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PROPAGATION OF ALGEBRAIC DEPENDENCE OF MEROMORPHIC MAPPINGS

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Abstract. In this paper we give some criteria for the propagation of algebraic dependence of dominant meromorphic mappings from an analytic finite covering space X over the complex m-space into a projective algebraic manifold and give their applications. We study this problem under a condition on the existence of meromorphic mappings separating the fibers of X.

0. Introduction

Let $\pi : X \to \mathbb{C}^m$ be a finite analytic covering space and M a projective algebraic manifold. Let $f_1, \cdots, f_l$ be dominant meromorphic mappings from X into M. Suppose that they have the same inverse images of given divisors on M. In this paper, we give conditions under which $f_1, \cdots, f_l$ are algebraically related. We study this problem from the point of view in Nevanlinna theory. Roughly our result says that if these mappings satisfy the same algebraic relation at all points of the set of the inverse images of divisors and if the given divisors are sufficiently ample, then they must satisfy this relationship identically. The theorems on the propagation of dependence from an analytic subset to the whole space was first studied by L. Smiley in his doctoral thesis [12]. There have been several studies on the propagation of dependence (cf. [6] and [5]). In the past, this problem has been studied under the conditions on the growth of meromorphic mappings and the ramification divisor B of $\pi : X \to \mathbb{C}^m$. For example, W. Stoll proved some theorems on the propagation of dependence of meromorphic mappings $f : X \to M$ under a condition on the growth of mappings (cf. [15]). In his results, at least one of the mappings f_j

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must grow quicker than the ramification divisor B. In this paper we give criteria for algebraic dependence under another condition, that is, under a condition on the existence of meromorphic mappings separating the fibers of $\pi : X \to \mathbb{C}^m$. Thanks to the theorem on algebroid reduction of meromorphic mappings proved by J. Noguchi [8], we can always find such a mapping. In some of our criteria, we assume complicated conditions, but they have wider ranges of applicability. These criteria are actually corollaries of two theorems, which are Fundamental Lemmas for our study. The method for the proofs of the Fundamental Lemmas uses the Second Main Theorem due to Noguchi [8] in an essential computational way. In applications of his Second Main Theorem, the ramification estimate for B and the theorem on algebroid reduction of meromorphic mappings are especially important. We give some remarks on algebroid reduction of Nevanlinna theory in §1. We give our Fundamental Lemmas and criteria for algebraic dependence in §2. In §§3–5, we give applications of results in §2. In particular, we give some conditions under which two holomorphic mappings are related by an endomorphism of elliptic curves. Details will be published elsewhere (cf. [4]).

§1. Algebroid Reduction of Nevanlinna Theory

Let $z = (z_1, \cdots, z_m)$ be the natural coordinate system in $\mathbb{C}^m$, and set

$$\|z\|^2 = \sum_{\nu=1}^{m} z_\nu \overline{z}_\nu, \quad X(r) = \pi^{-1}(\{z \in \mathbb{C}^m : \|z\| < r\}) \quad \text{and} \quad \alpha = \pi^* dd^c \|z\|^2,$$

where $d^c = (\sqrt{-1}/4\pi)(\overline{\partial} - \partial)$. For a (1,1)-current φ of order zero on X, we set

$$N(r, \varphi) = \frac{1}{s_0} \int_1^r \langle \varphi \wedge \alpha^{m-1}, \chi_{X(t)} \rangle \frac{dt}{t^{2m-1}},$$

where $\chi_{X(r)}$ denotes the characteristic function of $X(r)$. Let M be a compact complex manifold and $L \to M$ a line bundle over M. Denote by $|\cdot|$ a hermitian fiber metric in L, and by ω its Chern form. Let $f : X \to M$ be a meromorphic mapping. We set

$$T_f(r, L) = N(r, f^*\omega),$$

and call it the characteristic function of f with respect to L. Then we have the following inequality of Nevanlinna for meromorphic mappings:

Theorem 1.1. *Let $L \to M$ be a line bundle over M and let $f : X \to M$ be a nonconstant meromorphic mapping. Then*

$$N(r, f^*D) \leq T_f(r, L) + O(1)$$

for $D \in |L|$ *with* $f(X) \not\subseteq \operatorname{Supp} D$, *where* $O(1)$ *stands for a bounded term as* $r \to +\infty$.

