TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 3, pp. 443-469, September 2001
This paper is available online at http://www.math.nthu.edu.tw/tjm/

BLOOMFIELD-WATSON-KNOTT TYPE INEQUALITIES FOR EIGENVALUES

T. Ando*

Abstract. This paper is largely of expository nature. We generalize the determinantal and tracial inequalities, originating from Bloomfield-Watson and Knott, from the standpoint of majorization of eigenvalues, and observe the results as estimates of singular values of modified off-diagonal blocks of a block matrix representation in terms of the eigenvalues of the original matrix.

1. Introduction

In this paper of expository nature we treat, in principle, matrices. But let us start with C^*-algebras (with unit), of which a simplest example is the space of matrices. We shall use capital letters, $A, B, \ldots$ to denote general elements of a C^*-algebra, and identify a scalar with the unit multiplied by this scalar.

For C^*-algebras there are natural notions of *norm*, *positivity*, and *spectrum* etc. For the space of matrices the norm is *spectral norm*, positivity is *positive semidefiniteness*, and spectrum is *eigenvalue*.

The following identities in a C^*-algbera are well-known:

$$\|A\| = \lambda_{\max}(A) \equiv \text{ maximum spectrum of } |A| \stackrel{\text{def}}{=} (A^*A)^{1/2},$$

and for invertible A

$$\|A^{-1}\|^{-1} = \lambda_{\min}(A) \equiv \text{minimum spectrum of } |A|.$$

A linear map Φ from a C^*-algbera to another is said to be *positive* if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be *unital* if it is unit-preserving. (See [20] for C^*-algebras.)

Received July 26, 2000; revised September 4, 2000.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: 15A42, 15A45, 47A63.
Key words and phrases: Eigenvalue inequality, majorization, unital positive linear map, Kantorovich inequality, Bloomfield-Watson-Knott inequality, Khatri-Rao inequality, Rao inequality.
*Research supported in part by Grand-in-Aid for Scientific Research (C) 10640183.

The following are basic inequalities for a unital positive linear map Φ (see, e.g., [6])

$$\Phi(A^2) \geq \Phi(A)^2 \quad (\text{Hermitian } A), \quad \text{and} \quad \Phi(A^{-1}) \geq \Phi(A)^{-1} \quad (A > 0).$$

Since the square-root formation, $A \longmapsto A^{1/2}$, on the cone of positive elements, is *order-preserving* and the inverse formation, $A \longmapsto A^{-1}$, is *order-inverting*, the above inequalities imply

$$\Phi(A^2)^{1/2} \geq \Phi(A) \quad \text{and} \quad \Phi(A^{-1})^{-1} \leq \Phi(A) \qquad (A > 0).$$

In additive forms these inequalities for $A > 0$ become

$$\Phi(A^2) - \Phi(A)^2 \geq 0 \quad \text{and} \quad \Phi(A^2)^{1/2} - \Phi(A) \geq 0,$$

and

$$\Phi(A^{-1}) - \Phi(A)^{-1} \geq 0 \quad \text{and} \quad \Phi(A) - \Phi(A^{-1})^{-1} \geq 0.$$

Inequalities, complementary to the above ones, especially to the one for $\Phi(A^{-1})^{-1}$, are usually called *Kantorovich type inequalities*. More precisely, with these words we mean an upper estimate of $\Phi(A^2)$ by a scalar multiple of $\Phi(A)^2$ and an upper estimate of $\Phi(A)$ by a scalar multiple of $\Phi(A^{-1})^{-1}$. In additive forms, we require upper estimates of $\Phi(A^2) - \Phi(A)^2$ and $\Phi(A) - \Phi(A^{-1})^{-1}$ by scalars, and also upper estimates of $\Phi(A^2)^{1/2} - \Phi(A)$ and $\Phi(A^{-1}) - \Phi(A)^{-1}$ by scalars. Here those scalars are required to be determined by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$.

In Section 2, we formulate the Kantorovich type inequalities as upper estimates of the maximum spectra of $\Phi(A)^{-1}\Phi(A^2)\Phi(A)^{-1}$ and $\Phi(A^{-1})^{1/2}\Phi(A)\Phi(A^{-1})^{1/2}$ as well as those of $\Phi(A^2) - \Phi(A)^2$ and $\Phi(A) - \Phi(A^{-1})^{-1}$ and related ones.

In the case of a matrix $A > 0$ of order n, say, in addition to the maximum and the minimum spectrum, we can speak of the eigenvalues of A. The eigenvalues of A are arranged in *nonincreasing order* :

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

Here of course

$$\lambda_1(A) = \lambda_{\max}(A) \quad \text{and} \quad \lambda_n(A) = \lambda_{\min}(A).$$

Then for a unital positive linear map Φ from $\mathbb{M}_n$, the space of $n \times n$ matrices, to $\mathbb{M}_m$, we can consider the problem of estimating eigenvalues of $\Phi(A^{-1})^{1/2}\Phi(A)$ $\Phi(A^{-1})^{1/2}$, for instance, in terms of the eigenvalues of A.

It is known (e.g., [7]) that a unital positive linear map Φ from $\mathbb{M}_n$ to $\mathbb{M}_m$ at $A > 0$ is written in the form

$$\Phi(A) = \sum_{j=1}^{N} V_j^* A V_j,$$

where the V_j's are $n \times m$ matrices such that

$$\sum_{j=1}^{N} V_j^* V_j = I_m \quad \text{(identity matrix of order } m).$$

(Under more restrictive requirements of *complete positivity*, V_j, $j = 1, 2, \ldots, N$, are taken common for all matrices A.)

For a general unital positive linear map, it seems difficult to find compact estimates of the eigenvalues of $\Phi(A^{-1})^{1/2}\Phi(A)\Phi(A^{-1})^{1/2}$. Therefore we have to concentrate on the special case of a single V, that is,

$$\Phi(A) = V^* A V, \quad \text{where } V^* V = I_m. \tag{1.1}$$

With the *orthoprojection* $P \equiv VV^*$ and $P^{\perp} \equiv I_n - P$, according to the decompositon $I_n = P + P^{\perp}$, represent each $X \in \mathbb{M}_n$ in a block matrix form

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

To consider a unital linear map of the form (1.1) is equivalent, up to unitary similarity, to treat the *compression* $\Phi_P(X)$ defined by

$$\Phi_P(X) = X_{11}. \tag{1.2}$$

Now the problem is to find estimates of the eigenvalues of $B \equiv \Phi_P(A^{-1})^{1/2}$ $\Phi_P(A)\Phi_P(A^{-1})^{1/2}$, for instance. It seems difficult to find nontrivial direct estimates of $\lambda_j(B)$ (except $j = 1$) of the form

$$\lambda_j(B) \leq \mu_j \qquad (j = 1, 2, \ldots, m),$$

where

$$\mu_1 \geq \mu_2 \ldots \geq \mu_m \ (\geq 0)$$

are some scalars, determined from $\lambda_j(A)$, $j = 1, 2, \ldots, n$.

Instead, we expect the so-called *majorization* estimates (of *additive form*)

$$\sum_{j=1}^{k} \lambda_j(B) \leq \sum_{j=1}^{k} \mu_j \qquad (k = 1, 2, \ldots, m), \tag{1.3}$$

or those (of *multiplicative form*)

$$\prod_{j=1}^{k} \lambda_j(B) \leq \prod_{j=1}^{k} \mu_j \qquad (k = 1, 2, \ldots, m). \tag{1.4}$$

According to a basic result of majorization theory (e.g., [4, 16]), (1.3) implies that for any increasing convex function $f(t)$ on $(0, \infty)$,

$$\sum_{j=1}^{k} f(\lambda_j(B)) \leq \sum_{j=1}^{k} f(\mu_j) \qquad (k = 1, 2, \ldots, m). \tag{1.5}$$

Since (1.4) means

$$\sum_{j=1}^{k} f\big(\log(\lambda_j(B))\big) \leq \sum_{j=1}^{k} f\big(\log(\mu_j)\big) \qquad (k = 1, 2, \ldots, m),$$

it follows from the above comment that (1.4) implies that for any function $g(s)$ on $(-\infty, \infty)$ such that $g(\log t)$ is increasing and convex,

$$\sum_{j=1}^{k} g\big(\lambda_j(B)\big) \leq \sum_{j=1}^{k} g(\mu_j) \qquad (k = 1, 2, \ldots, m). \tag{1.6}$$

In particular, (1.3) follows from (1.4) with $g(s) = e^s$.

