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ON MINIMAL DISPLACEMENT PROBLEM AND RETRACTIONS OF BALLS ONTO SPHERES

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Abstract. We present here various facts and examples concerning the evaluation of the minimal displacement $d(T) = \inf \|Tx - x\|$ for some classes of Lipschitz self mappings of balls in Banach spaces.

Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with the unit ball B and the unit sphere S. The term *minimal displacement problem* has been first used by the present author in [4]. It came from the observation that while looking for the fixed points of a mapping $T : B \to B$ we often exercise the evaluation of the quantity

$$d(T) = \inf_{x \in B} \|x - Tx\|$$

called *the minimal displacement of* T.

Sometimes we are able to establish some uniform estimates for T's belonging to certain class $\mathcal{T}$. The minimal displacement for the whole class is defined as

$$d(\mathcal{T}) = \sup_{T \in \mathcal{T}} d(T) = \sup_{T \in \mathcal{T}} \inf_{x \in B} \|x - Tx\|.$$

The class $\mathcal{T}$ under concern is often divided into subclasses $\mathcal{T}(k)$ indexed by a certain parameter k. In this case we can also investigate the function of this parameter, namely,

$$\psi(k) = d(\mathcal{T}(k)).$$

Our aim here is to present some aspects of this problem via relatively simple examples. The whole theory can be presented in more general settings but here we restrict ourselves to the case of subclasses of Lipschitz maps defined on the ball.

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We show some connections to another problem of finding Lipschitz retractions of B onto its sphere S. For more details we refer the reader to [6].

We shall denote here by $\mathcal{L}$ the class of all Lipschitz maps. $\mathcal{L}$ is naturally divided into subclasses $\mathcal{L}(k)$ indexed by $k \in [0,\infty)$ consisting of all mappings satisfying Lipschitz condition with constant k:

$$\|Tx - Ty\| \leq k\|x - y\|.$$

This notation will be used regardless of the domain of T and other restrictions. Subclasses under concern and their natural scaling by Lipschitz constant k will be indexed $\mathcal{L}_0, \mathcal{L}_1$... and $\mathcal{L}_0(k), \mathcal{L}_1(k) \ldots$.

There are two basic facts lying in the background of this theory

A. *For any $k > 1$, there exists a mapping $T : B \to B$ of class $\mathcal{L}(k)$ such that $d(T) > 0$.*

This is an outcome of a more general theorem of a P. K. Lin and Y. Sternfeld [7] saying that the same holds for mappings $T : C \to C$, where C is an arbitrary closed and convex but noncompact subset of X.

B. *The unit sphere S is the lipschitzian retract of B.*

This means that there exists a mapping $R : B \to S$ such that $Rx = x$ for all $x \in S$ and such that $R \in \mathcal{L}(k)$ for certain (sufficiently large) k. It has been proved first by B. Nowak [9] for some Banach spaces and then in the general case by Y. Benyamini and Y. Sternfeld [2]. The above leads to the formulation of the "optimal" retraction problem.

For any space X, denote by

$$k_0(X) = \inf\{k : \text{there exists a retraction of class } \mathcal{L}(k)\}.$$

So far the constant $k_0(X)$ is unknown for any space X. The mentioned problem reads: find or give a good estimate for $k_0(X)$.

Finally, let us recall that the unit ball B is the lipschitzian retract of the whole space X. The standard retraction is the so called radial projection

$$Px = \begin{cases} x & \text{for } x \in B, \\ \frac{x}{\|x\|} & \text{for } x \notin B. \end{cases}$$

In general, $P \in \mathcal{L}(2)$ and for some spaces we have better evaluations. Especially for Hilbert space, $P \in \mathcal{L}(1)$.

