(u) - v_i \nabla g_i(u)]\right]\eta(x,u) \geqq 0. \tag{4.12}$$

Since P is an r-invex function for $r \neq 0$, there exists $p : (X^\circ \cup pr_X\Gamma) \times (X^\circ \cup pr_X\Gamma) \longrightarrow \mathbb{R}_+\backslash\{0\}$ such that

$$\frac{1}{r}e^{rP(x)} - \frac{1}{r}e^{rP(u)} \geqq e^{rP(u)} p(x,u)\nabla P(u)\eta(x,u). \tag{4.13}$$

From (4.12) and (4.13), we get

$$P(x) \geqq P(u).$$

Again if $\phi(x) \leq v$, then (4.8) holds and it would deduce a contradiction. If $r = 0$, we can get the same result similarly. The proof is now completed. ■

Theorem 4. (Strong Duality) *Let x^* be an efficient solution to problem (FP) and let h satisfy the constraint qualification* (2.4) *at x^*. Then there exist $y^* \in I \subset \mathbb{R}^{\urcorner}$, $z^* \in \mathbb{R}^{>}$ and $v^* \in \mathbb{R}^{\urcorner}$ such that (x^*, y^*, z^*, v^*) is (DFP_v1)-feasible. If the hypotheses of Theorem 3 are fulfilled, then (x^*, y^*, z^*, v^*) is an efficient solution to problem (DFP_v1) and their efficient values of (FP) and (DFP_v1) are equal.*

Proof. Let x^* be an efficient solution to problem (FP). Then there exist $y^* \in I \subset \mathbb{R}^{\urcorner}$,$z^* \in \mathbb{R}^{>}$,$v^* \in \mathbb{R}^{\urcorner}$ such that (x^*, y^*, z^*, v^*) satisfies (3.1)∼(3.5). Hence we get that (x^*, y^*, z^*, v^*) is feasible for (DFP_v1).

If (x^*, y^*, z^*, v^*) were not an efficient solution to (DFP_v1), then for feasible solution (x, y, z, v) of (DFP_v1) we have

$$v_i \geqq v_i^* = \frac{f_i(x^*)}{g_i(x^*)} \qquad \text{for all} \quad i = 1, 2, \cdots, k,$$

and

$$v_t > v_t^* = \frac{f_t(x^*)}{g_t(x^*)} \qquad \text{for some} \quad t \in \underline{k}.$$

It follows that $\phi(x^*) \leqq v$ which contradicts the weak duality (Theorem 3). Hence (x^*, y^*, z^*, v^*) is an efficient solution to (DFP_v1) and the efficient values of (FP) and (DFP_v1) are clearly equal at their respective efficient solution points. ■

Theorem 5. (Strict Converse Duality) *Let* x^* *and* (u^*, y^*, z^*, v^*) *be efficient solutions of* (FP) *and* (DFP_v1), *respectively with* $v_i^* = \frac{f_i(x^*)}{g_i(x^*)}$ *for all* $i = 1, 2, \cdots, k$. *Let*

$$A(\cdot) = \sum_{i=1}^{k} y_i^*[f_i(\cdot) - v_i^* g_i(\cdot)] + \sum_{j=1}^{m} z_j^* h_j(\cdot).$$

If A *is a strictly* r*-invex function with respect to* η *at* u^* *on* $X^\circ \cup pr_X\Gamma$. *Then* $x^* = u^*$.

Proof. We assume that $x^* \neq u^*$. By (4.1),

$$\left[\sum_{i=1}^{k} y_i^*[\nabla f_i(u^*) - v_i^* \nabla g_i(u^*)] + \sum_{j=1}^{m} z_j^* \nabla h_j^*(u^*)\right] \eta(x^*, u^*) = 0, \tag{4.14}$$

and by (4.2), (4.3) and (4.4), we get

$$A(u^*) = \sum_{i=1}^{k} y_i^*[f_i(u^*) - v_i^* g_i(u^*)] + \sum_{j=1}^{m} z_j^* h_j(u^*) \geqq 0. \tag{4.15}$$

Since A is a strictly r-invex function for $r \neq 0$, then there exists $a : (X^\circ \cup pr_X\Gamma) \times (X^\circ \cup pr_X\Gamma) \longrightarrow \mathbb{R}_+ \backslash \{0\}$ such that

$$\frac{1}{r} e^{rA(x^*)} - \frac{1}{r} e^{rA(u^*)} > e^{rA(u^*)} a(x^*, u^*) \nabla A(u^*) \eta(x^*, u^*). \tag{4.16}$$

