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STRONG CONVERGENCE THEOREMS BY THE VISCOSITY APPROXIMATION METHOD FOR A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we extend Moudafi's result in a Hilbert space to that in a Banach spaces. Then, we introduce implicit and explicit sequences for an infinite family of nonexpansive mappings in Banach spaces and prove strong convergence theorems for finding a common fixed point of the family of mappings.

1. Introduction

Let E be a Banach space and let C be a closed convex subset of E. Then a mapping T from C into itself is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

For a mapping T of C into itself, we denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. Let f be a function of C into itself. Then, f is said to be a-contractive on C if there exists a constant $a \in (0, 1)$ such that $\|f(x) - f(y)\| \leq a\|x - y\|$ for all $x, y \in C$. We denote that $Cont(C)$ is the set of all contractions on C. In 1967, Browder [3] obtained the following:

Theorem 1.1. *Let H be a Hilbert space and let C be a closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let x_0 be an arbitrary point of C and define $S_n : C \to C$ by*

$$S_n x = (1 - \alpha_n)Tx + \alpha_n x_0$$

for all $x \in C$ and $n \in \mathbb{N}$, where $0 < \alpha_n < 1$. Then the following hold:

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(i) *S_n has a unique fixed point $u_n \in C$;*

(ii) *if $\alpha_n \to 0$, then the sequence $\{u_n\}$ converges strongly to $P_{F(T)}x_0$ where $P_{F(T)}$ is the metric projection onto $F(T)$.*

After Browder's result, such a problem has been investigated by many authors: for instance, see Marino and Trombetta [8] and Takahashi and Kim [17]. In 2000, Moudafi [9] proved the following strong convergence theorem:

Theorem 1.2. *Let H be a Hilbert space and let C be a closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty and let f be a-contractive of C into itself. Let*

$$x_n = \frac{1}{1+\epsilon_n}Tx_n + \frac{\epsilon_n}{1+\epsilon_n}f(x_n), \tag{1}$$

where $\{\epsilon_n\}$ is a sequence in $(0,1)$ and $\epsilon_n \to 0$. Then $\{x_n\}$ converges strongly to the unique solution $\hat{x} \in C$ of the variational inequality

$$\hat{x} \in F(T) \text{ such that } \langle (I-f)\hat{x}, \hat{x}-x\rangle \le 0, \quad \forall x \in F(T),$$

i.e., $\hat{x} = P_{F(T)}f(\hat{x})$.

Further, in 2004, Xu [20] extended Moudafi's result in the framework of a Hilbert space to that in a uniformly smooth Banach space.

In this paper, motivated by Moudafi's result, we first extend Moudafi's result in a Hilbert space to that in a Banach space (Theorem 3.1). Next, we prove a strong convergence theorem for finding a common fixed point of an infinite family of nonexpansive mappings. Finally, using Theorem 3.1, we consider the problem of finding a zero of an accretive operator.

2. Preliminaries and Lemmas

We denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{R}$ and $\mathbb{R}^+$ the sets of all real numbers and all nonnegative real numbers, respectively. A Banach space E is called *uniformly convex* if for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|x_n + y_n\| = 2$, $\lim_{n\to\infty}\|x_n - y_n\| = 0$ holds. A closed convex sebset of C of a Banach space E is said to have *normal structure* if for each bounded closed convex subset of K of C which contains at least two points, there exists an element of K which is not a diametral point of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure. The following result was proved by Kirk [7].

Theorem 2.1. *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty.*

Let E be a Banach space and let E^* be its dual, that is, the space of all continuous linear functionals f on E. The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{f \in E^* : f(x) = \|x\|^2 = \|f\|^2\}$$

for every $x \in E$. The norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t \to \infty} \frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

exists for each $x, y \in S_E = \{x \in E : \|x\| = 1\}$. The norm of E said to be *uniformly Gâteaux differentiable* if for each y in S_E, the limit (2) is attained uniformly for $x \in S_E$. If a Banach space E has a Gâteaux differentiable norm, then the duality mapping J is single valued. Further, we have

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, J(y)\rangle$$

for every$x, y \in E$; see [15]. Let μ be a mean on $\mathbb{N}$, i.e., a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf_{n \in \mathbb{N}} a_n \leq \mu(f) \leq \sup_{n \in \mathbb{N}} a_n$$

for each $f = (a_1, a_2, \ldots,) \in l^\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(f)$. A Banach limit μ is a mean μ on $\mathbb{N}$ satisfying $\mu_n(a_n) = \mu_n(a_{n+1})$. Let $f = (a_1, a_2, \cdots,) \in l^\infty$ with $a_n \to a$ and let μ be a Banach limit on $\mathbb{N}$. Then, $\mu(f) = \mu_n(a_n) = a$; see [15] for more details. Further, we know the following result [16].