Radial projection is not necessarily optimal with respect of the value of its Lipschitz constant. For example, if $X = C[a,b]$, the Lipschitz constant of P equals 2 while there exists another retraction $Q : X \to B$ of class $\mathcal{L}(1)$. Indeed, put $Q(f)(t) = \alpha(f(t))$ or, in other words, $Q = \alpha \circ f$, where

$$\alpha(t) = \min\{1, \max[t, -1]\} = \begin{cases} -1 & \text{for } t \leq -1, \\ t & \text{for } -1 \leq t \leq 1, \\ 1 & \text{for } t \geq 1. \end{cases}$$

Obviously, for any ball $B_r = rB$ of radius r the retraction $P_r : X \to B_r$, given by $P_r x = rP(x/r)$, has the same Lipschitz constant as P. The same holds for $Q_r(f) = rQ(f/r)$.

Let us pass to examples illustrating the above notions. We believe certain observations can be new even for specialists.

Example I. Consider the whole family $\mathcal{L}_0$ of Lipschitz maps $T : B \to B$. Take $T \in \mathcal{L}_0(k)$ for certain k. If $k < 1$, then T being a contraction has a fixed point and $d(T) = 0$. If $k \geq 1$, then according to Statement A we may have $d(T) > 0$. But for any $\varepsilon > 0$, the equation

$$x = \frac{1}{k+\varepsilon} Tx \tag{1}$$

has a solution. This is because the right hand side of (1) is a contraction. Thus we have

$$\|x - Tx\| = \left(1 - \frac{1}{k+\varepsilon}\right) \|Tx\| \leq 1 - \frac{1}{k+\varepsilon},$$

implying as $\varepsilon \to 0$,

$$d(T) \leq 1 - \frac{1}{k}. \tag{2}$$

On the other hand,

$$d(\mathcal{L}_0) = 1. \tag{3}$$

This can be shown using B. Let $R : B \to S$ be a retraction of class $\mathcal{L}(k)$, where $k \geq k_0(X)$. Take any $\varepsilon > 0$ and consider the mapping $T_\varepsilon : B \to B$ defined by

$$T_\varepsilon x = \begin{cases} -R\left(\dfrac{x}{\varepsilon}\right) & \text{for } \|x\| \leq \varepsilon, \\ -\dfrac{x}{\|x\|} = -P\left(\dfrac{x}{\varepsilon}\right) & \text{for } \|x\| \geq \varepsilon. \end{cases} \tag{4}$$

It is easy to observe that $k_0(X) \geq 2$ (better evaluations will follow) and then notice that $T_\varepsilon \in \mathcal{L}(k/\varepsilon)$ and $d(T_\varepsilon) \geq 1 - \varepsilon$.

The estimate (3) shows that $d(\mathcal{L})$ is not achieved on any $T \in \mathcal{L}$. This is a partial observation of the fact that for any continuous mapping $T : B \to B$ we have $d(T) < 1$ [3].

Example II. Consider again the same class of all mappings $T : B \to B$ of class $\mathcal{L}_0$ but look on $\mathcal{L}_0$ as the union

$$\mathcal{L}_0 = \bigcup_{k \geq 0} \mathcal{L}_0(k).$$

Put

$$\psi_{0,X}(k) = d(\mathcal{L}_0(k)).$$

The estimate (2) shows that

$$\psi_{0,X}(k) \leq 1 - \frac{1}{k} \tag{5}$$

for $k \geq 1$. The case $k < 1$ is trivial. We shall often leave the indicator of the space writing ψ_0 instead of $\psi_{0,X}$. However this indicator is important since there are spaces with different $\psi_{0,X}$'s. Let us call the space *extremal* if

$$\psi_{0,X}(k) = 1 - \frac{1}{k}.$$

Such spaces do exist. Take for example $X = C[-1, 1]$ and for $x \in B$, define

$$(Tx)(t) = Q(k(x(t) + 2t)). \tag{6}$$

It is easy to check that $T \in \mathcal{L}_0(k)$ with $d(T) = 1 - (1/k)$. Similar examples can be constructed in many spaces as, for example, c_0, l^∞, spaces of differentiable functions with standard norms, and in all subspaces of $C[a, b]$ of finite codimension. This last recent result is not published yet and is due to Bolibok [3].

