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THE F. AND M. RIESZ THEOREM ON GROUPS

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Abstract. This is a survey work or an overview of recent developements of generalizations of the F. and M. Riesz theorem and Riesz sets on groups, emphasizing recent works on non-Abelian cases.

1. The Helson-Lowdenslager's Theorem

Let G be a locally compact Abelian group and $\hat{G}$ be the dual group of G, and let μ be a complex-valued regular measure on G. A subset $M \subset \hat{G}$ is called a Riesz set, if the condition that the Fourier transform $\hat{\mu}(\gamma)$, $\gamma \in \hat{G}$, vanishes off M (i.e., the spectrum of $\hat{\mu}$ is contained in M) implies that μ is absolutely continuous with respect to the Haar measure on G. The celebrated F. and M. Riesz's theorem is the following :

If G is the one-dimensional torus group T, then the subset $M = \{0, 1, 2, ...\}$ of the dual group of T is a Riesz set.

In other words, if μ is a measure on $T = [0, 2\pi)$ and the Fourier transform

$$\hat{\mu}(n) = \int_0^{2\pi} e^{-nx} d\mu(x) = 0$$

for $n < 0$, then μ is absolutely continuous respect to the Lebesgue measure on T.

The first generalization of the F. and M. Riesz theorem was given by Helson and Lowdenslager [8], by which the original F. and M. Riesz theorem is easily proved.

Let G be a compact Abelian group whose dual group $\hat{G}$ is an ordered group, i.e., there exists in $\hat{G}$ a subsemigroup P with $P \cap (-P) = \{0\}$ and $P \cup (-P) = \hat{G}$. If we define $x \geq y$ for $x - y \in P$, then the relation $\geq$ is a total order relation, by which $\hat{G}$ is a totally ordered group.

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Theorem 1 (Helson and Lowdenslager).

If μ is a measure on G and the Fourier transform $\hat{\mu}$ vanishes off P, then the absolutely continuous part μ_a and the singular part μ_s for the Haar measure m on G have the same property.

This theorem was further generalized by R. Doss [4] to the case of arbitrary locally compact Abelian groups.

Theorem 2 (Doss). *Let G be a locally compact Abelian group such that $\hat{G}$ is algebraically ordered, i.e., there exists a semigroup P in $\hat{G}$ with*

(1) $P \cup (-P) = \hat{G}$ *and* (2) $P \cap (-P) = \{0\}$.

Let μ be a measure on G such that $\hat{\mu}(\gamma) = 0$ for $\gamma < 0$. Then

(I) $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ *for* $\gamma < 0$

(II) $\hat{\mu}_s(0) = 0$.

A reasonable proof of this theorem is found in Hewitt, Koshi and Takahashi [10]. On the other hand, the original proof by Doss is not perfect.

E. Hewitt, S. Koshi and Y. Takahashi [10] generalized this theorem in the following :

Theorem 3. *Let G be a locally compact Abelian group and P a subsemigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$ and $\mu \in M_P(G)$. Then $\mu_a \in M_P(G)$ and $\mu_s \in M_P(G)$. $\mu \in M_P(G)$ means that μ is a measure on G with $\hat{\mu}(r) = 0$ for $r \notin P$.*

The proof of this theorem is found in [10]. We must mention that under the conditions of the above theorem it is proved that if $\mu \in M_{P^c}(G)$, then μ_a and $\mu_s \in M_{P^c}(G)$.

In this note, we shall explain generalizations of these theorems in the case of non-Abelian groups.

2. Riesz Sets in Dual Object of Compact Group

Let K be a compact group which is not necessarily Abelian and Σ_K be its dual object. Let $M(K)$ be the space of complex-valued bounded regular measures on K. m_K stands for the Haar measure of K and $Z(K)$ is the center of K. Let G be a closed subgroup of $Z(K)$. Naturally, G is an Abelian group. Let $\hat{G}$ be the dual group of G. For $\sigma \in \Sigma_K$, $U^{(\sigma)}$ denotes a continuous irreducible representation of K in σ with the representation space H_σ. It follows from Schur's lemma that there exists a map $\gamma : \Sigma_K \to \hat{G}$ such that

$$U_x^{(\sigma)} = (x, \gamma(\sigma))I \tag{2.1}$$

for $x \in G$ and $\sigma \in \Sigma_K$, where I is the identity operator on H_σ.

