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REMARKS ON POINCARÉ INEQUALITY AND ITS APPLICATIONS TO ELLIPTIC BOUNDARY VALUE PROBLEMS

Shyuh-yaur Tzeng†

Abstract. We study Poincaré inequality on unbounded domains and its applications to the semilinear elliptic equation$-\triangle u = u^p$.

1. Introduction

In the study of elliptic boundary value problems, the Poincaré inequality has frequently been used to obtain various estimates for the solutions. Such kinds of inequalities are usually more difficult to obtain for unbounded domains. Our aim in this note is to investigate under what conditions the Poincaré inequality holds, that is, there is a constat $C = C(\Omega)$ such that

$$\int_{\Omega} |u(x)|^2 dx \leq C \int_{\Omega} |\nabla u(x)|^2 dx \tag{1}$$

for all $u \in H_0^1(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$.

The detailed analysis is given in the next section. Some applications to elliptic boundary value problems will be discussed in Section 3.

2. Poincaré Inequality

Definition 1. An open subset of $\mathbb{R}^N$ is called a *P-domain* if (1) holds. For any open subset of $\mathbb{R}^N$, define

$$C(\Omega) = \sup_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_{\Omega} |u(x)|^2 dx}{\int_{\Omega} |\nabla u(x)|^2 dx}. \tag{2}$$

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It is known that $\mathbb{R}^N$ is not a P-domain. On the other hand, any bounded open set in $\mathbb{R}^N$ is a P-domain. For unbounded domains, it has been shown [9] that $\{(x_1, x'_N) \in \mathbb{R}^N : x_1 \in \mathbb{R}, |x'_N| < 1\}$ is a P-domain. Nevertheless, for a general unbounded set, it is no easy matter to verify if such a set is a P-domain. The first result of this paper is to give a criterion for an unbounded set to be a P-domain.

Theorem 1. *Let Ω_1 and Ω_2 be two P-domains. If $\Omega_1 \cap \Omega_2$ is bounded, then $\Omega = \Omega_1 \cup \Omega_2$ is a P-domain.*

Proof. Since $\Omega_1 \cap \Omega_2$ is bounded, $\Omega_1 \cap \Omega_2$ is contained in an open ball B of radius r with center at 0. Consider a smooth function ϕ which satisfies

$$\phi(t) = \begin{cases} 1, & |x| \le r, \\ 0, & |x| \ge 2r. \end{cases}$$

Let $C_0 = C(B)$, $C_1 = C(\Omega_1)$ and $C_2 = C(\Omega_2)$. A straightforward calculation yields

$$\int_\Omega u^2 dx \le 2\left(\int_\Omega (\phi u)^2 dx + \int_\Omega [(1-\phi)u]^2 dx\right) \tag{3}$$

and

$$\begin{aligned} &\int_\Omega (\phi u)^2 dx = \int_B (\phi u)^2 dx \le C_0 \int_B |\nabla(\phi u)|^2 dx \\ &\le 5C_0 \int_B |\nabla u|^2 dx + \frac{8C_0}{r}\int_B u^2 \le 5C_0 \int_\Omega |\nabla u|^2 dx + \frac{8C_0}{r}\int_\Omega u^2. \end{aligned} \tag{4}$$

Similarly, we obtain

$$\begin{aligned} &\int_\Omega [(1-\phi)u]^2 dx = \int_{\Omega_1} [(1-\phi)u]^2 dx + \int_{\Omega_2} [(1-\phi)u]^2 dx \\ &\le C(\Omega_1)\int_{\Omega_1} |\nabla[(1-\phi)u]|^2 dx + C(\Omega_2)\int_{\Omega_2} |\nabla[(1-\phi)u]|^2 dx \\ &\le 5(C_1 + C_2)\int_\Omega |\nabla u|^2 dx + \frac{8(C_1+C_2)}{r}\int_\Omega u^2. \end{aligned} \tag{5}$$

Putting (3)-(5) together yields

$$\int_\Omega u^2 dx \le 5(C_0 + C_1 + C_2)\int_\Omega |\nabla u|^2 dx + \frac{8(C_0+C_1+C_2)}{r}\int_\Omega u^2.$$

If r is large enough, then

$$\int_\Omega u^2 dx \le 5(C_0 + C_1 + C_2)\int_\Omega |\nabla u|^2 dx,$$

which completes the proof. ■

Next, we give a criterion for sets which are not P-domains.

Definition 2. An open subset Ω in $\mathbb{R}^N$ is called a *B-domain* if for any $r > 0$, there exists an open ball of radius r contained in Ω.

Theorem 2. *Any B-domain is not a P-domain.*

In the proof of Theorem 2, we will use the following lemma.

