TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 3, pp. 497-506, September 2001
This paper is available online at http://www.math.nthu.edu.tw/tjm

ON THE SHAPE OF NUMERICAL RANGES ASSOCIATED WITH LIE GROUPS

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Dedicated to Ky Fan on the occasion of his 85th birthday

Abstract. A survey of some recent results on the shape of the numerical ranges associated with Lie groups, mainly convexity and star-shapedness, is given. Some questions are asked.

1. Introduction

The classical numerical range of $A \in \mathbb{C}_{n\times n}$ is defined as the following subset of $\mathbb{C}$:

$$W(A) := \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

The celebrated Toeplitz-Hausdorff theorem [27, 13] asserts that it is convex. It is remarkable for it states that the image of the unit sphere in $\mathbb{C}^n$ (a hollow object) is convex under the nonlinear map, $x \mapsto x^*Ax$. Perhaps it is the most interesting geometric property of the set. Various generalizations have been considered in the literature and the development has been very active in the last decades [12, 20]. Our focus will be on the numerical ranges arising from Lie groups. Though the study is fruitful, it is still a new development and by no means covers all generalizations. In this note, we give a brief survey of some recent results on the shape of the numerical ranges, mainly convexity and star-shapedness. Some questions are asked. Our general references for Lie theory are [14, 18, 23].

Received February 9, 2000; revised October 16, 2000.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: Primary 15A60, 22E99.
Key words and phrases: Numerical range, convexity, star-shaped, Lie group, reductive Lie algebra.
The author thanks Professor Bit-Shun Tam for the arrangement of his visit in Taiwan, which is made possible by a grant from NSC of Taiwan and also thanks the Department of Mathematics of Tamkang University for the support and warm hospitality. The paper is based on a talk of the author given in ICMAA 2000, Taiwan, ROC, Jan. 17-21 2000. He also thanks Professor Pei-Yuan Wu for inviting him to the session of Numerical Ranges and Radii of the conference.

Halmos introduced the k-numerical range of $A \in \mathbb{C}_{n\times n}$:

$$W_k(A) := \left\{ \sum_{i=1}^{k} x_i^* A x_i : x_1, \ldots, x_k \text{ o.n. in } \mathbb{C}^n \right\}, \quad k = 1, \ldots, n.$$

He conjectured and Berger [8] proved that $W_k(A)$ is always convex. Then Westwick [29] considered the c-numerical range of A, where $c \in \mathbb{C}^n$:

$$W_c(A) := \left\{ \sum_{i=1}^{n} c_i x_i^* A x_i : x_1, \ldots, x_n \text{ o.n. in } \mathbb{C}^n \right\}.$$

By spectral decomposition, it can be formulated as

$$W_C(A) := \{\mathrm{tr}\, CUAU^{-1} : U \in U(n)\},$$

where $U(n)$ denotes the unitary group and C is normal with eigenvalues $c \in \mathbb{C}^n$. He proved that $W_C(A)$ is always convex for real c, i.e., C is Hermitian, and this is known as Westwick's convexity theorem, but $W_C(A)$ fails to be convex for complex c when $n \geq 3$. The main idea of Westwick's proof is the application of Morse theory on the homogeneous space $U(n)/\triangle(n)$ where $\triangle(n) \subset U(n)$ is the subgroup of diagonal matrices. Poon [24] gave the first elementary proof to Westwick's result. The result was later rediscovered by Ginsburg [2, p. 8].

2. Numerical Range and Compact Connected Lie Group

Let us elaborate on Westwick's setting. If $A = A_1 + iA_2$ is the Hermitian decomposition of $A \in \mathbb{C}_{n\times n}$, where A_1, A_2 are $n \times n$ Hermitian matrices, and C is an $n \times n$ Hermitian matrix, then $W_C(A)$ may be identified as the following subset of $\mathbb{R}^2$:

$$(1) \qquad W_C(A_1, A_2) := \{(\mathrm{tr}\, CUA_1U^{-1}, \mathrm{tr}\, CUA_2U^{-1}) : U \in U(n)\}.$$