It is known (e.g., [4, 16]) that the majorizations (1.3) and (1.4) are equivalent to the existence of $\alpha_\sigma \geq 0$, indexed by permutations $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ of the set $\{1, 2, \ldots, m\}$, with $\sum_\sigma \alpha_\sigma = 1$ such that

$$\lambda_j(B) \leq \sum_\sigma \alpha_\sigma \mu_{\sigma_j} \qquad (j = 1, 2, \ldots, m)$$

and respectively

$$\lambda_j(B) \leq \prod_\sigma \mu_{\sigma_j}^{\alpha_\sigma} \qquad (j = 1, 2, \ldots, m).$$

Notice that (1.3) and (1.4) for $k = m$ take the forms of tracial and determinantal inequalities:

$$\operatorname{tr}(B) \equiv \sum_{j=1}^{m} \lambda_j(B) \leq \sum_{j=1}^{m} \mu_j \tag{1.7}$$

and

$$\det(B) \equiv \prod_{j=1}^{m} \lambda_j(B) \leq \prod_{j=1}^{m} \mu_j. \tag{1.8}$$

A discovery of meaningful estimates of $\det(B)$ was made by Bloomfield-Watson [5] and Knott [10]. In this respect, we shall call related majorization estimates *Bloomfield-Waton-Knott type inequalities*.

Estimates of determinants and traces in a similar line have been developped subsequently by Khati-Rao [8, 9] and Rao [21]. Khatri-Rao [9] even mentions majorization estimates.

In Section 3, we give a unified treatment of majorization estimates in multiplicative form of the eigenvalues of $\Phi_P(A)^{-1}\Phi_P(A)\Phi_P(A)^{-1}$ and $\Phi_P(A^{-1})^{1/2}$ $\Phi_P(A)\Phi_P(A^{-1})^{1/2}$.

In Section 4, we give a unified treatment of majorization estimates in additive form of the eigenvalues of $\Phi_P(A) - \Phi_P(A^{-1})^{-1}$ and $\Phi_P(A^2) - \Phi_P(A)^2$.

Before closing this introduction, let us describe the matrices, which appear in connection with a compression, in terms of blocks of the block matrix respresention. Then it will be clear that the observations in Section 2 to 4 present estimates of the singular values of modified off-diagonal blocks in terms of the eigenvalues of the original matrix. Here recall that the *singular values* of a (rectangular) matrix X are the eigenvalues of its *modulus* $|X| \equiv (X^*X)^{1/2}$.

Let A be a positive definite matrix. Then the following relation is easily checked:

(1.9) $$\Phi_P(A)^{-1}\Phi_P(A^2)\Phi_P(A)^{-1} = 1 + |A_{21}A_{11}^{-1}|^2.$$

Further, since $\Phi_P(A^{-1})^{\frac{1}{2}}\Phi_P(A)\Phi_P(A^{-1})^{\frac{1}{2}}$ is unitarily similar to $\Phi_P(A)^{1/2}\Phi_P(A^{-1})$ $\Phi_P(A)^{1/2}$, there is a unitary matrix W such that

(1.10) $$\Phi_P(A^{-1})^{\frac{1}{2}}\Phi_P(A)\Phi_P(A^{-1})^{\frac{1}{2}} = W^*\{1 - |A_{22}^{-1/2}A_{21}A_{11}^{-1/2}|^2\}^{-1}W.$$

Also, the following relations are easily checked:

(1.11) $$\Phi_P(A^2) - \Phi_P(A)^2 = |A_{21}|^2.$$

and

(1.12) $$\Phi_P(A) - \Phi_P(A^{-1})^{-1} = |A_{22}^{-1/2}A_{21}|^2.$$

2. Estimates of Maximum Spectra

In this section, we observe the Kantorovich type inequalities as estimates of maximum spectra. There are many approaches for Kantorovich type inequalities. See, for instance, [3, 11, 12, 14, 15, 17, 18, 19].

Our starting point for deriving those estimates is the following rather trivial inequality (e.g., [11]) :

$$\{\lambda_{\max}(A) - A\}\{A - \lambda_{\min}(A)\} \geq 0 \qquad (A > 0). \tag{2.1}$$

Equivalent forms of this inequality are

$$\{\lambda_{\max}(A) + \lambda_{\min}(A)\}A - \lambda_{\max}(A) \cdot \lambda_{\min}(A) \geq A^2 \qquad (A > 0)$$

and

$$\{\lambda_{\max}(A) + \lambda_{\min}(A)\} - \{\lambda_{\max}(A) \cdot \lambda_{\min}(A)\}A^{-1} \geq A \quad (A > 0).$$

Then apply a unital positive linear map Φ to these inequalities to get our basic inequalities:

$$\{\lambda_{\max}(A) + \lambda_{\min}(A)\}\Phi(A) - \lambda_{\max}(A) \cdot \lambda_{\min}(A) \geq \Phi(A^2) \quad (A > 0), \tag{2.2}$$

and

$$\{\lambda_{\max}(A) + \lambda_{\min}(A)\} - \{\lambda_{\max}(A) \cdot \lambda_{\min}(A)\}\Phi(A^{-1}) \geq \Phi(A) \quad (A > 0). \tag{2.3}$$

Though $\Phi(A)$ (resp., $\Phi(A^{-1})$) does not commute with $\Phi(A^2)$ (resp., $\Phi(A)$) in general, on the left-hand side of (2.2) (resp., (2.3)) appear the single Hermitian $\Phi(A)$ (resp., $\Phi(A^{-1})$) and a scalar. Therefore, to get upper estimates of the left-hand side, we can compute as in the numerical case.

Let us consider, in general, two scalars $\alpha, \beta > 0$ and a positive variable t. Then the following identities are easily checked:

$$\begin{aligned}(\alpha+\beta)t - \alpha\beta &= \frac{(\alpha+\beta)^2}{4\alpha\beta}t^2 - \left\{\frac{\alpha+\beta}{2\sqrt{\alpha\beta}}t - \sqrt{\alpha\beta}\right\}^2 \\ &= t^2 + \frac{(\alpha-\beta)^2}{4} - \left\{t - \frac{\alpha+\beta}{2}\right\}^2 \\ &= \left\{\frac{(\alpha-\beta)^2}{4(\alpha+\beta)} + t\right\}^2 - \left\{t - \frac{(\alpha+\beta)^2 + 4\alpha\beta}{4(\alpha+\beta)}\right\}^2.\end{aligned}$$

With $\alpha = \lambda_{\max}(A),\ \beta = \lambda_{\min}(A)$ and $\Phi(A)$ in place of the positive scalar t, these identities yield

$$\begin{aligned}\frac{\{\lambda_{\max}(A) + \lambda_{\min}(A)\}^2}{4\lambda_{\max}(A) \cdot \lambda_{\min}(A)}\Phi(A)^2 \geq \{\lambda_{\max}(A) + \lambda_{\min}(A)\}\Phi(A) \\ -\lambda_{\max}(A) \cdot \lambda_{\min}(A),\end{aligned}$$

$$\Phi(A)^2 + \frac{\{\lambda_{\max}(A) - \lambda_{\min}(A)\}^2}{4} \geq \{\lambda_{\max}(A) + \lambda_{\min}(A)\}\Phi(A) - \lambda_{\max}(A) \cdot \lambda_{\min}(A),$$

and

$$\left\{ \frac{\{\lambda_{\max}(A) - \lambda_{\min}(A)\}^2}{4\{\lambda_{\max}(A) + \lambda_{\min}(A)\}} + \Phi(A) \right\}^2 \geq \{\lambda_{\max}(A) + \lambda_{\min}(A)\}\Phi(A) - \lambda_{\max}(A) \cdot \lambda_{\min}(A).$$

Then in view of the basic inequality (2.2), the above inequalities lead to the following

$$\frac{\{\lambda_{\max}(A) + \lambda_{\min}(A)\}^2}{4\lambda_{\max}(A) \cdot \lambda_{\min}(A)} \Phi(A)^2 \ \geq \ \Phi(A^2),$$

$$\frac{\{\lambda_{\max}(A) - \lambda_{\min}(A)\}^2}{4} \geq \Phi(A^2) - \Phi(A)^2,$$

and

$$\left\{ \frac{\{\lambda_{\max}(A) - \lambda_{\min}(A)\}^2}{4\{\lambda_{\max}(A) + \lambda_{\min}(A)\}} + \Phi(A) \right\}^2 \geq \Phi(A^2).$$

Via the order-preserving property of the square-root formation, the last inequality implies further

$$\frac{\{\lambda_{\max}(A) - \lambda_{\min}(A)\}^2}{4\{\lambda_{\max}(A) + \lambda_{\min}(A)\}} \geq \Phi(A^2)^{1/2} - \Phi(A).$$

We can formulate these inequalities in the following form.

