TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 4, pp. 725-730, December 2001
This paper is available online at http://www.math.nthu.edu.tw/tjm/

PRIME RINGS WITH ANNIHILATOR CONDITIONS ON POWER VALUES OF DERIVATIONS ON MULTILINEAR POLYNOMIALS

Vincenzo De Filippis

Abstract. Let R be a prime algebra over a commutative ring K, d a nonzero derivation of R, $f(x_1,..,x_n)$ a multilinear polynomial over K in n non-commuting variables, $a \in R$ and $m \geq 1$ a fixed integer. Suppose that $f(x_1,..,x_n)$ is not central on R. If $a(d(f(r_1,..,r_n)))^m = 0$, for any $r_1,..,r_n \in R$, then $a = 0$.

Throughout this paper R always denotes a prime ring with center $Z(R)$ and with extended centroid C, Q its Martindale quotient ring. We will consider some related problems concerning annihilators of power values of derivations in prime rings.

In [2] M. Bresar proved that if R is a semiprime ring, d a nonzero derivation of R and $a \in R$ such that $ad(x)^m = 0$, for all $x \in R$, where m is a fixed integer, then $ad(R) = 0$ when R is $(m-1)!$-torsion free. In [9] T. K. Lee and J. S. Lin proved Bresar's result whitout the assumption of $(m-1)!$-torsion free on R. They studied the Lie ideal case and, for the prime case, they showed that if R is a prime ring with a derivation $d \neq 0$, L a Lie ideal of R, $a \in R$ such that $ad(u)^m = 0$, for all $u \in L$, where m is fixed, then $ad(L) = 0$ unless the case when $char(R) = 2$ and $dim_C RC = 4$. In addition, if $[L, L] \neq 0$, then $ad(R) = 0$.

Recently in [3] C. M. Chang and T. K. Lee established a unified version of the previous results for prime rings. More precisely they proved the following theorem: let R be a prime ring, ϱ a nonzero right ideal of R, d a nonzero derivation of R, $a \in R$ such that $ad([x, y])^m \in Z(R)$ $(d([x, y])^m a \in Z(R))$. If $[\varrho, \varrho]\varrho \neq 0$ and $\dim_C RC > 4$, then either $ad(\varrho) = 0$ $(a = 0$ resp.) or d is the inner derivation induced by some $q \in Q$ such that $q\varrho = 0$.

Here we shall continue the investigation about the properties of a subset S of R related to its left annihilator $Ann_R(S) = \{x \in R : xS = (0)\}$. More precisely

Received June 10, 1999; revised Oct. 19, 1999.
Communicated by Pjek-Hwee Lee.
2000 *Mathematics Subject Classification*: 16N60, 16W25.
Key words and phrases: Prime rings, Differential identity, generalieed polynomial idenntity.

we shall study the case when $S = \{d(f(x_1,..,x_n))^m : x_1,..,x_n \in R\}$, where $f(x_1,..,x_n)$ is a multilinear polynomial in n non-commuting variables and m is a fixed integer. We shall prove:

Theorem 1. *Let R be a prime algebra over a commutative ring K, d a nonzero derivation of R, $f(x_1,..,x_n)$ a multilinear polynomial over K in n non-commuting variables, $a \in R$, $m \geq 1$ a fixed integer. Suppose that $f(x_1,..,x_n)$ is not central on R. If $a(d(f(r_1,..,r_n)))^m = 0$, for any $r_1,..,r_n \in R$, then $a = 0$.*

We first dispose of the case that R is not a domain. In fact, if R is a domain, by supposing $a \neq 0$, we get $(d(f(r_1,..,r_n)))^m = 0$, for any $r_1,..,r_n \in R$. In this situation, by [13], $f(x_1,..,x_n)$ must be central on R.

In all that follows let $T = Q *_C C\{X\}$ be the free product over C of the C-algebra Q and the free C-algebra $C\{X\}$, with X the countable set consisting of non-commuting indeterminates $x_1, x_2, \ldots, x_n, \ldots$. We refer the reader to [4] for the definitions and the related properties of these objects. Moreover we must remark that the main tool will be the theory of differential identities, initiated by Kharchenko in [6].