Let $\mu \in M(K)$. We shall denote by $\hat{\mu}$ the Fourier transform of μ, i.e., for $\sigma \in \Sigma_K$ and ξ, $\eta \in H_\sigma$,

$$\langle \hat{\mu}(\sigma)\xi, \eta\rangle = \int_K < \bar{U}_x^{(\sigma)}\xi, \eta > d\mu(x), \tag{2.2}$$

where $\bar{U}_x^{(\sigma)} = D_\sigma U_x^{(\sigma)} D_\sigma$ and D_σ is a conjugation on H_σ. (cf. [11]).

For a measure μ in $M(K)$, we can define spec(μ) as usual, i.e., for a subset $\triangle$ of Σ_K, spec$(\mu) \subset \triangle$ means that $\hat{\mu}(\sigma) = 0$ for $\sigma \in \triangle^c$. We shall define a Riesz set Δ in Σ_K (= dual object of K). A subset $\triangle$ of Σ_K is called a Riesz set if any measure μ in $M(K)$ such that spec$(\mu) \subset \triangle$ becomes absolutely continuous with respect to the Haar measure on K.

Brummelhuis [2, 3] showed the following F. and M. Riesz theorem by using methods of Shapiro [4].

Theorem 4. *Let K be a metrizable compact group, and let $Z(K)$, the center of K, contain the circle group $\mathbf{T}$ as a closed subgroup. Let $\triangle \subset \Sigma_K$ satisfy the following two conditions.*

(i) *for each $m \in \mathbf{Z} = \hat{\mathbf{T}}$, $\{\sigma \in \triangle : \gamma(\sigma) = m\}$ is finite,*

(ii) *the set $\{\gamma(\sigma) : \sigma \in \triangle\}$ is bounded from below.*

Then $\triangle$ is a Riesz set.

H. Yamaguchi [18] has extended Theorem 4 in the following :

Theorem 5. *Let $\triangle$ be a subset of Σ_K with the following conditions* :

(i) *For each $w \in \hat{G}$, $\{\sigma \in \triangle : \gamma(\sigma) = w\}$ is a Riesz set in $\hat{G}$.*

(ii) *The set $\{\gamma(\sigma) : \sigma \in \triangle\}$ is a Riesz set in $\hat{G}$.*

Then $\triangle$ is a Riesz set.

We use and quote many notations from the book of Hewitt and Ross [11]. For a compact group K and its dual object Σ_K and for a closed subgroup of the center of K, we get a transformation group (G, K) in a natural way. For $0 < p < \infty$, $L^p(K)$ and $\|f\|_p$ for a function f on K are defined as usual.

By the Radon-Nikodym theorem, we can identify $L^1(K)$ with the space of absolutely continuous measures in $M(K)$ (= the space of measures on K). $C(K)$ denotes the space of continuous funtions on K. For $\mu \in M(K)$, let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to m_K. For $\tau \in \Sigma_K$, $\mathbf{T}_\tau(K)$ is the linear span of all functions $x \mapsto \langle U_x^{(\tau)}\xi, \eta\rangle$ for $\xi, \eta \in H_\tau$. Let $\mathbf{T}(K)$ be the space of functions generated by all $\mathbf{T}_\tau(K)$, whose elements are called trigonometric polynomials on K. For a subset E of Σ_K, let $M_E(K) = \{\mu \in M(K) : \text{spec}(\mu) \subset E\}$. $\mathbf{T}_E(K)$ and $L^1_E(K)$ are similarly defined. Here, we shall give some definitions.

A subset E of Σ_K is a Riesz set if and only if $M_E(K) \subset L^1_E(K)$. Let $0 < p < \infty$. A subset E of Σ_K is called a $\Lambda(p)$-set if for some $0 < q < p$, there exists a constant $C > 0$ such that $\|f\|_p \leq C\|f\|_q$ for all $f \in \mathbf{T}_E(K)$.

It is known that any $\Lambda(1)$-set in Σ_K is a Riesz set.

For the Helson-Lowdenslager theorem (Theorem 1) in the Abelian case, H. Yamaguchi [18] succeeds to extend it to the non-Abelian case.