From (4.14) and (4.16), we can get

$$A(x^*) > A(u^*). \tag{4.17}$$

This is also valid for $r = 0$.
Since

$$v_i^* = \frac{f_i(x^*)}{g_i(x^*)} \quad \text{for all} \quad i = 1, 2, \cdots, k,$$

from (2.3) and (4.4), we have

$$\sum_{j=1}^{m} z_j^* h_j(x^*) \leqq 0.$$

Therefore,

$$A(x^*) = \sum_{i=1}^{k} y_i^* [f_i(x^*) - v_i^* g_i(x^*)] + \sum_{j=1}^{m} z_j^* h_j(x^*) \leqq 0. \tag{4.18}$$

From (4.17) and (4.18), we have $A(u^*) < 0$ which contradicts (4.15). Hence $x^* = u^*$. ■

Remark 2. The function $A(\cdot)$ in Theorem 5 is expressed by the sum of the modified objective part $B(\cdot)$ of (FP) and its constraint part $L(\cdot)$. We do not know whether both of such two parts are strictly r-invex functions if $A(\cdot) = B(\cdot) + L(\cdot)$ is a strictly r-invex function. However if $B(\cdot)$ is a strictly r-invex function and $L(\cdot)$ is an r-invex function then the (strict converse duality) Theorem 5 is still valid. We state this situation in the following theorem.

Theorem 6. (Strict Converse Duality) *Let x^* and (u^*, y^*, z^*, v^*) be efficient solutions of (FP) and (DFP_v1), respectively, with $v_i^* = \frac{f_i(x^*)}{g_i(x^*)}$ for all $i = 1, 2, \cdots, k$. Let*

$$B(\cdot) = \sum_{i=1}^{k} y_i^* [f_i(\cdot) - v_i^* g_i(\cdot)]$$

be a strictly r-invex function
and let

$$L(\cdot) = \sum_{j=1}^{m} z_j^* h_j(\cdot)$$

be an r-invex function with respect to η at u^* on $X^\circ \cup pr_X\Gamma$. Then $x^* = u^*$.

Proof. We assume that $x^* \neq u^*$. By (4.1),

$$\left[\sum_{i=1}^{k} y_i^* [\nabla f_i(u^*) - v_i^* \nabla g_i(u^*)] + \sum_{j=1}^{m} z_j^* \nabla h_j^*(u^*)\right] \eta(x^*, u^*) = 0, \tag{4.19}$$

and by (4.2) and (4.4), we get

$$\sum_{i=1}^{k} y_i^*[f_i(u^*) - v_i^* g_i(u^*)] \geq 0. \tag{4.20}$$

From (2.3), (4.3) and (4.4), we obtain

$$\sum_{j=1}^{m} z_j^* h_j(x^*) \leq \sum_{j=1}^{m} z_j^* h_j(u^*). \tag{4.21}$$

Since L is an r-invex function for $r \neq 0$, then there exists $l : (X^\circ \cup pr_X\Gamma) \times (X^\circ \cup pr_X\Gamma) \longrightarrow \mathbb{R}_+ \backslash \{0\}$ such that

$$\frac{1}{r}e^{rL(x^*)} - \frac{1}{r}e^{rL(u^*)} \geq e^{rL(u^*)} l(x^*, u^*) \nabla L(u^*) \eta(x^*, u^*). \tag{4.22}$$

By (4.21) and (4.22), we obtain

$$\nabla L(u^*) \eta(x^*, u^*) \leq 0. \tag{4.23}$$

From (4.19) and (4.23), we know

$$\left[\sum_{i=1}^{k} y_i^*[\nabla f_i(u^*) - v_i^* \nabla g_i(u^*)]\right] \eta(x^*, u^*) \geq 0. \tag{4.24}$$

Since B is a strictly r-invex function for $r \neq 0$, then there exists $b : (X^\circ \cup pr_X\Gamma) \times (X^\circ \cup pr_X\Gamma) \longrightarrow \mathbb{R}_+ \backslash \{0\}$ such that

$$\frac{1}{r}e^{rB(x^*)} - \frac{1}{r}e^{rB(u^*)} > e^{rB(u^*)} b(x^*, u^*) \nabla B(u^*) \eta(x^*, u^*). \tag{4.25}$$

From (4.24) and (4.25), we can get

$$B(x^*) > B(u^*). \tag{4.26}$$

Since $v_i^* = \dfrac{f_i(x^*)}{g_i(x^*)}$ for all $i = 1, 2, \cdots, k$, from (4.4) we get

$$B(x^*) = \sum_{i=1}^{k} y_i^*[f_i(x^*) - v_i^* g_i(x^*)] = 0. \tag{4.27}$$

From (4.26) and (4.27), we obtain $B(u^*) < 0$ which contradicts (4.20). If $r = 0$, we can obtain the same result similarly. Hence the proof is completed. ■

Remark 3. In Theorem 6, if L is a strictly r-invex function and B is an r-invex function with respect to η, then the conclusion of Theorem 6 also holds.