Theorem 2.1. *For a unital positive linear map* Φ *and* $A > 0$, *the following estimates hold for the maximum spectra* :

(a)

$$\lambda_{\max}\left(\Phi(A)^{-1}\Phi(A^2)\Phi(A)^{-1}\right) \leq \frac{\{\lambda_{\max}(A) + \lambda_{\min}(A)\}^2}{4\lambda_{\max}(A) \cdot \lambda_{\min}(A)},$$

(b)

$$\lambda_{\max}\left(\Phi(A^2) - \Phi(A)^2\right) \leq \frac{\{\lambda_{\max}(A) - \lambda_{\min}(A)\}^2}{4},$$

(c)

$$\lambda_{\max}\left(\Phi(A^2)^{1/2} - \Phi(A)\right) \leq \frac{\{\lambda_{\max}(A) - \lambda_{\min}(A)\}^2}{4\{\lambda_{\max}(A) + \lambda_{\min}(A)\}}.$$

Notice that for A to be Herimitian is enough to derive **(b)** because for a scalar $\gamma > 0$ with $A + \gamma > 0$,

$$\Phi(A^2) - \Phi(A)^2 = \Phi\left((A+\gamma)^2)\right) - \Phi(A+\gamma)^2$$

and

$$\lambda_{\max}(A) - \lambda_{\min}(A) = \lambda_{\max}(A+\gamma) - \lambda_{\min}(A+\gamma).$$

Let us pause to see how these inequalities look like for the simplest case of a unital positive linear map from $\mathbb{M}_2$ to the scalars, more precisely, when Φ is defined as

$$\Phi : \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \longmapsto x_{11}.$$

Let $\lambda \geq \mu$ be the eigenvalues of a 2×2 positive definite matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0.$$

Then inequalities **(a)**, **(b)** and **(c)** become

$$1 + \left\{\frac{|a_{21}|}{a_{11}}\right\}^2 \leq \frac{(\lambda+\mu)^2}{4\lambda\mu}, \quad \text{so} \quad \frac{|a_{21}|}{a_{11}} \leq \frac{\lambda-\mu}{2\sqrt{\lambda\mu}}, \tag{2.4}$$

$$|a_{21}|^2 \leq \frac{(\lambda-\mu)^2}{4}, \quad \text{so} \quad |a_{21}| \leq \frac{\lambda-\mu}{2}, \tag{2.5}$$

and

$$\sqrt{a_{11}^2 + |a_{21}|^2} - a_{11} \leq \frac{(\lambda-\mu)^2}{4(\lambda+\mu)}. \tag{2.6}$$

Next let us turn to the observation of $\Phi(A^{-1})$. The following scalar identities are easily checked:

$$\begin{aligned}(\alpha+\beta) - \alpha\beta t &= \frac{(\alpha+\beta)^2}{4\alpha\beta}t^{-1} - \left\{\frac{\alpha+\beta}{2\sqrt{\alpha\beta}}t^{-1/2} - \sqrt{\alpha\beta}t^{1/2}\right\}^2 \\ &= t^{-1} + (\sqrt{\alpha} - \sqrt{\beta})^2 - \{t^{-1/2} - \sqrt{\alpha\beta}t^{1/2}\}^2.\end{aligned}$$

With $\alpha = \lambda_{\max}(A)$, $\beta = \lambda_{\min}(A)$ and $\Phi(A^{-1})$ in place of the positive scalar t, these identities yield

$$\begin{aligned}\frac{\{\lambda_{\max}(A) + \lambda_{\min}(A)\}^2}{4\lambda_{\max}(A)\cdot\lambda_{\min}(A)}\Phi(A^{-1})^{-1} \geq \{\lambda_{\max}(A) + \lambda_{\min}(A)\} \\ -\lambda_{\max}(A)\cdot\lambda_{\min}(A)\Phi(A^{-1}),\end{aligned}$$

and

$$\left\{\sqrt{\lambda_{\max}(A)}-\sqrt{\lambda_{\min}(A)}\right\}^2+\Phi(A^{-1})^{-1}\geq\{\lambda_{\max}(A)+\lambda_{\min}(A)\}$$
$$-\lambda_{\max}(A)\cdot\lambda_{\min}(A).\Phi(A^{-1}).$$

Then in view of the basic inequality (2.3), these inequalities lead to the following

$$\frac{\{\lambda_{\max}(A)+\lambda_{\min}(A)\}^2}{4\lambda_{\max}(A)\cdot\lambda_{\min}(A)}\Phi(A^{-1})^{-1}\geq\ \Phi(A),$$

and

$$\left\{\sqrt{\lambda_{\max}(A)}-\sqrt{\lambda_{\min}(A)}\right\}^2\geq\Phi(A)-\Phi(A^{-1})^{-1}.$$

We can formulate these inequalities in the following form.

Theorem 2.2. *For a unital positive linear map* Φ *and* $A>0$, *the following estimates hold for the maximum spectra* :

(a)

$$\lambda_{\max}\left(\Phi(A^{-1})^{1/2}\Phi(A)\Phi(A^{-1})^{1/2}\right)\leq\frac{\{\lambda_{\max}(A)+\lambda_{\min}(A)\}^2}{4\lambda_{\max}(A)\cdot\lambda_{\min}(A)},$$

(b)

$$\lambda_{\max}\left(\Phi(A)-\Phi(A^{-1})^{-1}\right)\leq\left\{\sqrt{\lambda_{\max}(A)}-\sqrt{\lambda_{\min}(A)}\right\}^2.$$

To estimate $\Phi(A^{-1})-\Phi(A)^{-1}\geq 0$ from above, we can use the following modified form of the basic inequality (2.3),

$$\frac{1}{\lambda_{\max}(A)\cdot\lambda_{\min}(A)}\left\{\{\lambda_{\max}(A)+\lambda_{\min}(A)\}-\Phi(A)\right\}\geq\Phi(A^{-1}).$$

Further, to estimate the left-hand side of the above, we use an easily checked identity

$$\frac{(\alpha+\beta)-t}{\alpha\beta}=\frac{(\sqrt{\alpha}-\sqrt{\beta})^2}{\alpha\beta}+t^{-1}-\frac{(t^{\frac{1}{2}}-\sqrt{\alpha\beta}t^{-\frac{1}{2}})^2}{\alpha\beta}.$$

With $\alpha=\lambda_{\max}(A)$, $\beta=\lambda_{\min}(A)$ and $\Phi(A)$ in place of the positive scalar t, this inequality implies

$$\frac{\left\{\sqrt{\lambda_{\max}(A)}-\sqrt{\lambda_{\min}(A)}\right\}^2}{\lambda_{\max}(A)\cdot\lambda_{\min}(A)}+\Phi(A)^{-1}$$
$$\geq\frac{1}{\lambda_{\max}(A)\cdot\lambda_{\min}(A)}\left\{(\lambda_{\max}(A)+\lambda_{\min}(A))-\Phi(A)\right\}$$

so that
$$\frac{\left\{\sqrt{\lambda_{\max}(A)}-\sqrt{\lambda_{\min}(A)}\right\}^2}{\lambda_{\max}(A)\cdot\lambda_{\min}(A)}+\Phi(A)^{-1}\geq\Phi(A^{-1}).$$

We can formulate this inequality in the following form.

Theorem 2.3. *For a unital positive linear map* Φ *and* $A > 0$, *the following estimate holds for the maximum spectrum* :

$$\lambda_{\max}\left(\Phi(A^{-1})-\Phi(A)^{-1}\right)\leq\frac{\left\{\sqrt{\lambda_{\max}(A)}-\sqrt{\lambda_{\min}(A)}\right\}^2}{\lambda_{\max}(A)\cdot\lambda_{\min}(A)}.$$

Before closing this section, let us see again how the inequalities in Theorems 2.2 and 2.3 look like for the simplest case mentioned before.