It is well-known that $U(n)$ is a compact connected Lie group whose Lie algebra $\mathfrak{u}(n)$ is the set of skew Hermitian matrices. Notice that

$$\mathrm{tr}\, CU^{-1}BU = \mathrm{tr}\, BUCU^{-1} = -\mathrm{tr}\,(iB)U(iC)U^{-1}$$

and thus we may assume that $A_1, A_2, C \in \mathfrak{u}(n)$ if convexity is the main concern, and (1) can be written as $W_C(A_1, A_2) = \{(\mathrm{tr}\, A_1L, \mathrm{tr}\, A_2L) : L \in \mathrm{Ad}\,(U(n))C\}$, where $\mathrm{Ad}\,(U(n))C := \{UCU^{-1} : U \in U(n)\}$ is the adjoint orbit of C. This orbital point of view turns out to be very useful in our study. The consideration of

Raïs [25] is then natural: Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ which is equipped with a G-invariant inner product $\langle \cdot, \cdot \rangle$, i.e.,

$$\langle \mathrm{Ad}\,(g)X, \mathrm{Ad}\,(g)Y \rangle = \langle X, Y \rangle, \qquad X, Y \in \mathfrak{g}, \quad g \in G.$$

For $A_1, A_2, C \in \mathfrak{g}$, the C-numerical range of the pair (A_1, A_2) is defined to be the following subset of $\mathbb{R}^2$:

$$W_C(A_1, A_2) := \{(\langle A_1, \mathrm{Ad}\,(g)C \rangle, \langle A_2, \mathrm{Ad}\,(g)C \rangle) \ : \ g \in G\}. \tag{2}$$

It can be rewritten as

$$W_C(A_1, A_2) = \{(\langle A_1, L \rangle, \langle A_2, L \rangle) \ : \ L \in \mathrm{Ad}\,(G)C\}, \tag{3}$$

where $\mathrm{Ad}\,(G)C := \{\mathrm{Ad}\,(g)C : g \in G\}$ is the adjoint orbit of C in $\mathfrak{g}$.

By using a result of Atiyah [1] on a smooth function whose Hamiltonian vector field generates a torus action on a compact connected symplectic manifold, and the well-known result of Kirillov-Kostant-Souriau: the co-adjoint orbit of a compact connected Lie group has a natural symplectic structure [17], we have

Theorem 2.1. [26] *Let G be a compact connected Lie group. For $A_1, A_2, C \in \mathfrak{g}$, the generalized numerical range $W_C(A_1, A_2)$ defined by* (2) *is convex.*

Corollary 2.2.

(1) (Westwick [27]) *Let $G = U(n)$ or $SU(n)$. The C-numerical range $W_C(A_1, A_2) = \{(\mathrm{tr}\, A_1UCU^{-1}, \mathrm{tr}\, A_2UCU^{-1}) : U \in G\}$ is convex, where A_1, A_2 and C are Hermitian matrices.*

(2) *The set $W_C(A_1, A_2) = \{(\mathrm{tr}\, A_1OCO^{-1}, \mathrm{tr}\, A_2OCO^{-1}) : O \in SO(n)\}$ is convex, where A_1, A_2, and C are real skew symmetric matrices.*

(3) *The set $W_C(A_1, A_2) = \{(\mathrm{tr}\, A_1OCO^{-1}, \mathrm{tr}\, A_2OCO^{-1}) : O \in O(2n+1)\}$ is convex and is equal to $\{(\mathrm{tr}\, A_1OCO^{T}, \mathrm{tr}\, A_2OCO^{T}) : O \in SO(2n+1)\}$, where A_1, A_2, and C are real skew symmetric matrices.*

(4) *The set $W_C(A_1, A_2) = \{(\mathrm{tr}\, A_1UCU^{-1}, \mathrm{tr}\, A_2UCU^{-1}) : U \in Sp(n)\}$ is convex, where $A_1, A_2, C \in \mathfrak{sp}(n)$ and the symplectic group $Sp(n) \subset U(2n)$ consists of*

$$\begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix} \in U(2n).$$

Remark 2.3. Theorem 2.1 is best possible in the sense that $W_C(A_1, \ldots, A_p)$ may fail to be convex if $p \geq 3$. Indeed, when $G = U(n)$ and $C = \mathrm{diag}\,(1, 0, \ldots, 0)$, $W_C(A_1, \ldots, A_p)$ fails to be convex [3] for some choice of A's when $p \geq 3$ or $n = 2$ while $p = 3$. But it is convex when $p = 3$ and $n \geq 3$. Also see [6].

