TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 3, pp. 623-633, June 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

OSCILLATION THEOREM FOR SECOND-ORDER DIFFERENCE EQUATIONS

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Abstract. Sufficient and necessary conditions are established for the second-order difference equations

$$\Delta(r_{n-1}\Delta x_{n-1}) + p_n x_n^{\gamma} = 0, n = 1, 2, \ldots$$

where γ is the quotient of odd positive integers. Our results extend the well known oscillation theorem which was proved in [1,JMAA,91:9-29,1983], and answer an open problem in [2] when $r_n = 1, \gamma = 1$.

1. Introduction

Consider the second order difference equations

$$\Delta(r_{n-1}\Delta x_{n-1}) + p_n x_n^{\gamma} = 0, n = 1, 2, \ldots \tag{1}$$

where $\Delta x_n = x_{n+1} - x_n$,$\gamma$ is the quotient of odd positive integers and $p_n, r_n \in (0, \infty)$ for $n = 1, 2, \ldots$ with p_n not eventually equal to zero. Denote

$$R_n = \sum_{s=0}^{n-1} \frac{1}{r_s} \tag{2}$$

and assume

$$\lim_{n\to\infty} R_n = \sum_{s=0}^{\infty} \frac{1}{r_s} = \infty. \tag{3}$$

Received March 4, 2006, accepted June 30, 2006.
Communicated by Sze-Bi Hsu.
2000 *Mathematics Subject Classification*: 39A11.
Key words and phrases: Oscillation, Nonoscillatory solution, Difference equations, Contraction principle.
This research is supported by NNSF of P. R. China (No. 10601040) and 973 project of P. R. China (No. 2006CB708304).

A solution x_n of Eq. (1) is oscillatory if x_n are neither eventually all positive nor eventually all negative. Otherwise, it is called nonoscillatory.

Eq. (1) is a discrete analogue of the second order equation

$$(r(t)y')' + p(t)y^{\gamma} = 0 \tag{4}$$

which appears in astrophysics, relativistic mechanics,nuclear physics,chemical reactions,etc (See [2]).

we say that Eq. (1) and Eq.(4) is strictly superlinear if $\gamma > 1$; strictly sublinear if $0 < \gamma < 1$; and linear if $\gamma = 1$.

When $r_n = 1$,Eq. (1) was reduced to

$$\Delta^2 x_{n-1} + p_n x_n^{\gamma} = 0, n = 1, 2, \dots \tag{5}$$

which is the discrete analogue of the Emden-Fowler equation ([2])

$$y'' + p(t)y^{\gamma} = 0. \tag{6}$$

In the past years, the theory of the oscillatory behavior of second-order differential and difference equations have been investigated by many authors, and numerous oscillation criteria have been obtained (see [1-9]). For example, for the linear differential equations, that is, when $\gamma = 1$, numerous oscillation criteria have been obtained. when $\gamma = 1, r(t) = 1$, most important conditions that guarantee Eq.(4) is oscillatory as follows:

$(A1)$ $\lim\limits_{t\to\infty} \int_{t_0}^{t} p(s)ds = \infty.$ $\quad (Fite[3])$

$(A2)$ $\lim\limits_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} p(u)duds = \infty.$ $\quad (W\text{int}ner[4])$

$(A3)$ $-\infty < \lim\limits_{t\to\infty} \inf \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} p(u)duds < \lim\limits_{x\to\infty} \sup \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} p(u)duds \leq \infty.$ (Hartman[5]).