Remark 1. Recall that d can be extended uniquely to a derivation on Q [8] which will be also denoted by d. Since by [8] R and Q satisfy the same differential identities, we have that $a(d(f(x_1,..,x_n)))^m = 0$ also in Q. Moreover Q is prime, by the primeness of R, and replacing R by Q we may assume, without loss of generality, $C = Z(R)$ and R is a C-algebra centrally closed.

From now on let K be a commutative ring, R a prime K-algebra, $f(x_1,..,x_n)$ a multilinear polynomial over K in n non-commuting variables, $a \in R$ and $m \geq 1$.

Moreover $f(x_1,..,x_n)$ is not central on R and, for all $r_1,..,r_n \in R$, $a(d(f(r_1,.., r_n)))^m = 0$. We will use the following notation:

$$f(x_1,..,x_n) = x_1 x_2..x_n + \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(n)}.$$

We begin with the following:

Lemma 1. *If d is an outer derivation of R then $a = 0$.*

Proof. Suppose on the contrary that $a \neq 0$. We denote by $f^d(x_1,..,x_n)$ the polynomial obtained from $f(x_1,..,x_n)$ by replacing each coefficient α_σ with $\delta(\alpha_\sigma \cdot 1)$. Thus we write $d(f(x_1,..,x_n)) = f^d(x_1,..,x_n) + \sum_i f(x_1,..,d(x_i),..,x_n)$. Since R satisfies the generalized differential identity

$$a\,(d(f(x_1,..,x_n)))^m =$$

$$a\left(f^d(x_1,..,x_n)+\sum_i f(x_1,..,d(x_i),..,x_n)\right)^m$$

and d is an outer derivation, by [6] R satisfies the generalized polynomial identity

$$a\left(f^d(x_1,..,x_n)+\sum_i f(x_1,..,y_i,..,x_n)\right)^m$$

and in particular R satisfies $a\,(f(y_1,x_2,..,x_n))^m$. As a consequence of [5], since R is prime and $f(x_1,..,x_n)$ is not an identity for R, we get $a=0$, a contradiction. ■

In all that follows we will consider the only case when d is an inner derivation in R. This means that there exists $q\in R$ such that $a[q,f(r_1,..,r_n)]^m=0$, for any $r_1,..,r_n\in R$.

Lemma 2. *If R does not satisfy any non-trivial generalized polynomial identity, then $a=0$.*

Proof. Since R does not satisfy any non-trivial generalized polynomial identity, we have that

$$a[q,f(x_1,..,x_n)]^m$$

is the zero element in the free product $T=Q*_C C\{x_1,..,x_n\}$, that is

$$a[q,f(x_1,..,x_n)]^{m-1}(qf(x_1,..,x_n)-f(x_1,..,x_n)q)=0\in T.$$

Since $q\notin C$, it follows that $a[q,f(x_1,..,x_n)]^{m-1}f(x_1,..,x_n)q=0\in T$ and so $a[q,f(x_1,..,x_n)]^{m-1}=0\in T$. Continuing this process, we obtain that $a=0$. ■

Lemma 3. *If R is a dense ring of linear transformations over an infinite dimensional right vector space V over a division ring D, then $a=0$.*

Proof. Since $f(x_1,..,x_n)$ is a multilinear polynomial and $a[q,f(r_1,..,r_n)]^m=0$, for all $r_1,..,r_n\in R$, by [13, Lemma 2] we have $a[q,r]^m=0$, for all $r\in R$. Hence $a=0$ follows from [9, Theorem 1]. ■

Now we are ready to prove the following:

Theorem 1. *Let R be a prime K-algebra, d a nonzero derivation of R, $f(x_1,..,x_n)$ a multilinear polynomial over K in n non-commuting variables, $a\in R$ and $m\geq 1$. Suppose that $f(x_1,..,x_n)$ is not central on R. If $a(d(f(r_1,..,r_n))^m=0$, for any $r_1,..,r_n\in R$, then $a=0$.*

Proof. By Lemma 1, we assume that d is the inner derivation induced by $q \in R$, moreover by remark 1, $C = Z(R)$ and R is a C-algebra centrally closed, that is $R = RC$. If R does not satisfy any non-trivial generalized polynomial identity then, by Lemma 2, $a = 0$. Thus we may suppose that R satisfies a non-trivial generalized polynomial identity. By Martindale's theorem in [11], R is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D. If $\dim_D V = \infty$, then, by Lemma 3, we get the conclusion required.