3. Numerical Range and Reductive Lie Algebra

Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{z}$ be a real reductive Lie algebra, where $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$ is semisimple and $\mathfrak{z}$ is the center of $\mathfrak{g}$. Let $K \subset G_0$ (it is unique once we fix the analytic group G for $\mathfrak{g}$ [14, p. 112]) be the analytic group of $\mathfrak{k}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a given Cartan decomposition of $\mathfrak{g}$. Here $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $B(\cdot, \cdot)$. For $A_1, \ldots, A_p, C \in \mathfrak{p}$, the C-numerical range of $(A_1, \ldots, A_p)$ is defined [26, 21] as the following subset of $\mathbb{R}^p$:

$$W_C(A_1, \ldots, A_p) = \{(B(A_1, Z), \ldots, B(A_p, Z)) : Z \in \mathrm{Ad}\,(K)C\}, \tag{4}$$

where $\mathrm{Ad}\,(K)C = \{\mathrm{Ad}\,(k)C : k \in K\}$ is the orbit of C in $\mathfrak{p}$ under the adjoint action of K. Once we fix the Lie algebra $\mathfrak{g}$, the C-numerical range is independent of the choice of analytic group G associated with it [21]. Moreover, the choice of Cartan decomposition of $\mathfrak{g}$ does not affect the convexity or the nonconvexity of the numerical range. The above definition was motivated by a result of Au-Yeung and Tsing [6]: $W_C(A_1, A_2, A_3)$ is convex when $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ ($\mathfrak{gl}(n, \mathbb{H})$) and C, A_1, A_2, A_3 are Hermitian matrices over $\mathbb{C}$ ($\mathbb{H}$) with $n \geq 3$.

Indeed, the setting (4) is more general than (3) if the invariant inner product is $-B(\cdot, \cdot)$. To see this, it is sufficient to consider semisimple compact connected Lie group G in (3). It is because for every compact connected Lie group G, G is the commuting product $G_s Z_0$ and $\mathfrak{g} = \mathfrak{g}_s + \mathfrak{z}$, where G_s is the analytic subgroup of G with semisimple Lie algebra [14, p. 132], $\mathfrak{g}_s = [\mathfrak{g}, \mathfrak{g}]$ and Z_0 is the identity component of the center Z of G whose Lie algebra is $\mathfrak{z}$. Now $\mathrm{Ad}\,(Z)$ is trivial and $\mathrm{Ad}\,(G)$ acts trivially on $\mathfrak{z}$. So for any $C = C_s + C_z$, where $C_s \in \mathfrak{g}_s$, $C_z \in \mathfrak{z}$, we have $\mathrm{Ad}\,(G)C = \mathrm{Ad}\,(G_s)C_s + C_z$. So $W_C(A_1, A_2)$ in (3) can be written as

$$\{(\langle A_{1s}, L\rangle, \langle A_{2s}, L\rangle) : L \in \mathrm{Ad}\,(G_s)C_s\} + H,$$

where $A_i = A_{is} + A_{iz}$, $i = 1, 2$, and

$$H := (\langle A_{1s}, C_z\rangle, \langle A_{2s}, C_z\rangle) + (\langle A_{1z}, C_s\rangle, \langle A_{2z}, C_s\rangle) + (\langle A_{1z}, C_z\rangle, \langle A_{2z}, C_z\rangle)$$

is a constant since $\langle \cdot, \cdot \rangle$ is Ad-invariant and the adjoint action is trivial on $\mathfrak{z}$. Thus it suffices to consider the semisimple G_s. Now $\mathfrak{g} = \mathfrak{g}_s + i\mathfrak{g}_s$ is complex semisimple which is viewed as a real semisimple Lie algebra. Identifying $\mathfrak{p} = i\mathfrak{g}_s$ with $\mathfrak{g}_s$ in (4), we get (3).

It is known [21] that $\mathfrak{sl}_2(\mathbb{R})$ is the only one giving nonconvex $W_C(A_1, A_2)$ among simple classical real Lie algebras (up to isomorphism). Concerning the convexity of $W_C(A_1, A_2, A_3)$ we have the following table and the proofs involve delicate computation.