5. Wolfe Type Dual Model

In order to propose the Wolfe type dual model, it is convenient to restate the necessary conditions in Theorem 1 as the following form. At first, by the expressions (3.1) and (3.2), we get

$$\begin{aligned} 0 &= \sum_{i=1}^{k} y_i^*[\nabla f_i(x^*) - v_i^* \nabla g_i(x^*)] + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*) \\ &= \sum_{i=1}^{k} y_i^*[\nabla f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*)] + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*). \end{aligned}$$

Replace y_i^* by $\overline{y_i} g_i(x^*), i = 1, 2, \cdots, k$ and from (3.5), we obtain

$$\sum_{i=1}^{k} \overline{y_i} g_i(x^*)[\nabla f_i(x^*) + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*)] + \sum_{i=1}^{k} \overline{y_i} f_i(x^*) \nabla(-g_i(x^*)) = 0. \tag{5.1}$$

Then (3.3) and (5.1) imply

$$\begin{aligned} &\sum_{i=1}^{k} \overline{y_i} g_i(x^*)[\nabla f_i(x^*) + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*)] \\ &\quad + \sum_{i=1}^{k} \overline{y_i}[f_i(x^*) + \sum_{j=1}^{m} z_j^* h_j(x^*)] \nabla(-g_i(x^*)) = 0. \end{aligned}$$

Consequently, Theorem 1 can be rewritten as the following theorem:

Theorem 7. (Necessary Optimality Conditions) *Let x^* be an efficient solution to (FP). Assume that h satisfies the constraint qualification (2.4) at x^*. Then there exist $y^* \in \mathbb{R}_+^k$ and $z^* \in \mathbb{R}^m$ such that (x^*, y^*, z^*) satisfies,*

$$\begin{aligned} &\sum_{i=1}^{k} y_i^* g_i(x^*)[\nabla f_i(x^*) + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*)] \\ &\quad + \sum_{i=1}^{k} y_i^*(-\nabla g_i(x^*))[f_i(x^*) + \sum_{j=1}^{m} z_j^* h_j(x^*)] = 0, \end{aligned} \tag{5.2}$$

$$z_j^* h_j(x^*) = 0 \quad \textit{for all} \quad j = 1, 2, \cdots, m, \tag{5.3}$$

$$h_j(x^*) \leqq 0 \quad \textit{for all} \quad j = 1, 2, \cdots, m, \tag{5.4}$$

$$y^* \in I \quad \textit{and} \quad z^* \in \mathbb{R}_+^{\geqslant}. \tag{5.5}$$

The Wolfe type dual model to (FP) is given as follows

$$(DFP2) \qquad \text{Maximize} \left(\frac{f_1(u) + \sum_{j=1}^{m} z_j h_j(u)}{g_1(u)}, \cdots, \frac{f_k(u) + \sum_{j=1}^{m} z_j h_j(u)}{g_k(u)} \right)$$

subject to

$$\begin{aligned} &\sum_{i=1}^{k} y_i g_i(u)[\nabla f_i(u) + \sum_{j=1}^{m} z_j \nabla h_j(u)] \\ &\quad + \sum_{i=1}^{k} y_i [f_i(u) + \sum_{j=1}^{m} z_j h_j(u)] \nabla(-g_i(u)) = 0, \end{aligned} \tag{5.6}$$

$$u \in X, y \in I \subset \mathbb{R}^{\urcorner}, z \in \mathbb{R}^{\geqq}_{+}. \tag{5.7}$$

Let $\widetilde{\Gamma}$ denote the set of all feasible points of $(DFP2)$. Moreover, we denote by $pr_X\widetilde{\Gamma}$ the projection of the set $\widetilde{\Gamma}$ on X. Denote by $\Psi_i(u, z) = \dfrac{f_i(u) + \sum_{j=1}^{m} z_j h_j(u)}{g_i(u)}$ and $\Psi(u, z) = (\Psi_1(u, z), \Psi_2(u, z), \cdots, \Psi_k(u, z))$.