Let E be an effective divisor on $\mathbb{C}^m$. Set $N_1(r, E) = N(r, \operatorname{Supp} E)$. A meromorphic mapping $f : X \to M$ is said to be *dominant* if rank $f = \dim M$. The following second main theorem for dominant meromorphic mappings gives us an essential computational technique for the Fundamental Lemmas (cf. [8, Theorem 1]):

Theorem 1.2. *Let* M *be a projective algebraic manifold with* $m \geq \dim M$ *and* $L \to M$ *an ample line bundle. Suppose that* $D_1, \cdots, D_q$ *are divisors in* $|L|$ *such that* $D_1 + \cdots + D_q$ *has only simple normal crossings. Let* $f : X \to M$ *be a dominant meromorphic mapping. Then*

$$q\, T_f(r, L) + T_f(r, K_M) \leq \sum_{j=1}^{q} N_1(r, f^*D_j) + N(r,\ B) + S_f(r),$$

where $S_f(r) = O(\log T_f(r, L)) + o(\log r)$ *except on a Borel subset* $E \subseteq [1, +\infty)$ *with finite measure.*

In applications of Theorem 1.2, it is essential to give the estimate for $N(r,\ B)$ by the characteristic function of f. For meromorphic mappings $f : X \to M$, we have the following ramification estimate proved by J. Noguchi:

Definition 1.3. Let Y be a compact complex manifold. We say that a meromorphic mapping $f : X \to Y$ *separates the fibers of* $\pi : X \to \mathbb{C}^m$ if there exists a point z in $\mathbb{C}^m - (\operatorname{Supp} \pi_* B \cup \pi(I(f)))$ such that $f(x) \neq f(y)$ for any distinct points $x,\ y \in \pi^{-1}(z)$. Here $I(f)$ denotes the locus of indeterminacy of f.

Assume that $f : X \to M$ separates the fibers of $\pi : X \to M$ and that L is ample. Since L is ample, there exist the least positive integer μ_0 and a pair of sections $\sigma_0, \sigma_1 \in H^0(M,\ \mu_0 L)$ such that a meromorphic function $f^*(\sigma_0/\sigma_1)$ separates the fibers of $\pi : X \to \mathbb{C}^m$. Let s_0 be the sheet number of $\pi : X \to \mathbb{C}^m$. Then we have the following ramification estimate due to J. Noguchi (cf. [8, p. 277]):

Theorem 1.4 (Noguchi). *Suppose that* $L \to M$ *is ample and* $f : X \to M$ *separates the fibers of* $\pi : X \to M$. *Let* μ_0 *be as above. Then*

$$N(r,\ B) \leq 2\mu_0(s_0 - 1)\, T_f(r,\ L) + O(1).$$

In the case where f does not separate fibers of $\pi : X \to M$, we cannot estimate the growth of the ramification divisor in general. However, we have the following reduction theorem proved by J. Noguchi (cf. [8, p. 272]):

Theorem 1.5 (Noguchi). *Let $f : X \to M$ be a meromorphic mapping. Then there exist a finite analytic covering space $\varpi : \underline{X} \to \mathbb{C}^m$, a surjective proper holomorphic mapping $\lambda : X \to \underline{X}$ and a meromorphic mapping $\underline{f} : \underline{X} \to M$ which separates the fibers of $\varpi : \underline{X} \to \mathbb{C}^m$ such that the following diagram*

$$\begin{tikzcd}
\mathbb{C}^m \arrow[d, "\mathrm{id}"'] & X \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\lambda"'] & M \arrow[d, "\mathrm{id}"] \\
\mathbb{C}^m & \underline{X} \arrow[l, "\varpi"] \arrow[r, "\underline{f}"'] & M
\end{tikzcd}$$

is commutative. Furthermore, if f is dominant, so is $\underline{f}$.

Remark 1.6. By making use of Theorem 1.4, we can easily obtain the following equalities (cf. [8, p. 273]):

$$T_f(r,\ L) = T_{\underline{f}}(r,\ L), \quad \text{and} \quad N(r,\ f^*D) = N(r,\ \underline{f}^*D).$$

We also have

$$N(r,\ \underline{B}) \leq 2\mu_0(s_0 - 1)\, T_f(r,\ L) + O(1),$$

where $\underline{B}$ is the ramification divisor of $\varpi : \underline{X} \to \mathbb{C}^m$. Therefore we can apply Theorems 1.2 and 1.4 for an arbitrary dominant meromorphic mapping $f : X \to M$. For the theory of algebroid reduction of Nevanlinna theory, see also [14].

§2. Fundamental Lemmas

We first give a definition of algebraic dependence of meromorphic mappings. For a positive integer l, set $M^l = M \times \cdots \times M$ (l-times). For meromorphic mappings $f_1, \cdots, f_l : X \to M$, we define a meromorphic mapping $f_1 \times \cdots \times f_l : X \to M^l$ by

$$(f_{\times} \cdots \times f_l)(z) = (f_{(}z), \cdots, f_l(z)), \quad z \in X - (I(f_1) \cup \cdots \cup I(f_l)),$$

where, for each j, $I(f_j)$ is the indeterminacy locus of f_j. A proper algebraic subset Σ of M^l is said to be *decomposable* if, for some positive integer s not greater than l, there exist positive integers $l_1, \cdots, l_s$ with $l = l_1 + \cdots + l_s$ and algebraic subsets $\Sigma_j \subseteq M^{l_j}$ such that $\Sigma = \Sigma_1 \times \cdots \times \Sigma_s$.

