TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 3, pp. 655-665, September 2001
This paper is available online at http://www.math.nthu.edu.tw/tjm/

MULTIPLICATIVE PRINCIPAL-MINOR INEQUALITIES FOR TRIDIAGONAL SIGN-SYMMETRIC P-MATRICES

Shaun M. Fallat* and Charles R. Johnson†

Abstract. The question of which ratios of products of principal minors are bounded over all matrices in a given class has been of interest historically. This question is settled herein for the class of tridiagonal sign-symmetric P-matrices, which essentially lies in each of the classes: positive definite, invertible totally nonnegative and M-matrices. It happens that all bounded ratios are bounded by one.

1. Introduction

An n-by-n real matrix $A = [a_{ij}]$ is called a *P-matrix* if each of its principal minors are positive, and A is called *sign-symmetric* if $a_{ij}a_{ji} > 0$ or $a_{ij} = a_{ji} = 0$ for each pair $1 \leq i, j \leq n$. Further, A is called *weakly sign-symmetric* if $a_{ij}a_{ji} \geq 0$ for each pair $1 \leq i, j \leq n$. The tridiagonal sign-symmetric P-matrices (TSP) lie at the crossroads of the three most important classes that generalize positivity among matrices: the positive definite, the invertible totally nonnegative (all minors nonnegative and positive determinant) and the M-matrices. Sign-symmetric tridiagonal matrices may be symmetrized by positive diagonal equivalence (or similarity), and $A \in$ TSP if and only if its symmetrization is positive definite. A signature matrix is a diagonal matrix S of ± 1's. A sign-symmetric tridiagonal matrix A is signature similar to a (unique) nonnegative matrix and $A \in$ TSP if and only if that nonnegative matrix is a totally nonnegative matrix. A sign-symmetric tridiagonal matrix A is also signature similar to a (unique) matrix with nonpositive off-diagonal entries, and $A \in$ TSP if and only if this matrix is an M-matrix. We also note that diagonal

Received April 14, 2000; revised February 28, 2001.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: 15A15, 15A48.
Key words and phrases: Tridiagonal matrix, P-matrix, determinant, principal minor.
*Research supported in part by an NSERC research grant.
†Research supported in part by the Office of Navel Research contract N00014-90-J-1739 and NSF grant 92-00899.

similarity does not change the value of any principal minor, so that inequalities among products of principal minors of a TSP matrix are unchanged by the diagonal similarities mentioned above. Tridiagonal matrices arise in a variety of applications including orthogonal polynomials, continued fractions, recurrence relations and special reductions in numerical analysis. Thus the TSP matrices are a natural class to study.

Our interest here lies in understanding all ratios of products of principal minors that are bounded (independent of the matrix) over the class of TSP matrices. (We note that, by an elementary closure argument, if "weakly sign-symmetric" were substituted for "sign-symmetric" in the definition of TSP, the answer to our question would be the same.) Such questions have been addressed (though not always fully settled) in the case of each of the classes we have mentioned: positive definite [2], invertible totally nonnegative [3] and M-matrices [4] (see also [5]). Since, essentially, the TSP matrices lie inside each of these classes (discussion above), the inequalities/bounded ratios for TSP matrices are a superset of those for each of the three classes, and, thus, may be a possible source of insight for those problems. We give a complete solution to the problem for TSP matrices, and, like the M-matrix case, though unlike the positive definite case, all bounded ratios are bounded by 1.

2. Preliminaries

For an n-by-n matrix $A = [a_{ij}]$, $\alpha, \beta \subseteq \mathbb{N} \equiv \{1, 2, \ldots, n\}$, the submatrix of A lying in rows indexed by α and the columns indexed by β will be denoted by $A[\alpha|\beta]$. Similarly, $A(\alpha|\beta)$ is the submatrix obtained from A by deleting the rows indexed by α and columns indexed by β. If $\alpha = \beta$, then the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$, and the complementary principal submatrix is $A(\alpha)$. Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ denote a collection of index sets, in which $\alpha_i \subseteq \mathbb{N}$, $i = 1, 2, \ldots, p$. Then we define $\alpha(A) = \det A[\alpha_1] \det A[\alpha_2] \cdots \det A[\alpha_p]$. If, further, $\beta = \{\beta_1, \beta_2, \ldots, \beta_q\}$ is another collection of index sets with $\beta_i \subseteq \mathbb{N}$, for all i, then we write $\alpha \leq \beta$ with respect to TSP if $\alpha(A) \leq \beta(A)$ for every n-by-n TSP matrix A.