Assume throughout this section that $f_i(u) + \sum_{j=1}^{m} z_j h_j(u) \geqq 0$ and $g_i(u) > 0$, for all $i = 1, 2, \cdots, k$.

Theorem 8. (Weak Duality) *Let x be (FP)-feasible and let (u, y, z) be $(DFP2)$-feasible. Let*

$$S(\cdot) = \sum_{i=1}^{k} y_i g_i(u)[f_i(\cdot) + \sum_{j=1}^{m} z_j h_j(\cdot)] - \sum_{i=1}^{k} y_i g_i(\cdot)[f_i(u) + \sum_{j=1}^{m} z_j h_j(u)].$$

If S is an r-invex function with respect to η at u on $X^\circ \cup pr_X\widetilde{\Gamma}$, then $\phi(x) \nleq \Psi(u, z)$.

Proof. Let x be (FP)-feasible and let (u, y, z) be $(DFP2)$-feasible. From (5.6), we get

$$\begin{aligned} &\Bigg[\sum_{i=1}^{k} y_i g_i(u)[\nabla f_i(u) + \sum_{j=1}^{m} z_j \nabla h_j(u)] + \sum_{i=1}^{k} y_i [f_i(u) \\ &\quad + \sum_{j=1}^{m} z_j h_j(u)] \nabla(-g_i(u)) \Bigg] \eta(x, u) = 0. \end{aligned} \tag{5.8}$$

Since S is an r-invex function with respect to η at u for $r \neq 0$, there exists $s : (X^\circ \cup pr_X\widetilde{\Gamma}) \times (X^\circ \cup pr_X\widetilde{\Gamma}) \longrightarrow \mathbb{R}_+\backslash\{0\}$ such that

$$\frac{1}{r}e^{rS(x)} - \frac{1}{r}e^{rS(u)} \geqq e^{rS(u)}s(x,u)\nabla S(u)\eta(x,u). \tag{5.9}$$

From (5.8) and (5.9), we obtain

$$S(x) \geqq S(u) = 0. \tag{5.10}$$

Note that (5.10) is also valid for $r = 0$.
Suppose on the contrary that $\phi(x) \leq \Psi(u,z)$. Then

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(u) + \sum_{j=1}^{m} z_j h_j(u)}{g_i(u)} \qquad \text{for} \qquad i = 1, 2, \cdots, k,$$

and

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(u) + \sum_{j=1}^{m} z_j h_j(u)}{g_t(u)} \qquad \text{for some} \qquad t \in \underline{k}.$$

It follows that

$$\sum_{i=1}^{k} y_i[f_i(x)g_i(u)] < \sum_{i=1}^{k} y_i g_i(x)[f_i(u) + \sum_{j=1}^{m} z_j h_j(u)]$$

which is equivalently to

$$\begin{aligned} &\sum_{i=1}^{k} y_i[f_i(x) + \sum_{j=1}^{m} z_j h_j(x)]g_i(u) - \sum_{i=1}^{k} y_i g_i(x)[f_i(u) \\ &\quad + \sum_{j=1}^{m} z_j h_j(u)] < \sum_{i=1}^{k} y_i g_i(u) \sum_{j=1}^{m} z_j h_j(x). \end{aligned} \tag{5.11}$$

Since $g_i(u) > 0$ and $h_j(x) \leqq 0$, $\sum_{i=1}^{k} y_i g_i(u) \sum_{j=1}^{m} z_j h_j(x) \leqq 0$. Therefore (5.11), implies

$$\sum_{i=1}^{k} y_i g_i(u)[f_i(x) + \sum_{j=1}^{m} z_j h_j(x)] - \sum_{i=1}^{k} y_i g_i(x)[f_i(u) + \sum_{j=1}^{m} z_j h_j(u)] < 0.$$

This contradicts (5.10) and the proof is completed. ■

Theorem 9. (Strong Duality) *Let x^* be an efficient solution to problem (FP) and let h satisfy the constraint qualification (2.4) at x^*. Then there exist $y^* \in I \subset \mathbb{R}^{\urcorner}$ and $z^* \in \mathbb{R}^{>}$ such that $(x^*, y^*, z^*) \in (DFP2)$-feasible. If the hypotheses of Theorem 8 are fulfilled, then (x^*, y^*, z^*) is an efficient solution to problem $(DFP2)$ and the two efficient values of (FP) and $(DFP2)$ are equal.*