Definition 2.1. Let S be an analytic subset of X. Nonconstant meromorphic mappings $f_1, \cdots, f_l : X \to M$ are said to be *algebraically dependent on S* if there exists a proper algebraic subset Σ of M^l such that $(f_1 \times \cdots \times f_l)(S) \subseteq \Sigma$ and Σ is not decomposable. In this case, we also say that $f_1, \cdots, f_l$ *are Σ-related on S.*

For a line bundle L over M, we denote by $H^0(M, L)$ the space of all holomorphic sections of $L \to M$.

Definition 2.2. A line bundle L over M is said to be *big* provided that

$$\dim H^0(M, \nu L) \geq C\nu^{\dim M}$$

for all sufficiently large positive integers ν and for some positive constant C.

Let $\mathrm{Pic}(M)$ be the Picard group over M. Let $F \in \mathrm{Pic}(M) \otimes \mathbb{Q}$ and $\gamma \in \mathbb{Q}$. We simply write γF for $F^{\otimes \gamma}$. Then F is said to be *big* (resp. *ample*) provided that a line bundle $\nu F \in \mathrm{Pic}(M)$ is big (resp. ample) for some positive integer ν. We fix an ample line bundle $L \to M$. We assume that there exists at least one dominant meromorphic mapping $f_0 : X \to M$.

Let $D_1, \cdots, D_q$ be divisors in $|L|$ such that $D_1 + \cdots + D_q$ has only simple normal crossings, where $|L|$ is the complete linear system defined by L. Let $S_1, \cdots, S_q$ be hypersurfaces in X such that $\dim S_i \cap S_j \leq m - 2$ for any $i \neq j$. We define a hypersurface S in X by $S = S_1 \cup \cdots \cup S_q$. Let E be an effective divisor on X, and let k be a positive integer. If $E = \sum_j \nu_j E'_j$ for distinct irreducible hypersurfaces E'_j in X and for nonnegative integers ν_j, then we define the support of E with order at most k by

$$\mathrm{Supp}_k E = \bigcup_{0 < \nu_j \leq k} E'_j.$$

Assume that $\mathrm{Supp}_{k_j} f_0^* D_j$ coincides with S_j for all j with $1 \leq j \leq q$, where k_j is a fixed positive integer. Let $\mathfrak{F}$ be the set of all *dominant* meromorphic mappings $f : X \to M$ such that $\mathrm{Supp}_{k_j} f^* D_j$ is equal to S_j for each j with $1 \leq j \leq q$. Let $F_1, \cdots, F_l$ be big line bundles over M. We define a line bundle $\tilde{F}$ over M^l by

$$\tilde{F} = \pi_1^* F_1 \otimes \cdots \otimes \pi_l^* F_l,$$

where $\pi_j : M^l \to M$ is the natural projection on the jth factor. Let $\tilde{L}$ be a big line bundle over M^l. Note that, in general,

$$\tilde{L} \notin \pi_1^* \mathrm{Pic}(M) \oplus \cdots \oplus \pi_l^* \mathrm{Pic}(M).$$

In the case of $\tilde{L} \neq \tilde{F}$, we assume that there exists a positive rational number $\tilde{\gamma}$ such that $\tilde{\gamma}\tilde{F} \otimes \tilde{L}^{-1}$ is big. If $\tilde{L} = \tilde{F}$, then we take $\tilde{\gamma} = 1$. Let $\mathfrak{R}$ be the set of all hypersurfaces Σ in X such that $\Sigma = \mathrm{Supp}\ \tilde{D}$ for some $\tilde{D} \in |\tilde{L}|$ and Σ is not decomposable.

Let s_0 be the sheet number of $\pi : X \to \mathbb{C}^m$. Assume that $f : X \to M$ separates the fibers of $\pi : X \to M$. Since L is ample, there exist a positive integer μ and a pair of sections $\sigma_0, \sigma_1 \in H^0(M, \mu L)$ such that a meromorphic function $f^*(\sigma_0/\sigma_1)$

separates the fibers of $\pi : X \to \mathbb{C}^m$ for all such mappings f. We denote by μ_0 the least positive integer among those μ's. We assume that there exists a line bundle, say, F_0 in $\{F_1, \cdots, F_l\}$ such that $F_0 \otimes F_j^{-1}$ is either big or trivial for all j. Set $k_0 = \max_{1 \le j \le q} k_j$. We define $L_0 \in \mathrm{Pic}(M) \otimes \mathbb{Q}$ by

$$L_0 = \left(\sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2\mu_0(s_0 - 1) \right) L \otimes \left(-\frac{\tilde{\gamma} l k_0}{k_0 + 1} F_0 \right).$$