Lemma 2.2. *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm, let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on $\mathbb{N}$. Let $z \in C$. Then*

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if $\mu_n \langle y - z, J(x_n - z)\rangle \leq 0$ for all $y \in C$, where J is the duality mapping of E.

We also know the following lemma [13].

Lemma 2.3. *Let a be a real number and let $(a_1, a_2, \cdots) \in l^\infty$ such that $\mu_n(a_n) \le a$ for all Banach limits μ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \le 0$. Then, $\limsup_{n\to\infty} a_n \le a$.*

Let $T_1, T_2, \ldots$ be infinite mappings of C into itself and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 \le \lambda_i \le 1$ for every $i \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, Takahashi [14] (see also [12, 18 and 6]) defined a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned}$$

Such a mapping W_n is called the *W-mapping* generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Using [12] and [1], we obtain the following two lemmas.

Lemma 2.4. *Let C be a nonempty closed convex subset of a Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^\infty F(T_i)$ is nonempty and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_1 \le 1$ and $0 < \lambda_i \le b < 1$ for any $i = 2, 3, \ldots$. Then for every $x \in C$ and $k \in \mathbb{N}$, the $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.4, for $k \in \mathbb{N}$, we define mappings $U_{\infty,k}$ and U of C into itself as follows:

$$U_{\infty,k}x = \lim_{n\to\infty} U_{n,k}x$$

and

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a U is called the *W-mapping* generated by $T_1, T_2, \ldots,$ and $\lambda_1, \lambda_2, \ldots$.

Lemma 2.5. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that*

$\bigcap_{i=1}^{\infty} F(T_i)$ *is nonempty and let* $\lambda_1, \lambda_2, \ldots$ *be real numbers such that* $0 < \lambda_1 \leq 1$ *and* $0 < \lambda_i \leq b < 1$ *for any* $i = 2, 3, \ldots$. *Let* $W_n (n = 1, 2, \ldots)$ *be the* W*-mappings of* C *into itself generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ *and let* U *be the* W*-mapping generated by* $T_1, T_2, \ldots$ *and* $\lambda_1, \lambda_2, \ldots$. *Then* $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ *and* $F(U) = \bigcap_{i=1}^{\infty} F(T_i)$.

An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be *accretive* if for any $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. A mapping $B \subset E \times E$ is said to be c-strongly accretive if for any $x_i \in D(B)$ and $y_i \in Bx_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq c\|x_1 - x_2\|^2$, where $c > 0$. If A is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$$

for all $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, and $r > 0$. An accretive operator A is said to be *m-accretive* if $R(I + rA) = E$ for all $r > 0$. If A is accretive, then we can define, for any $r > 0$, a nonexpansive single valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the *resolvent* of A. We also define the *Yosida approximation* A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know that for an m-accretive operator A, we have $A^{-1}0 = F(J_r)$ for all $r > 0$. See [15] for more details.

3. Strong Convergence Theorem

We first prove the following strong convergene theorem which generalizes the Browder's convergence theorem. Our proof employs the methods of Reich [11], Takahashi and Kim [17]; see [15].

Theorem 3.1. *Let* E *be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let* C *be a closed convex subset of* E, *and let* T *be a nonexpansive mapping of* C *into itself such that* $F(T) \neq \emptyset$. *Let* f *be an* a*-contractive mapping of* C *into itself. For* $n \in \mathbb{N}$, *define* $S_n : C \to C$ *by*

$$S_n x = (1 - \alpha_n)Tx + \alpha_n f(x)$$

for each $x \in C$, *where* $0 < \alpha_n < 1$. *Then the following hold:*

(i) S_n *has a unique fixed point* u_n *in* C;

(ii) *if* $\alpha_n \to 0$, *then the sequence* $\{u_n\}$ *converges strongly to* $u \in F(T)$.