We shall also consider ratios of products of principal minors. For two given collections α and β of index sets we shall interpret α/β as both a numerical ratio $\alpha(A)/\beta(A)$ for a given TSP matrix A and also as a formal ratio to be manipulated according to natural rules. When interpreted numerically, such ratios are well-defined because the class of TSP is preserved under extraction of principal submatrices, so that $\beta(A) \neq 0$ whenever A is a TSP matrix. Since, by convention, $\det A[\phi] = 1$, we also assume, without loss of generality, that in any ratio α/β both collections α and β have the same number of index sets, since if there is a disparity in the total number of index sets between α and β, the one with the fewer sets may be augmented with copies of ϕ. Either α or β may include repeated index sets.

The following multiplicative principal minor inequalities are classical for n-by-n positive definite, totally nonnegative or M-matrices (see [1, 6, 7, 8, 10, 11]):

$$\text{Hadamard: } \det A \leq \prod_{i=1}^{n} a_{ii};$$

$$\text{Fischer: For } S \subseteq \mathbb{N},\ \det A \leq \det A[S] \cdot \det A[S^c];$$

$$\text{Koteljanskii: Let } S, T \subseteq \mathbb{N},$$

$$\det A[S \cup T] \cdot \det A[S \cap T] \leq \det A[S] \cdot \det A[T].$$

Here, as throughout, S^c is the complement of S relative to $\mathbb{N}$. Since the class TSP may be viewed as lying in any of the classes: invertible totally nonnegative, M-matrices, or positive definite, it follows that each inequality (Hadamard, Fischer, and Koteljanskii) holds for every matrix in TSP. Each of the above classical inequalities may be written in our form $\alpha \leq \beta$. For example, Hadamard's inequality, $\det A \leq \prod_{i=1}^{n} a_{ii}$, has $\alpha = \{N, \phi, \dots, \phi\}$ and $\beta = \{\{1\}, \{2\}, \dots, \{n\}\}$, and Koteljanskii's inequality has the collections $\alpha = \{S \cup T, S \cap T\}$ and $\beta = \{S, T\}$. Our main and most general problem of interest is to characterize, via set-theoretic conditions, all pairs of collections of index sets α, β such that

$$\frac{\alpha(A)}{\beta(A)} \leq K$$

for some constant $K \geq 0$ (which depends on n) and for all n-by-n TSP matrices A. If such a constant exists for all TSP matrices A, we say that the ratio α/β is *bounded* with respect to the class of TSP matrices. Whenever $\alpha/\beta \leq 1$ for a given class of matrices we say that α/β is an *inequality* with respect to the class.

This problem was resolved for the classes of M- and inverse M-matrices [4], and has received much attention for the class of positive definite and totally nonnegative matrices.

Let α be any given collection of index sets. For $i \in \{1, 2, \dots, n\}$, we define $f_\alpha(i)$ to be the number of index sets in α that contain the element i. In other words, $f_\alpha(i)$ counts the multiplicity of the index i in the collection α (see also [2, 4]). The next proposition demonstrates a simple necessary (and by no means sufficient) condition for a given ratio of principal minors to be bounded with respect to TSP matrices.

Proposition 2.1. *Let α and β be two collections of index sets. If α/β is bounded with respect to the class of TSP matrices, then $f_\alpha(i) = f_\beta(i)$ for every $i = 1, 2, \dots, n$.*

Proof. Suppose there exists an index i for which $f_\alpha(i) > f_\beta(i)$ (if $f_\alpha(i) < f_\beta(i)$, consider the inverse of the matrix used in the argument to follow). For

$k \geq 1$, let $D_k = \text{diag}(1, \ldots, 1, k, 1, \ldots, 1)$, where the number k occurs in the (i, i)-entry of D_k. Then D_k is a TSP matrix for every value k, and $\alpha(A)/\beta(A) = k^{(f_\alpha(i) - f_\beta(i))} = k^t$, where $t \geq 1$. Hence α/β is not a bounded ratio. ■

If a given ratio α/β satisfies the condition $f_\alpha(i) = f_\beta(i)$ for every $i = 1, 2, \ldots, n$, then we say that the ratio satisfies ST0, or set-theoretic zero (see also [2, 4]).

Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be a given collection of index sets. We let $f_\alpha(J)$ denote the number of index sets in α that contain J, where $J \subseteq \mathbb{N}$. In [2], it is shown that if a ratio α/β is bounded with respect to the class of positive definite matrices, then

$$f_\alpha(J) \geq f_\beta(J) \text{ for every subset } J \subseteq \mathbb{N}. \tag{1}$$

If a ratio α/β satisfies (1), then we say that the ratio α/β satisfies (ST1). In [5], it was also noted that this condition is not sufficient for a ratio to be bounded with respect to the class of positive definite matrices.

In [4], one of the main results is as follows.

Theorem 2.2. *Let α/β be a given ratio of index sets. Then the following are equivalent* :

(*i*) *α/β satisfies* (*ST0*) *and* (*ST1*);

(*ii*) *α/β is bounded with respect to M-matrices*;

(*iii*) *$\alpha \leq \beta$ with respect to M-matrices.*

Thus a ratio α/β is bounded with respect to the class of M-matrices if and only if it satisfies (ST0) and (ST1), and any such bounded ratio is an inequality with respect to M-matrices.

We note here that the condition (ST1) is neither necessary nor sufficient (see [3]) for a ratio to be bounded with respect to the class of totally nonnegative matrices. However, the following result holds for bounded ratios with respect to totally nonnegative matrices. The dispersion of a given nonempty set $S = \{i_1, i_2, \ldots, i_k\}$, where $i_j < i_{j+1}$ ($j = 1, 2, \ldots, k-1$), is given by $d(S) = i_k - i_1 - (k-1)$ for $k > 1$ and $d(S) = 0$ for $k = 1$ (see [1, p. 170]).

Proposition 2.3. ([3]). *Let α/β be a given ratio. If α/β is bounded with respect to the class of totally nonnegative matrices*, *then* $f_\alpha(J) \geq f_\beta(J)$ for all $J \subseteq \mathbb{N}$, with $d(J) = 0$, *that is*, *for all contiguous subsets J of* $\mathbb{N}$.

If a ratio α/β satisfies the condition in Proposition 2.3, we say that α/β satisfies (ST1)$'$. We note that (ST1)$'$ will play an integral role here.

Before we come to our main results we consider one more concept, which we illustrate with an example. If A is a tridiagonal matrix (not necessarily in TSP), then it follows that

$$A[\{1,2,4,6,7\}] = A[\{1,2\}] \oplus A[\{4\}] \oplus A[\{6,7\}],$$

where $\oplus$ denotes direct sum. In particular,

$$\det A[\{1,2,4,6,7\}] = \det A[\{1,2\}] \det A[\{4\}] \det A[\{6,7\}].$$

It is clear that to compute a principal minor of a tridiagonal matrix it is enough to compute principal minors based on associated contiguous index sets. Let S and T be any two ordered index sets of $\{1, 2, \dots, n\}$. We say that $S \preceq T$ if the the maximum element of S is less than the minimum element of T. For example, if $S = \{1, 2, 3\}$ and $T = \{4, 6, 7, 8, 9\}$, then $S \preceq T$, but if $S = \{1, 2, 3\}$ and $T = \{3, 5, 6, 7\}$, then $S \npreceq T$.

Definition 2.4. Let α be any index set. Then we say that $\beta_1, \beta_2, \dots, \beta_p$ is the contiguous set decomposition of α if

1. *$\beta_1, \beta_2, \dots, \beta_p$ is a partition of α,*
2. *β_i is an ordered contiguous set, i.e., $d(\beta_i) = 0$ for $i = 1, 2, \dots, p$,*
3. *$\beta_i \preceq \beta_j$ for $i < j$, and*
4. *$\beta_i \cup \beta_j$, $i \neq j$, is not an ordered contiguous set.*

Often we may refer to $\beta_1, \beta_2, \dots, \beta_p$ as the *contiguous components* of α.

For example, the contiguous set decomposition of $\{1, 2, 4, 6, 7\}$ is $\{1, 2\}$, $\{4\}$, $\{6, 7\}$. The above definition is motivated by the fact that if α is any index set and $\beta_1, \beta_2, \dots, \beta_p$ are its contiguous components, then

$$\det A[\alpha] = \prod_{i=1}^{p} \det A[\beta_i]$$

for any tridiagonal matrix A.