Kamenev [6] gave another condition for the oscillation, i.e.,

$(A4)$ $\lim\limits_{t\to\infty} \sup \frac{1}{t^n} \int_0^t (t-s)^n p(s)ds = \infty, n > 1.$

The Kamenev criterion has been extended by several authors. Among them, when $\gamma = 1, r(t) \neq 1$, Philos [7], Q.Kong [8] obtain results on the oscillation by replacing the kerned function $(t-s)^n$ by a general class of functions $H(t, s)$ satisfying certain assumptions. This class of functions is as follows: Let $H : D \equiv \{(t, s) : t \geq s \geq t_0\} \to R$ be a continuous function such that

$$H(t,t) = 0 \quad for \quad t \geq t_0 \quad and \quad H(t,s) > 0 \quad for \quad t > s \geq t_0,$$

and has a continuous and nonpositive partial derivative $H_s(t, s)$ on D with respect to the second variable. Moreover, let $h : D \to R$ be a continuous function with

$$H_s(t,s) = -h(t,s)\sqrt{H(t,s)} \quad for \quad all \quad (t,s) \in D,$$

some related concrete results can be found in [7-8] and the references therein. However, most of these articles only discussed the sufficient conditions by use of the Raccati technique and integral averaging technique. Just a few of them have studied the necessary and sufficient conditions for the oscillatory behavior. In [1], the authors studied oscillatory criteria for Eq.(5) and obtained very important results. To the best of our knowledge, paper [1] is probably the only one publications on the necessary and sufficient conditions for the oscillatory of solution of second-order difference equations.

The following theorem was established by Hooker and Patula [1].

Theorem A. (a) *Assume that*

$$\sum_{n=1}^{\infty} np_n < \infty \tag{7}$$

Then Eq. (5) *has a bounded nonoscillatory solution.*

(b) *Assume that* $\gamma > 1$. *Then every solution of Eq.* (5) *oscillates if and only if*

$$\sum_{n=1}^{\infty} np_n = \infty. \tag{8}$$

(c) *Assume that* $\gamma = 1$ *and*

$$\sum_{n=1}^{\infty} p_n = \infty. \tag{9}$$

Then every solution of Eq. (5) *oscillates.*

(d) *Assume that* $0 < \gamma < 1$, *then every solution of Eq.* (5) *oscillates if and only if*

$$\sum_{n=1}^{\infty} n^{\gamma} p_n = \infty. \tag{10}$$

It is interesting to note that when $\gamma \neq 1$, that is when Eq.(5) is nonlinear, we know necessary and sufficient conditions for oscillation. However, for the linear equation, that is when $\gamma = 1$, necessary and sufficient conditions for oscillation are not known. It follows from Theorem A (a) that when $\gamma = 1$,

$$\sum_{n=1}^{\infty} np_n = \infty$$

is a necessary condition for all bounded solutions to oscillate. But is it sufficient? (see[2])

Motived by these results, our aim in this paper is to investigate the oscillatory of the solution of so-called generalized Emden-Fowler difference equation (1) and establish several sufficient and necessary conditions. Our following results extent Theorem A and answer the above problem .

Our main results are:

Theorem. *Assume that* (3) *holds, then the following statements are true:*

(a) *Assume that*

$$\sum_{n=1}^{\infty} R_n p_n < \infty \tag{11}$$

Then Eq. (1) *has a bounded nonoscillatory solution.*

(b) *Assume that* $\gamma > 1$*, then every solution of Eq.* (1) *oscillates if and only if*

$$\sum_{n=1}^{\infty} R_n p_n = \infty. \tag{12}$$

(c) *Assume that* $\gamma = 1$*, then every bounded solution of Eq.* (1) *oscillates if and only if*

$$\sum_{n=1}^{\infty} R_n p_n = \infty. \tag{13}$$

(d) *Assume that* $0 < \gamma < 1$*, then every solution of Eq.* (1) *oscillates if and only if*

$$\sum_{n=1}^{\infty} R_n^{\gamma} p_n = \infty. \tag{14}$$

Remark 1. When $r_n = 1$, then $R_n = n$, our theorem extend and improve Theorem A.

2. The Proof of the Theorem

Before we present the proof of the theorem we need the following lemma.