Proof. Let x^* be an efficient solution to problem (FP). Then there exist $y^* \in I \subset \mathbb{R}^{\urcorner}$ and $z^* \in \mathbb{R}^{>}$ such that (x^*, y^*, z^*) satisfies (5.2)~(5.5). Hence (x^*, y^*, z^*) is feasible to $(DFP2)$.
If (x^*, y^*, z^*) were not an efficient solution to $(DFP2)$, then for feasible solution (x, y, z) of $(DFP2)$ we have

$$\frac{f_i(x^*) + \sum_{j=1}^{m} z_j^* h_j(x^*)}{g_i(x^*)} \leqq \frac{f_i(x) + \sum_{j=1}^{m} z_j h_j(x)}{g_i(x)} \quad \text{for} \quad i = 1, 2, \cdots, k,$$

$$\frac{f_t(x^*) + \sum_{j=1}^{m} z_j^* h_j(x^*)}{g_t(x^*)} < \frac{f_t(x) + \sum_{j=1}^{m} z_j h_j(x)}{g_t(x)} \quad \text{for some} \quad t \in \underline{k}.$$

From the above inequalities, (5.3) and (5.5), we can get $\phi(x^*) \leq \Psi(x, z)$ which contradicts the weak duality (Theorem 8). Hence (x^*, y^*, z^*) is an efficient solution to $(DFP2)$ and the efficient values of (FP) and $(DFP2)$ are clearly equal at their respective efficient solution points. ■

Theorem 10. (Strict Converse Duality) *Let x^* and (u^*, y^*, z^*) be efficient solutions to (PF) and $(DFP2)$, respectively.*
Let

$$W(\cdot) = \sum_{i=1}^{k} y_i^* g_i(u^*)[f_i(\cdot) + \sum_{j=1}^{m} z_j^* h_j(\cdot)] - \sum_{i=1}^{k} y_i^* g_i(\cdot)[f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)].$$

If W is a strictly r-invex function with respect to η at u^* on $X^\circ \cup pr_X\widetilde{\Gamma}$. Then $x^* = u^*$.

Proof. Suppose that $x^* \neq u^*$. By (5.6), we have

$$\left[\sum_{i=1}^{k} y_i^* g_i(u^*)[\nabla f_i(u^*) + \sum_{j=1}^{m} z_j^* \nabla h_j(u^*)] + \sum_{i=1}^{k} y_i^* [f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)] \nabla(-g_i(u^*)) \right] \eta(x^*, u^*) = 0. \tag{5.12}$$

From Theorem 9, we know that there exist $\overline{y}$ and $\overline{z}$ such that $(x^*, \overline{y}, \overline{z})$ is the efficient solution to $(DFP2)$ and

$$\frac{f_i(x^*) + \sum_{j=1}^{m} \overline{z_j} h_j(x^*)}{g_i(x^*)} = \frac{f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)}{g_i(u^*)}. \tag{5.13}$$

By (5.3), (5.5) and (5.13), we obtain

$$\frac{f_i(x^*)}{g_i(x^*)} = \frac{f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)}{g_i(u^*)}.$$

Hence

$$f_i(x^*) g_i(u^*) = [f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)] g_i(x^*). \tag{5.14}$$

From (5.7) and (5.14), we have

$$W(x^*) = \sum_{i=1}^{k} y_i^* g_i(u^*) \sum_{j=1}^{m} z_j^* h_j(x^*).$$

It follows from the above equality together with (2.3), (5.7) and $g_i(u^*) > 0$ that $W(x^*) \leqq 0$. Since $W(x^*) \leqq 0$ and $W(u^*) = 0$, we know

$$W(x^*) \leqq W(u^*). \tag{5.15}$$

From the strictly r-invexity of W with respect to η at u^* for $r \neq 0$, there exists $w : (X^\circ \cup pr_X \widetilde{\Gamma}) \times (X^\circ \cup pr_X \widetilde{\Gamma}) \longrightarrow \mathbb{R}_+ \backslash \{0\}$ such that

$$\frac{1}{r} e^{rW(x^*)} - \frac{1}{r} e^{rW(u^*)} > e^{rW(u^*)} w(x^*, u^*) \nabla W(u^*) \eta(x^*, u^*). \tag{5.16}$$

Therefore, (5.15) and (5.16) yield

$$\left[\sum_{i=1}^{k} y_i^* g_i(u^*) [\nabla f_i(u^*) + \sum_{j=1}^{m} z_j^* \nabla h_j(u^*)] + \sum_{i=1}^{k} y_i^* [f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)] \nabla(-g_i(u^*)) \right] \eta(x^*, u^*) < 0. \tag{5.17}$$

Observe that (5.17) is also valid for $r = 0$.