Let $\lambda \geq \mu$ be again the eigenvalues of a 2×2 positive definite matrix

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0.$$

Then by Theorems 2.2 and 2.3 we have the following:

$$\frac{1}{1-\left\{\frac{|a_{21}|}{\sqrt{a_{11}a_{22}}}\right\}^2}\leq\frac{(\lambda+\mu)^2}{4\lambda\mu},\quad\text{so}\quad\frac{|a_{21}|}{\sqrt{a_{11}a_{22}}}\leq\frac{\lambda-\mu}{\lambda+\mu},\tag{2.7}$$

$$\left\{\frac{|a_{21}|}{\sqrt{a_{22}}}\right\}^2\leq\{\sqrt{\lambda}-\sqrt{\mu}\}^2,\quad\text{so}\quad\frac{|a_{21}|}{\sqrt{a_{22}}}\leq\sqrt{\lambda}-\sqrt{\mu},\tag{2.8}$$

and

$$\frac{1}{a_{11}-\left\{\frac{|a_{21}|}{\sqrt{a_{22}}}\right\}^2}-\frac{1}{a_{11}}\leq\frac{(\sqrt{\lambda}-\sqrt{\mu})^2}{\lambda\mu},\quad\text{so}\quad\frac{|a_{21}|}{\sqrt{a_{11}}}\leq\sqrt{\lambda}-\sqrt{\mu}.\tag{2.9}$$

3. Estimates in Multiplicative Form

Our next interest is in finding estimates of the eigenvalues of

$$\Phi(A^{-1})^{1/2}\Phi(A)\Phi(A^{-1})^{1/2}\quad\text{and}\quad\Phi(A)^{-1}\Phi(A)\Phi(A)^{-1}$$

in multiplicative majorization form for a unital positive linear map Φ on the space of $n \times n$ matrices and positive definite $A \in \mathbb{M}_n$. But we have to restrict the observation to the case of a compression.

Recall that the *compression* Φ_P with respect to an orthoprojection P of rank m is the map from $\mathbb{M}_n$ to $\mathbb{M}_m$, defined as

$$\Phi_P(X) \stackrel{\text{def}}{=} X_{11} \equiv PXP, \tag{3.1}$$

where

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

is the block representation of $X \in \mathbb{M}_n$ according to the the decomposition $I = P + P^{\perp}$.

As mentioned in Section 1,

$$\Phi_P(A^{-1})^{1/2}\Phi_P(A)\Phi_P(A^{-1})^{1/2} \quad \text{and} \quad \Phi_P(A)^{-1}\Phi_P(A)\Phi_P(A)^{-1}$$

are described in terms of the blocks of A_{ij} and since, for a (rectangular) matrix X, the eigenvalues of X^*X are the same as those of XX^*, moduls 0, considering $\Phi_{P^{\perp}}$ if necessary, we may and do assume, in the rest of the paper, that

$$m \ \leq \ n - m. \tag{3.2}$$

Recall that for a positive semidefinite matrix X (of order m), its eigenvalues are arranged in nondecreasing order

$$\lambda_1(X) \ \geq \ \lambda_2(X) \ \geq \ \cdots \ \geq \ \lambda_m(X).$$

There will be no confusion if we denote by X^{-1} the (Moore-Penrose) *generalized inverse* of X, and by $\det(X)$ the product of the positive eigenvalues of X.

Lemma. *For a compression Φ_P with respect to an orthoprojection P of rank m and $0 < A \in \mathbb{M}_n$,*

(a)

$$\prod_{j=1}^{k} \lambda_j \left(\Phi_P(A^{-1})^{1/2}\Phi_P(A)\Phi_P(A^{-1})^{1/2}\right) \leq \det\left(\Phi_{Q_1}(A^{-1})^{1/2}\Phi_{Q_1}(A)\Phi_{Q_1}(A^{-1})^{1/2}\right),$$

where Q_1 is the orthoprojection to the subspace spanned by the eigenvectors of

$$B_1 \equiv \Phi_P(A^{-1})^{1/2}\Phi_P(A)\Phi_P(A^{-1})^{1/2},$$

corresponding to its largest k eigenvalues;

(b)

$$\prod_{j=1}^{k} \lambda_j \left(\Phi_P(A)^{-1}\Phi_P(A^2)\Phi_P(A)^{-1}\right) \le \det\left(\Phi_{Q_2}(A)^{-1}\Phi_{Q_2}(A^2)\Phi_{Q_2}(A)^{-1}\right),$$

where Q_2 is the orthoprojection to the subspace spanned by the eigenvectors of

$$B_2 \equiv \Phi_P(A)^{-1}\Phi_P(A^2)\Phi_P(A)^{-1},$$

corresponding to its largest k eigenvalues.

Proof. Notice first that by definition

$$\prod_{j=1}^{k} \lambda_j \left(\Phi_P(A^{-1})^{\frac{1}{2}}\Phi_P(A)\Phi_P(A^{-1})^{\frac{1}{2}}\right) = \det(Q_1B_1Q_1). \tag{3.3}$$

Since Q_1 commutes with B_1,

$$Q_1B_1Q_1 \le B_1 = (PA^{-1}P)^{1/2} \cdot (PAP) \cdot (PA^{-1}P)^{1/2},$$

which implies

$$(PA^{-1}P)^{-1/2} \cdot (Q_1B_1Q_1) \cdot (PA^{-1}P)^{-1/2} \le PAP.$$

Since $Q_1 \le P$, we have then

$$Q_1(PA^{-1}P)^{-1/2}Q_1 \cdot (Q_1B_1Q_1) \cdot Q_1(PA^{-1}P)^{-1/2}Q_1 \le Q_1AQ_1,$$

and hence

$$Q_1B_1Q_1 \le \{Q_1(PA^{-1}P)^{-1/2}Q_1\}^{-1} \cdot (Q_1A_1Q_1) \cdot \{Q_1(PA^{-1}P)^{-1/2}Q_1\}^{-1}.$$

This implies

$$\det(Q_1B_1Q_1) \le \det\left(\{Q_1(PA^{-1}P)^{-1/2}Q_1\}^{-2}\right) \cdot \det(Q_1AQ_1). \tag{3.4}$$

Since it is known (e.g., [2]) that

$$\begin{bmatrix} (PA^{-1}P)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \max\{X\,;\, 0 \le X \le A, \text{and } \operatorname{ran}(X) \subset \operatorname{ran}(P)\},$$

where $\mathrm{ran}(X)$ denotes the range subspace of X, and similarly

$$\begin{bmatrix} (Q_1A^{-1}Q_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \max\left\{X\,;\, 0 \le X \le A, \text{and } \mathrm{ran}(X) \subset \mathrm{ran}(Q_1)\right\},$$

we can see, by $\mathrm{ran}(Q_1) \subset \mathrm{ran}(P)$, that

$$(Q_1A^{-1}Q_1)^{-1} \le (PA^{-1}P)^{-1}.$$

Then since the square-root formation preserves order relation, we have

$$(Q_1A^{-1}Q_1)^{-1/2} \le Q_1(PA^{-1}P)^{-1/2}Q_1,$$

which implies

$$\det\left(\{Q_1(PA^{-1}P)^{-1/2}Q_1\}^{-2}\right) \le \det(Q_1A^{-1}Q_1).$$

Therefore we can conclude by (3.4) that

$$\begin{aligned}\det(Q_1B_1Q_1) &\le \det(Q_1A^{-1}Q_1)\cdot\det(Q_1AQ_1)\\ &= \det\left(\Phi_{Q_1}(A^{-1})^{1/2}\Phi_{Q_1}(A)\Phi_{Q_1}(A^{-1})^{1/2}\right),\end{aligned}$$

which proves **(a)** by (3.3).

The proof of **(b)** is quite similar to the above proof. In fact,

$$\prod_{j=1}^{k}\lambda_j(B_2) = \det(Q_2B_2Q_2)$$

and

$$Q_2B_2Q_2 \le B_2 = (PAP)^{-1}\cdot PA^2P\cdot(PAP)^{-1},$$

which implies

$$(PAP)(Q_2B_2Q_2)(PAP) \le PA^2P.$$

Then since $Q_2 \le P$, we have

$$(Q_2AQ_2)(Q_2B_2Q_2)(Q_2AQ_2) \le Q_2A^2Q_2,$$

which implies

$$\begin{aligned}Q_2B_2Q_2 &\le (Q_2AQ_2)^{-1}\cdot(Q_2A^2Q_2)\cdot(Q_2AQ_2)^{-1}\\ &\equiv \Phi_{Q_2}(A)^{-1}\Phi_{Q_2}(A)\Phi_{Q_2}(A)^{-1},\end{aligned}$$

which leads to **(b)**. ■

The following is a majorization version of the result of Bloomfield–Watson [5] and Knott [10].

Theorem 3.1. *For a compression Φ_P with respect to an orthoprojection P of rank $m \le n-m$ and $0 < A \in \mathbb{M}_n$, the following majorization estimates in multipicative form hold* :

(a)

$$\prod_{j=1}^{k} \lambda_j \left(\Phi_P(A^{-1})^{1/2} \Phi_P(A) \Phi_P(A^{-1})^{1/2} \right) \le \prod_{j=1}^{k} \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \quad (k = 1, 2, \dots, m),$$

and

(b)

$$\prod_{j=1}^{k} \lambda_j \left(\Phi_P(A)^{-1} \Phi_P(A^2) \Phi_P(A)^{-1} \right) \le \prod_{j=1}^{k} \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \quad (k = 1, 2, \dots, m).$$

First of all, notice that since $1 \le \lambda_j(A)/\lambda_{n-j+1}(A)$, $j = 1, 2, \dots, m$, is a nonincresing sequence, so is the sequence

$$\frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} = \frac{1}{4} \left\{ \sqrt{\frac{\lambda_j(A)}{\lambda_{n-j+1}(A)}} + \sqrt{\frac{\lambda_{n-j+1}(A)}{\lambda_j(A)}} \right\}^2$$

because the function $t^{1/2} + t^{-1/2}$ is increasing on $[1, \infty)$.