Further, for each $f \in Cont(C)$, define P with $P(f) = \lim_{n\to\infty} u_n$. Then $P(f)$ solves the variational inequality

$$\langle (I-f)P(f), J(P(f)-x)\rangle \geq 0 \quad \textit{for any } x \in F(T). \tag{3}$$

Proof. (i) Let $x, y \in C$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \|S_n x - S_n y\| &\leq (1-\alpha_n)\|Tx - Ty\| + \alpha_n \|f(x) - f(y)\| \\ &\leq (1-\alpha_n)\|x-y\| + a\alpha_n\|x-y\| \\ &= (1-\alpha_n(1-a))\|x-y\|. \end{aligned}$$

Then, since S_n is a contraction of C into itself, there exists a unique fixed point u_n of S_n in C.

(ii) Let $z \in F(T)$. Since

$$\begin{aligned} \|u_n - z\| &= \|(1-\alpha_n)(Tu_n - z) + \alpha_n(f(u_n) - z)\| \\ &\leq (1-\alpha_n)\|u_n - z\| + \alpha_n\|f(u_n) - z\| \\ &\leq (1-\alpha_n)\|u_n - z\| + \alpha_n\{\|f(u_n) - f(z)\| + \|f(z) - z\|\} \\ &\leq (1-\alpha_n)\|u_n - z\| + a\alpha_n\|u_n - z\| + \alpha_n\|f(z) - z\|, \end{aligned}$$

we have

$$\|u_n - z\| \leq \frac{1}{1-a}\|f(z) - z\|.$$

Since $\{u_n\}$ is bounded, for any subsequence $\{u_{n_i}\}$ of $\{u_n\}$, we can define a real valued function g on C given by

$$g(z) = \mu_i \|u_{n_i} - z\|$$

for any $z \in C$, where μ is a Banach limit. Define the set

$$M = \{v \in C : g(v) = \inf_{z\in C} g(z)\}.$$

Then M is nonempty, bounded, convex and closed; for more details, see [15]. Further, since

$$\begin{aligned} \|u_n - Tu_n\| &\leq \|(1-\alpha_n)Tu_n + \alpha_n f(u_n) - Tu_n\| \\ &= \alpha_n\|Tu_n - f(u_n)\| \end{aligned}$$

and Tu_n and $f(u_n)$ are bounded, we obtain

$$\lim_{n\to\infty}\|u_n - Tu_n\| = 0. \tag{4}$$

For any $v \in M$, from (4), we have

$$\begin{aligned}\mu_i\|u_{n_i} - Tv\| &\leq \mu_i\{\|u_{n_i} - Tu_{n_i}\| + \|Tu_{n_i} - Tv\|\} \\ &\leq \mu_i\|u_{n_i} - v\|.\end{aligned}$$

This implies that M is T-invariant. Therefore, from Theorem 2.1, we have a fixed point z_0 of T in M. Next, we show that $\{u_n\}$ converges strongly to a fixed point of T. Since z_0 is a minimizer of the function g on C, by Lemma 2.2, we have

$$\mu_i\langle z - z_0, J(u_{n_i} - z_0)\rangle \leq 0 \tag{5}$$

for all $z \in C$. Putting $z = f(z_0)$ in (5), we have

$$\mu_i\langle f(z_0) - z_0, J(u_{n_i} - z_0)\rangle \leq 0. \tag{6}$$

Since

$$\begin{aligned}&\|u_{n_i} - z_0\|^2 \\ &= \langle u_{n_i} - z_0, J(u_{n_i} - z_0)\rangle \\ &= (1 - \alpha_{n_i})\langle Tu_{n_i} - z_0, J(u_{n_i} - z_0)\rangle + \alpha_{n_i}\langle f(u_{n_i}) - z_0, J(u_{n_i} - z_0)\rangle \\ &\leq (1 - \alpha_{n_i})\|u_{n_i} - z_0\|^2 + \alpha_{n_i}\langle f(u_{n_i}) - z_0, J(u_{n_i} - z_0)\rangle,\end{aligned}$$

