TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 3, pp. 555-563, September 2001
This paper is available online at http://www.math.nthu.edu.tw/tjm/

WEIGHTED COMPOSITION OPERATORS BETWEEN H^∞ AND THE BLOCH SPACE

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Dedicated to Professor Kôzô Yabuta on his sixtieth birthday

Abstract. We study bounded and compact weighted composition operators, induced by a fixed analytic function and an analytic self-map of the open unit disk, between H^∞ and the Bloch space.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. We can define a linear operator uC_φ on the space of analytic functions on $\mathbb{D}$, called a *weighted composition operator*, by

$$uC_\varphi f := u \cdot (f \circ \varphi)$$

for a function f analytic on $\mathbb{D}$. We can regard this operator as a generalization of a multiplication operator and a composition operator. Each operator has been investigated on various Banach spaces of analytic functions on $\mathbb{D}$. See [6] and [2] for information on composition operators. In this paper we study the boundedness and the compactness of weighted composition operators between the classical Hardy space and the Bloch space.

We recall that the Hardy space H^∞ is the algebra of bounded analytic functions on $\mathbb{D}$. We also recall that the Bloch space $\mathcal{B}$ consists of all analytic functions f on $\mathbb{D}$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

Received April 14, 2000, revised July 18, 2000.
Communicated by P. Y. Wu.
2001 *Mathematics Subject Classification*: Primary 47B33, 30D55, 30H05.
Key words and phrases: Weighted composition operator, Hardy space, Bloch space.
The author was supported by the Grant-in-Aid for Scientific Research (C) No.11640179, Japan Society for the Promotion of Science and by Nippon Institute of Technology No.217.

Let $\mathcal{B}_o$ denote the subspace of $\mathcal{B}$ consisting of those $f \in \mathcal{B}$ for which $(1 - |z|^2)f'(z) \to 0$ as $|z| \to 1$. This space is called a little Bloch space.

Under the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}}(1 - |z|^2)|f'(z)|,$$

the Bloch space $\mathcal{B}$ becomes a Banach space and $H^\infty \subset \mathcal{B}$. For more information on Hardy spaces and the Bloch space, see [3] and [7, Chap. 5] respectively.

To characterize the compactness of weighted composition operators between H^∞ and $\mathcal{B}$, we will need the following result, whose proof is an easy modification of that of Proposition 3.11 in [2].

Proposition 1. *Let X and Y be H^∞ or $\mathcal{B}$. Then uC_φ is compact from X to Y if and only if whenever $\{f_n\}$ is bounded in X and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $uC_\varphi f_n \to 0$ in Y.*

The author would like to thank the referee for his careful reading, comments and suggestions that led to many improvements in the final version.

2. The case $uC_\varphi : \mathcal{B} \to H^\infty$

In this section we characterize bounded and compact weighted composition operators in the case $uC_\varphi : \mathcal{B} \to H^\infty$. The following theorem describes such properties, including the case that uC_φ is acting on the little Bloch space $\mathcal{B}_o$.

Theorem 1. *Let u be an analytic function on the unit disk $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then the following are equivalent*:

- (i) $uC_\varphi : \mathcal{B} \to H^\infty$ *is bounded*;
- (ii) $uC_\varphi : \mathcal{B} \to H^\infty$ *is compact*;
- (iii) $uC_\varphi : \mathcal{B}_o \to H^\infty$ *is bounded*;
- (iv) $uC_\varphi : \mathcal{B}_o \to H^\infty$ *is compact*;
- (v) $u \in H^\infty$ *and for a sequence* $\{z_n\}$ *in* $\mathbb{D}$ *such that* $|\varphi(z_n)|$ *tends to* 1 *as* $n \to \infty$, $u(z_n)$ *goes to* 0;
- (vi) $uC_\varphi : H^\infty \to H^\infty$ *is compact.*

Proof. The equivalence of (v) and (vi) is known (see [1, Proposition 2.3]). The implications (ii)⇒(i), (i)⇒(iii) and (ii)⇒(iv)⇒(iii) are clear.

(iii)⇒(v). It is clear that $u \in H^\infty$. For $\lambda \in \mathbb{D}$, let $f(z) = -\log(1 - \overline{\varphi(\lambda)}z)$. Then $f \in \mathcal{B}_o$ and $\|f\|_{\mathcal{B}} \leq 2$. So

$$\begin{aligned} 2\|uC_\varphi\| &\geq \|uC_\varphi f\|_\infty \\ &= \sup_{z \in \mathbb{D}} \left| u(z) \log \frac{1}{1 - \overline{\varphi(\lambda)}\varphi(z)} \right| \\ &\geq |u(\lambda)| \log \frac{1}{1 - |\varphi(\lambda)|^2} \geq 0. \end{aligned}$$

Thus if there is a sequence $\{\lambda_n\}$ in $\mathbb{D}$ such that $|\varphi(\lambda_n)|$ tends to 1, the inequality above implies $u(\lambda_n) \to 0$.