Lemma 1. *Let B_r be an open ball of radius r. Then $C(B_r) \to \infty$ as $r \to \infty$.*

Proof. By change of variables, we get $C(B_r) = r^2 C(B_1)$, from which the result follows. ■

Proof of Theorem 2. For any $r > 0$, Ω contains an open ball B_r. Since $H_0^1(B_r) \subseteq H_0^1(\Omega)$, it follows from Lemma 1 that $C(\Omega) = \infty$. Thus the proof is complete. ■

Let $B_r(0)$ be the open ball of radius r with center at 0. As more concrete examples of P-domains, we have

Example 1. Let

$$D_1 = \{(x_1, x'_N) \in \mathbb{R}^N : |x'_N| < 1, x_1 > 0\}. \tag{6}$$

Then by Theorem 1, $D_1 \cup B_r(0)$ is a P-domain.

Example 2. Let

$$D_2 = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : x_1^2 + \cdots + x_{N-1}^2 < 1, x_N > 0\}. \tag{7}$$

Then $D_1 \cup D_2$ is a P-domain.

3. Applications to Elliptic Boundary Value Problems

As an application of our previous results, we now study the existence of positive solutions of

$$\text{(E)} \qquad \begin{aligned} -\triangle u = u^p, u > 0 \text{ in } \Omega, \\ u \in H_0^1(\Omega), \end{aligned}$$

where $p > 1$ if $N = 1, 2$ and $p \in (1, (N+2)/(N-2))$ if $N > 2$. To seek solutions of (E), we define

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx$$

and look for the minimizers of J on the manifold:

$$M = M(\Omega) = \{u \in H_0^1(\Omega)\backslash\{0\} : J'(u)u = 0\}.$$

Set

$$\alpha = \alpha(\Omega) = \inf_{u \in M} J(u).$$

It has been shown [2, 3, 14] that if u is a minimizer of J on M, then $|u|$ is a positive solution of (E). We are going to investigate under what conditions there is a minimizer of J on M.

Definition 3. A P-domain Ω is *periodic* if there exist a partition $\{Q_m\}$ and a set $\{y_m\}$ in $\mathbb{R}^N$ which satisfy the following conditions:
(a) $\{y_m\}$ forms a subgroup of $\mathbb{R}^N$,
(b) Q_0 is bounded,
(c) $Q_m = Q_0 + y_m$, and
(d) there exists a constant $C = C(\Omega, Q_m)$ such that

$$\int_{Q_m} |u(x)|^2 dx \le C \int_{Q_m} |\nabla u(x)|^2 dx$$

for all $u \in H_0^1(\Omega)$ and $m \in \mathbb{Z}$.

Theorem 3. *If Ω is a periodic P-domain, then J has a minimizer on M.*

Proof. Let $\{u_n\} \subset M(\Omega)$ be a sequence which satisfies

$$\lim_{n\to\infty} J(u_n) = \alpha(\Omega). \tag{8}$$

Let

$$d_n = \max_m \left(\int_{Q_m} |u_n|^{p+1} dx\right)^{1/(p+1)}. \tag{9}$$

Then

$$
\begin{aligned}
\alpha(\Omega)+o(1) &= J(u_n)=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega |u_n|^{p+1}dx \\
&= \left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_m\int_{Q_m} |u_n|^{p+1}dx \\
&\le d_n^{p-1}\left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_m\left(\int_{Q_m} |u_n|^{p+1}dx\right)^{\frac{2}{p+1}} \\
&\le \mathrm{c}d_n^{p-1}\left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_m\int_{Q_m} |\nabla u_n|^{2}dx \\
&= \mathrm{c}d_n^{p-1}\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega |\nabla u_n|^{2}dx=\mathrm{c}d_n^{p-1}\alpha(\Omega)+o(1)
\end{aligned}
\tag{10}
$$

for some positive constant c which is independent of m. Therefore, there is a $\delta > 0$ such that $d_n > \delta$ for all n.

Next, we claim that there exists a $u \in M(\Omega)$ such that

$$J'(u) = 0 \text{ and } J(u) = \alpha(\Omega). \tag{11}$$

For each n, find a Q_n such that

$$\int_{Q_n} |u_n|^{p+1}dx > \frac{\delta}{2}. \tag{12}$$

Let $\nu_n(x) = u_n(x + y_n)$. It follows from (8) that

$$\lim_{n\to\infty} J(\nu_n) = \alpha(\Omega). \tag{13}$$

Using the standard deformation theory [2, 8] yields

$$J'(\nu_n) \to 0 \text{ as } n \to \infty. \tag{14}$$

Along a subsequence, $\nu_n \to u$ weakly in $H_0^1(\Omega)$. Then using (12) and Sobolev compact imbedding theorem yields

$$\int_{Q_0} |u|^{p+1}dx \ge \frac{\delta}{2}.$$

Consequently, $u \not\equiv 0$. Moreover, by (14),

$$J'(u) = 0,$$

which shows $u \in M(\Omega)$. Now (13) together with Fatou's Lemma gives

$$\begin{aligned}\alpha(\Omega) \leq J(u) &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega |u|^{p+1} dx \\ &\leq \liminf \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega |\nu_n|^{p+1} dx = \alpha(\Omega),\end{aligned}$$

which completes the proof. ■

Example 3. The following are some simple examples of periodic P-domains.