There are spaces which are not extremal. Such are all uniformly convex spaces and l^1. The known estimates are (see [3, 5])

$$\psi_{0,H}(k) \leq \left(1 - \frac{1}{k}\right) \sqrt{\frac{k}{k+1}} \tag{7}$$

for Hilbert space H (sketch of the proof will be given in the next section) and

$$\psi_{0,l^1}(k) \leq \begin{cases} \dfrac{2+\sqrt{3}}{4}\left(1 - \dfrac{1}{k}\right) & \text{for } 1 \leq k \leq 3 + 2\sqrt{3}, \\ \dfrac{k+1}{k+3} & \text{for } k > 3 + 2\sqrt{3}. \end{cases} \tag{8}$$

It is not known whether these estimates are sharp. Actually we do not know the exact formula for $\psi_{0,X}$ for any nonextremal space.

To finish this case, let us list some basic properties of ψ_0 :

a) $\psi_0(k)$ is nondecreasing,
b) $\lim_{k\to\infty}\psi_0(k) = 1$,
c) $\psi(1-\alpha+\alpha k) \geq \alpha\psi_0(k)$ for all $0 \leq \alpha \leq 1$,
d) $\psi_0(k)/(k-1)$ is nonincreasing,
e) $k\psi_0(k)/(k-1)$ is nondecreasing,
f) $\psi_0'(1) = \lim_{k\to 1^+}\psi_0(k)/(k-1)$ exists and $\psi_0'(1) > 0$,
g) $\psi_0(k) \geq \psi_0'(1)(1-(1/k))$,
h) X is extremal if and only if $\psi_0'(1) = 1$.

Property a) is trivial, b) follows from the fact that $d(\mathcal{L}) = 1$, and c) follows from the simple observation that if $T \in \mathcal{L}_0(k)$ and $\alpha \in [0,1]$, then $T_\alpha = (1-\alpha)I + \alpha T \in \mathcal{L}_0(1-\alpha+\alpha k)$ with $\|x - T_\alpha x\| = \alpha\|x - Tx\|$ for all $x \in B$. Next, d) is a reformulation of c). Property e) requires a proof. f), g), h) then follow easily.

Take any $A > k$. Let $T \in \mathcal{L}_0(k)$. Fix $x \in B$ and consider the equation

$$y = \left(1 - \frac{1}{A}\right)x + \frac{1}{A}Ty.$$

In view of Banach Contraction Principle, it has exactly one solution y depending on x. Put $y = Fx$. We have the implicit formula

$$Fx = \left(1 - \frac{1}{A}\right)x + \frac{1}{A}TFx \tag{9}$$

from which we get $F \in \mathcal{L}_0((A-1)/(A-k))$ and $TF \in \mathcal{L}_0(k(A-1)/(A-k))$. Also if $\|x - Tx\| \geq d > 0$ for all $x \in B$, then

$$\|x - TFx\| = \frac{A}{A-1}\|Fx - TFx\| \geq \frac{Ad}{A-1}.$$

Consequently,

$$\psi_0\left(\frac{k(A-1)}{A-k}\right) \geq \frac{A}{A-1}\psi_0(k).$$

Denoting $k(A-1)/(A-k)$ by l (observe that $l > k$), we obtain

$$\frac{l\psi_0(l)}{l-1} \geq \frac{k\psi_0(k)}{k-1},$$

which ends the proof.

Let us finish this section by listing a few open problems. The most intriguing in our opinion are

Problem 1. *Find exact formulas for* $\psi_{0,H}$ *and* ψ_{0,l^1}.

Problem 2. *Does there exist a Banach space* Y *such that* $\psi_{0,Y} \leq \psi_{0,X}$ *for all Banach spaces* X? *Is this a Hilbert space or* l^1 ?

The estimates (7) and (8) are probably not sharp and do not indicate which space is "better". Estimate (7) is smaller for k close to 1 while (8) is better for large k.

Two more related questions are:

Problem 3. *Does there exist a space* Y *such that* $\psi'_{0,Y}(1) \leq \psi'_{0,X}(1)$ *for all spaces* X ? *Is this a Hilbert space*?