Theorem 6. *Let K be a compact group (not necessarily Abelian) and let G be a closed subgroup of $Z(K)$. Suppose $\hat{G}$ is ordered and let P be a semigroup of non-negative elements by the order in $\hat{G}$ with $P \cup (-P) = \hat{G}$ and $P \cap (-P) = \{0\}$. Let $\mu \in M(K)$, and let E be a $\Lambda(1)$-set in Σ_K. Suppose, given $\epsilon > 0$, there exists a finite set $F_\epsilon \subset \Sigma_K$ such that whenever $p(k)$ is a trigonometric polynomial in $\mathbf{T}_A$ with $\| p \|_\infty \leq 1$, where $A = (F_\epsilon \cup E \cup \gamma^{-1}(P))^c$, we have $| \int_K p(k^{-1}) d\mu(k)| \leq \epsilon$. Then we have :*

(i) spec$(\mu_s) \subset \gamma^{-1}(-P)$;

(ii) *if, in addition, $\mu \in M(K)$ has the property that $\lambda \star \mu$ is absolutely continuous with repect to the Haar measure m_K on K for all $\lambda \in L^1(G)$, then $\hat{\mu}_s = 0$ on $\gamma^{-1}(0)$.*

In view of the condition (ii) of Theorem 6, we introduce the notation : $N(m_K) = \{\mu \in M(G) : \lambda \star \mu$ is absolutely continuous with respect to m_K for all $\lambda \in L^1(G)\}$. For any measure $\nu \in M(K)$, we can similarly define the set of measures $N(\nu)$.

Yamaguchi [18] also obtained the following theorem :

Theorem 7. *Let E be a $\Lambda(1)$-set in Σ_K. Let μ be a measure in $M(K)$ such that* spec$(\mu) \subset E \cup \gamma^{-1}(-P)$. *Then the following hold :*

(i) spec$(\mu_a) \subset E \cup \gamma^{-1}(-P)$.

(ii) spec$(\mu_s) \subset \gamma^{-1}(-P)$.

(iii) *If in addition $\mu \in N(m_K)$, then $\hat{\mu} = 0$ on $\gamma^{-1}(0)$.*

Theorem 7 is considered as a generalization of the theorem given by Pigno [12].

3. Key Lemma and Exmples

For the proof of Theorems 5, 6, and 7, it is better to start from Theorem 1 and Shapiro's method. Key lemmas of the proof are due to the disintegration method given by Bourbaki [1]. We shall show Lemma 1 as follows :

Lemma 1. *Let μ be a measure on a compact metrizable group K and G be a closed subgroup of the center of K. Let π be the natural homomorphism from K onto K/G. Let $\eta = \pi(\mu)$ (= continuous image under π). Then there exists a family $\{\lambda_{\dot{x}}\}_{\dot{x} \in K/G}$ consisting of measures in $M(K)$ with the following properties :*

(1) *$\dot{x} \to \lambda_{\dot{x}}(f)$ is a Borel measurable function for each bounded Borel measurable function f on K,*

(2) supp$(\lambda_{\dot{x}}) \subset \pi^{-1}(\{\dot{x}\})$,

(3) $\|\lambda_{\dot{x}}\| \leq 1$,

(4) $\mu(g) = \int_{K/G} \lambda_{\dot{x}}(g) d\eta(\dot{x})$ *for each measurable function* g *on* G.

Conversely, let $\{\lambda'\}_{\dot{X} \in K/G}$ *be a family of measures in* $M(K)$ *which satisfies* (1), (2), *and* (4). *Then we have*

(5) $\lambda_{\dot{x}} = \lambda'_{\dot{x}}$ *for almost all* $\dot{x}(\eta)$.

This lemma is due to Bourbaki [1]. The same kind of lemma in the transformation group case is also obtained [15].

Next, we shall show some examples of Riesz sets in non-Abelian compact groups.

Example 1. Let $\mathbf{U(2)}$ and $\mathbf{SU(2)}$ be the unitary group and special unitary group of dimension 2, repectively. Let ρ be the map of $\mathbf{T} \times \mathbf{SU(2)}$ into $\mathbf{U(2)}$ with $\rho(\alpha, u) = \alpha u$.

Then ρ is an onto continuous homomorphism with kernel $\{(1, E), (-1, -E)\}$, where E is the unit matrix in $\mathbf{SU(2)}$. Evidently, $\mathbf{T} \cong \{e^{i\theta}E : 0 \leq \theta < 2\pi\} \subset Z(\mathbf{U(2)})$. Let $\mathbf{T}^{(\ell)}$ be as in [12, (29.13)] ($\ell = 0, 1/2.1, 3/2, ...$). Let $\mathbf{U(2)}^{\wedge}$ and $\mathbf{SU(2)}^{\wedge}$ be the dual objects of $\mathbf{U(2)}$ and $\mathbf{SU(2)}$, respectively. Then $\mathbf{SU(2)}^{\wedge} = \{\mathbf{T}^{(\ell)} : \ell = 0, 1/2, 1, 3/2, ...\}$ and $\mathbf{U(2)}^{\wedge} = \{\tau_{n,\ell} : n = 1, 2, ...$ and $\ell = 0, 1/2, 1, 3/2, ..., n + 2\ell$ is an even integer$\}$, where $\tau_{n,\ell}(v) = e^{in\theta}\mathbf{T}_u^{(\ell)}$ for $v = e^{i\theta}u \in \mathbf{U(2)}$ for $u \in \mathbf{U(2)}$. (cf. [12, (29.48)]).