(v)⇒(ii). Suppose that $uC_\varphi : \mathcal{B} \to H^\infty$ is not compact. Then, by Proposition 1, there exists a bounded sequence $\{f_n\}$ in $\mathcal{B}$ such that $f_n \to 0$ uniformly on compacts but $\|uC_\varphi f_n\|_\infty \not\to 0$, i.e., there exist a $\delta > 0$ such that

$$\|uC_\varphi f_n\|_\infty \geq \delta \quad \text{for all } n.$$

Then we can choose a sequence $\{z_n\}$ in $\mathbb{D}$ such that

$$(1) \qquad |u(z_n)C_\varphi f_n(z_n)| \geq \delta/2.$$

If $\varphi(z_n)$ tends to a point ζ, then ζ is in $\partial\mathbb{D}$. By (v), $u(z_n) \to 0$. This contradicts (1). ■

3. THE CASE $uC_\varphi : H^\infty \to \mathcal{B}$

At first we will characterize the boundedness.

Theorem 2. *Let u be an analytic function on the unit disk $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then $uC_\varphi : H^\infty \to \mathcal{B}$ is bounded if and only if the following* (i) *and* (ii) *are satisfied*:

(i) $u \in \mathcal{B}$;

(ii) $\sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |u(z)\varphi'(z)| < \infty$.

Proof. Suppose that $uC_\varphi : H^\infty \to \mathcal{B}$ is bounded. Then it is evident that

$$(2) \qquad u \in \mathcal{B}$$

and

$$(3) \qquad \sup_{z \in \mathbb{D}} (1 - |z|^2)|u(z)\varphi'(z)| < \infty.$$

For $\lambda \in \mathbb{D}$, let $f(z) = (1-|\varphi(\lambda)|^2)/(1-\overline{\varphi(\lambda)}z)$. Then $f \in H^\infty$ and $\|f\|_\infty \le 2$. So

$$\begin{aligned} 2\|uC_\varphi\| &\ge \|uC_\varphi f\|_{\mathcal{B}} \\ &\ge \frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)| - (1-|\lambda|^2)|u'(\lambda)|. \end{aligned}$$

Since $u \in \mathcal{B}$,

$$\sup_{\lambda\in\mathbb{D}} \frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)| < \infty. \tag{4}$$

Thus, for a fixed δ, $0 < \delta < 1$, by (4),

$$\sup\left\{\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\varphi'(\lambda)| : \lambda \in \mathbb{D}, |\varphi(\lambda)| > \delta\right\} < \infty. \tag{5}$$

For $\lambda \in \mathbb{D}$ such that $|\varphi(\lambda)| \le \delta$, we have

$$\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\varphi'(\lambda)| \le \frac{1}{1-\delta^2}(1-|\lambda|^2)|u(\lambda)\varphi'(\lambda)|$$

and so, by (3),

$$\sup\left\{\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\varphi'(\lambda)| : \lambda \in \mathbb{D}, |\varphi(\lambda)| \le \delta\right\} < \infty. \tag{6}$$

Consequently, by (5) and (6),

$$\sup_{\lambda\in\mathbb{D}} \frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\varphi'(\lambda)| < \infty.$$

Conversely, suppose that conditions (i) and (ii) hold. For a function $f \in H^\infty$, we have the following inequality:

$$\begin{aligned} &(1-|z|^2)|(uC_\varphi f)'(z)| \\ &= (1-|z|^2)|u'(z)f(\varphi(z)) + u(z)f'(\varphi(z))\varphi'(z))| \\ &\le (1-|z|^2)|u'(z)f(\varphi(z))| \\ &\quad + \frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)|(1-|\varphi(z)|^2)|f'(\varphi(z))| \\ &\le \|u\|_{\mathcal{B}}\|f\|_\infty + \frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)|\|f\|_{\mathcal{B}}. \end{aligned}$$

By the definition of $\|\cdot\|_{\mathcal{B}}$ and the Schwarz lemma, we have $\|f\|_{\mathcal{B}} \le 2\|f\|_\infty$ for $f \in H^\infty$. So the conditions imply that the right-hand side above is bounded by some constant times $\|f\|_\infty$. Consequently, $uC_\varphi : H^\infty \to \mathcal{B}$ is bounded. ■

Next we will consider the compactness.