Problem 4. *From* (7) *we have* $\psi'_{0,H}(1) \leq 1/\sqrt{2}$. *Is this the best estimate*?

Example III. Let us consider the subclass $\mathcal{L}_1$ of $\mathcal{L}$ consisting of mappings T transforming B into S. Obviously,

$$\mathcal{L}_1 = \bigcup_{k \geq 1} \mathcal{L}_1(k),$$

where

$$\mathcal{L}_1(k) = \{T : B \to S \ : \ T \in \mathcal{L}(k)\}.$$

For this family of mappings, let us define

$$\psi_{1,X}(k) = d(\mathcal{L}_1(k)) = \sup\{d(t) : T \in \mathcal{L}_1(k)\}.$$

As before, we shall sometimes skip the space indicator writing $\psi_1(k)$ instead of $\psi_{1,X}(k)$. Obviously, $\psi_{1,X}$ is nondecreasing, and of course

$$\psi_{1,X}(k) \leq \psi_{0,X}(k) \leq 1 - \frac{1}{k}.$$

Construction (4) from Example I shows that

$$\lim_{k \to \infty} \psi_1(k) = 1,$$

while construction (6) from Example II indicates that for some spaces (e.g., $C[a,b]$) we may have

$$\psi_{1,X}(k) = \psi_{0,X}(k) = 1 - \frac{1}{k}.$$

First we show that in general it is not true. Let H be a Hilbert space, $T : B \to S$ be of class $\mathcal{L}(k)$ and such that for all $x \in B$,

$$\|x - Tx\| \geq (1 - \varepsilon)\psi_{1,H}(k).$$

Take $A > k$ and let y, z be respectively solutions of the two equations

$$y = \frac{1}{A} Ty$$

and

$$z = \left(1 - \frac{1}{A}\right) y + \frac{1}{A} Tz.$$

We have

$$\begin{aligned} k^2\|z-y\|^2 &\geq \|Tz - Ty\|^2 = \|Az - (A-1)y - Ay\|^2 \\ &= \|(Az - Ay) - (A-1)y\|^2 \\ &= A^2\|z-y\|^2 - 2A(A-1)(z-y, y) + (A-1)^2\|y\|^2, \end{aligned}$$

implying

$$2A(z-y, y) \geq (A-1)\|y\|^2.$$

But $\|Tz\| = \|Ty\| = 1$. Also, $z - y = (Tz - z)/(A-1)$, and $Tz - y = A(z-y)$. Thus we have

$$\begin{aligned} 1 &= \|Tz\|^2 = \|(Tz - y) + y\|^2 = \|A(z-y) + y\|^2 \\ &= A^2\|z-y\|^2 + 2A(z-y, y) + \|y\|^2 \\ &= \left(\frac{A}{A-1}\right)^2 \|z - Tz\|^2 + A\|y\|^2 \\ &\geq \left(1 - \frac{1}{A}\right)^{-2} (1-\varepsilon)^2 \psi_{1,H}^2(k) + \frac{1}{A}. \end{aligned}$$

And finally,

$$(1-\varepsilon)^2 \psi_{1,H}^2(k) \leq \left(1 - \frac{1}{A}\right)^3,$$

from which letting $\varepsilon \to 0$ and $A \to k$ we obtain

$$\psi_{1,H}(k) \leq \left(1 - \frac{1}{k}\right)^{\frac{3}{2}} = \left(1 - \frac{1}{k}\right)\sqrt{\frac{k-1}{k}}. \tag{10}$$

This is a better evaluation than (7):

$$\psi_{0,H}(k) \leq \left(1 - \frac{1}{k}\right)\sqrt{\frac{k}{k+1}}.$$

Here let us indicate that the proof of (7) can be obtained in the same way as above if we only skip the fact that $\|Tz\| = \|Ty\| = 1$ and observe that $\|y\| = \|Ty - y\|/(A-1)$. The conclusion that

$$\psi_{1,H}(k) < \psi_{0,H}(k)$$

at least for small k can be drawn from the fact that (10) implies $\psi'_{1,H}(1) = 0$, while $\psi'_{0,H}(1) > 0$.