$$\begin{aligned}&D_3 = O \times \mathbb{R}^m, \text{ where } O \text{ is a bounded domain in } \mathbb{R}^p, m \geq 1, p \geq 1,\\ &D_4 = \{(x, y) \in \mathbb{R}^2 : \sin x < y < 1 + \sin x\},\\ &D_5 = \{(x, y, z) \in \mathbb{R}^3 : x = (\cos\theta)(2 + \sin z), y = (\sin\theta)(2 + \sin z), z \in R, \theta \in \mathbb{R}\}.\end{aligned}$$

Remark 1. As proved in [10], the only solution of (E) is $u \equiv 0$ if $\Omega = D_1$, where D_1 was defined in Example 1. Thus we cannot expect that (E) has positive solutions on all P-domains.

Theorem 4. *Let Ω_1 and Ω_2 be two P-domains such that $\Omega_1 \cap \Omega_2$ is bounded. If $\alpha(\Omega_1) \leq \alpha(\Omega_2)$ and J has minimizers on $M(\Omega_1)$, then there is a positive solution on Ω.*

Proof. Let $\alpha_1 = \alpha(\Omega_1)$ and $\alpha_2 = \alpha(\Omega_2)$. It is not difficult to show, by using the Maximum Principle, that

$$\alpha(\Omega) < \alpha_1. \tag{18}$$

Let $\{u_n\} \subset M(\Omega)$ be a sequence which satisfies

$$J(u_n) \to \alpha(\Omega) \text{ as } n \to \infty. \tag{19}$$

Since $\Omega_1 \cap \Omega_2$ is bounded, there is an $r > 0$ such that $\Omega_1 \cap \Omega_2 \subset B_r(0)$.

We now claim there is a $d > 0$ such that

$$\int_{|x| \leq 2r} u_n^2 dx \geq d \tag{20}$$

for all n. For otherwise, there exists a subsequence $\{u_n\}$ such that

$$\lim_{n\to\infty} \int_{|x| \leq 2r} u_n^2 dx = 0. \tag{21}$$

Let $\Omega_1' = \Omega_1 \backslash \overline{B_r(0)}$ and $\Omega_2' = \Omega_2 \backslash \overline{B_r(0)}$. Then by (21), there exist $\{u_{n,1}\} \subset H_0^1(\Omega_1')$ and $\{u_{n,2}\} \subset H_0^1(\Omega_2')$ such that

$$\lim_{n\to\infty} \|u_n - u_{n,1} - u_{n,2}\| = 0$$

and

$$u_{n,1} + u_{n,2} \in M(\Omega)$$

for all n. Consequently, $\alpha(\Omega) \geq \min(\alpha_1, \alpha_2)$, which contradicts (18). The rest of the proof is essentially the same as that of (11). ■

The proof of Theorem 4 also yields

Corollary 1. *If Ω_1 and Ω_2 are two periodic P-domains and $\Omega_1 \cap \Omega_2$ is bounded, then there is a positive solution on $\Omega_1 \cup \Omega_2$.*

A consequence of Theorem 4 is the following:

Example 4. Let $\Omega = D_1 \cup B_r(0)$. If r is large enough, then J has a minimizer on $M(\Omega)$.

Proof. It suffices to show that

$$\lim_{r\to\infty} \alpha(B_r) = 0.$$

For any $r > 1$, take $u_r \in M(B_r)$ such that

$$\sup_{r>1} \int_{B_r} u_r^2 dx < \infty$$

and

$$C(B_r) < \frac{2\int_{B_r} |u_r(x)|^2 dx}{\int_{B_r} |\nabla u_r(x)|^2 dx}.$$

Applying the Hölder inequality and the Sobolev inequality yields

$$\begin{aligned} \alpha(B_r) &\leq \frac{1}{2}\int_{B_r} |\nabla u_r|^2 dx - \frac{1}{p+1}\int_{B_r} |u_r|^{p+1} dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{B_r} |\nabla u_r|^2 dx \leq \frac{c}{C(B_r)} \end{aligned}$$

for some constant c. Then using Lemma 1 yields

$$\lim_{r\to\infty} \alpha(B_r) = 0.$$

■

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Department of Mathematics, National Changhua University of Education
Changhua, Taiwan, R.O.C.