Table 3.1. [21]

$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}),\ n \geq 2$	:	Yes if $n > 2$ (best possible)
$\mathfrak{h} = \mathfrak{sl}_n(\mathbb{R})$	:	No
$\mathfrak{h} = \mathfrak{sl}_m(\mathbb{H}),\ n = 2m$	:	Yes if $n > 2$ (best possible)
$\mathfrak{h} = \mathfrak{su}_{p,q}\,(p = 0, 1, \ldots, [n/2],\ p + q = n)$	:	Yes if $p \neq q$ (best possible); No if $p = q$
$\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C}),\ n \geq 2$	:	Yes if $n > 2$ (best possible)
$\mathfrak{h} = \mathfrak{so}_{p,q}\,(p = 0, 1, \ldots, n,\ p + q = 2n + 1)$	:	No
$\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C}),\ n = 2m,\ m \geq 3$	:	Yes (best possible)
$\mathfrak{h} = \mathfrak{sp}_n(\mathbb{R}),\ n = 2m$	:	No
$\mathfrak{h} = \mathfrak{sp}_{p,q},\ (p = 0, 1, \ldots, [m/2],\ p + q = m)$	:	No
$\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C}), n \geq 4$	:	Yes (best possible)
$\mathfrak{h} = \mathfrak{so}_{p,q},\ (p = 0, 1, \ldots, n,\ p + q = 2n)$	:	No
$\mathfrak{h} = \mathfrak{so}^*(2n)$	:	No if n is even. Yes if n is odd.

The following is the only case in the above list without an answer.

Problem 3.2 [21]. For the case $\mathfrak{so}^*(2n)$ with an odd integer n, what is the largest $m \geq 3$ so that $W_C(A_1, \ldots, A_m)$ is always convex? It is known that $m \leq 5$.

Remark 3.2 [21]. The exceptional simple Lie algebras are [23]: 3 for $\mathfrak{g}_2$; 4 for $\mathfrak{f}_4$; 6 for $\mathfrak{e}_6$; 5 for $\mathfrak{e}_7$ and 4 for $\mathfrak{e}_8$. The total number of cases is 22. Among them 5 are compact Lie algebras and the corresponding numerical ranges are trivial. For those 5 complex simple Lie algebras of exceptional type, when we consider them as real Lie algebras, Theorem 2.1 yields the convexity of $W_C(A_1, A_2)$. Hence 12 cases are left open.

4. Generalized Numerical Range and Normality

Westwick's convexity result asserts (after a suitable translation and rotation) that $W_C(A)$ is convex if C is normal and has collinear eigenvalues, for all $A \in \mathbb{C}_{n\times n}$. Given a normal C, Marcus [22] further conjectured that if $W_C(A)$ is convex for all $A \in \mathbb{C}_{n\times n}$, then the eigenvalues of C are collinear. Au-Yeung and Tsing [7] proved Marcus' conjecture affirmatively and their result is even stronger: $W_c([c]^*) = \{\mathrm{tr}\,[c]U[c]^*U^{-1} : U \in U(n)\}$ is not convex if the entries of c are not collinear, where $[c] = \mathrm{diag}\,(c_1, \ldots, c_n)$. Also see [9, 10].

Now we have the following setting. Let $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ be the Cartan decomposition of a *complex* semisimple Lie algebra and let $B(\cdot, \cdot)$ be the Killing form on $\mathfrak{g}$. Let

θ be the Cartan involution, i.e., $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $x + y \mapsto x - y$ if $x \in \mathfrak{k}$ and $\mathfrak{p} = i\mathfrak{k}$. Then θ and the Killing form induce an inner product on $\mathfrak{g}$:

$$(x, y)_\theta = -B(x, \theta y), \qquad x, y \in \mathfrak{g}.$$

Given $x, y \in \mathfrak{g}$, we define the x-numerical range of y as the following subset of $\mathbb{C}$:

$$W_x(y) := \{(x, z)_\theta : z \in \mathrm{Ad}\,(K)y\}.$$

The numerical range for the complex reductive case is similarly defined. When $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, the Cartan decomposition is the usual Hermitian decomposition, $K = SU(n)$ and $\theta(A) = -A^*$, $A \in \mathfrak{gl}(n, \mathbb{C})$. Thus if $A, C \in \mathfrak{gl}(n, \mathbb{C})$, then $W_C(A) = \{\mathrm{tr}\, CUA^*U^{-1} : U \in SU(n)\}$. The only difference between this setting and the usual setting in the literature is that A is replaced by A^* and this yields no difficulty.