The function $\psi_1(k)$ has an application to the optimal retraction problem. Suppose $R : B \to S$ is a retraction having Lipschitz constant $k \geq k_0(X)$. Define the mapping $T = -R$. Then $T : B \to S$, $T \in \mathcal{L}(k)$ and $T^2 = R$. Take $\varepsilon > 0$ and select $x \in B$ such that

$$\|x - Tx\| = \|x + Rx\| \leq d(T) + \varepsilon.$$

Now, define a curve $\gamma : [0, 1] \to S$ by

$$\gamma(t) = T((1 - t)x + tTx).$$

Then γ being lipschitzian is rectifiable and joins the two antipodal points $-Rx$ and Rx of S. Let $g(X)$ denote the infimum of the lengths of such curves. The number $g(X)$ is sometimes called the "girth" of the sphere and obviously $g(X) \geq 2$ for any space X, while $g(H) = \pi$ for Hilbert space H (see [10]). Since γ is k-lipschitzian, its length $l(\gamma)$ satisfies

$$g(X) \leq l(X) \leq k\|Tx - x\| \leq k(d(T) + \varepsilon).$$

Moreover, $d(T) \leq \psi_{1,X}(k)$ and letting ε pass to zero we obtain

$$k\psi_{1,X}(k) \geq g(X).$$

In view of $\psi_1(k) \leq 1 - (1/k)$, we get $k \geq g(X) + 1$, and consequently,

$$k_0(X) \geq g(X) + 1 \geq 3$$

for all spaces X. Similar reasoning for a Hilbert space leads to the inequality

$$k\left(1 - \frac{1}{k}\right)^{\frac{3}{2}} \geq \pi,$$

which when solved numerically shows that $k_0(H) \geq 4.55\ldots$.

Most of the questions we asked about ψ_0 can be raised as well in connection with ψ_1.

Example IV. *This time let us look for the subclass $\mathcal{L}_2$ of $\mathcal{L}$ consisting of all mappings $T : B \to X$ (the image is not neccesarily contained in B) sending all the points $x \in S$ to the origin. This means that $\|x\| = 1$ implies $\|Tx\| = 0$ or, in other words, $T(S) = \{0\}$. This class is also naturally divided into subclasses $\mathcal{L}_2(k)$.*

Our interest in this class is connected too with the optimal retraction problem. If T is of class $\mathcal{L}_2(k)$ with $d(T) > 0$, then we can easily construct a retraction $R : B \to S$ by putting

$$Rx = \frac{x - Tx}{\|x - Tx\|} = P\left(\frac{x - Tx}{d(T)}\right). \tag{11}$$

Obviously, R is lipschitzian and

$$R \in \mathcal{L}_1\left(2\frac{k+1}{d(T)}\right). \tag{12}$$

Having estimations for $d(T)$, we can evaluate $k_0(X)$ (an example will follow in the next section). It is also worthwhile to observe that any lipschitzian retraction $R : B \to S$ can be obtained via formula (11). If $R \in \mathcal{L}(k)$, $k \geq k_0(X)$, then putting $T = I - R$ we get the mapping of class $\mathcal{L}_2(k+1)$ satisfying $\|Tx - x\| = d(T) = 1$ for all $x \in B$. Applying (11) to T we reconstruct R.

In view of the above it seems to be worthwhile to define analogously as before the function

$$\psi_2(k) = \sup\{d(T) : T \in \mathcal{L}_2(k)\} = d(\mathcal{L}_2(k)).$$

Obviously, $\psi_2(k) \leq 1$ for $k \geq 1$ and in view of the above remarks for $k > k_0(X)+1$ we have $\psi_2(k) = 1$.