By making use of Theorems 1.1, 1.2 and 1.4, we have our basic result as follows:

Fundamental Lemma I. *Let $f_1, \cdots, f_l$ be arbitrary mappings in $\mathcal{F}$ and $\Sigma \in \mathcal{R}$. Suppose that $f_1, \cdots, f_l$ are Σ-related on S. If $L_0 \otimes K_M$ is big, then $f_1, \cdots, f_l$ are Σ-related on X.*

Now, let us consider a more general case. Let $L_1, \cdots, L_l$ be ample line bundles over M. Let $q_1, \cdots, q_l$ be positive integers and assume that $D_j = D_{j1} + \cdots + D_{jq_j} \in |q_j L|$ has only normal crossings, where $D_{jk} \in |Lj$. Let Z be a hypersurface in X. Let $\mathcal{G}$ be a family of dominant meromorphic mappings $f : X \to M$ such that

$$\mathrm{Supp}_{k_j} f^* D_j = Z$$

for some $1 \le j \le l$. In the case where $L_j = L$ for all j, we define $G_0 \in \mathrm{Pic}(M) \otimes \mathbb{Q}$ by

$$G_0 = \left(\min_{1 \le j \le l} \left\{ \frac{q_j k_j}{k_j + 1} \right\} - 2\mu_0(s_0 - 1) \right) L \otimes \left(-\frac{\tilde{\gamma} l k_0}{k_0 + 1} F_0 \right).$$

Then we have one more Fundamental Lemma for our study.

Fundamental Lemma II. *Let $f_1, \cdots, f_l$ be arbitrary mappings in $\mathcal{G}$ and $\Sigma \in \mathcal{R}$. Suppose that $f_1, \cdots, f_l$ are Σ-related on Z. If $G_0 \otimes K_M$ is big, then $f_1, \cdots, f_l$ are Σ-related on X.*

Now, we will give criteria for the propagation of algebraic dependence of dominant meromorphic mappings, which are corollaries of Fundamental Lemma I. For $F \in \mathrm{Pic}(M) \otimes \mathbb{Q}$, we define $[F/L]$ by

$$[F/L] = \inf\{\gamma \in \mathbb{Q};\ \gamma L \otimes F^{-1} \text{ is big}\}.$$

Set

$$p_0 = \sum_{j=1}^{q} \frac{k_j}{k_j + 1} - [K_M^{-1}/L] - 2\mu_0(s_0 - 1).$$

We also set

$$m_1 = q - [K_M^{-1}/L] - 2\mu_0(s_0 - 1) \quad \text{and} \quad m_j = q - [K_M^{-1}/L] \ (2 \le j \le l).$$

Then we have the following criterion for the propagation of algebraic dependence:

Proposition 2.3. *Let $f_1, \cdots, f_l \in \mathcal{F}$. Suppose that they are Σ-related on S. If m_j are positive and if*

$$p_0 - \frac{\tilde{\gamma} l k_0}{k_0 + 1}[F_1/L] + m_1 \sum_{j=2}^{l} \left(p_0 - \frac{\tilde{\gamma} l k_0}{k_0 + 1}[F_j/L] \right) > 0,$$

then $f_1, \cdots, f_l$ are Σ-related on X.

We also have the following criteria similar to Proposition 2.3. Set

$$n_1 = q_1 - [K_M^{-1}/L_1] - 2\mu_0(s_0 - 1) \quad \text{and} \quad n_j = q_j - [K_M^{-1}/L_j] \ (2 \le j \le l).$$

We also set

$$p_j = \frac{q_j k_j}{1 + k_j} - [K_M^{-1}/L_j] - 2\mu_0(s_0 - 1)$$

for all j with $1 \le j \le l$. Then we have the following criterion:

Proposition 2.4. *Let $f_1, \cdots, f_l$ be arbitrary mappings in $\mathcal{G}$ and $\Sigma \in \mathcal{R}$. Suppose that $f_1, \cdots, f_l$ are Σ-related on Z. If all $n_j > 0$ and if*

$$p_1 - \frac{\tilde{\gamma} l k_0}{k_0 + 1}[F_1/L_1] + n_1 \sum_{j=2}^{l} \left(p_j - \frac{\tilde{\gamma} l k_0}{k_0 + 1}[F_j/L_j] \right) > 0,$$

then $f_1, \cdots, f_l$ are Σ-related on X.