3. Main Results

If α is a collection of index sets, then we let α' denote the collection obtained from α by replacing each index set of α by its contiguous components. Observe that by the remarks following Definition 2.4, $\alpha(A) = \alpha'(A)$ for any tridiagonal matrix A. Hence we may view the contiguous decomposition of a collection of index sets as evaluation, namely, $\alpha(A)$, at any tridiagonal matrix A and vice-versa.

This connection will be utilized throughout this section. It is easy to verify that if α/β satisfies (ST0), then the ratio α'/β' also satisfies (ST0).

Lemma 3.1. *The ratio of products of index sets α/β satisfies (ST1)$'$ if and only if α'/β' satisfies (ST1).*

Proof. First observe that if α/β satisfies (ST1)$'$, then it is clear that α'/β' also satisfies (ST1)$'$. Let $J \subset \mathbb{N}$, and let J' be the smallest contiguous set that contains J. (Note, by definition, $J = J'$ if and only if J is contiguous.) We claim that $f_{\alpha'}(J) = f_{\alpha'}(J')$. To verify the claim, first note that since $J \subset J'$, it is clear that $f_{\alpha'}(J) \geq f_{\alpha'}(J')$. To prove $f_{\alpha'}(J) \leq f_{\alpha'}(J')$, observe that if s_i is a contiguous index set in α' that contains J, then s_i also contains J'. This completes the proof of the claim. Next, observe that

$$f_{\alpha'}(J) = f_{\alpha'}(J') \geq f_{\beta'}(J') = f_{\beta'}(J),$$

where the inequality follows since α'/β' satisfies (ST1)$'$. Hence α'/β' satisfies (ST1). The converse follows easily from the fact that for any contiguous index set L,

$$f_{\alpha}(L) = f_{\alpha'}(L) \geq f_{\beta'}(L) = f_{\beta}(L)$$

(here the inequality follows since α'/β' satisfies (ST1)). Thus if α'/β' satisfies (ST1), then α/β satisfies (ST1)$'$. ■

We are now in a position to prove our main observations.

Theorem 3.2. *Let α/β be a given ratio of index sets. Then the following are equivalent* :

(*i*) *α/β satisfies (ST0) and (ST1)$'$*;

(*ii*) *α/β is bounded with respect to TSP*;

(*iii*) *$\alpha \leq \beta$ with respect to TSP.*

Proof. $(i) \Rightarrow (ii)$: Assume that α/β satisfies (ST0) and (ST1)$'$. Let α' (β') be the collection of index sets obtained from α (β) by replacing each index set in α (β) by its contiguous components. Then for any tridiagonal matrix A, we have $\alpha(A)/\beta(A) = \alpha'(A)/\beta'(A)$. By Lemma 3.1, the new ratio α'/β' satisfies (ST0) and (ST1). Thus the ratio α'/β' is bounded with respect to the M-matrices (Theorem 2.2). Since any TSP matrix A is signature similar to an M-matrix, and signature similarity does not affect the value of a ratio, we have

$$\frac{\alpha(A)}{\beta(A)} = \frac{\alpha'(A)}{\beta'(A)} = \frac{\alpha'(SAS)}{\beta'(SAS)} \leq K \tag{2}$$

for some positive constant K (by Theorem 2.2). Hence the ratio α/β is bounded with respect to TSP.
(*ii*) $\Rightarrow$ (*iii*): By Theorem 2.2, the constant K in (2) can be chosen to equal 1, since SAS is an M-matrix. Hence any bounded ratio with respect to TSP is, in fact, an inequality.
(*iii*) $\Rightarrow$ (*i*): We have already verified the necessity of (ST0) in Proposition 2.1. Suppose that the ratio α/β does not satisfy (ST1)$'$, say $f_\alpha(J) < f_\beta(J)$ for some contiguous index set J with $|J| = k > 1$. We will construct a sequence of TSP matrices A_i for which the set $\{\alpha(A_i)/\beta(A_i), i = 1, 2, \dots\}$ is unbounded above. Let $A = [a_{ij}]$ be defined by

$$a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -x & \text{if } j = i+1 \text{ or } j = i-1 \text{ and } i, j \in J, \\ 0 & \text{otherwise,} \end{cases}$$

where $x = (2\cos(\pi/k+1))^{-1}$. It is well-known that the largest eigenvalue of the (0,1) matrix of order k with ones on the superdiagonal and subdiagonal and zeros elsewhere is $2\cos(\pi/k+1)$. Hence $A[J]$ is singular and all of its proper principal submatrices are nonsingular (in fact, $A[J]$ is an irreducible singular M-matrix). Then, let $A_i = A + (1/i)I$, $i = 1, 2, \dots$. It is easily seen that A_i is a TSP matrix and the set $\{\alpha(A_i)/\beta(A_i), i = 1, 2, \dots\}$ is unbounded above. ■