Let us find the first evaluation of $\psi_2(k)$ for smaller k's. Without loss of generality we can assume that our mappings are defined on the whole space X and take value zero outside of B ($\|x\| \geq 1$ implies $Tx = 0$). Observe that for all $x \in B$, $x \neq 0$,

$$\|Tx\| = \left\|Tx - T\left(\frac{x}{\|x\|}\right)\right\| \leq k(1 - \|x\|)$$

and

$$\|Tx - T0\| \leq k\|x\|.$$

Consequently, $\|T0\| \leq k$ and for all $x \in X$ we have

$$\begin{aligned}\left\|Tx - \frac{1}{2}T0\right\| &\leq \frac{1}{2}\|Tx\| + \frac{1}{2}\|Tx - T0\| \\ &\leq \frac{k}{2}(1 - \|x\|) + \frac{k}{2}\|x\| = \frac{k}{2}.\end{aligned}$$

This means that T transforms the ball $B((T0)/2, k/2)$ into itself. We leave to the reader the justification of the simple observation that

$$d(T) \leq \frac{k}{2}\psi_{0,X}(k) \leq \frac{k-1}{2}.$$

Finally,

$$\psi_{2,X}(k) \leq \min\left\{1, \frac{k}{2}\psi_{0,X}(k)\right\} \leq \min\left\{1, \frac{k-1}{2}\right\}$$

and in consequence

$$\psi'_{2,X}(1) \leq \frac{1}{2}\psi'_{0,X}(1).$$

It is not known whether the above evaluations are sharp. If there is an *extremal* space for which

$$\psi_{2,X}(k) = \min\left\{1, \frac{k-1}{2}\right\} = \begin{cases} \frac{k-1}{2} & \text{for } k \leq 3, \\ 1 & \text{for } k > 3, \end{cases}$$

then using (12) with $k = 3$ we would get $k_0(X) \leq 8$. Let us recall that the best estimate for k_0, known so far, is $k_0(L^1(0,1)) \leq 9.43...$ (see [6]).

Example V. Again let us consider mappings sending S into 0, $T(S) = \{0\}$. However, let us restrict our attention to the subclass of $\mathcal{L}_3$ of $\mathcal{L}_2$ consisting of those mappings for which $T(B) \subset B$. Analogously as before, let us scale $\mathcal{L}_3$ into $\mathcal{L}_3(k)$ and define $\psi_{3,X}(k)$. In view of the above restrictions we have

$$\psi_{3,X}(k) \leq \psi_{0,X}(k) \leq 1 - \frac{1}{k}$$

and

$$\psi_{3,X}(k) \leq \psi_{2,X}(k).$$

It is expected that $\psi_{3,X}(k) < \psi_{2,X}(k)$ for all $k > 1$. Nevertheless ψ_3 can also be used to obtain some estimates for $k_0(X)$ via formula (12). This can be illustrated by the following construction.

Let $X = C[0,1]$. It is not difficult to check that for any $r_1, r_2 > 0$ and $f, g \in C[0,1]$,

$$\|Q_{r_1}f - Q_{r_2}g\| \leq \max\{\|f-g\|, |r_1 - r_2|\},$$

where Q_r denotes the projection of class $\mathcal{L}(1)$ defined at the beginning. As we have already shown there exists a mapping $T_1 : B \to B$ of class $\mathcal{L}_0(k)$ such that $d(T) = 1 - (1/k)$. Consider the ball of radius 2 ($B_2 = 2B$) and extend T_1 to the mapping $T_2 : 2B \to B$ by putting

$$T_2 f = \begin{cases} Tf & \text{for } \|f\| \leq 1, \\ T(Qf) & \text{for } 1 \leq \|f\| \leq 2 - \frac{1}{k}, \\ Q_{k(2-\|f\|)}(T(Qf)) & \text{for } 2 - \frac{1}{k} \leq \|f\| \leq 2. \end{cases}$$

Again $T_2 \in \mathcal{L}(k)$ and for all $f \in S_2$ ($\|f\| = 2$) we have $T_2 f = 0$. Moreover, it can be observed that $\|f - T_2 f\| \geq 1 - (1/k)$ for all $f \in B_2$. Now denoting

$$Tf = \frac{1}{2}T_2(2f),$$

we get the mapping $T \in \mathcal{L}_3(k)$ with $d(T) = \frac{1}{2}(1 - (1/k))$. This leads to the observation that

$$\frac{1}{2}\left(1 - \frac{1}{k}\right) \leq \psi_{3,C[0,1]}(k) \leq \psi_{2,C[0,1]}(k) \leq \frac{k-1}{2},$$

implying

$$\psi'_{3,C[0,1]}(1) = \psi'_{2,C[0,1]}(1) = \frac{1}{2}.$$

However it can be proved that the left part of the last inequality is not sharp since $\lim_{k\to\infty} \psi_{3,X}(k) = 1$ for all spaces X.