Let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{p} = i\mathfrak{k}$ and thus $i\mathfrak{a} + \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Now an element $x \in \mathfrak{g}$ is said to be *normal* if $\mathrm{Ad}\,(k)x \in i\mathfrak{a} + \mathfrak{a}$ for some $k \in K$. Motivated by the result of Au-Yeung and Tsing [7] and some computer generated figures, we have

Conjecture 4.1. Let $\mathfrak{g}$ be a complex simple Lie algebra. If $x \in \mathfrak{g}$ is normal and there does not exist a $\xi \in \mathbb{C}$ such that $\xi x \in \mathfrak{a}$, then $W_x(x)$ is not convex.

For example, if $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, then the conjecture is that the set

$$\{\mathrm{tr}\, COC^*O^{-1} : \ O \in SO(n)\}$$

is not convex, where

$$C = \begin{bmatrix} 0 & a_1 + ib_1 \\ -(a_1 + ib_1) & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & a_m + ib_m \\ -(a_m + ib_m) & 0 \end{bmatrix} (\oplus 0)$$
$$\in \mathbb{C}_{n \times n}, \ m = [n/2],$$

if $a_1 + ib_1, \ldots, a_m + ib_m$ are not collinear. We remark that

$$\begin{aligned} |(x, \mathrm{Ad}\,(k)x)_\theta|^2 &\le (x, x)_\theta (\mathrm{Ad}\,(k)x, \mathrm{Ad}\,(k)x)_\theta \quad \text{(by Cauchy-Schwarz inequality)} \\ &= -(x, x)_\theta B(\mathrm{Ad}\,(k)x, \theta \mathrm{Ad}\,(k)x) \\ &= -(x, x)_\theta B(\mathrm{Ad}\,(k)x, \mathrm{Ad}\,(k)\theta x) \quad (\text{by } \theta \mathrm{Ad}\,(k) = \mathrm{Ad}\,(k)\theta) \\ &= -(x, x)_\theta B(x, \theta x) \quad (\text{since } B(\cdot, \cdot) \text{ is } \mathrm{Ad}\,(K)\text{-invariant}) \\ &= (x, x)_\theta^2. \end{aligned}$$

Note that θ and $\mathrm{Ad}\,(k)$ commute since $\mathrm{Ad}\,(K)$ leaves $\mathfrak{k}$ and $\mathfrak{p} = i\mathfrak{k}$ invariant. So $(x, x)_\theta \in W_x(x)$ is positive and has the largest magnitude. (The boundary of $W_c(c)$

near this point is concave as shown in the proof of Au-Yeung and Tsing [7] when $c's$ are not collinear for the $\mathfrak{gl}_n(\mathbb{C})$ case). Moreover $W_x(x)$ is symmetric about the origin for if $w \in W_x(x)$, then $w = (x, \mathrm{Ad}(k)x)_\theta$ and $\overline{w} = \overline{(x, \mathrm{Ad}(k)x)_\theta} = (\mathrm{Ad}(k)x, x)_\theta = (x, \mathrm{Ad}(k^{-1})x) \in W_x(x)$.

A related problem is concerning Kostant's convexity theorem [19] for complex reductive Lie algebras. Kostants's result claims that if $\mathfrak{g}$ is a real reductive Lie algebra, then

$$\pi(\mathrm{Ad}\,(K)x) = \mathrm{conv}\, Wx, \qquad x \in \mathfrak{a},$$

where $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subalgebra in $\mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$, W is the Weyl group of $(\mathfrak{g}, \mathfrak{a})$, $\pi : \mathfrak{p} \to \mathfrak{a}$ is the orthogonal projection with respect to the Killing form and $\mathrm{conv}\, S$ denotes the convex hull of the set S. This generalizes a classical result of Schur and Horn, namely,

$$\mathcal{W}(\lambda) := \{\mathrm{diag}\, U\Lambda U^{-1} : U \in U(n)\} = \mathrm{conv}\, S_n\lambda,$$

where $\Lambda = \mathrm{diag}\,(\lambda)$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and S_n is the symmetric group. Au-Yeung and Sing [4] proved that if $\lambda \in \mathbb{C}^n$ with $\lambda's$ not collinear, then $\mathcal{W}(\lambda)$ is not convex. Nevertheless, Tsing [26] proved that $\mathcal{W}(\lambda)$ is star-shaped with respect to the star center $(\sum_{i=1}^n \lambda_i)e$, where $e = (1, 1, \ldots, 1)$. Here we say that a nonempty subset X of a vector space is star-shaped with respect to a star center s if $tx + (1-t)s \in X$ whenever $x \in X$ and $t \in [0, 1]$. Thus it is natural to ask the following questions.