Lemma. *Assume that (3) holds and the Eq. (1) has nonoscillatory solution. Then Eq. (1) has a solution* $\{x_n\}$ *such that for some* $N > 0$*,*

$$x_n > 0, \Delta x_n > 0, and \quad \Delta(r_n \Delta x_n) < 0 \quad for \quad n \geq N. \tag{15}$$

Proof. As γ is the quotient of odd integers, the opposite of a solution is also a solution and so Eq. (1) has a solution $\{x_n\}$ which is eventually positive. That is, there exists $N > 0$ such that

$$x_n > 0 \quad for \quad n \geq N.$$

From Eq. (1) we see that

$$\Delta(r_{n-1}\Delta x_{n-1}) = -p_n x_n^\gamma < 0, \quad for \quad n \geq N$$

Therefore, $r_{n-1}\Delta x_{n-1}$ is monotone decreasing for $n \geq N$. The next cases are possible:

Case 1. $r_{n-1}\Delta x_{n-1} < 0, n \geq N$. then we have $r_{n-1}\Delta x_{n-1} \leq r_{N-1}\Delta x_{N-1} = c_{N-1} < 0, n \geq N$, and obtain $\Delta x_{n-1} \leq c_{N-1}\frac{1}{r_{n-1}}, x_n \leq x_{N-1}+c_{N-1}\sum_{s=N}^{n}\frac{1}{r_{s-1}} \to -\infty$, which contradicts the assumption that $x_n > 0$.

Case 2. $r_{n-1}\Delta x_{n-1} > 0, n \geq N$. In this case, we have

$$\Delta x_n > 0, n \geq N.$$

The proof of the lemma is complete.

Proof of the Theorem.

(a) Assume that (11) holds. We must prove that Eq. (1) has a bounded nonoscillatory solution.

Observe that if $\{x_n\}$ satifies the equation

$$x_n = 1 - \sum_{i=n+1}^{\infty}(R_i - R_n)p_i x_i^\gamma, \tag{16}$$

then $\{x_n\}$ is a solution of Eq. (1). Therefore it suffices to show that Eq. (16) has bounded nonoscillatory solution. To this end, choose N so large that

$$\max\{\sum_{i=N}^{\infty} R_i p_i, 2\gamma\sum_{i=N}^{\infty} R_i p_i\} < \frac{1}{2}. \tag{17}$$

Consider the Banach space l_∞^N of all bounded, real sequences $z = \{z_n\}_{n\geq N}$ with the norm defined by $||z|| = \sup_{n\geq N}|z_n|$.

Set

$$S = \{z \in l_\infty^N : \frac{1}{2} \leq z_n \leq 1, n \geq N\}. \tag{18}$$

Clearly S is a closed subset of l_∞^N. Define the mapping T on S by

$$(Tz)_n = 1 - \sum_{i=n+1}^{\infty} (R_i - R_n) p_i z_i^\gamma \quad for\ n \geq N. \tag{19}$$

Note that $z_i^\gamma \leq 1$ and so

$$(Tz)_n \geq 1 - \sum_{i=n+1}^{\infty} (R_i - R_n) p_i \geq \frac{1}{2} \quad for\ n \geq N.$$

Also clearly, $(Tz)_n \leq 1$. Thus $T : S \to S$. We now claim that T is a contraction on S. Set $f(x) = x^\gamma$, we find for $x_1, x_2 \in (\frac{1}{2}, 1)$,

$$|x_1^\gamma - x_2^\gamma| \leq |f'(\xi)||x_1 - x_2|,\ where\ \xi \in (\min\{x_1, x_2\}, \max\{x_1, x_2\}).$$

But

$$|f(x)| = |\gamma \xi^{\gamma-1}| \leq \begin{cases} \gamma & if\ \gamma \geq 1\,, \\ 2\gamma & if\ 0 < \gamma < 1\,. \end{cases}$$

And so

$$|x_1^\gamma - x_2^\gamma| \leq 2\gamma |x_1 - x_2| \quad for\ x_1, x_2 \in (\frac{1}{2}, 1).$$