Proof of Theorem 3.1. In view of Lemma, writing Q_1 or Q_2 as P anew, it suffices to show the following two inequalities for any orthoprojection P of rank $m \le n-m$:

$$\det \left(\Phi_P(A^{-1})^{1/2} \Phi_P(A) \Phi_P(A^{-1})^{1/2} \right) \le \prod_{j=1}^{m} \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)}, \tag{3.5}$$

and

$$\det \left(\Phi_P(A)^{-1} \Phi_P(A^2) \Phi_P(A)^{-1} \right) \le \prod_{j=1}^{m} \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)}. \tag{3.6}$$

To prove (3.5), we may assume that P is an orthoprojection of rank m, at which the left-hand side of (3.5), which is

$$\det(PAP) \cdot \det(PA^{-1}P) \equiv \det\left(\Phi_P(A^{-1})^{1/2}\Phi_P(A)\Phi_P(A^{-1})^{1/2}\right),$$

takes the *maximum* over the set of orthoprojections of rank m.

Bloomfield–Watson [5], and Knott [10] solved the equivalent extremal problems by a Lagrange multiplier method. Let us take the method of variation of orthoprojections, as used in Alpargu [1].

Fix an orthoprojection P of rank m, which satisfies the above extremal condition. For each Hermitian $H \in \mathbb{M}_m$, consider the one-parameter group of unitary matrices defined by

$$P_H(t) \equiv \exp(-itH) \cdot P \cdot \exp(itH) \qquad (-\infty < t < \infty). \tag{3.7}$$

Then by the assumption on P, the function

$$f_H(t) \equiv \det\left(P_H(t)AP_H(t)\right) \cdot \det\left(P_H(t)A^{-1}P_H(t)\right)$$

takes its maximum at $t = 0$. Therefore we have

$$\frac{d}{dt} f_H(t)|_{t=0} = 0 \qquad \text{(for all Hermitian } H). \tag{3.8}$$

It is easy to see that

$$P_H(t)AP_H(t) = \exp(-itH)\{P(A + it[H, A])P\}\exp(itH) + o(t)$$

and

$$P_H(t)A^{-1}P_H(t) = \exp(-itH)\left\{P(A^{-1} + it[H, A^{-1}])P\right\}\exp(itH) + o(t),$$

and that for any matrix X,

$$\det(I + tX) = 1 + t\mathrm{tr}(X) + o(t),$$

where $[X, Y]$ is the *commutator* of X and Y, that is,

$$[X, Y] \stackrel{\text{def}}{=} XY - YX,$$

and $o(t)$ is a matrix or scalar fucntion for small t such that $\|o(t)\|/t \to 0$ as $t \to 0$.

Then we can see

$$\frac{d}{dt}\det\left(P_H(t)AP_H(t)\right)|_{t=0} = i\det(PAP) \cdot \mathrm{tr}\left((PAP)^{-1}[H, A]\right)$$

and

$$\frac{d}{dt}\det\left(P_H(t)A^{-1}P_H(t)\right)|_{t=0} = i\det(PA^{-1}P)\cdot \mathrm{tr}\left((PA^{-1}P)^{-1}[H,A^{-1}]\right).$$

Therefore we have

$$\begin{aligned}\frac{d}{dt}f_H(t)|_{t=0} &= i\mathrm{tr}\left((PAP)^{-1}[H,A] + (PA^{-1}P)^{-1}[H,A^{-1}]\right)\cdot\det(PAP)\cdot\det(PA^{-1}P)\\ &= i\mathrm{tr}\left(H[A,(PAP)^{-1}] + H[A^{-1},(PA^{-1}P)^{-1}]\right)\cdot\det(PAP)\cdot\det(PA^{-1}P).\end{aligned}$$

Then (3.8) (for all Hermitian H) is possible only when

$$A(PAP)^{-1} - (PAP)^{-1}A + A^{-1}(PA^{-1}P)^{-1} - (PA^{-1}P)^{-1}A^{-1} = 0.$$

Since

$$A^{-1} = \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}\end{bmatrix},$$

the above identity means

$$-A_{11}^{-1}A_{12} + A_{12}A_{22}^{-1} = 0 \text{ or, equivalently, } A_{11}A_{12} = A_{12}A_{22},$$

which implies

$$A_{11}\cdot A_{12}A_{21} = A_{12}A_{22}A_{21} \quad\text{and}\quad A_{21}A_{11}A_{12} = A_{21}A_{12}\cdot A_{22}. \tag{3.9}$$

Since A_{11}, $A_{12}A_{21}$ and the right-hand side of the first identity of (3.9) are Hermitian, we can conclude that A_{11} commutes with $A_{12}A_{21}$, and hence with $|A_{21}| \equiv (A_{12}A_{21})^{1/2}$. Similarly we can see from the second identity of (3.9) that A_{22} commutes with $|A_{12}| \equiv (A_{21}A_{12})^{1/2}$.

Since $m \le n-m$ by assumption, there is an $(n-m)\times m$ isometric matrix U such that $A_{21} = U\cdot|A_{21}|$. Then since A_{22} commutes with $|A_{12}| = U\cdot|A_{21}|\cdot U^*$, the matrix $|A_{21}|$ commutes with $U^*A_{22}U$.

Define four $m\times m$ positive semidefinite matrices $\tilde{A}_{ij}$ $(i,j=1,2)$ by

$$\tilde{A} \stackrel{\text{def}}{=} A_{11},\ \tilde{A}_{21} = \tilde{A}_{12} \stackrel{\text{def}}{=} |A_{21}|, \quad\text{and}\quad \tilde{A}_{22} \stackrel{\text{def}}{=} U^*A_{22}U,$$

and a $(2m)\times(2m)$ matrix $\tilde{A}$ by

$$\tilde{A} \stackrel{\text{def}}{=} \begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix} = \begin{bmatrix} I_m & 0\\ 0 & U^*\end{bmatrix}\cdot\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\cdot\begin{bmatrix} I_m & 0\\ 0 & U\end{bmatrix}. \tag{3.10}$$

Since

$$\begin{bmatrix} I_m & 0 \\ 0 & U \end{bmatrix}^* \begin{bmatrix} I_m & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix},$$

according to the min-max theorem for eigenvalues of Hermitian matrices we can see from (3.10) that

$$\lambda_j(\tilde{A}) \le \lambda_j(A) \text{ and } \lambda_{2m-j+1}(\tilde{A}) \ge \lambda_{n-j+1}(A) \qquad (j = 1, 2, \cdots, m),$$

so that

$$\frac{\lambda_j(\tilde{A})}{\lambda_{2m-j+1}(\tilde{A})} \le \frac{\lambda_j(A)}{\lambda_{n-j+1}(A)} \qquad (j = 1, 2, \dots, m).$$

Since the function $(t^{1/2} + t^{-1/2})^2$ is increasing for $t \ge 1$, this implies

$$\prod_{j=1}^{m} \frac{\left\{\lambda_j(\tilde{A}) + \lambda_{2m-j+1}(\tilde{A})\right\}^2}{4\lambda_j(\tilde{A}) \cdot \lambda_{2m-j+1}(\tilde{A})} \le \prod_{j=1}^{m} \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)}. \tag{3.11}$$