Therefore consider the case $\dim_D(V) = k$, with k finite positive integer ≥ 2, because R is not a domain. In this condition R is a simple ring which satisfies a non-trivial generalized polynomial identity. By [7, Lemma 2; 12 theorem 2.3.29] $R \subseteq M_t(F)$, for a suitable field F and $t \geq 2$, moreover $M_t(F)$ satisfies the same generalized identity of R. Since $f(x_1, .., x_n)$ is not central on R then, by [10], there exist $u_1, .., u_n \in M_t(F)$, such that $f(u_1, .., u_n) = \beta e_{ij}$, for some distinct i, j, with $\beta \in F - \{0\}$ and e_{ij} the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. Moreover, since the set $\{f(x_1, .., x_n) : x_1, .., x_n \in M_t(F)\}$ is invariant under the action of all F-automorphisms of $M_t(F)$, then for any $i \neq j$ there exist $r_1, .., r_n \in M_t(F)$ such that $f(r_1, .., r_n) = \beta e_{ij}$.

Suppose on the contrary that the matrix $a = \sum a_{hl} e_{hl}$ is not zero. Let $q = \sum q_{hl} e_{hl}$, with $q_{hl} \in F$ and fix i and $j \neq i$. Then

$$\begin{aligned} 0 &= a[q, f(r_1, .., r_n)]^m = a\,(qf(r_1, .., r_n) - f(r_1, .., r_n)q)^m \\ &= a\,(q\beta e_{ij} - \beta e_{ij} q)^m . \end{aligned}$$

In particular, right multiplying by $e_{ij}q$ we have

$$0 = a\,(q\beta e_{ij} - \beta e_{ij} q)^m \, e_{ij} q = a(-\beta)^m (e_{ij} q)^{m+1}.$$

Then, for all $j \neq i$, either $q_{ji} = 0$ or the i-th column of the matrix a is zero, a desired contradiction.

<u>Case 1:</u> **<u>$t = 2$.</u>**

Since $f(x_1, .., x_n)$ is not central on R, by [10, lemmas 2 and 9], there exists a sequence of matrices $r = (r_1, .., r_n)$ such that $f(r) = \beta e_{21}$ is not zero.

Suppose that q is not diagonal, say $q_{12} \neq 0$, then the 2-nd column of a is zero. In other words the following hold:

$$q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \quad q_{12} \neq 0$$

$$a = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}$$

$$f(r) = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}.$$

By calculation it follows that

$$[q, f(r)]^{2m} = \begin{bmatrix} (q_{12}\beta)^{2m} & 0 \\ 0 & (q_{12}\beta)^{2m} \end{bmatrix}.$$

Then $0 = a[q, f(r)]^{2m} = a(q_{12}\beta)^{2m} = 0$ and so $a = 0$.

Moreover we get the same conclusion if suppose $q_{21} \neq 0$. Thus we conclude that if $k = 2$, either q is a diagonal matrix or $a = 0$.

Case 2: $t \geq 3$.

Also in this case we want to prove that if a is not zero then q is a diagonal matrix. Suppose there exists $q_{ji} \neq 0$, $i \neq j$, then the i-th column of a is zero. For all $l \neq i, j$ let $\varphi_{li} \in Aut_F(M_t(F))$ such that $\varphi_{li}(x) = (1 + e_{li})x(1 - e_{li})$. Consider the following valutations of $f(x_1, .., x_n)$:

$$f(r) = \gamma e_{ij}, \quad f(s) = \varphi_{li}(f(r)) = \gamma e_{ij} + \gamma e_{lj}, \quad \gamma \neq 0.$$

Since the i-th column of a is zero, by $a[q, f(s)]^m = 0$ and right multiplying by $e_{ij} + e_{lj}$, we have:

$$0 = a[q, f(s)]^m(e_{ij} + e_{lj}) = a(-\gamma)^m(q_{ji} + q_{jl})^m(e_{ij} + e_{lj}) \quad (\mathbf{1}).$$

Notice that if $q_{ji} + q_{jl} = 0$, then $q_{jl} = -q_{ji} \neq 0$ and, as in the first part of the proof, the l-th column of a is zero. On the other hand, if $q_{ji} + q_{jl} \neq 0$, by (1), for all k, $a_{kl} = -a_{ki}$ and, since the i-th column of a is zero, it follows again that the l-th one is also zero. Hence we can say that the matrix a has at most one nonzero column, the j-th one.