Let $z, w \in S$. Then for $n \geq N$,

$$\begin{aligned} |(Tz)_n - (Tw)_n| &\leq \sum_{i=n+1}^{\infty} (R_i - R_n) p_i |z_i^\gamma - w_i^\gamma| \\ &\leq 2\gamma \sum_{i=n+1}^{\infty} (R_i - R_n) p_i |z_i - w_i| \\ &\leq 2\gamma |z - w| \sum_{i=n+1}^{\infty} (R_i - R_n) p_i \leq \frac{1}{2} ||z - w||. \end{aligned}$$

Hence

$$||Tz - Tw|| \leq \frac{1}{2} ||z - w||. \tag{20}$$

and so T is a contraction on S. The (unique) fixed point of T is the desired bounded, nonoscillatory solution of Eq. (16).

(b) Assume that (12) holds,we must prove every solution of Eq. (1) oscillates. Otherwise by lemma , Eq. (1) has a solution $\{x_n\}$ such that (15) holds. By multiplying both sides of Eq. (1) by $R_n x_n^{-\gamma}$ and then by summing up we obtain

$$\sum_{n=N}^{k-1} R_n x_n^{-\gamma} \Delta(r_{n-1}\Delta x_{n-1}) + \sum_{n=N}^{k-1} R_n p_n = 0.$$

By using the summation-by-parts formula we find

$$R_k x_k^{-\gamma} r_{k-1}\Delta x_{k-1} - R_N x_N^{-\gamma} r_{N-1}\Delta x_{N-1} - \sum_{n=N}^{k-1} r_n \Delta x_n \Delta(R_n x_n^{-\gamma}) + \sum_{n=N}^{k-1} R_n p_n = 0.$$

Hence

$$\sum_{n=N}^{k-1} r_n \Delta x_n \Delta(R_n x_n^{-\gamma}) = \infty. \tag{21}$$

Now observe that

$$\Delta(R_n x_n^{-\gamma}) = \frac{1}{r_n} x_{n+1}^{-\gamma} + R_n \Delta(x_n^{-\gamma}) \leq \frac{1}{r_n} x_{n+1}^{-\gamma}.$$

and so

$$\sum_{n=N}^{k-1} r_n \Delta x_n \Delta(R_n x_n^{-\gamma}) \leq \sum_{n=N}^{k-1} r_n \Delta x_n \frac{1}{r_n} x_{n+1}^{-\gamma} = \sum_{n=N}^{k-1} \Delta x_n x_{n+1}^{-\gamma} \tag{22}$$

Set $f(x) = x_n + (\Delta x_n)(x-n)$ for $n \leq x \leq n+1$ and $n \geq N$, then f is continuous and increasing for $x \geq N$, and so

$$\begin{aligned} x_{n+1}^{-\gamma}\Delta x_n &= \int_n^{n+1} x_{n+1}^{-\gamma}\Delta x_n dx = \int_n^{n+1} f(n+1)^{-\gamma} f'(x) dx \\ &< \int_n^{n+1} f(x)^{-\gamma} f'(x) dx = \frac{1}{\gamma-1}[f(n+1)^{1-\gamma} - f(n)^{1-\gamma}] \end{aligned}$$

By summing up from $n = N$ to $n = k-1$ we obtain

$$\sum_{n=N}^{k-1} x_{n+1}^{-\gamma}\Delta x_n \leq \frac{1}{\gamma-1}[f(k)^{1-\gamma} - f(N)^{1-\gamma}] \leq \frac{f(N)^{1-\gamma}}{\gamma-1}$$

which because of (22) contradicts (21).

Conversely we must prove that if every solution of Eq. (1) oscillates and $\gamma > 1$, then (12) holds. Otherwise (11) holds and by (a) we obtain the contraction that Eq. (1) has a nonoscillatory solution.