Since $\tilde{A}_{12} = \tilde{A}_{21}$ commutes with $\tilde{A}_{11}$ and $\tilde{A}_{22}$, and again by (3.9),

$$\tilde{A}_{11}\tilde{A}_{12} = \tilde{A}_{12}\tilde{A}_{22},$$

there are $m \times m$ diagonal positive semidefinite matrices D_1, D_2 and D_3, and a unitary matrix V such that

$$\begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} \cdot \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \cdot \begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix} = \begin{bmatrix} D_1 & D_3 \\ D_3 & D_2 \end{bmatrix}$$

with

$$D_1 = \mathrm{diag}(\alpha_1, \cdots, \alpha_m), \quad D_2 = \mathrm{diag}(\beta_1, \cdots, \beta_m), \text{ and } D_3 = \mathrm{diag}(\gamma_1, \cdots, \gamma_m),$$

where $\alpha_j, \beta_j > 0$ $(j = 1, 2, \dots, m)$ and $\gamma_j \ge 0$ $(j = 1, 2, \dots, m)$. Then we can write

$$\begin{aligned} \det(PAP) \cdot \det(PA^{-1}P) &= \det\left(A_{11}^{1/2}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{11}^{1/2}\right) \\ &= \det\left(I - |A_{22}^{-1/2}A_{21}A_{11}^{-1/2}|^2\right)^{-1} \\ &= \det\left(I - |\tilde{A}_{22}^{-1/2}\tilde{A}_{21}\tilde{A}_{11}^{-1/2}|^2\right)^{-1} \\ &= \det\left(I - |D_2^{-1/2}D_3D_1^{-1/2}|^2\right)^{-1} \\ &= \prod_{j=1}^{m} \frac{\alpha_j\beta_j}{\alpha_j\beta_j - \gamma_j^2}, \end{aligned}$$

that is

$$\det(PAP)\cdot\det(PA^{-1}P)=\prod_{j=1}^{m}\frac{\alpha_j\beta_j}{\alpha_j\beta_j-\gamma_j^2}. \tag{3.12}$$

Denote by $\lambda_1^{(j)}\geq\lambda_2^{(j)}$ the eigenvalues of the 2×2 matrix $\begin{bmatrix}\alpha_j & \gamma_j\\ \gamma_j & \beta_j\end{bmatrix}$. Since by definition of the D_j's,

$$\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix}\quad\text{is unitarily similar to}\quad\oplus\sum_{j=1}^{m}\begin{bmatrix}\alpha_j & \gamma_j\\ \gamma_j & \beta_j\end{bmatrix},$$

we have the identity

$$\{\lambda_1^{(j)},\lambda_2^{(j)}\ ;\ j=1,2,\ldots,m\}=\{\lambda_j(\tilde{A})\ ;\ j=1,2,\cdots,2m\},$$

so that for any choice of $1\leq j_1\leq j_2\leq\cdots\leq j_k$ $(k\leq m)$,

$$\prod_{i=1}^{k}\frac{\lambda_1^{(j_i)}}{\lambda_2^{(j_i)}}=\frac{\prod_{i=1}^{k}\lambda_1^{(j_i)}}{\prod_{i=1}^{k}\lambda_2^{(j_i)}}\leq\frac{\prod_{i=1}^{k}\lambda_i(\tilde{A})}{\prod_{i=1}^{k}\lambda_{2m-i+1}(\tilde{A})}=\prod_{i=1}^{k}\frac{\lambda_i(\tilde{A})}{\lambda_{2m-i+1}(\tilde{A})}.$$

Then since the function

$$\log\left(\frac{\exp(\frac{t}{2})+\exp(-\frac{t}{2})}{2}\right)^2\qquad(t>0)$$

is increasing and convex, according to a basic theorem on majorization (1.6), we can conclude

$$\prod_{j=1}^{m}\frac{\{\lambda_1^{(j)}+\lambda_2^{(j)}\}^2}{4\lambda_1^{(j)}\cdot\lambda_2^{(j)}}\leq\prod_{j=1}^{m}\frac{\left\{\lambda_j(\tilde{A})+\lambda_{2m-j+1}(\tilde{A})\right\}^2}{4\lambda_j(\tilde{A})\cdot\lambda_{2m-j+1}(\tilde{A})}. \tag{3.13}$$

Finally, since the Kantorovich type inequality (2.7), applied to the 2×2 matrix $\begin{bmatrix}\alpha_j & \gamma_j\\ \gamma_j & \beta_j\end{bmatrix}$, yields

$$\frac{\alpha_j\beta_j}{\alpha_j\beta_j-\gamma_j^2}\leq\frac{\{\lambda_1^{(j)}+\lambda_2^{(j)}\}^2}{4\lambda_1^{(j)}\cdot\lambda_2^{(j)}}\qquad(j=1,2,\ldots,m)$$

and hence

$$\prod_{j=1}^{m}\frac{\alpha_j\beta_j}{\alpha_j\beta_j-\gamma_j^2}\leq\prod_{j=1}^{m}\frac{\{\lambda_1^{(j)}+\lambda_2^{(j)}\}^2}{4\lambda_1^{(j)}\cdot\lambda_2^{(j)}}, \tag{3.14}$$

combining (3.11), (3.12), (3.13) and (3.14), we arrive at the expected inequality (3.5). This completes the proof of **(a)**.

A proof of **(b)** is parallel to the above. Suppose that at P the left-hand side of (3.6), which is

$$\frac{\det(PA^2P)}{\det(PAP)^2} \equiv \det\left(\Phi_P(A)^{-1}\Phi_P(A^2)\Phi_P(A)^{-1}\right),$$

attains its maximum over the set of all orthoprojections of rank m.

Khatri-Rao [9] solved an equivalent extremal problem by a Lagrange multiplier method. Let us again take the method of variation of orthoprojections.

For each Hermitian H, consider again the projections $P_H(t)$, defined by (3.7), and a function

$$g_H(t) \stackrel{\text{def}}{=} \frac{\det(P_H(t)A^2P_H(t))}{\det(P_H(t)AP_H(t))^2}.$$

Then $g_H(t)$ takes its maximum at $t = 0$, so that

$$\frac{d}{dt}g_H(t)|_{t=0} = 0 \qquad \text{(for all Hermitian } H). \tag{3.15}$$

Since

$$\frac{d}{dt}\det\left(P_H(t)A^2P_H(t)\right)|_{t=0} = i\det(PA^2P)\cdot \operatorname{tr}\left((PA^2P)^{-1}[H, A^2]\right),$$

and

$$\frac{d}{dt}\det\left(P_H(t)AP_H(t)\right)^2|_{t=0} = 2i\det(PAP)^2\cdot \operatorname{tr}\left((PAP)^{-1}[H, A]\right),$$

we have

$$\begin{aligned}\frac{d}{dt}g_H(t)|_{t=0} &= i\frac{\det(PA^2P)}{\det(PAP)^2}\cdot \operatorname{tr}\left((PA^2P)^{-1}[H, A^2] - 2(PAP)^{-1}[H, A]\right)\\ &= i\frac{\det(PA^2P)}{\det(PAP)^2}\cdot \operatorname{tr}\left(\{[A^2, (PA^2P)^{-1}] - 2[A, (PAP)^{-1}]\}H\right).\end{aligned}$$

Now (3.15) (for all Hermitian H) is possible only when

$$[A^2, (PA^2P)^{-1}] - 2[A, (PAP)^{-1}] = 0.$$

This is equivalent to the relation

$$(A_{21}A_{11} + A_{22}A_{21})(A_{11}^2 + A_{12}A_{21})^{-1} - 2A_{21}A_{11}^{-1} = 0,$$

which implies

$$A_{22}A_{21} \;=\; A_{21}A_{11} + 2A_{21}A_{11}^{-1}A_{12} \cdot A_{21}.$$

From this we can derive the following two identities

$$A_{12}A_{22}A_{21} = A_{12}A_{21} \cdot A_{11} + 2A_{12} \cdot A_{21}A_{11}^{-1}A_{12} \cdot A_{21} \tag{3.16}$$

and

$$A_{22} \cdot (A_{21}A_{11}^{-1}A_{12}) = A_{21}A_{12} + 2(A_{21}A_{11}^{-1}A_{12})^2. \tag{3.17}$$

Then (3.16) implies the commutativity of A_{11} and $|A_{21}|$ while (3.17) does that of $|A_{12}|$ and A_{22}. Now just as in the proof of **(a)**, there are positive numbers $\alpha_j, \beta_j > 0$ and nonnegative numbers γ_j, $j = 1, 2, \ldots, m$, such that

$$\frac{\det(PA^2P)}{\det(PAP)^2} = \prod_{j=1}^{m} \frac{\alpha_j^2 + \gamma_j^2}{\alpha_j^2}, \tag{3.18}$$

and with the help of the Kantorovich type inequality (2.4), applied to the 2×2 matrix,

$$\prod_{j=1}^{m} \frac{\alpha_j^2 + \gamma^2}{\alpha_j^2} \le \prod_{j=1}^{m} \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)}. \tag{3.19}$$

Now combining (3.18) and (3.19), we arrive at the inequality (3.6). This completes the proof of Theorem. ■

As mentioned in the introduction, it will be useful to express the results of Theorem 3.1 as estimates of the eigenvalues of the matrices appearing in the block matrix representation.