We now discuss the notion of a *basic k-ratio* $R(i_1, \dots, i_k)$ on k indices $i_1, \dots, i_k \in \mathbb{N}$ (see also [4]). In $R(i_1, \dots, i_k)$, the full set $\{i_1, \dots, i_k\}$ appears in the numerator, all of its $(k-1)$-membered subsets appear in the denominator, all $(k-2)$-membered subsets in the numerator, and so on, until ϕ appears in the denominator when k is odd and in the numerator when k is even. For example, a basic 2-ratio, or $R(1, 2)$, is

$$\frac{\{1,2\}\phi}{\{1\}\{2\}},$$

and a basic 3-ratio, or $R(2, 3, 5)$, is

$$\frac{\{2,3,5\}\{2\}\{3\}\{5\}}{\{2,3\}\{2,5\}\{3,5\}\phi},$$

and so-on. The class of basic n-ratios were essential in [4] as they turned out to be the "generators" of all of the bounded ratios for M-matrices. In particular, it was shown in [4, Lemma 4.1] that any ratio which satisfies (ST1) can be written as a product of basic ratios and conversely. Hence, by Theorem 2.2, any bounded ratio with respect to the M-matrices is a product of basic ratios. For TSP matrices we have the following.

Lemma 3.3. *Let α/β be any ratio of index sets satisfying (ST0). Then α/β is bounded with respect to TSP if and only if α'/β' is a product of basic ratios.*

Proof. If α/β is bounded, then, by Theorem 3.2, α/β satisfies (ST1)$'$ and hence by Lemma 3.1 α'/β' satisfies (ST1). Thus using [4, Lemma 4.1], α'/β' can be written as a product of basic ratios. On the other hand, suppose α/β is a ratio satisfying (ST0) such that α'/β' is a product of basic ratios. Then by [4, Lemma 4.1] and Theorem 2.2, α'/β' is bounded with respect to the M-matrices. Therefore α/β is bounded with respect to TSP. ■

Consider the following illustrative example. Let $\alpha = \{\{1,2,3\}, \{2,3,4\}, \{1\}, \{4\}\}$ and let $\beta = \{\{1,2,4\}, \{1,3,4\}, \{2\}, \{3\}\}$. Then α/β satisfies $(ST1)'$ but not $(ST1)$. Furthermore, after canceling repeated sets:

$$\frac{\alpha'}{\beta'} = \frac{\{1,2,3\}, \{2,3,4\}, \emptyset, \emptyset}{\{1,2\}, \{3,4\}, \{2\}, \{3\}}.$$

It then follows that $\alpha'/\beta' = R(1,2,3)R(2,3,4)R(1,3)R^2(2,3)R(2,4)$.

Let A be an n-by-n matrix partitioned as follows:

$$A = \begin{bmatrix} A_{11} & a_{12} \\ a_{21} & a_{nn} \end{bmatrix},$$

where a_{nn} is a scalar and A_{11} is $(n-1)$-by-$(n-1)$. Then we let A/a_{nn} denote the *Schur complement of* a_{nn} *in* A, which is equal to $A/a_{nn} = A_{11} - (a_{21}a_{12})/a_{nn}$; see also [9, p. 22]. It is well-known that

$$\det A[S \cup \{n\}] = a_{nn} \det(A/a_{nn}[S]), \tag{3}$$

where $S \subseteq \{1, 2, \ldots, n-1\}$. Also, it follows from (3) and the definition of A/a_{nn} that if A is TSP so is A/a_{nn}.