(c) Assume that $\gamma = 1$, We will prove that every bounded solution of Eq. (1) oscillates. Otherwise by lemma , Eq. (1) has a solution $\{x_n\}$ such that (15) holds, then $\lim_{n\to\infty} x_n = c > 0$, and there exists $N > 0$ such that

$$\frac{c}{2} \leq x_n \leq c, \quad for \quad n > N.$$

By means of Eq. (1), we have

$$\Delta(r_{n-1}\Delta x_{n-1}) + \frac{c}{2}p_n \leq 0, for \quad n > N, \tag{23}$$

multiplying both sides of inequality (23) by R_n and then by summing up we obtain

$$\sum_{n=N}^{k-1} R_n\Delta(r_{n-1}\Delta x_{n-1}) + \frac{c}{2}\sum_{n=N}^{k-1} R_n p_n \leq 0.$$

By the summation-by-parts formula we have

$$R_k r_{k-1}\Delta x_{k-1} - R_N r_{N-1}\Delta x_{N-1} - \sum_{n=N}^{k-1} r_n\Delta x_n\Delta R_n + \frac{c}{2}\sum_{n=N}^{k-1} R_n p_n \leq 0,$$

Observe that $\sum_{n=N}^{k-1} r_n\Delta x_n\Delta R_n = \sum_{n=N}^{k-1} \Delta x_n = x_k - x_N$ then we get

$$R_k r_{k-1}\Delta x_{k-1} - R_N r_{N-1}\Delta x_{N-1} - x_k + x_N + \frac{c}{2}\sum_{n=N}^{k-1} R_n p_n \leq 0,$$

from above inequality we have $x_k \to \infty$, which obviously contradicts the bounded behavior of x_n .

Conversely, we should prove that if every bounded solution of Eq. (1) oscillates then (13) holds. Otherwise (11) holds and by (a) we obtain the contraction that Eq. (1) has a bounded nonoscillatory solution.

(d) Assume that (14) holds, we shall prove that every solution of Eq. (1) oscillates. Otherwise by lemma, Eq. (1) has a solution $\{x_n\}$ such that (15) holds.

Set $g_n = r_n\Delta x_n$, then $\Delta g_n = \Delta(r_n\Delta x_n) < 0$, g_n is decreasing for $n \geq N$. Observe that for $n \geq N$,

$$x_n - x_N = \sum_{s=N}^{n-1} \Delta x_s = \sum_{s=N}^{n-1} \frac{g_s}{r_s} \geq g_n \sum_{s=N}^{n-1} \frac{1}{r_s}$$

and so

$$\frac{x_n}{g_n} = \frac{x_n}{r_n\Delta x_n} \geq \sum_{s=N}^{n-1} \frac{1}{r_s}, \quad for \quad n \geq N. \tag{24}$$

By dividing both terms of Eq. (1) by $(r_n\Delta x_n)^\gamma$,and then applying (24), and finally by summing up from $n = N$ to $n = k$, we obtain

$$\sum_{n=N}^{k} \frac{\Delta(r_n\Delta x_n)}{(r_n\Delta x_n)^\gamma} + \sum_{n=N}^{k} R_n^\gamma p_n \le 0. \tag{25}$$

In view of (25) it follows that

$$\sum_{n=N}^{k} \frac{\Delta(r_n\Delta x_n)}{(r_n\Delta x_n)^\gamma} = -\infty. \tag{26}$$

Set $g(x) = r_n\Delta x_n + \Delta(r_n\Delta x_n)(x-n) \quad for \quad n \le x \le n+1$ and $n \ge N$,then $g(x)$ is continuous and decreasing for $n \ge N$.

$$g(x) \le g(n) = r_n\Delta x_n, \quad for \quad n \le x \le n+1.$$

Then we have

$$\frac{\Delta(r_n\Delta x_n)}{(r_n\Delta x_n)^\gamma} = \int_n^{n+1} \frac{\Delta(r_n\Delta x_n)}{(r_n\Delta x_n)^\gamma} dx \ge \int_n^{n+1} \frac{g'(x)}{g^\gamma(x)} dx$$

By summing up from $n = N$ to $n = k$, this implies that

$$\sum_{n=N}^{k} \frac{\Delta(r_n\Delta x_n)}{(r_n\Delta x_n)^\gamma} \ge \frac{g^{1-\gamma}(k) - g^{1-\gamma}(N)}{1-\gamma} \ge \frac{-g^{1-\gamma}(N)}{1-\gamma},$$

which, as $k \to \infty$ contradicts (26).