Set $e_0 = 2\mu_0(s_0 - 1) + 1$. Then we also have the following:

Proposition 2.5. *Let $f_1, \cdots, f_l$ be as in Proposition* 2.4. *If all $n_j > 0$ and if*

$$p_1 - \frac{\tilde{\gamma} l k_0}{k_0 + 1}[F_1/L_j] + \sum_{j=2}^{l} \left(n_1 p_j - \frac{\tilde{\gamma} l e_0 k_0}{n_j(k_0 + 1)}[F_j/L_j] \right) > 0,$$

then $f_1, \cdots, f_l$ are Σ-related on X.

Remarks 2.6. (1) The cases, when either all $k_j = 1$ or all $k_j = +\infty$, are especially important from the viewpoint of Nevanlinna theory. We now consider the case where $k_j = +\infty$ for some j. We first note that $\mathrm{Supp}\, f^*D = \mathrm{Supp}_{k_j} f^*D$ if $k_j = +\infty$. Set $k_j/(k_j+1) = 1$ and $1/(k_j+1) = 0$ for $k_j = +\infty$. Then it is easy to see that the proofs of the Fundamental Lemmas also work in the case where $k_j = +\infty$ for some j. Hence the conclusions of the above propositions are still valid for the case where some of the k_j is ∞. We also note that the proof of Fundamental Lemma I also works in the case where some of the S_j are empty sets.

(2) We give a remark on the assumptions in Fundamental Lemmas I and II. In Fundamental Lemma I, we assume that $D_1, \cdots, D_q$ are linearly equivalent. We consider the case where D_i and D_j are not linearly equivalent for some pair $(i,\ j)$ but all the Chern classes $c_1([D_j])$ are identical. In this case, the conclusion of Fundamental Lemma I remains valid provided that the line bundle

$$\left(\sum_{j=1}^{q}\frac{k_j}{k_j+1} - 2\mu_0(s_0-1)\right)[D_1]\otimes\left(-\frac{\tilde{\gamma}lk_0}{k_0+1}F_0\right)$$

is ample. We next consider Fundamental Lemma II. In the case where L_i and L_j are not the same for some i and j but all the Chern classes $c_1(L_j)$ are identical, the conclusion of Fundamental Lemma II is still valid if the line bundle

$$\left(\min_{1\le j\le l}\left\{\frac{q_jk_j}{k_j+1}\right\} - 2\mu_0(s_0-1)\right)L_1\otimes\left(-\frac{\tilde{\gamma}lk_0}{k_0+1}F_0\right)$$

is ample.

§3. Unicity Theorems for Meromorphic Mappings

In this section we give some unicity theorems as an application of the criteria for dependence by taking line bundles F_j of a special type. For the details of this direction, see [2, 3, 5, 13]. We keep the same notation as in §2. Let $\Phi : M \to \mathbb{P}_n(\mathbb{C})$ be a meromorphic mapping with $\mathrm{rank}\,\Phi = \dim\ M$. We denote by H the hyperplane bundle over $\mathbb{P}_n(\mathbb{C})$. Now let $l = 2$ and take $F_1 = F_2 = \Phi^*H$. We also take $\tilde{L} = \tilde{F}$. Then we see

$$L_0 = \left(\sum_{j=1}^{q}\frac{k_j}{k_j+1} - 2\mu_0(s_0-1)\right)L\otimes\left(-\frac{2k_0}{k_0+1}\Phi^*H\right).$$

Set

$$\Omega_0 = M - (\{w\in M - I(\Phi) :\ \mathrm{rank}\, d\Phi(w) < \dim\ M\}\cup I(\Phi)),$$

where $I(\Phi)$ is the locus of indeterminacy of Φ. A set $\{D_j\}_{j=1}^q$ of divisors is said to be *generic with respect to* f_0 *and* Φ provided that

$$f_0(\mathbf{C}^m - I(f_0))\cap \mathrm{Supp}\, D_j\cap\Omega_0 \neq \emptyset$$

for at least one $1 \leq j \leq q$, where $I(f_0)$ denotes the locus of indeterminacy of f_0. We assume that $\{D_j\}_{j=1}^q$ is generic with respect to f_0 and Φ in what follows. Let $\mathcal{F}_1$ be the set of all mappings $f \in \mathcal{F}$ such that $f = f_0$ on S. Then we have the following unicity theorems by Fundamental Lemma I and by the uniqueness of analytic continuation (cf. [2, Theorem 2.1]):

Theorem 3.1. *Suppose that $L_0 \otimes K_M$ is big. Then the family $\mathcal{F}_1$ contains just one mapping f_0.*

We next consider the case $\dim\ M = 1$. Assume that M is a compact Riemann surface with genus g_0. In the case $g_0 = 0$, we have the following unicity theorem for meromorphic functions on X by Theorem 3.1, which is closely related to the uniqueness problem of algebroid functions (cf. [1, Theorem 3.3]).