Hence we get (5.17) which contradicts (5.12). Thus the proof is completed. ■

6. Mond-weir Duality

In this section, we consider the Mond-Weir type dual problem as follows:

$$(DFP3) \qquad \text{Maximize} \qquad \left(\frac{f_1(u)}{g_1(u)}, \cdots, \frac{f_k(u)}{g_k(u)}\right)$$

subject to

$$\begin{aligned} &\sum_{i=1}^{k} y_i g_i(u)[\nabla f_i(u) + \sum_{j=1}^{m} z_j \nabla h_j(u)] \\ &\quad + \sum_{i=1}^{k} y_i(-\nabla g_i(u))[f_i(u) + \sum_{j=1}^{m} z_j h_j(u)] = 0, \end{aligned} \tag{6.1}$$

$$\sum_{j=1}^{m} z_j h_j(u) \geqq 0, \tag{6.2}$$

$$u \in X, y \in I \subset \mathbb{R}^{\urcorner}, z \in \mathbb{R}^{\gtrless}_{+}. \tag{6.3}$$

Let $\overline{\Gamma}$ denote the set of all feasible points of $(DFP3)$. Moreover, we denote by $pr_X\overline{\Gamma}$ the projection of the set $\overline{\Gamma}$ on X.
Denoted by

$$\Phi_i(u) = \frac{f_i(u)}{g_i(u)} \quad \text{and} \quad \Phi(u) = (\Phi_1(u), \Phi_2(u), \cdots, \Phi_k(u).$$

Theorem 11. (Weak Duality) *Let x be (FP)-feasible and let (u, y, z) be $(DFP3)$-feasible. Define*

$$S(\cdot) = \sum_{i=1}^{k} y_i g_i(u)[f_i(\cdot) + \sum_{j=1}^{m} z_j h_j(\cdot)] - \sum_{i=1}^{k} y_i g_i(\cdot)[f_i(u) + \sum_{j=1}^{m} z_j h_j(u)].$$

If S is an r-invex function with respect to η at u on $X^\circ \cup pr_X\overline{\Gamma}$, then $\phi(x) \nleqq \Phi(u)$.

Proof. Let x be (FP)-feasible and let (u, y, z) be $(DFP3)$-feasible. By (6.1), we get

$$\begin{bmatrix} \sum_{i=1}^{k} y_i g_i(u)[\nabla f_i(u) + \sum_{j=1}^{m} z_j \nabla h_j(u)] \\ + \sum_{i=1}^{k} y_i [f_i(u) + \sum_{j=1}^{m} z_j h_j(u)] \nabla(-g_i(u)) \end{bmatrix} \eta(x, u) = 0. \tag{6.4}$$

Since S is an r-invex function with respect to η at u for $r \neq 0$, then there exists $s : (X^\circ \cup pr_X\overline{\Gamma}) \times (X^\circ \cup pr_X\overline{\Gamma}) \longrightarrow \mathbb{R}_+ \backslash \{0\}$ such that

$$\frac{1}{r}e^{rS(x)} - \frac{1}{r}e^{rS(u)} \geqq e^{rS(u)} s(x,u)\nabla S(u)\eta(x,u). \tag{6.5}$$

From (6.4) and (6.5), we obtain

$$S(x) \geq S(u) = 0. \tag{6.6}$$

Note that (6.6) is also valid for $r = 0$.
Suppose on the contrary that $\phi(x) \leq \Phi(u)$. Then

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(u)}{g_i(u)} \qquad \text{for all} \qquad i = 1, 2, \cdots, k,$$

and

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(u)}{g_t(u)} \qquad \text{for some} \qquad t \in \underline{k}.$$

It follows that

$$\sum_{i=1}^{k} y_i[f_i(x)g_i(u)] < \sum_{i=1}^{k} y_i g_i(x) f_i(u). \tag{6.7}$$

From (2.3), (6.2), (6.3) and (6.7), we obtain

$$\sum_{i=1}^{k} y_i g_i(u)[f_i(x) + \sum_{j=1}^{m} z_j h_j(x)] < \sum_{i=1}^{k} y_i g_i(x)[f_i(u) + \sum_{j=1}^{m} z_j h_j(u)].$$