Theorem 3. *Let* u *be an analytic function on the unit disk* $\mathbb{D}$ *and* φ *an analytic self-map of* $\mathbb{D}$. *Suppose that* $uC_\varphi : H^\infty \to \mathcal{B}$ *is bounded. Then* uC_φ *is compact if and only if the following* (i) *and* (ii) *are satisfied*:

(i) $\lim_{|\varphi(z)|\to 1}(1-|z|^2)|u'(z)| = 0$;

(ii) $\lim_{|\varphi(z)|\to 1}\frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)| = 0.$

Proof. Suppose $uC_\varphi : H^\infty \to \mathcal{B}$ is compact. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Let $f_n(z) = (1-|\varphi(z_n)|^2)/(1-\overline{\varphi(z_n)}z)$. Then $f_n \in H^\infty$, $\|f_n\|_\infty \le 2$ and f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$.

Since uC_φ is compact, we have

$$\|uC_\varphi f_n\|_{\mathcal{B}} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$\begin{aligned}\|uC_\varphi f_n\|_{\mathcal{B}} &\ge \sup_{z\in\mathbb{D}}(1-|z|^2)|(uC_\varphi f_n)'(z)| \\ &\ge (1-|z_n|^2)\left|u'(z_n) + u(z_n)\frac{1-|\varphi(z_n)|^2}{(1-|\varphi(z_n)|^2)^2}\overline{\varphi(z_n)}\varphi'(z_n)\right| \\ &\ge \left|(1-|z_n|^2)|u'(z_n)| - \frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|\right|.\end{aligned}$$

So we get

$$\lim_{|\varphi(z_n)|\to 1}(1-|z_n|^2)|u'(z_n)| = \lim_{|\varphi(z_n)|\to 1}\frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}|u(z_n)\varphi'(z_n)|. \tag{7}$$

Next let

$$g_n(z) = \frac{1-|\varphi(z_n)|^2}{1-\overline{\varphi(z_n)}z} - \left(\frac{1-|\varphi(z_n)|^2}{1-\overline{\varphi(z_n)}z}\right)^{1/2}$$

for a sequence $\{z_n\}$ in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Then $\{g_n\}$ is a bounded sequence in H^∞ and $g_n(z) \to 0$ uniformly on every compact subset of $\mathbb{D}$. Moreover, we notice that $g_n(\varphi(z_n)) = 0$ and

$$g_n'(\varphi(z_n)) = \frac{\overline{\varphi(z_n)}}{2(1-|\varphi(z_n)|^2)}.$$

By the similar method as above,

$$
\begin{aligned}
0 \leftarrow & \|uC_\varphi g_n\|_{\mathcal{B}} \\
& \geq \frac{1-|z_n|^2}{2(1-|\varphi(z_n)|^2)}|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|.
\end{aligned}
$$

Thus we can get condition (ii) and so, by (7),

$$\lim_{|\varphi(z_n)|\to 1}(1-|z_n|^2)|u'(z_n)| = 0.$$

Conversely, suppose that conditions (i) and (ii) hold. We will use Proposition 1. Let $\{f_n\}$ be a sequence in H^∞ with $\|f_n\|_\infty \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, By the assumption, for any $\varepsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that $\delta < |\varphi(z)| < 1$ implies

$$(1-|z|^2)|u'(z)| < \frac{\varepsilon}{2}$$

and

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)| < \frac{\varepsilon}{2}.$$

Let $K = \{w \in \mathbb{D} : |w| \leq \delta\}$. Note that K is a compact subset of $\mathbb{D}$. Then

$$
\begin{aligned}
& \|uC_\varphi f_n\|_{\mathcal{B}} \\
& = \sup_{z\in\mathbb{D}}(1-|z|^2)|(uC_\varphi f_n)'(z)| \\
& \leq \sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)f_n(\varphi(z))| \\
& \quad + \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)|(1-|\varphi(z)|^2)|f_n'(\varphi(z))| \\
& \leq \sup_{\{z\in\mathbb{D}:\varphi(z)\in K\}}(1-|z|^2)|u'(z)f_n(\varphi(z))| \\
& \quad + \sup_{\{z\in\mathbb{D}:\varphi(z)\in K\}} \frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)|(1-|\varphi(z)|^2)|f_n'(\varphi(z))| + \varepsilon \\
& \leq \|u\|_{\mathcal{B}} \sup_{w\in K}|f_n(w)| + M \sup_{w\in K}(1-|w|^2)|f_n'(w)| + \varepsilon,
\end{aligned}
$$

where $M = \sup\{(1-|z|^2)|u(z)\varphi'(z)|/(1-|\varphi(z)|^2) : z \in \mathbb{D}\}$.

As $n \to \infty$,

$$\|uC_\varphi f_n\|_{\mathcal{B}} \leq \varepsilon.$$

Consequently, uC_φ is compact. ■

Next we will characterize bounded and compact weighted composition operators in the case $uC_\varphi : H^\infty \to \mathcal{B}_o$.