Thus $a = ae_{jj}$ and so

$$\begin{aligned} 0 &= a[q, f(r)]^m = ae_{jj}[q, \gamma e_{ij}]^m = ae_{jj}(q\gamma e_{ij} - \gamma e_{ij}q)[q, \gamma e_{ij}]^{m-1} \\ &= ae_{jj}q\gamma e_{ij}[q, \gamma e_{ij}]^{m-1} = ... = ae_{jj}(q\gamma e_{ij})^m = a(q_{ji}\gamma)^m. \end{aligned}$$

Hence $a = 0$.

The previous two cases show that if a is the nonzero matrix then q is a diagonal one, $q = \sum q_{kk}e_{kk}$. Now let $\varphi_{ij} \in Aut_F(M_t(F))$ such that $\varphi_{ij}(x) = (1 + e_{ij})x(1 - e_{ij})$, with $i \neq j$. Since $0 = \varphi_{ij}(a)[\varphi_{ij}(q), \varphi(f(x_1, .., x_n))]^m = \varphi_{ij}(a)[\varphi_{ij}(q), f(y_1, .., y_n)]^m$ and $a \neq 0$, we have that $\varphi(q)$ is also diagonal. On the other hand $\varphi_{ij}(q) = q + (q_{jj} - q_{ii})e_{ij}$, i.e. $q_{jj} = q_{ii}$ and q is central in $M_t(F)$, which is a contradiction. Therefore must be $a = 0$. ∎

Acknowledgement

The author wishes to thank the referee; by following his valuable suggestions, he has been able to shorten the proof of Lemmas 2 and 3 and to correct the proof of case 2 in Theorem 1.

REFERENCES

1. K. I Beidar, W. S. Martindale III, A. V. Mikhalev, *Rings with generalized identities*, Pure and Applied Math., Dekker, New York, 1996.
2. M. Bresar, *A note on derivations*, Math. J. Okayama Univ. **32** (1990), 83-88.
3. C. M. Chang and T. K. Lee, *Annihilators of power values in prime rings*, Comm. in Algebra, **26(7)** (1998), 2091-2113.
4. C. L. Chuang, GPIs having coefficients in Utumi quotient rings, *Proc. Amer. Mat. Soc.* **103(3)**, (1988), 723-728.
5. C. L. Chuang and T. K. Lee, Rings with annihilators conditions on multilinear polynomials, *Chin. J. Math.* **24(2)** (1996), 177-185.
6. V. K. Kharchenko, *Differential identities of prime rings*, Algebra and Logic **17** (1978), 155-168.
7. C. Lanski, An Engel condition with derivation, *Proc. Amer. Math. Soc.* **118(3)** (1993), 731-734.
8. T. K. Lee, Semiprime ring with differential identities, *Bull. Inst. Math. Acad. Sinica*, **20(1)** (1992), 27-38.
9. T. K. Lee and J. S. Lin, A result on derivations, *Proc. Amer. Math. Soc.* **124(6)** (1996), 1687-1691.
10. U. Leron, Nil and power central polynomials in rings, *Trans. Amer. Math. Soc.* **202** (1975), 97-103.
11. W. S. Martindale III, Prime rings satisfying a generalized polynomial identity, *J. Algebra* **12** (1969), 576-584.
12. L. Rowen, *Polynomial identities in Ring Theory*, Pure and Applied Math. vol. 84 (1980), Academic Press, New York.
13. T. L. Wong, Derivations with power central values on multilinear polynomials, *Algebra Colloq.* **3(4)** (1996), 369-378.

Dipartimento di Matematica, Universitá di Messina
Salita Sperone, contrada Papardo
98166, Messina, ITALIA.
E-mail: enzo@dipmat.unime.it