This implies

$$\sum_{i=1}^{k} y_i g_i(u)[f_i(x) + \sum_{j=1}^{m} z_j h_j(x)] - \sum_{i=1}^{k} y_i g_i(x)[f_i(u) + \sum_{j=1}^{m} z_j h_j(u)] < 0$$

which contradicts (6.6). Thus the proof is complete. ∎

Theorem 12. (Strong Duality) *Let x^* be an efficient solution to problem (FP) and let h satisfy the constraint qualification* (2.4) *at x^*. Then there exist $y^* \in I \subset \mathbb{R}^{\urcorner}$ and $z^* \in \mathbb{R}^{\geqslant}$ such that $(x^*, y^*, z^*) \in (DFP3)$-feasible. If the hypotheses of Theorem 11 are fulfilled, then (x^*, y^*, z^*) is an efficient solution to problem $(DFP3)$ and both efficient values of (FP) and $(DFP3)$ are equal.*

Proof. Let x^* be an efficient solution to problem (FP). Then there exist $y^* \in I \subset \mathbb{R}^{\urcorner}$ and $z^* \in \mathbb{R}^{\geqslant}$ such that (x^*, y^*, z^*) satisfies $(5.2) \sim (5.5)$. Hence we

get (x^*, y^*, z^*) is feasible for $(DFP3)$.
If (x^*, y^*, z^*) were not an efficient solution to $(DFP3)$, then for any feasible solution (x, y, z) of $(DFP3)$ we have

$$\frac{f_i(x^*)}{g_i(x^*)} \leqq \frac{f_i(x)}{g_i(x)} \quad \text{for} \quad i = 1, 2, \cdots, k,$$
$$\frac{f_t(x^*)}{g_t(x^*)} < \frac{f_t(x)}{g_t(x)} \quad \text{for some} \quad t \in \underline{k}.$$

It follows that $\phi(x^*) \leq \Phi(x)$ which contradicts the weak duality (Theorem 11). Hence (x^*, y^*, z^*) is an efficient solution to $(DFP3)$ and both efficient values of (FP) and $(DFP3)$ are clearly equal at their respective efficient solution points. ■

Theorem 13. (Strict Converse Duality) *Let x^* and (u^*, y^*, z^*) be efficient solutions of (PF) and $(DFP3)$, respectively. Define*

$$W(\cdot) = \sum_{i=1}^{k} y_i^* g_i(u^*)[f_i(\cdot) + \sum_{j=1}^{m} z_j^* h_j(\cdot)] - \sum_{i=1}^{k} y_i^* g_i(\cdot)[f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)].$$

If W is a strictly r-invex function with respect to η at u^* on $X^\circ \cup pr_X\overline{\Gamma}$. Then $x^* = u^*$.

Proof. Suppose that $x^* \neq u^*$. By (6.1), we have

$$\left[\sum_{i=1}^{k} y_i^* g_i(u^*)[\nabla f_i(u^*) + \sum_{j=1}^{m} z_j^* \nabla h_j(u^*)] + \sum_{i=1}^{k} y_i^* [f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)] \nabla(-g_i(u^*))\right] \eta(x^*, u^*) = 0. \tag{6.8}$$

By Theorem 12, there exist $\overline{y}$ and $\overline{z}$ such that $(x^*, \overline{y}, \overline{z})$ is an efficient solution to $(DFP3)$ and for each i

$$\frac{f_i(x^*)}{g_i(x^*)} = \frac{f_i(u^*)}{g_i(u^*)}. \tag{6.9}$$

By (6.2) and (6.9), we have

$$\frac{f_i(x^*)}{g_i(x^*)} \leqq \frac{f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)}{g_i(u^*)}. \tag{6.10}$$

Hence

$$f_i(x^*)g_i(u^*) \leqq [f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)]g_i(x^*). \tag{6.11}$$

From (2.3), (6.3), (6.11) and $g_i > 0$, we get

$$W(x^*) \leqq 0 \quad \text{and} \quad W(u^*) = 0. \tag{6.12}$$

Since W is a strictly r-invex function with respect to η at u^* for $r \neq 0$, then there exists $w : (X^\circ \cup pr_X\overline{\Gamma}) \times (X^\circ \cup pr_X\overline{\Gamma}) \longrightarrow \mathbb{R}_+\backslash\{0\}$ such that

$$\frac{1}{r}e^{rW(x^*)} - \frac{1}{r}e^{rW(u^*)} > e^{rW(u^*)}w(x^*,u^*)\nabla W(u^*)\eta(x^*,u^*). \tag{6.13}$$