we have $\|u_{n_i} - z_0\|^2 \leq \langle f(u_{n_i}) - z_0, J(u_{n_i} - z_0)\rangle$ and hence

$$\mu_i\|u_{n_i} - z_0\|^2 \leq \mu_i\langle f(u_{n_i}) - z_0, J(u_{n_i} - z_0)\rangle, \tag{7}$$

where μ is a Banach limit. From (6) and (7), we have

$$\begin{aligned}&\mu_i\|u_{n_i} - z_0\|^2 \\ &\leq \mu_i\langle f(u_{n_i}) - f(z_0), J(u_{n_i} - z_0)\rangle + \mu_i\langle f(z_0) - z_0, J(u_{n_i} - z_0)\rangle \\ &\leq \mu_i\langle f(u_{n_i}) - f(z_0), J(u_{n_i} - z_0)\rangle \\ &\leq a\mu_i\|u_{n_i} - z_0\|^2.\end{aligned}$$

This implies $\mu_i\|u_{n_i} - z_0\|^2 = 0$. So, we can choose a subsequence $\{u_{n_j}\}$ of $\{u_{n_i}\}$ such that $\{u_{n_j}\}$ converges strongly to z_0. In order to prove $\{u_n\}$ converges strongly to a fixed point of T, we assume that $\{u_{n_k}\} \to z$ and $\{u_{n_l}\} \to \hat{z}$. Then, from

$$\begin{aligned}\|z - Tz\| &\leq \|z - u_{n_k}\| + \|u_{n_k} - Tz\| \\ &\leq \|z - u_{n_k}\| + \|(1 - \alpha_{n_k})Tu_{n_k} + \alpha_{n_k}f(u_{n_k}) - Tz\| \\ &\leq 2\|z - u_{n_k}\| + \alpha_{n_k}\|Tu_{n_k} - f(u_{n_k})\|,\end{aligned}$$

we obtain $z = Tz$. Similarly, we have $\hat{z} = T\hat{z}$. Since, $I - T$ is accretive, we have for any $w \in F(T)$, $\langle u_n - Tu_n, J(u_n - w)\rangle \geq 0$. From $u_n = (1-\alpha_n)Tu_n + \alpha_n f(u_n)$, we have

$$\langle Tu_n - f(u_n), J(u_n - w)\rangle \leq 0. \tag{8}$$

From (8), we have

$$\langle Tu_{n_k} - f(u_{n_k}), J(u_{n_k} - \hat{z})\rangle \leq 0$$

and

$$\langle Tu_{n_l} - f(u_{n_l}), J(u_{n_l} - z)\rangle \leq 0.$$

So, we have $\langle Tz - f(z), J(z - \hat{z})\rangle \leq 0$ and $\langle T\hat{z} - f(\hat{z}), J(\hat{z} - z)\rangle \leq 0$. Since $Tz = z$ and $T\hat{z} = \hat{z}$, we have

$$\langle z - f(z), J(z - \hat{z})\rangle \leq 0$$

and

$$\langle \hat{z} - f(\hat{z}), J(\hat{z} - z)\rangle \leq 0.$$

This implies

$$\|z - \hat{z}\|^2 \leq \langle f(z) - f(\hat{z}), J(z - \hat{z})\rangle \leq a\|z - \hat{z}\|^2.$$

So, we obtain $z = \hat{z}$. Therefore, $\{u_n\}$ converges strongly to a fixed point of T. Now, we define a mapping $P : Cont(C) \to F(T)$ by $P(f) = \lim_{n\to\infty} u_n$. Since $(I - f)u_n = -\frac{1-\alpha_n}{\alpha_n}(I - T)u_n$, we have

$$\begin{aligned}\langle (I - f)u_n, J(u_n - x)\rangle &= -\tfrac{1-\alpha_n}{\alpha_n}\langle (I - T)u_n, J(u_n - x)\rangle \\ &\leq 0\end{aligned}$$

for all $x \in F(T)$. Taking the limit, we obtain

$$\langle (I - f)P(f), J(P(f) - x)\rangle \leq 0. \qquad \blacksquare$$

Further, using the *W-mapping*, we obtain the following theorem.