Theorem 3.2. *Let $f_1,\ f_2 : X \to \mathbb{P}_1(\mathbb{C})$ be nonconstant holomorphic mappings. Let $a_1, \cdots, a_d$ be distinct points in $\mathbb{P}_1(\mathbb{C})$. The following hold.*

(1) *Suppose that* $\mathrm{Supp} f_1{}^* a_j = \mathrm{Supp} f_2{}^* a_j$ *for all j. If $d \geq 2s_0 + 3$, then f_1 and f_2 are identical on X.*

(2) *Suppose that* $\mathrm{Supp}_1 f_1{}^* a_j = \mathrm{Supp}_1 f_2{}^* a_j$ *for all j. If $d \geq 4s_0 + 3$, then f_1 and f_2 are identical on X.*

Note that H. Ueda gave an example which shows that the above theorem is sharp in the case $X = \mathbb{C}$ (cf. [16, p. 458]).

Example 3.3. We consider the integral

$$z = \varphi(w) := \int_0^w (1 - t^4)^{-\frac{1}{2}} dt$$

on the unit disc in $\mathbb{C}$. Set $z_1 = \varphi(1)$, $z_2 = \varphi(\sqrt{-1})$, $z_3 = \varphi(-1)$ and $z_4 = \varphi(-\sqrt{-1})$. Then φ maps the unit disc onto the square $z_1\, z_2\, z_3\, z_4$. By Schwarz's reflection principle, the inverse function of $z = \varphi(w)$ can be analytically continued over the complex plane $\mathbb{C}$, and the resulting function $w = f(z)$ is doubly periodic. Let $a_1 = 1$, $a_2 = \sqrt{-1}$, $a_3 = -1$, $a_4 = -\sqrt{-1}$, $a_5 = 0$ and $a_6 = \infty$. Set $f_1 = f$ and $f_2 = \sqrt{-1} f$. Then $\mathrm{Supp}_1 f_1{}^* a_j = \mathrm{Supp}_1 f_2{}^* a_j$ for all j, but $f_1 \not\equiv f_2$.

The uniqueness problem of holomorphic mappings into a compact Riemann surface with positive genus is not well studied (cf. [1, 5, 6, 10]. In the case of $g_0 = 1$, we will discuss the uniqueness for holomorphic mappings into smooth elliptic curves in §5. We now consider the case where $g_0 \geq 2$. Note that Riemann-Roch's theorem shows that $\mu_0 \leq g_0 + 1$. In this case, by making use of Theorem 3.1, we have the following unicity theorem (cf. [1, Theorem 3.6]):

Theorem 3.4. *Let $f_1,\ f_2 : X \to M$ be nonconstant holomorphic mappings. Let $a_1, \cdots, a_d$ be distinct points in M. The following hold.*

(1) *Suppose that* $\mathrm{Supp} f_1^* a_j = \mathrm{Supp} f_2^* a_j$ *for all* j. *If* $d > \max\{4g_0,\ 2(g_0+1)(s_0-1)\}$, *then* f_1 *and* f_2 *are identical on* X.

(2) *Suppose that* $\mathrm{Supp}_1 f_1^* a_j = \mathrm{Supp}_1 f_2^* a_j$ *for all* j. *If* $d > \max\{4g_0,\ 2(g_0+1)(2s_0+1) - 8g_0\}$, *then* f_1 *and* f_2 *are identical on* X.

Under the condition of Theorems 3.2 and 3.4, at least one $\mathrm{Supp}_1 f_1^* a_j$ is not empty. We note that some examples of holomorphic mappings into compact Riemann surfaces satisfying our condition were constructed (cf. [9, §5]).

§4. Meromorphic Mappings into Complex Projective Spaces

In this section, we investigate meromorphic mappings into complex projective spaces. Note that if $f : X \to \mathbb{P}_n(\mathbb{C})$ separates the fibers of $\pi : X \to \mathbb{C}^m$, we can always take $\mu_0 = 1$. Let H be the hyperplane bundle over $\mathbb{P}_n(\mathbb{C})$. Then $\mathrm{Pic}(\mathbb{P}_n(\mathbb{C})) \cong \mathbb{Z}$ and H is the generator of $\mathrm{Pic}(\mathbb{P}_n(\mathbb{C}))$ with $c_1(H) = 1$. Let $K_{\mathbb{P}_n}(\mathbb{C})$ be the canonical bundle of $\mathbb{P}_n(\mathbb{C})$. Then, as is well-known, we have $K_{\mathbb{P}_n}(\mathbb{C}) = -(n+1)H$. We note that

$$\mathrm{Pic}(\mathbb{P}_n(\mathbb{C})^2) = \pi_1^* \mathrm{Pic}(\mathbb{P}_n(\mathbb{C})) \oplus \pi_2^* \mathrm{Pic}(\mathbb{P}_n(\mathbb{C})).$$