If we use (12) to generate the retraction

$$Rf = \frac{f - Tf}{\|f - Tf\|} = P\left(\frac{f - Tf}{\frac{1}{2}\left(1 - \frac{1}{k}\right)}\right),$$

we get $R \in \mathcal{L}(4k(k+1)/(k-1))$ and therefore

$$k_0(C[0,1]) \leq 4 \min_{k>1} \frac{k(k+1)}{k-1} = 4\left(1 + \sqrt{2}\right)^2 = 23.31\ldots .$$

This estimate is probably far from being sharp but according to our knowledge it is so far the best known.

The last two examples are of a little different nature and are less connected to the optimal retraction problem but are close to the classical theory of nonexpansive mappings.

Recall that a mapping T is said to be *nonexpansive* if $T \in \mathcal{L}(1)$ or, in other words, if

$$\|Tx - Ty\| \leq \|x - y\|.$$

If T is nonexpansive, so are all of its iterates T^n, $n = 0, 1, 2, 3, \ldots$. (T^0 = Id). There is a class of mappings sharing a similar property. A mapping T is said to be *uniformly lipschitzian* or T *has the uniform Lipschitz constant* k if there exists $k \geq 1$ such that

$$\|T^n x - T^n y\| \leq k\|x - y\|$$

for $n = 0, 1, 2, \ldots$. The class $\mathcal{UL}$ of uniformly lipschitzian mappings is naturally scaled into subclasses $\mathcal{UL}(k)$. If $T \in \mathcal{UL}(k)$, then defining an equivalent metric by

$$r(x, y) = \sup\{\|T^n x - T^n y\| : n = 0, 1, 2, \ldots\},$$

we observe that

$$r(Tx, Ty) \leq r(x, y),$$

which means that T is nonexpansive with respect to this metric. On the other hand, if $r(\cdot,\cdot)$ is any metric equivalent to the one induced by the norm, which means that there are two constants $a, b > 0$ such that

$$a\|x-y\| \le r(x,y) \le b\|x-y\|,$$

then any r-nonexpansive T satisfies

$$\|T^n x - T^n y\| \le \frac{b}{a}\|x-y\|$$

for $n = 0, 1, 2, \ldots$. Thus the class $\mathcal{UL}$ can be viewed as the class of all mappings nonexpansive with respect to equivalent metrics.

A basic fact about uniformly lipschitzian mappings is the following.

C. *If the space X is uniformly convex, then there exists a constant $\gamma > 1$ such that each mapping $T : B \to B$ of class $\mathcal{UL}(k)$ with $k < \gamma$ has a fixed point.*

This fact has been first observed by W. A. Kirk and the present author [5] not only for balls but for all convex, bounded and closed subsets of X. Then it has been investigated and extended to more general or special cases by many authors. For example, it is known that in the case of Hilbert space all mappings $T \in \mathcal{UL}(k)$ with $k < \sqrt{2}$ have fixed points while there exists a mapping $T : B \to B$, $T \in \mathcal{UL}(\frac{\pi}{2})$ with $d(T) = 0$ but without fixed points (the example has been given by J.-B. Baillon [1]).