Theorem 3.2. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a closed convex subset of E, and let $\{T_n\}$ be a countable family of nonexpansive mappings of C into itself suth that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty. Let f be a-contractive mapping of C into itself. Let b be a real number with $0 < b < 1$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_1 \leq 1$ and $0 < \lambda_i \leq b < 1$ for every $i = 2, 3, \ldots$. Let $W_n (n = 1, 2, \ldots)$ be W-mappings* *of C into itself generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. Let U be the W-mapping generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$, i.e.,*

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$$

for every $x \in C$. For $n \in \mathbb{N}$, define $S_n : C \to C$ by

$$S_n x = (1 - \frac{1}{n})Ux + \frac{1}{n} f(x)$$

for each $x \in C$. Then the following hold:

- (i) *S_n has a unique fixed point u_n in C;*
- (ii) *the sequence $\{u_n\}$ converges strongly to $u \in F(U)$. Further, for each $f \in Cont(C)$, define $P(f) = \lim_{n\to\infty} u_n$.*

Then $P(f)$ solves the variational inequality

$$\langle (I - f)P(f), J(P(f) - x)\rangle \geq 0 \quad \text{for any } x \in F(U). \tag{9}$$

Next, using Theorem 3.2, we prove the following strong convergence theorem for finding a common fixed point of a countable family of nonexpansive mappings.

Theorem 3.3. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm . Let C be a closed convex subset of E, and let $\{T_n\}$ be a countable family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty. Let f be a-contractive mapping of C into itself. Let b be a real number with $0 < b < 1$ and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 < \lambda_1 \leq 1$ and $0 < \lambda_i \leq b < 1$ for every $i = 2, 3, \dots$. Let $W_n (n = 1, 2, \dots)$ be W-mappings of C into itself generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. Let U be the W-mapping generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$, i.e.,*

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$$

for every $x \in C$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) W_n x_n, \quad n = 1, 2, \dots, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Then, $\{x_n\}$ converges strongly to $z = P_{F(U)} f(z)$, where $P_{F(U)}$is the sunny nonexpansive retraction of C onto $F(U)$.

Proof. From Lemma 2.5, we obtain $\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(W_n) = F(U)$. For any $u \in \bigcap_{i=1}^{\infty} F(T_i)$, we have

$$\begin{aligned}\|x_{n+1} - u\| &\le \alpha_n \|f(x_n) - u\| + (1-\alpha_n)\|W_n x_n - u\| \\ &\le \alpha_n \{\|f(x_n) - f(u)\| + \|f(u) - u\|\} + (1-\alpha_n)\|x_n - u\| \\ &\le (1 - \alpha_n(1-a))\|x_n - u\| + \alpha_n(1-a)\tfrac{1}{1-a}\|f(u) - u\|.\end{aligned}$$

If $\|x_n - u\| \le \frac{1}{1-a}\|f(u) - u\|$, we obtain

$$\|x_{n+1} - u\| \le \frac{1}{1-a}\|f(u) - u\|.$$

If $\|x_n - u\| \ge \frac{1}{1-a}\|f(u) - u\|$, we obtain

$$\|x_{n+1} - u\| \le \|x_n - u\|.$$

So, we have $\{x_n\}$ is bounded. We also obtain $\{W_n x_n\}$ and $\{f(x_n)\}$ are bounded. Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. We have

$$\begin{aligned}\|W_n x_{n-1} - W_{n-1}x_{n-1}\| &= \|U_{n,1}x_{n-1} - U_{n-1,1}x_{n-1}\| \\ &\le \lambda_1 \|U_{n,2}x_{n-1} - U_{n-1,2}x_{n-1}\| \\ &\quad \vdots \\ &\le \prod_{i=1}^{n} \lambda_i \|T_n x_{n-1} - x_{n-1}\| \\ &\le K(\prod_{i=1}^{n} \lambda_i)\end{aligned}$$

where $K = 2\sup_{x\in C}\|x\|$.