Theorem 4. *Let u be an analytic function on the unit disk $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then the following are equivalent*:

(i) $uC_\varphi : H^\infty \to \mathcal{B}_o$ *is bounded*;

(ii) $uC_\varphi : H^\infty \to \mathcal{B}_o$ *is compact*;

(iii) $u \in \mathcal{B}_o$ *and*

$$\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)| = 0.$$

Proof. The implication (ii)⇒(i) is clear.

(i)⇒(iii). Suppose that $uC_\varphi : H^\infty \to \mathcal{B}_o$ is bounded. Then it is evident that

$$u \in \mathcal{B}_o \tag{8}$$

and

$$\lim_{|z|\to 1}(1-|z|^2)|u(z)\varphi'(z)| = 0. \tag{9}$$

In the case that $\|\varphi\|_\infty < 1$, (8) and (9) give (iii). And so we suppose $\|\varphi\|_\infty = 1$. Let $\{\varphi(z_n)\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Of course, $|z_n| \to 1$. We can take a subsequence (we denote by the same $\{\varphi(z_n)\}$) of $\{\varphi(z_n)\}$ that is an interpolating sequence in $\mathbb{D}$. A sequence $\{\zeta_n\}$ in $\mathbb{D}$ is called an *interpolating sequence* if they are "hyperbolically apart", i.e., there exists a $\delta > 0$ so that

$$\prod_{m\neq n} \frac{|\zeta_m - \zeta_n|}{|1-\overline{\zeta_m}\zeta_n|} > \delta.$$

It is known that $\{\zeta_n\}$ is an interpolating sequence in $\mathbb{D}$ if and only if the infinite Blaschke product

$$\prod_{n=1}^{\infty} \frac{z-\zeta_n}{1-\overline{\zeta_n}z}$$

converges uniformly on compacts to a $f \in H^\infty$ with $\|f\|_\infty = 1$, and evidently $f(\zeta_n) = 0$ for all n (the readers can find a detailed treatment on the subject of interpolating sequences in, for example, [3]). Let b be an interpolating Blaschke product with zeros $\{\varphi(z_n)\}$. Then

$$\begin{aligned}(1-|\varphi(z_n)|^2)|b'(\varphi(z_n))| &= \prod_{m\neq n}\left|\frac{\varphi(z_n)-\varphi(z_m)}{1-\overline{\varphi(z_n)}\varphi(z_m)}\right| \\ &\geq \delta\end{aligned}$$

for some $\delta > 0$.

By (i), $uC_\varphi b \in \mathcal{B}_o$. Then

$$\begin{aligned}&(1-|z|^2)|(uC_\varphi b)'(z)|\\ &\geq (1-|z|^2)|u(z)\varphi'(z)b'(\varphi(z))| - (1-|z|^2)|u'(z)b(\varphi(z))|.\end{aligned}$$

So

$$(1-|z|^2)|u(z)\varphi'(z)b'(\varphi(z))| \leq (1-|z|^2)|(uC_\varphi b)'(z)| + (1-|z|^2)|u'(z)|.$$

As $|z| \to 1$,

$$(1-|z|^2)|u(z)\varphi'(z)b'(\varphi(z))| \to 0.$$

Here

$$\begin{aligned}&(1-|z_n|^2)|u(z_n)\varphi'(z_n)b'(\varphi(z_n))|\\ &= \frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}|u(z_n)\varphi'(z_n)|(1-|\varphi(z_n)|^2)|b'(\varphi(z_n))|\\ &\geq \frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}|u(z_n)\varphi'(z_n)|\delta.\end{aligned}$$

Consequently,

$$\lim_{|z_n|\to 1} \frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}|u(z_n)\varphi'(z_n)| = 0.$$

The implication (iii)⇒(ii) would be proved by using Lemma 1 in [5]. ■

Remark. The author and Zhao studied the boundedness and the compactness of uC_φ on the Bloch and the little Bloch spaces in [4] and obtained the following result [4, Theorem 2]: Let u be an analytic function on the unit disk $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Suppose that uC_φ is bounded on $\mathcal{B}$. Then uC_φ is compact on $\mathcal{B}$ if and only if the following (i) and (ii) are satisfied:

(i) $\displaystyle\lim_{|\varphi(z)|\to 1}(1-|z|^2)|u'(z)|\log\frac{2}{1-|\varphi(z)|^2} = 0;$

(ii) $\displaystyle\lim_{|\varphi(z)|\to 1}\frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)| = 0.$

By this result and Theorem 3 above, we get that if $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded (resp. compact), then $uC_\varphi : H^\infty \to \mathcal{B}$ is bounded (resp. compact).

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