Before we come to our final lemma we first consider the following instructive example. Let $\gamma/\delta = R(1,2,3)$. It is easy to verify directly that the contiguous decomposition of $R(1,2,3)$ is

$$\gamma'/\delta' = \frac{\{1,2,3\}\{2\}}{\{1,2\}\{2,3\}}, \tag{4}$$

a Koteljanskii ratio. In other words, for any 3-by-3 TSP matrix A, we have

$$\gamma(A)/\delta(A) = \frac{\det A[\{1,2,3\}] \det A[\{2\}]}{\det A[\{1,2\}] \det A[\{2,3\}]}.$$

Now consider $\alpha/\beta = R(1,2,3,4)$. Observe that

$$(5) \qquad \alpha/\beta = (R(1,2,3))^{-1}\left[\frac{\{1,2,3,4\}\{1,4\}\{2,4\}\{3,4\}}{\{1,2,4\}\{1,3,4\}\{2,3,4\}\{4\}}\right].$$

Our goal is to describe α'/β', the contiguous decomposition of $R(1,2,3,4)$. Equivalently, we are interested in describing $\alpha(A)/\beta(A)$ for any TSP matrix A. By (4), we already know the contiguous decomposition of $R(1,2,3)$. Notice that the ratio in the second factor of (5) is a basic 3-ratio on the index set $\{1,2,3\}$ in which each index set has been appended by the index 4. This is where Schur complements come in as a device to "decompose" this second factor. Namely, for any 4-by-4 TSP matrix A, let $C = A/a_{44}$. Then by (3), the second factor evaluated at A can be written as

$$\frac{\det C[\{1,2,3\}]\det C[\{1\}]\det C[\{2\}]\det C[\{3\}]}{\det C[\{1,2\}]\det C[\{1,3\}]\det C[\{2,3\}]\phi},$$

which is a basic 3-ratio on the index set $\{1,2,3\}$ evaluated at the TSP matrix C. Hence this ratio reduces to

$$(6) \qquad \begin{aligned}\frac{\det C[\{1,2,3\}]\det C[\{2\}]}{\det C[\{1,2\}]\det C[\{2,3\}]} &= \frac{\det A[\{1,2,3,4\}]\det A[\{2,4\}]}{\det A[\{1,2,4\}]\det A[\{2,3,4\}]} \text{ (by (3))}\\ &= \frac{\det A[\{1,2,3,4\}]\det A[\{2\}]}{\det A[\{1,2\}]\det A[\{2,3,4\}]},\end{aligned}$$

since A is tridiagonal. Combining (4) and (6) gives

$$\alpha(A)/\beta(A) = \frac{\det A[\{1,2,3,4\}]\det A[\{2,3\}]}{\det A[\{1,2,3\}]\det A[\{2,3,4\}]}$$

for any 4-by-4 TSP matrix A. In other words,

$$\alpha'/\beta' = \frac{\{1,2,3,4\}\{2,3\}}{\{1,2,3\}\{2,3,4\}}.$$

This argument represents the essence of our proof for the next lemma.

Lemma 3.4. *Let $\alpha/\beta = R(i+1, i+2, \ldots, i+k)$ be a basic k-ratio on the contiguous index set $\{i+1, i+2, \ldots, i+k\}$. Then*

$$\alpha'/\beta' = \frac{\{i+1,i+2,\ldots,i+k\}\{i+2,i+3,\ldots,i+k-1\}}{\{i+1,i+2,\ldots,i+k-1\}\{i+2,i+3,\ldots,i+k\}}.$$

Proof. The proof is by induction on k, the number of indices. If $k = 2$, then we are done. Assume that the result is true for all basic ratios on contiguous index sets with fewer than k indices. Consider $R(i+1, i+2, \ldots, i+k)$. Without loss of generality, we may assume that $\{i+1, i+2, \ldots, i+k\} = \{1, 2, \ldots, k\}$. Observe that

$$R(1,2,\ldots,k) = (R(1,2,\ldots,k-1))^{-1}\frac{\gamma}{\delta}, \tag{7}$$

where γ/δ is a basic $(k-1)$-ratio on the index set $\{1, 2, \ldots, k-1\}$ in which each index set is appended with the index k. Let A be any k-by-k TSP matrix, and define $C = A/a_{kk}$. Then, applying (3) it follows that

$$\frac{\gamma(A)}{\delta(A)} = \frac{S(C)}{T(C)},$$

where $S/T = R(1, 2, \ldots, k-1)$. Thus the contiguous decomposition of γ/δ coincides with the contiguous decomposition of $R(1, 2, \ldots, k-1)$. Applying the induction hypothesis to the ratio S/T, we have

$$S'/T' = \frac{\{1,2,\ldots,k-1\}\{2,3,\ldots,k-2\}}{\{1,2,\ldots,k-2\}\{2,3,\ldots,k-1\}}.$$