So, we have

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &= \|\alpha_n f(x_n) + (1-\alpha_n)W_n x_n - (\alpha_{n-1}f(x_{n-1}) + (1-\alpha_{n-1})W_{n-1}x_{n-1})\| \\ &\le (1 - \alpha_n + a\cdot\alpha_n)\|x_n - x_{n-1}\| + K|\alpha_n - \alpha_{n-1}| \\ &\qquad + (1-\alpha_{n-1})\|W_n x_{n-1} - W_{n-1}x_{n-1}\| \\ &\le (1 - \alpha_n + a\cdot\alpha_n)\|x_n - x_{n-1}\| + K|\alpha_n - \alpha_{n-1}| + (1-\alpha_{n-1})K\cdot\prod_{i=1}^{n}\lambda_i.\end{aligned}$$

For all $m, n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\| x_{n+m+1} - x_{n+m}\| \\
&\quad \leq (1-\alpha_{n+m} + a\cdot\alpha_{n+m})\|x_{n+m} - x_{n+m-1}\| + K|\alpha_{n+m} - \alpha_{n+m-1}| \\
&\quad\quad +(1-\alpha_{n+m-1})K\cdot\prod_{i=1}^{n+m}\lambda_i \\
&\quad \leq (1-(1-a)\alpha_{n+m})\{(1-(1-a)\alpha_{n+m-1})\|x_{n+m-1} - x_{n+m-2}\| \\
&\quad\quad +K|\alpha_{n+m-1} - \alpha_{n+m-2}| + (1-\alpha_{n+m-2})K\cdot\prod_{i=1}^{n+m-1}\lambda_i\} \\
&\quad\quad +K|\alpha_{n+m} - \alpha_{n+m-1}| + (1-\alpha_{n+m-1})K\cdot\prod_{i=1}^{n+m}\lambda_i \\
&\quad\quad\quad \vdots \\
&\quad \leq \prod_{k=m}^{n+m-1}(1-(1-a)\alpha_{k+1})\|x_{m+1} - x_m\| \\
&\quad\quad +K\sum_{k=m}^{n+m-1}|\alpha_{k+1} - \alpha_k| + K\sum_{l=m}^{n+m-1}(\prod_{i=1}^{l+1}\lambda_i) \\
&\quad \leq \prod_{k=m}^{n+m-1}(1-(1-a)\alpha_{k+1})\|x_{m+1} - x_m\| \\
&\quad\quad +K\sum_{k=m}^{n+m-1}|\alpha_{k+1} - \alpha_k| + K\frac{b^{m+1}(1-b^n)}{1-b}.
\end{aligned}
$$

Therefore, from $\sum_{n=1}^{\infty}\alpha_n = \infty$, we obtain

$$
\begin{aligned}
\limsup_{n\to\infty}\|x_{n+1} - x_n\| &= \limsup_{n\to\infty}\|x_{n+m+1} - x_{n+m}\| \\
&\leq K\sum_{k=m}^{\infty}|\alpha_{k+1} - \alpha_k| + K\frac{b^{m+1}}{1-b}
\end{aligned}
$$

for all $m \in \mathbb{N}$. Moreover, since $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, we have

$$
\begin{aligned}
\limsup_{n\to\infty}\|x_{n+1} - x_n\| &\leq K\lim_{m\to\infty}\sum_{k=m}^{\infty}|\alpha_{k+1} - \alpha_k| + K\lim_{m\to\infty}\frac{b^{m+1}}{1-b} \\
&= 0,
\end{aligned}
$$

and hence

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{10}$$

For each $k \in \mathbb{N}$, let $u_k = \frac{1}{k}f(u_k) + (1-\frac{1}{k})Uu_k$. From Theorem 3.2, we know that u_k converges strongly to $u = P_{F(U)}f(u)$ as $k \to \infty$. We obtain, for every $n, k \in \mathbb{N}$,

$$\begin{aligned}
&\|x_{n+1} - Ux_k\| \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)W_n x_n - Uu_k\| \\
&\le \alpha_n\|f(x_n) - Uu_k\| + (1-\alpha_n)\{\|W_n x_n - W_n u_k\| + \|W_n u_k - Uu_k\|\} \\
&\le K\cdot\alpha_n + \|x_n - u_k\| + \|W_n u_k - Uu_k\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\|W_n u_k - Uu_k\| = 0$, for each $k \in \mathbb{N}$, we have

$$\mu_n\|x_n - Uu_k\|^2 = \mu_n\|x_{n+1} - Uu_k\|^2 \le \mu_n\|x_n - u_k\|^2, \tag{11}$$