Hence we may assume that $\tilde{L} = \tilde{F}$. Since $\mathrm{Pic}(\mathbb{P}_n(\mathbb{C})) \cong \mathbb{Z}$, there exists a positive integer d such that $L = dH$. There also exist positive integers d_j such that $F_j = d_j H$ for $j = 1, 2$. Thus a holomorphic section of $\tilde{L} \to \mathbb{P}_n(\mathbb{C})^2$ is a homogeneous polynomial $P(\xi; \zeta)$ of degree d_1 in $\xi = (\xi_0, \cdots, \xi_n)$ and degree d_2 in $\zeta = (\zeta_0, \cdots, \zeta_n)$. Let S be a hypersurface in X. In [6], S. J. Drouilhet dealt with the case of $n = 1$ and obtained some theorems on the dependence of meromorphic functions. By making use of Proposition 2.3, we have the following:

Theorem 4.1. *Let* D *and* D' *be hypersurfaces of degree* d *which have only simple normal crossings. Let* $f_1,\ f_2 : \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ *be dominant meromorphic mappings such that* $\mathrm{Supp}_k\, f_1^* D = \mathrm{Supp}_k\, f_2^* D' = Z$ *as point sets* $(1 \le k \le +\infty)$. *Let* $P(\xi; \zeta)$ *be as above. Suppose that* $P(f_1; f_2) = 0$ *on* Z. *If* $d \ge d_1 + d_2 + 1 + (1 + k^{-1})(n+1)$, *then* $P(f_1; f_2) = 0$ *on* $\mathbb{C}^m$.

If $X = \mathbb{C}$, $n = 1$ and $k = +\infty$, then we have Drouilhet's theorem (cf. [6, p. 495]). He also gave some examples which show that his result is sharp.

§5. Holomorphic Mappings into Smooth Elliptic Curves

In this section we consider the case where M is a smooth elliptic curve E. The uniqueness problem of holomorphic mappings into elliptic curves was first studied by E. M. Schmid [10]. Schmid obtained the following unicity theorem: Let f, $g : \mathbb{C} \to E$ be nonconstant holomorphic mappings. If $f^{-1}(a_j) = g^{-1}(a_j)$ as point sets for distinct five points $a_1, \cdots, a_5$ in E, then f and g are identical. In this section, we consider the problem to determine the condition which yields $g = \varphi(f)$ for an endomorphism φ of the abelian group E. We first note the following fact: If $f : X \to E$ separates the fibers of $\pi : X \to \mathbb{C}^m$, then we can take $\mu_0 = 2$ (cf. [9, p. 286]). Note that the canonical bundle K_E of E is trivial. It is well-known that

$$\mathrm{Pic}(E^2) \neq \pi_1^* \mathrm{Pic}(E) \oplus \pi_2^* \mathrm{Pic}(E).$$

We denote by $[p]$ the point bundle determined by $p \in E$. Let $F_1 = F_2 = [p]$. Let $f,\ g : X \to E$ be nonconstant holomorphic mappings. We denote by $\mathrm{End}(E)$ the ring of endomorphisms of E. Let $\varphi \in \mathrm{End}\,(E)$ and consider a curve

$$\tilde{S} = \{(x,\ y) \in E \times E;\ y = \varphi(x)\}$$

in $E \times E$. Let $\tilde{L}$ be the line bundle $[\tilde{S}]$ determined by $\tilde{S}$. In this section, $\tilde{\gamma}$ denotes the infimum of rational numbers such that $\gamma \tilde{F} \otimes [\tilde{S}]^{-1}$ is ample. Then we have $\tilde{\gamma} = \deg \varphi + 1$. This result is proved by T. Katsura. For the proof of Katsura's theorem, see [4, §6]. Note that if φ is an endomorphism defined by $\varphi(x) = nx$ for some $n \in \mathbb{Z}$, then $\tilde{\gamma} = n^2 + 1$ (cf. [11, p. 89]). By making use of Fundamental Lemma II, we have the following theorem:

Theorem 5.1. *Let $f,\ g$ and φ be as above. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and φ an endomorphism of E. Set $D_2 = \varphi(D_1)$. Assume that the number of points in D_2 is also d. Suppose that* $\mathrm{Supp}_k\ f^* D_1 = \mathrm{Supp}_k\ g^* D_2$ *for some k. If $d > 2(\deg \varphi + 1) + 8(s_0 - 1)(1 + k^{-1})$, then $g = \varphi(f)$.*

In the case where $\sharp D_2 < d$, we have the following theorem by Proposition 2.5:

Theorem 5.2. *Let $f,\ g : \mathbb{C}^m \to E$ be nonconstant holomorphic mappings. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and $\varphi \in \mathrm{End}(E)$. Set $D_2 = \varphi(D_1)$. Assume that the number of points in D_2 is d'. Suppose that* $\mathrm{Supp}_1\ f^* D_1 = \mathrm{Supp}_1\ g^* D_2$. *If $dd' > (d + d')(\deg \varphi + 1)$, then $g = \varphi(f)$.*

In the case where $m = 1,$ $\mathrm{Supp}\ f^* D_1 = \mathrm{Supp}\ g^* D_2$ and E has no complex multiplication, we have Drouilhet's theorem (cf. [6, Theorem 6]). We finally give the following unicity theorem, which is a direct conclusion of Theorem 5.1:

Theorem 5.3. *Let $a_1, \cdots, a_d$ be distinct points in E. Let $f_1,\ f_2 : X \to E$ be nonconstant holomorphic mappings. Suppose that* $\mathrm{Supp}_1\ f_1^* a_j = \mathrm{Supp}_1\ f_2^* a_j$ *for all j. If $d > 16 s_0 - 12$, then f_1 and f_2 are identical.*

Remark 5.4. Note that Theorem 5.3 is sharp. We give an example. Let f_1, $f_2 : \mathbb{C}^m \to E$ be nonconstant holomorphic mappings. We first note that each f_j has no defect and has no totally ramified value. Let $a_1, \cdots, a_4$ be two-torsion points in E and $\wp$ the Weierstrass $\wp$ function. If $f_1^* a_j = f_2^* a_j$ for $j = 1, \cdots, 4$, then we also have $(\wp \circ f_1)^* a_j = (\wp \circ f_2)^* a_j$. Now, applying Nevanlinna's four points theorem for meromorphic functions $\wp \circ f_1$, $\wp \circ f_2 : \mathbb{C}^m \to \mathbb{P}_1(\mathbb{C})$, we can easily see that $\wp \circ f_1$ and $\wp \circ f_2$ are identical (cf. [7, p. 122]). Since $\wp$ is an even function, we have $f_1 \equiv f_2$ or $f_1 \equiv -f_2$. In contrast to the case $M = \mathbb{P}_1(\mathbb{C})$, it seems that the structure of the function field of E affects strongly the uniqueness problem for holomorphic mappings.

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References

1. Y. Aihara, A unicity theorem for meromorphic mappings into compactified locally symmetric spaces, *Kodai Math. J.* **14** (1991), 392-405.
2. Y. Aihara, The uniqueness problem of meromorphic mappings with deficiencies, *Tohoku Math. J.* **51** (1999), 315-328.
3. Y. Aihara, Unicity theorems for meromorphic mappings with deficiencies, *Complex Variables Theory Appl.* **42** (2000), 259-268.
4. Y. Aihara, Algebraic dependence of meromorphic mappings in value distribution theory, preprint, 2000.
5. S. J. Drouilhet, A unicity theorem for meromorphic mappings between algebraic varieties, *Trans. Amer. Math. Soc.* **265** (1981), 349-358.
6. S. J. Drouilhet, Criteria for algebraic dependence of meromorphic mappings into algebraic varieties, *Illinois J. Math.* **26** (1982), 492-502.
7. R. Nevanlinna, *Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes*, Gauthier-Villars, Paris, 1929.
8. J. Noguchi, Meromorphic mappings of covering spaces over $\mathbf{C}^m$ into a projective variety and defect relations, *Hiroshima Math. J.* **6** (1976), 265-280.
9. J. Noguchi, Holomorphic mappings into closed Riemann surfaces, *Hiroshima Math. J.* **6** (1976), 281-290.
10. E. M. Schmid, Some theorems on value distribution of meromorphic functions, *Math. Z.* **23** (1971), 561-580.
11. J. Silverman, *The Arithmetic of Elliptic Curves*, Springer-Verlag, Berlin-Heidelberg-New York, 1991.

12. L. Smiley, *Dependence Theorems on Meromorphic Maps*, Ph.D. Thesis, Notre Dame Univ., 1979.

13. L. Smiley, Geometric conditions for the unicity of holomorphic curves, *Contemp. Math.* **25** (1983), 149-154.

14. W. Stoll, Algebroid reduction of Nevanlinna theory, in: *Complex Analysis III*, Proc. 1985-1986, C. A. Berenstein, ed. Lect. Notes in Math. **1277**, Springer-Verlag, Berlin-Heidelberg-New York, 1987, pp. 131-241.

15. W. Stoll, Propagation of dependences, *Pacific J. Math.* **139** (1989), 311-336.

16. H. Ueda, Unicity theorems for meromorphic or entire functions, *Kodai Math. J.* **3** (1980), 457-471.

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