Conversely, we should prove that if every solution of Eq. (1) oscillates, then (14) holds. Otherwise $\sum\limits_{n=1}^{\infty} R_n^\gamma p_n < \infty$.

Now choose N_0 so large that $\sum\limits_{n=N_0}^{\infty} R_n^\gamma p_n < \frac{1}{2}$. Let $\{x_n\}$ be the unique solution of solution of Eq. (1) with

$$x_{N_0} = 0, x_{N_0+1} = \frac{1}{r_{N_0}}.$$

That is

$$x_{N_0} = 0, g_{N_0} = r_{N_0}\Delta x_{N_0} = 1.$$

By induction, we can prove that

$$\frac{1}{2} \le g_n = r_n\Delta x_n \le 1 \ for \ all \ n \ge N_0. \tag{27}$$

In fact, assume that $\frac{1}{2} \leq r_n \Delta x_n \leq 1 \ for \ n \leq N-1$,where $N > N_0$, then it is obvious that $x_n \geq 0 \ for \ n \leq N-1$. We obtain

$$x_n = x_n - x_{N_0} = \sum_{s=N_0}^{n-1} \Delta x_s = \sum_{s=N_0}^{n-1} \frac{g_s}{r_s} \leq \sum_{s=N_0}^{n-1} \frac{1}{r_s} \quad for \quad n \leq N.$$

Therefore

$$\begin{aligned} 1 \geq r_N \Delta x_N &= r_{N_0} \Delta x_{N_0} - \sum_{s=N_0}^{N-1} p_s x_s^\gamma \\ &\geq r_{N_0} \Delta x_{N_0} - \sum_{s=N_0}^{\infty} p_s R_s^\gamma \geq 1 - \tfrac{1}{2} = \tfrac{1}{2}. \end{aligned}$$

That is: $\frac{1}{2} \leq r_n \Delta x_n \leq 1 \ for \ n = N$. So (27) holds. $\{x_n\}$ is a nonoscillatory solution of Eq. (1). This contradiction completes the proof of part (d).

The proof of the theorem is complete.

Remark 2. From the above proof of the theorem, we see that our results are also valid for following more general equation:

$$\Delta(r_{n-1}\Delta x_{n-1}) + p_n |x_n|^\gamma \mathrm{sgn} x_n = 0, n = 1, 2, ...$$

where $\gamma > 0$.

Acknowledgment

We would like to thank Prof. A.B.Dishliev for his valuable advices and very careful reading of the manuscript. We also thank the referee for his many valuable suggestions on our paper.

References

1. J. W. Hooker, and W. T. Patula, A second order nonlinear difference equation: Oscillation and asymptotic growth. *J. Math. Anal. Appl.*, **91** (1983), 9-29.
2. V. L. Kocic, and G. E. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, 1993.
3. W. B. Fite, Concerning the zeros of the solutions of certain differential equations, *Trans. Amer. Math. Soc.*, **19** (1918), 341-352.
4. A. Wintner, A criterion of oscillatory, *Quart. Appl. Math.*, **7** (1949), 115-117,.

5. P. Hartman, On nonoscillatory linear differential equations of second, *Amer. J. Math.*, **74** (1952), 389-400.

6. I. V. Kamenev, An integral criterion for oscillation of linear differential equations of second order (in Russian), *Mat. Zametki*, **23** (1978), 249-251.

7. C. G. Philos, Oscillation criteria for second order superlinear differential equations, *Canad. J. Math.*, **41** (1989), 321-340.

8. Q. Kong, Interval criteria for oscillation of second-order linear ordinary differential equations, *J. Math. Anal. Appl.*, **229** (1999), 258-270.

9. R. P. Agarwal, S. R. Grace and D. O'Regan, *Oscillation theory for difference and functional differential equations*, Kluwer, Dordrecht, 2000.

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