where μ is a Banach limit. On the other hand, from $x_n - u_k = \frac{1}{k}(x_n - f(u_k)) + (1-\frac{1}{k})(x_n - Uu_k)$, we also have

$$\begin{aligned}
(1-\frac{1}{k})^2\|x_n - Uu_k\|^2 &\ge \|x_n - u_k\|^2 - \frac{2}{k}\langle x_n - f(u_k), J(x_n - u_k)\rangle \\
&= \|x_n - u_k\|^2 - \frac{2}{k}\langle x_n - u_k + u_k - f(u_k), J(x_n - u_k)\rangle \\
&= (1-\frac{2}{k})\|x_n - u_k\|^2 + \frac{2}{k}\langle f(u_k) - u_k, J(x_n - u_k)\rangle.
\end{aligned}$$

So, from (11), we have

$$\begin{aligned}
(1-\frac{1}{k})^2\mu_n\|x_n - u_k\|^2 &\ge (1-\frac{1}{k})^2\mu_n\|x_n - Uu_k\|^2 \\
&\ge (1-\frac{2}{k})\mu_n\|x_n - u_k\|^2 + \frac{2}{k}\mu_n\langle f(u_k) - u_k, J(x_n - u_k)\rangle.
\end{aligned}$$

This implies that

$$\frac{1}{2k}\mu_n\|x_n - u_k\|^2 \ge \mu_n\langle f(u_k) - u_k, J(x_n - u_k)\rangle. \tag{12}$$

Since $\{u_k\}$ converges strongly to $u = P_{F(U)}f(u)$ as $k \to \infty$, from the uniformly Gâteaux differentiability of the norm of E and (12), we have

$$0 \ge \mu_n\langle f(u) - u, J(x_n - u)\rangle,$$

where $u = P_{F(U)}f(u)$. By (10), we have

$$\lim_{n\to\infty}|\langle f(u) - u, J(x_{n+1} - u)\rangle - \langle f(u) - u, J(x_n - u)\rangle| = 0$$

Hence, from Lemma 2.3, we obtain

$$\limsup_{n\to\infty}\langle f(u)-u, J(x_n-u)\rangle \leq 0. \tag{13}$$

From $x_{n+1}-u=\alpha_n(f(x_n)-u)+(1-\alpha_n)(W_nx_n-u)$, we have

$$(1-\alpha_n)^2\|W_nx_n-u\|^2 \geq \|x_{n+1}-u\|^2-2\alpha_n\langle f(x_n)-u, J(x_{n+1}-u)\rangle.$$

Hence,

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1-\alpha_n)^2\|W_nx_n-u\|^2+2\alpha_n\langle f(x_n)-u, J(x_{n+1}-u)\rangle\\ &\leq (1-\alpha_n)^2\|x_n-u\|^2+2\alpha_n\langle f(x_n)-f(u), J(x_{n+1}-u)\rangle\\ &\quad +2\alpha_n\langle f(u)-u, J(x_{n+1}-u)\rangle\\ &\leq (1-\alpha_n)^2\|x_n-u\|^2+2\alpha_n a\|x_n-u\|\|x_{n+1}-u\|\\ &\quad +2\alpha_n\langle f(u)-u, J(x_{n+1}-u)\rangle\\ &\leq (1-\alpha_n)^2\|x_n-u\|^2+\alpha_n a\{\|x_n-u\|^2+\|x_{n+1}-u\|^2\}\\ &\quad +2\alpha_n\langle f(u)-u, J(x_{n+1}-u)\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}&\|x_{n+1}-u\|^2\\ &\leq \frac{(1-\alpha_n)^2+a\alpha_n}{1-a\alpha_n}\|x_n-u\|^2+\frac{2\alpha_n}{1-a\alpha_n}\langle f(u)-u, J(x_{n+1}-u)\rangle\\ &\leq \frac{1-2\alpha_n+a\alpha_n}{1-a\alpha_n}\|x_n-u\|^2+\frac{\alpha_n^2}{1-a\alpha_n}M+\frac{2\alpha_n}{1-a\alpha_n}\langle f(u)-u, J(x_{n+1}-u)\rangle\\ &\leq (1-\frac{2(1-a)\alpha_n}{1-a\alpha_n})\|x_n-u\|^2\\ &\quad +\frac{2(1-a)\alpha_n}{1-a\alpha_n}\{\frac{\alpha_nM}{2(1-a)}+\frac{1}{1-a}\langle f(u)-u, J(x_{n+1}-u)\rangle\},\end{aligned}$$