Hence

$$\begin{aligned} S(C)/T(C) &= \frac{\det C[\{1,2,\ldots,k-1\}]\det C[\{2,3,\ldots,k-2\}]}{\det C[\{1,2,\ldots,k-2\}]\det C[\{2,3,\ldots,k-1\}]} \\ &= \frac{\det A[\{1,2,3,\ldots,k\}]\det A[\{2,3,\ldots,k-2,k\}]}{\det A[\{1,2,\ldots,k-2,k\}]\det A[\{2,3,\ldots,k\}]} \quad (\text{by } (3)) \\ &= \frac{\det A[\{1,2,3,\ldots,k\}]\det A[\{2,3,\ldots,k-2\}]}{\det A[\{1,2\ldots,k-2\}]\det A[\{2,3,\ldots,k\}]}, \end{aligned} \tag{8}$$

since A is tridiagonal. Thus using (7), (8), and induction, we obtain

$$\alpha(A)/\beta(A) = \frac{\det A[\{1,2,3,\ldots,k\}]\det A[\{2,3,\ldots,k-1\}]}{\det A[\{1,2\ldots,k-1\}]\det A[\{2,3,\ldots,k\}]}$$

for any k-by-k TSP matrix A. Hence the contiguous decomposition of α/β is given by

$$\alpha'/\beta' = \frac{\{1,2,3,\ldots,k\}\{2,3,\ldots,k-1\}}{\{1,2\ldots,k-1\}\{2,3,\ldots,k\}},$$

as desired. ■

Note that in the above lemma the ratio α'/β' is a **Koteljanskii ratio**. We remark here that a purely combinatorial proof would be interesting so as to avoid the need for Schur complements; nonetheless they are a useful tool in this case. Hence using Lemma 3.4 we have the next result.

Corollary 3.5. *Let α/β be any ratio of index sets. Then α/β is bounded with respect to TSP if and only if α'/β' is a product of Koteljanskii ratios.*

Proof. By Lemma 3.3, α/β is bounded with respect to TSP if and only if α'/β' is a product of basic ratios. By Lemma 3.4, it follows that α'/β' is a product of Koteljanskii ratios. This completes the proof. ∎

References

1. T. Ando, Totally positive matrices, *Linear Algebra Appl.* **90** (1987), 165-219.
2. W. W. Barrett and C. R. Johnson, Determinantal inequalities for positive-definite matrices, *Discrete Math.* **115** (1993), 97-106.
3. S. M. Fallat, M. I. Gekhtman and C. R. Johnson, Multiplicative principal-minor inequalities for totally nonnegative matrices, in preparation.
4. S. M. Fallat, H. T. Hall and C. R. Johnson, Characterization of product inequalities for principal minors of M-matrices and inverse M-matrices, *Quart. J. Math. Oxford Ser.* (*2*) **49** (1998), 451-458.
5. S. M. Fallat and C. R. Johnson, Determinantal inequalities: Ancient history and recent advances, *Contemp. Math.* **259** (2000) 199-212.
6. E. Fischer, Über den Hadamard'schen determinantensatz, *Arch. Math. Physik* (3), *13* (1908), 32-40.
7. F. R. Gantmacher and M. G. Krein, *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen Mechanischer Systeme*, Akademie-Verlag, Berlin, 1960.
8. J. Hadamard, Résoultion d'une question relative aux déterminants, *Bull. Sci. Math. Sér. 2*, **17** (1893), 240-246.
9. R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.
10. R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991.
11. D. M. Koteljanskiĭ, A property of sign-symmetric matrices (Russian), *Mat. Nauk* (*N.S.*) **8** (1953) 163-167; English transl.: *Translations of the AMS*, Series 2 **27** (1963), 19-24.

Shaun M. Fallat
Department of Mathematics and Statistics, University of Regina,
Regina, Saskatchewan, S4S 0A2, Canada
E-mail: sfallat@math.uregina.ca

Charles R. Johnson
Department of Mathematics, College of William and Mary, Williamsburg,
VA. 23187-8795, U.S.A.
E-mail: crjohnso@math.wm.edu