Question 4.2. Let $\mathfrak{g}$ be a complex simple Lie algebra. If $x \in \mathfrak{g}$ is normal and there does not exist a $\xi \in \mathbb{C}$ such that $\xi x \in \mathfrak{a}$, is it true that $\mathcal{W}(x) := \pi(\mathrm{Ad}\,(K)x)$ is not convex, where $\pi : \mathfrak{g} \to i\mathfrak{a} + \mathfrak{a}$ is the orthogonal projection with respect to the inner product $(\cdot, \cdot)_\theta$?

For example, if $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, then the question is whether the set

$$\mathcal{W}(C) := \{(a_{12}, a_{34}, a_{56}, \ldots, a_{2m+1,2m}) : A = OCO^{-1},\ O \in SO(n)\} \subset \mathbb{R}^m$$

is not convex where

$$C = \begin{bmatrix} 0 & a_1 + ib_1 \\ -(a_1 + ib_1) & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & a_m + ib_m \\ -(a_m + ib_m) & 0 \end{bmatrix} (\oplus 0)$$
$$\in \mathbb{C}_{n\times n},\ m = [n/2],$$

if $a_1 + ib_1, \ldots, a_m + ib_m$ are not collinear with the origin. We remark that if Conjecture 4.1 is true, then the answer to Question 4.2 is positive.

Question 4.3. Let $\mathfrak{g}$ be a complex reductive Lie algebra. If $x \in \mathfrak{g}$ is normal, is it true that $\mathcal{W}(x) := \pi(\mathrm{Ad}\,(K)x)$ is star-shaped with respect to the star center $\pi(x_z)$, where $x = x_s + x_z$, $x_z \in \mathfrak{z}$ and $x_s \in [\mathfrak{g}, \mathfrak{g}]$?

For the case $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, the question is whether the set $\mathcal{W}(C)$ is star-shaped or not for the above C with general $a_1 + ib_1, \ldots, a_m + ib_m$?

5. Star-Shapedness

When $C, A \in \mathbb{C}_{n\times n}$ with C normal, Straus conjectured and Tsing [28] proved that the C-numerical range

$$W_C(A) = \{\operatorname{tr} CUAU^{-1} : \ U \in U(n)\}$$

is star-shaped with star center $(1/n)\operatorname{tr} A \operatorname{tr} C$, a very interesting result on the shape of the numerical range. Later Hughes [15] proved an infinite-dimensional analog of Tsing's result: the closure of the set

$$W_C(T) := \left\{ \sum_{i,j=1}^{n} c_{ij}\langle Te_i, e_j\rangle : e_1, \ldots, e_n \text{ is o.n. in } H \right\}$$

is star-shaped with respect to the set $(\operatorname{tr} C)W_e(T)$, where H is an infinite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$, T is a bounded linear operator on H, and $W_e(T) = \{\lambda : \lambda = \lim_{m\to\infty}\langle Tf_m, f_m\rangle, \{f_m\} \text{ is o.n. in } H\}$. Jones [16] proved the same result without assuming that C is normal. However, as pointed out in [11], Hughes' proof could not be applied to prove the finite-dimensional result of Tsing and it seems that the proof of Jones cannot be modified to prove the star-shapedness of $W_C(T)$ when H is finite-dimensional. Recently, Cheung and Tsing [11] proved that $W_C(A)$ is star-shaped with the star center $\frac{1}{n}\operatorname{tr} A \operatorname{tr} C$. With the notations as before, we make the following

Conjecture 5.1. Let $\mathfrak{g}$ be a complex reductive Lie algebra. If $x, y \in \mathfrak{g}$, then the x-numerical range of y, $W_x(y) := \{(x, w)_\theta : w \in \operatorname{Ad}(K)y\}$ is star-shaped with respect to the star center $(x_z, y_z)_\theta$, where $x = x_s + x_z \in \mathfrak{g}$, $x_z \in \mathfrak{z}$ and $x_s \in [\mathfrak{g}, \mathfrak{g}]$.

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