where $M=\sup_n\|x_n-u\|^2$. Put $\beta_n=\frac{2(1-a)\alpha_n}{1-a\alpha_n}$. We obtain $\sum_{n=1}^{\infty}\beta_n=\infty$ and

$$\lim_{n\to\infty}\beta_n=0.$$

Let $\epsilon>0$. From (13), there exists $m\in\mathbb{N}$ such that $\frac{\alpha_nM}{2(1-a)}\leq\frac{\epsilon}{2}$ and

$$\frac{1}{1-a}\langle f(u)-u, J(x_n-u)\rangle \leq \frac{\epsilon}{2}$$

for all $n\geq m$. Then we have

$$\|x_{m+1}-u\|^2 \leq (1-\beta_m)\|x_m-u\|^2+(1-(1-\beta_m))\epsilon.$$

Similarly, we have

$$\|x_{m+n} - u\|^2 \leq \prod_{k=m}^{m+n-1} (1 - \beta_k)\|x_m - u\|^2 + (1 - \prod_{k=m}^{m+n-1} (1 - \beta_k))\epsilon.$$

We know that $\sum_{k=m}^{\infty} \beta_k = \infty$ implies $\prod_{k=m}^{\infty} (1 - \beta_k) = 0$. Therefore, we have

$$\limsup_{n\to\infty} \|x_n - u\|^2 = \limsup_{n\to\infty} \|x_{m+n} - u\|^2 \leq \epsilon.$$

Since $\epsilon > 0$ is arbitary, we obtain

$$\limsup_{n\to\infty} \|x_n - u\|^2 \leq 0.$$

So, we conclude that $\{x_n\}$ converges strongly to $u = P_{F(U)}f(u)$. ∎

4. Applications

Let E be a Banach space and let $A \subset E \times E$ be an m-accretive operator. In this section, we consider the problem of finding a point $v \in E$ such that $0 \in Av$. Many researchers have studied the convergence properties of such a problem; see, for instance, Bruck and Reich [4], Reich [10, 11], Kamimura and Takahashi [5].

On the other hand, there is the viscosity approximation method; for instance, see Tikhonov in 1963 [19]. This method provide an efficient approach to many problems of mathematical analysis; see, Attouch [2] and the references mentioned there. The abstract setting of the viscosity approximation method is as follows: Let $f : E \to (-\infty, \infty]$ be a real-valued function with some constraints. We consider the minimization problem

$$\min\{f(x); x \in E\}. \qquad \cdots \text{(MP)}$$

In order to find a point of solution set of (MP), for $\epsilon > 0$, we consider the approximate minimization problem

$$\min\{f(x) + \epsilon g(x); x \in E\}, \qquad \cdots \text{(AMP)}$$

where $g : E \to [0, \infty]$ called the viscosity function. Usually, the function g has some properties like strict convexity, continuity and coresiveness. Motivated by this method, we can prove the following theorem:

Theorem 4.1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let $A \subset E \times E$ be an m-accretive operator and* *let $B \subset E \times E$ be an m-accretive operator which is c-strongly accretive. Let $J_r^A = (I + rA)^{-1}$ and let $J_r^B = (I + rB)^{-1}$ for all $r > 0$. For $r > 0$, let x_r satisfying*

$$A_r(x_r) + rB_r(x_r) = 0, \tag{14}$$

where $A_r = \frac{1}{r}(I - J_r^A)$ and $B_r = \frac{1}{r}(I - J_r^B)$. Then $\{x_r\}$ converges strongly to $\hat{x}$ as $r \to 0$, where $\hat{x} = J_r^A(\hat{x})$

Proof. The viscosity formulation $0 = A_r(x_r) + rB_r(x_r)$ can be rewritten as

$$x_r = \frac{1}{1+r} J_r^A x_r + \frac{r}{1+r} J_r^B x_r.$$

Since J_r^A is a nonexpansive mapping and J_r^B is $\frac{1}{1+rc}$-contractive, by Theorem 3.1, we obtain $x_r \to \hat{x} \in F(J_r^A)$. ■

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