Theorem 3.2. *Let an $n \times n$ positive definite matrix A be represented in a block matrix form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11}, for instacnce, is an $m \times m$ matrix with $m \le n - m$. Then the following estimates hold :

(a)

$$\prod_{j=1}^{k} \frac{1}{1 - \lambda_j\left(|A_{22}^{-1/2}A_{21}A_{11}^{-1/2}|\right)^2} \le \prod_{j=1}^{k} \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \quad (k = 1, 2, \ldots, m),$$

and consequently

$$\sum_{j=1}^{k} \lambda_j \left(|A_{22}^{-1/2} A_{21} A_{11}^{-1/2}|\right)^2 \leq \sum_{j=1}^{k} \frac{\{\lambda_j(A) - \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \quad (k = 1, 2, \ldots, m).$$

(b)

$$\prod_{j=1}^{k} \{1 + \lambda_j \left(|A_{21} A_{11}^{-1}|\right)^2\} \leq \prod_{j=1}^{k} \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \quad (k = 1, 2, \ldots, m),$$

and consequently

$$\sum_{j=1}^{k} \lambda_j \left(|A_{21} A_{11}^{-1}|\right)^2 \leq \sum_{j=1}^{k} \frac{\{\lambda_j(A) - \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \quad (k = 1, 2, \ldots, m).$$

In fact, the first assertion of each of **(a)** and **(b)** is the restatement of the corresponding part in Theorem 3.1. To see the second assertion of **(a)**, notice that by the basic theorem on majorization (1.6) the first part implies

$$\sum_{j=1}^{k} \frac{1}{1 - \lambda_j \left(|A_{22}^{-1.2} A_{21} A_{11}^{-1/2}|\right)^2} \leq \sum_{j=1}^{k} \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \quad (k = 1, 2, \ldots, m).$$

Now the required assertion follows from the inequality

$$1 + t \leq \frac{1}{1 - t} \qquad (0 \leq t < 1),$$

and the identity

$$\frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} - 1 = \frac{\{\lambda_j(A) - \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \quad (k = 1, 2, \ldots, m).$$

In a similar way, the second assertion of **(b)** follows from the inequality

$$\sum_{j=1}^{k} \{1 + \lambda_j \left(|A_{21} A_{11}^{-1}|\right)^2\} \leq \sum_{j=1}^{k} \frac{\{\lambda_j(A) + \lambda_{n-j+1}(A)\}^2}{4\lambda_j(A) \cdot \lambda_{n-j+1}(A)} \quad (k = 1, 2, \ldots, m),$$

which, in turn, follows from the first part by the basic theorem on majorization (1.7).

4. Estimates in Additive Form

Our next interest is in finding estimates of

$$\Phi(A) - \Phi(A^{-1})^{-1} \quad \text{and} \quad \Phi(A^2) - \Phi(A)^2$$

in additive majorization form for a unital positive linear map Φ on the space of $n \times n$ matrices and positive definite $A \in \mathbb{M}_n$. But here also we have to restrict our observation to the case of a compression.

Lemma. *For a compression Φ_P with respect to an orthoprojection P of rank m and $0 < A \in \mathbb{M}_n$, the following inequalities hold for each $1 \leq k \leq m$:*

$$\sum_{j=1}^{k} \lambda_j \left(\Phi_P(A) - \Phi_P(A^{-1})^{-1}\right) \leq \operatorname{tr}\left(\Phi_Q(A) - \Phi_Q(A^{-1})^{-1}\right),$$

where Q is the orthoprojection to the subspace spanned by the eigenvectors of

$$B \equiv \Phi_P(A) - \Phi_P(A^{-1})^{-1},$$

corresponding to its largest k eigenvalues.

In fact, the assertion follows from the following two facts:

$$\sum_{j=1}^{k} \lambda_j \left(\Phi_P(A) - \Phi_P(A^{-1})^{-1}\right) = \operatorname{tr}(QBQ),$$

and

$$QBQ \leq \Phi_Q(A) - \Phi_Q(A^{-1})^{-1}$$

because, as in the proof of Lemma 3.1, $Q \leq P$ implies

$$Q\Phi_P(A)Q = QAQ \quad \text{and} \quad Q\Phi_P(A^{-1})^{-1}Q \geq (QA^{-1}Q)^{-1} = \Phi_Q(A^{-1})^{-1}.$$

The following is a majorization version of the result of Khatri–Rao [9].

Theorem 4.1. *For $0 < A \in \mathbb{M}_n$ and an orthoprojection P of rank $m \leq n-m$ the following majorization estimates in addtive form hold* :

$$\sum_{j=1}^{k} \lambda_j \left(\Phi_P(A) - \Phi_P(A^{-1})^{-1}\right) \leq \sum_{j=1}^{k} \left\{\sqrt{\lambda_j(A)} - \sqrt{\lambda_{n-j+1}(A)}\right\}^2 \quad (k = 1, 2, \ldots, m),$$

or, *equivalently*,

$$\sum_{j=1}^{k} \lambda_j \left(|A_{22}^{-1/2}A_{21}|\right)^2 \leq \sum_{j=1}^{k} \left\{\sqrt{\lambda_j(A)} - \sqrt{\lambda_{n-j+1}(A)}\right\}^2 \quad (k = 1, 2, \ldots, m).$$

Proof. In view of Lemma, it suffices, as in the proof of Theorem 3.1, to prove that for an orthoprojection P of rank $m \le n-m$,

$$\operatorname{tr}\left(\Phi_P(A) - \Phi_P(A^{-1})^{-1}\right) \le \sum_{j=1}^{m} \left\{ \sqrt{\lambda_j(A)} - \sqrt{\lambda_{n-j+1}(A)} \right\}^2. \tag{4.1}$$

Khatri-Rao [9] solved an equivalent extremal problem by a Lagrange multiplier method. Here let us take again the method of variation of orthoprojections as in the proof of Theorem 3.1.

To prove (4.1), we may and do assume that the maximum of the left-hand side of (4.1) over the class of orthoprojections of rank m is attained at this P.

For each Hermitian matrix H, consider again the one-parameter family of orthoprojections $P_{_H}(t)$ defined in (3.7). Then by the assumption on P, the function $h_{_H}(t)$ defined by

$$h_{_H}(t) \equiv \operatorname{tr}\left(\Phi_{P_{_H}(t)}(A) - \Phi_{P_{_H}(t)}(A^{-1})^{-1}\right)$$

attains its maximum at $t=0$, so

$$\frac{d}{dt} h_{_H}(t)|_{t=0} = 0 \qquad (\text{for all Hermitian } H). \tag{4.2}$$

We can compute as follows.

$$\frac{d}{dt} h_{_H}(t)|_{t=0} = i\operatorname{tr}(XH),$$

where

$$X = [A, (P^{\perp}AP^{\perp})^{-1}(P^{\perp}AP) + (PAP^{\perp})(P^{\perp}AP^{\perp})^{-1} \\ -(P^{\perp}AP^{\perp})^{-1}(P^{\perp}AP)(PAP^{\perp})(P^{\perp}AP^{\perp})^{-1}].$$

Then (4.2) (for all Hermitian H) is possible only when $X=0$, that is,

$$A_{11}A_{12}A_{22}^{-1} - A_{12} - A_{12}A_{22}^{-1}A_{21}A_{12}A_{22}^{-1} = 0,$$

or,

$$(A_{11} - A_{12}A_{22}^{-1}A_{21})A_{12}A_{22}^{-1} = A_{12},$$

which implies

$$(A_{11} - A_{12}A_{22}^{-1}A_{21})A_{12}A_{22}^{-1}A_{21} = A_{12}A_{21}. \tag{4.3}$$

Since all factors of (4.3) are Hermitian, A_{11} must commute with $A_{12}A_{22}^{-1}A_{21}$ and $|A_{21}|$. Then as in the proof of Theorem 3.2, there are $\alpha_j, \beta_j > 0$ and $\gamma_j \geq 0$, $j = 1, 2, \dots, m$, such that each 2×2 matrix $\begin{bmatrix} \alpha_j & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix}$ is positive definite and

$$\text{(4.4)} \qquad \operatorname{tr}\left(\Phi_P(A) - \Phi_P(A^{-1})^{-1}\right) = \sum_{j=1}^{m} \frac{\gamma_j^2}{\beta_j},$$

and, with its eigenvalues $\lambda_1^{(j)} \geq \lambda_2^{(j)}$,

$$\text{(4.5)} \qquad \sum_{j=1}^{m} \left\{ \sqrt{\lambda_1^{(j)}} - \sqrt{\lambda_2^{(j)}} \right\}^2 \leq \sum_{j=1}^{m} \left\{ \sqrt{\lambda_j(A)} - \sqrt{\lambda_{n-j+1}(A)} \right\}^2.$$

On the other hand, the Kantorovich type inequality (2.8), applied to the 2×2 matrix, yields

$$\text{(4.6)} \qquad \frac{\gamma_j^2}{\beta_j} \leq \left\{ \sqrt{\lambda_1^{(j)}} - \sqrt{\lambda_2^{(j)}} \right\}^2 \quad (j = 1, 2, \dots, m).$$

Finally combining (4.4), (4.5) and (4.6), we arrive at the inequality (4.1). This completes the proof. ∎

We can apply the same variational method to $\Phi_P(A^2) - \Phi_P(A)^2$ to get

$$\text{(4.7)} \quad \sum_{j=1}^{k} \lambda_j\left(\Phi_P(A^2) - \Phi_P(A)^2\right) \leq \sum_{j=1}^{k} \frac{\{\lambda_j(A) - \lambda_{n-j+1}(A)\}^2}{4} \quad (k = 1, 2, \dots, m)$$

or, equivalently,

$$\sum_{j=1}^{k} \lambda_j(|A_{21}|)^2 \leq \sum_{j=1}^{k} \frac{\{\lambda_j(A) - \lambda_{n-j+1}(A)\}^2}{4} \quad (k = 1, 2, \dots, m).$$

But a more careful consideration, similar to that in the proof of Theorem 3.1, due to Li and Mathias [13], can yield much sharper estimates.