From (6.12) and (6.13), we obtain

$$\begin{aligned}\Bigg[&\sum_{i=1}^{k} y_i^* g_i(u^*)[\nabla f_i(u^*) + \sum_{j=1}^{m} z_j^* \nabla h_j(u^*)] \\ &+ \sum_{i=1}^{k} y_i^* [f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)]\nabla(-g_i(u^*))\Bigg] \eta(x^*,u^*) < 0\end{aligned} \tag{6.14}$$

which is also valid for $r = 0$. But then (6.14) contradicts (6.8). Thus the proof is completed. ■

References

1. T. Antczak, Multiobjective Programming under d-invexity, *European Journal of Operational Research*, **137** (2002), 28-36.
2. T. Antczak, The Notion of V-r-Invexity in Differentiable Multiobjective Programming, *J. Appl. Anal.*, **11** (2005), 63-79.
3. D. Bhatia and S. Pandey, A Note on Multiobjective Fractional Programming, *Cahiers du Cero*, **33** (1991), 3-11.
4. D. Bhatia and P. K. Garg, Duality for Nonsmooth Nonlinear Fractional Multiobjective Programs Via (F, ρ)-Convexity, *Optimization*, **43** (1998), 185-197.
5. N. Datta, Efficiency in Multi-Objective Fractional Programming, *J. Inform. Optim. Sci.*, **3** (1982), 262-268.
6. A. M. Geoffrion, Proper Efficiency and Theory of Vector Maximization, *J. Math. Anal. Appl.*, **22** (1968), 618-630.
7. M. A. Hanson, On Sufficiency of the Kuhn-Tucker Conditions, *J. Math. Anal. Appl.*, **80** (1981), 545-550.

8. R. N. Kaul and V. Lyall, A Note on Nonlinear Fractional Vector Maximization, *Opsearch*, **26** (1989), 108-121.

9. Z. A. Khan and M. A. Hanson, On Ratio Invexity in Mathematical Programming, *J. Math. Anal. Appl.*, **205** (1997), 330-336.

10. J. C. Liu, Optimality and Duality for Multiobjectional Programming Involving Nonsmooth Pseudoinvex Functions, *Optimization*, **37** (1996), 27-39.

11. R. N. Mukherjee, Generalized Convex Duality for Multiobjective Fractional Programs, *J. Math. Anal. Appl.*, **162** (1991), 309-316.

12. C. Singh and M. A, Hanson, Saddlepoint Theory for Nondifferentiable Multiobjective Fractional Programming, *J. Inform. Optim. Sci.*, **7** (1986), 41-48.

13. C. Singh, Optimality Conditions in Multiobjective Differentiable Programming, *J. Optim. Theory Appl.*, **53** (1987), 115-123.

14. C. Singh and M. A, Hanson, Generalized Proper Efficiency in Multiobjective Programming, *J. Inform. Optim. Sci.*, **12** (1991), 139-144.

15. C. Singh, S. K. Suneja and N. G. Rueda, Preinvexity in Multiobjective Fractional Programming, *J. Inform. Optim. Sci.*, **13** (1992), 293-302.

16. S. K. Suneja and S. Gupta, Duality in Multiple Objective Fractional Programming Problems Involving Non-convex Functions, *Opsearch*, **27** (1990), 239-253.

17. S. K. Suneja and M. K. Srivastava, Duality in Multiobjective Fractional Programming Involving Generalized Invexity, *Operational Research Society of India*, **31** (1994), 127-143.

18. S. K. Suneja and C. S. Lalitha, Multiobjective Fractional Programming Involving ρ-Invex and Related Functions, *Opsearch*, **30** (1993), 1-14.

19. T. Weir, A Duality for a Multiple Objective Fractional Programming Problem, *J. Inform. Optim. Sci.*, **7** (1986), 261-269.

20. T. Weir, A Note on Invex Functions and Multiple Objective Optimization, *Opsearch*, **25** (1988), 98-104.

21. T. Weir, Duality for Nondifferentiable Multiple Objective Fractional Programming Problems, *Utilitas Math.*, **36** (1989), 53-64.

Jin-Chirng Lee and Shun-Chin Ho
Department of Applied Mathematics,
Chung Yung Christian University,
Chung-Li 320,
Taiwan
E-mail: jclee@math.cycu.edu.tw