Let $\gamma : \mathbf{U(2)}^{\wedge} \to \hat{\mathbf{T}} \cong \mathbf{Z}$ be the map which appears in (2.1). Then $\gamma(n, \ell) = n$. Using Theorems 5, 6, and 7, we have the following facts.

(i) For $\alpha > 0$, let $\triangle_\alpha = \{\tau_{n,\ell} \in \mathbf{U(2)}^{\wedge} : \ell < \alpha n\}$. Then $\triangle_\alpha$ is a Riesz set in $\mathbf{U(2)}^{\wedge}$.

(ii) Let $\triangle = \{\tau_{n,\ell} \in \mathbf{U(2)}^{\wedge} : n \geq 0\}$. Let $\mu \in M(\mathbf{U(2)})$, and suppose that spec$(\mu) \subset \triangle$. Then, spec(μ_a) and spec(μ_s) are contained in $\triangle$.

References

1. N. Bourbaki, *Integration*, *Elements de Mathematique*, Livre VI, Chapter 6, Paris, Herman, 1959.

2. R. G. M. Brummelhuis, An F. and M. Riesz theorem for bounded symmetric domain, *Ann. Inst. Fourier* (*Grenoble*) **37** (1987), 135-150.

3. R. G. M. Brummelhuis, A note on Riesz sets and lacunary sets, *J. Austral. Math. Soc.* **48** (1990), 57-65.

4. R. Doss, On the Fourier-Stieltijes transforms of singular or absolutely continuous measures, *Math. Z.* **97** (1967), 77-84.

5. R. Doss, On measures with small transforms, *Pacific J. Math*. **26** (1968), 258-263.

6. C. Finet and V. Tardivel-Nachef, A variant of a Yamaguchi's result, *Hokkaido Math. J.* **21** (1992), 483-489.

7. C. Finet and V. Tardivel-Nachef, Lacunary sets on transformation groups, *Hokkaido Math. J.* **23** (1994), 1-19.

8. H. Helson and D. Lowdenslager, Prediction theory and Fourier series in several variables, *Acta Math*. 99 (1958), 165-202.

9. E. Hewitt and S. Koshi, Orderings in locally compact Abelian groups and the theorem of F. and M. Riesz, *Math. Proc. Cambridge Philos. Soc.* **94** (1983), 441-457.

10. E. Hewitt, S. Koshi and Y. Takahashi, The F. and M. Riesz theorem revisited, *Math. Scand.* **60** (1987), 63-76.

11. E. Hewitt nd K. A. Ross, *Abstract Harmonic Analysis*, Vol. II, New York-Heidelberg-Berlin, Springer-Verlag, 1970.

12. L. Pigno, A variant of the F. and M. Riesz theorem, *J. London Math. Soc.* **9** (1974), 368-370.

13. W. Rudin, *Fourier Analysis on Groups*, Intersciences, New York, 1962.

14. J. H. Shapiro, Subpaces of $L^p(G)$ spaned by characters, $0 < p < 1$, *Israel J. Math*. **29** (1978), 248-264.

15. H. Yamaguchi, The F. and M. Riesz theorem on certain transformation groups, *Hokkaido Math. J.* **17** (1988), 289-332.

16. H. Yamaguchi, The F. and M. Riesz theorem on certain transformation groups II, *Hokkaido Math. J.* **19** (1990), 345-357.

17. H. Yamaguchi, Absolute continuity of measures on compact transformation groups, *Hokkaido Math. J.* **22** (1993), 25-33.

18. H. Yamaguchi, An F. and M. Riesz theorem on compact group, in: *Transaction of German-Japanese Symposium on 'Infinite-Dimenional Harmonic Analysis' held in Tübingen*, Verlag D.+M. Gräbner, Bamberg, 1996, pp. 249-273.

19. H. Yamaguchi, On the product of Riesz sets in dual objects of compact groups, preprint.

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