Theorem 4.2. *For $n \times n$ Hermitian A and an orthoprojection P of rank $m \leq n - m$, the following majorization estimates in addtive form hold* :

$$\sum_{j=1}^{k} \sqrt{\lambda_j\left(\Phi_P(A^2) - \Phi_P(A)^2\right)} \leq \sum_{j=1}^{k} \frac{\lambda_j(A) - \lambda_{n-j+1}(A)}{2} \quad (k = 1, 2, \dots, m)$$

or, *equivalently*,

$$\sum_{j=1}^{k} \lambda_j(|A_{21}|) \leq \sum_{j=1}^{k} \frac{\lambda_j(A) - \lambda_{n-j+1}(A)}{2} \quad (k = 1, 2, \ldots, m).$$

Proof. Denote for simplicity

$$\lambda_j \equiv \lambda_j(|A_{21}|) \quad (j = 1, 2, \ldots, m).$$

Then there exist an $(n-m) \times m$ matrix U with $U^*U = I_m$ and a unitary matrix V of order m such that

$$U^* A_{21} V = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m).$$

Let

$$\tilde{A}_{11} = V^* A_{11} V,\ \tilde{A}_{12} = V^* A_{12} U,\ \tilde{A}_{21} = U^* A_{21} V,\ \tilde{A}_{22} = U^* A_{22} U,$$

and consider the $(2m) \times (2m)$ Hermitian matrix

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \equiv [\tilde{a}_{i,j}]_{i,j=1}^{2m}.$$

Then, by assumption,

$$\tilde{A}_{21} = \tilde{A}_{12} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m).$$

As in the proof of Theorem 3.1, we can see

$$\lambda_j(A) \geq \lambda_j(\tilde{A}) \quad \text{and} \quad \lambda_{n-j+1}(A) \ \leq \ \lambda_{2m-j+1}(\tilde{A}),$$

so that

$$\lambda_j(A) - \lambda_{n-j+1}(A) \geq \lambda_j(\tilde{A}) - \lambda_{2m-j+1}(\tilde{A}) \qquad (j = 1, 2, \ldots, m).$$

For each $j = 1, 2, \ldots, m$, consider the 2×2 matrix

$$\begin{bmatrix} \tilde{a}_{jj} & \tilde{a}_{j,m+j} \\ \tilde{a}_{m+j,j} & \tilde{a}_{m+j,m+j} \end{bmatrix}$$

and its eigenvalues $\lambda_1^{(j)} \geq \lambda_2^{(j)}$. Then the Kantorovich type inequality (2.5), applied to the 2×2 matrix, yields

$$\lambda_j \leq \frac{\lambda_1^{(j)} - \lambda_2^{(j)}}{2}.$$

Though the direct sum

$$\oplus \sum_{j=1}^{m} \begin{bmatrix} \tilde{a}_{jj} & \tilde{a}_{j,m+j} \\ \tilde{a}_{m+j,j} & \tilde{a}_{m+j,m+j} \end{bmatrix}$$

is not unitarily similar to $\tilde{A}$, there are orthoprojections P_j, $j = 1, 2, \ldots, m$, such that $\sum_{j=1}^{m} P_j = I$ and $\sum_{j=1}^{m} P_j \tilde{A} P_j$ is unitarily similar to the direct sum in question. Then according to the so-called pinching theorem (e.g., [4, p. 50]), we can see

$$\sum_{j=1}^{k} \lambda_1^{(j)} \leq \sum_{j=1}^{k} \lambda_j(\tilde{A}) \quad (k = 1, 2, \ldots, m)$$

and

$$\sum_{j=1}^{k} \lambda_2^{(j)} \geq \sum_{j=1}^{k} \lambda_{2m-j+1}(\tilde{A}) \quad (k = 1, 2, \ldots, m).$$

Therefore we can conclude

$$\begin{aligned} \sum_{j=1}^{k} \lambda_j &\leq \sum_{j=1}^{k} \frac{\lambda_1^{(j)} - \lambda_2^{(j)}}{2} \\ &\leq \sum_{j=1}^{k} \frac{\lambda_j(\tilde{A}) - \lambda_{2m-j+1}(\tilde{A})}{2} \\ &\leq \sum_{j=1}^{k} \frac{\lambda_j(A) - \lambda_{n-j+1}(A)}{2}. \end{aligned}$$

This completes the proof. ■

Finally, notice that since t^2 is increasing and convex on $[0, \infty)$, inequality (4.7) follows from Theorem 4.2 by the basic theorem on majorization (1.5).

REFERENCES

1. G. Alpargu, *The Kantorovich Inequality, with some Extensions and with Statistical Applications*, MS Thesis, McGill University, 1996.
2. T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Linear Algebra Appl.* **26** (1979), 203-241.
3. T. Ando, Quasi-orders on the positive cone of a C^*-algebra, *Linear and Multilinear Algebra* **41** (1996), 81-93.
4. R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.

5. P. Bloomfield and G. S. Watson, The inefficiency of least square, *Biometrika* **62** (1975), 124-128.

6. M.-D. Choi, Schwarz inequalties for positive linear maps, *Illinois J. Math.* **18** (1974), 565-574.

7. M.-D. Choi, Completely positive linear maps on complex matrices, *Linear Algebra Appl.* **10** (1975), 285-290.

8. C. G. Khatri and C. R. Rao, Some extensions of the Kantorovich inequality and statistical applications, *J. Multivariate Anal.* **11** (1981), 498-505.

9. C. G. Khatri and C. R. Rao, Some generalizations of the Kantorovich inequality, *Sankhyā Ser. A*, **44** (1982), 91-102.

10. M. Knott, On the minimum efficiency of least squares, *Biometrika* **62** (1975), 129-132.

11. F. Kubo, unpublished note, (1997).

12. C.-K. Li and R. Mathias, Matrix inequalities involving a positive linear map, *Linear and Multilinear Algebra* **41** (1996), 221-231.

13. C.-K. Li and R. Mathias, Inequalities on the singular values of an off-diagonal block of a Hermitian matrix, *J. Inequal. Appl.* **3** (1999), 137-142.

14. S. Liu, *Contributions to Matrix Calculus and Applications in Econometrics*, Thesis Publisher, Amsterdam, 1995.

15. S. Liu and H. Neudecker, Some matrix Kantorovich-type inequalities, *J. Math. Anal. Appl.* **197** (1996), 23-26.

16. A. W. Marshall and I. Olkin, *Inequalities*: *Theory of Majorization and its Applications*, Academic Press, New York, 1979.

17. A. W. Marshall and I. Olkin, Matrix versions of the Cauchy and Kantorovich inequalities, *Aequationes Math.* **40** (1990), 89-93.

18. M. Mond and J. E. Pecaric, A matrix version of the Ky Fan generalization of the Kantorovich inequality, *Linear and Multilinear Algebra* **36** (1994), 217-221.

19. J. E. Pecaric, S. Puntanen and S. P. H. Styan, Some further matrix extensions of the Cauchy – Schwarz and Kantorovich inequalities with some statistical applications, *Linear Algebra Appl.* **237/238** (1996), 455-476.

20. G. K. Pedersen, *C^*-algebras and Their Automorphism Groups*, Academic Press, London, 1979.

21. C. R. Rao, The inefficiency of least squares; extensions of the Kantorovich inequalities, *Linear Algebra Appl.* **70** (1985), 249-255.

Hokusei Gakuen University
Sapporo 004–8631, Japan
E-mail: ando@hokusei.ac.jp