Example VI. Let us consider the class $\mathcal{L}_4$ divided into natural subclasses $\mathcal{L}_4(k)$ consisting of all $\mathcal{UL}$ mappings $T : B \to B$. For this setting define as before the function $\psi_{4,X}(k)$. There are two natural constants connected with ψ_4:

$$\gamma_0(X) = \sup\{k : \text{all mappings } T \in \mathcal{L}_4(k) \text{ have fixed points}\}$$

and

$$\begin{aligned}\gamma_1(X) &= \sup\{k : \text{ all mappings } T \in \mathcal{L}_4(k) \text{ satisfy } d(T) = 0\}\\ &= \sup\{k : \psi_{4,X}(k) = 0\}.\end{aligned}$$

Obviously, $\gamma_0(X) \le \gamma_1(X)$ and $\psi_{4,X}(k) = 0$ for $k \le \gamma_1(X)$. However, $\gamma_1(X) < \infty$, and even more, the construction (4) from Example I actually shows that $\lim_{k\to\infty} \psi_{4,X}(k) = 1$.

The results mentioned above say that for uniformly convex (and some other) spaces, $\gamma_0(X) > 1$. Especially for Hilbert spaces H we have $\sqrt{2} \le \gamma_0(H) \le \frac{\pi}{2}$. Practically nothing is known about γ_1. Here are some questions.

Problem 5. *Are there spaces with* $\gamma_0 < \gamma_1$?

Problem 6. *For which spaces* $\gamma_1 = 1$? *In other words, does there exist a space with* $\psi_{4,X}(k) > 0$ *for* $k > 1$?

We hope the answer to this last question can be affirmative but the example is unknown. Finally,

Problem 7. *Can one find a formula for* $\psi_{4,H}(k)$ *or give some good estimations*?

Example VII. Let our last class $\mathcal{L}_5$ consist of all $T : B \to B$ being 2-periodic (*involutions*), which means T satisfying the condition $T^2x = x$ (or $T^2 = \mathrm{Id}$). Obviously, $\mathcal{L}_5 \subset \mathcal{L}_4$ and for each k, $\mathcal{L}_5(k) \subset \mathcal{L}_4(k)$.

Take any $x \in B$ and let $y = (x + Tx)/2$, $T \in \mathcal{L}_5(k)$. We have

$$\|Ty - Tx\| \leq k\|x - y\| = \frac{k}{2}\|x - Tx\|,$$
$$\|Ty - x\| = \|Ty - T^2x\| \leq k\|y - Tx\| = \frac{k}{2}\|x - Tx\|.$$

The above implies

$$\|y - Ty\| \leq \frac{k}{2}\|x - Tx\|.$$

Now, it is a routine observation that if $k < 2$ then the sequence of consecutive iterates $x_0 = x$, $x_{n+1} = (x_n + Tx_n)/2$ converges to a fixed point of T. In other words, all involutions of class $\mathcal{L}_0(k)$ with $k < 2$ have fixed points. For spaces with more regular geometrical structure, uniformly convex spaces or Hilbert spaces, this estimate is even better. For example, it is known that for Hilbert spaces the same holds for $k < \sqrt{\pi^2 - 3} = 2.62\ldots$ (see [6]).

Define now our last function $\psi_{5,X}(k)$ in the usual manner. The above remarks say that $\psi_{5,X}(k) = 0$ for $k < 2$ and even further in regular spaces. Nothing more is known.

Problem 8. *Does there exist a lipschitzian involution* T *of* B *onto* B *without a fixed point? Does there exist such an involution with* $d(T) > 0$?

This last question is connected with the well-known problem of geometric nonlinear functional analysis concerning uniform classification of spheres. It is known that for any infinite-dimensional Banach space X, its unit ball B and its unit sphere S are homeomorphic. The question whether B and S are Lipschitz equivalent is open.

Assume that there exists a homeomorphism $h : B \to S$ such that $h(B) = S$ with $h \in \mathcal{L}(k)$ and $h^{-1} \in \mathcal{L}(k)$ for certain k. Then

$$\|h^{-1}x - h^{-1}y\| \geq \frac{1}{k}\|x - y\|$$

for all $x, y \in S$. Putting

$$Tx = h^{-1}(-hx),$$

we obtain an involution of class $\mathcal{L}(k^2)$ for which

$$\|x - Tx\| = \|x - h^{-1}(-hx)\| = \|h^{-1}(hx) - h^{-1}(-hx)\|$$

$$\geq \frac{1}{k}\|2hx\| = \frac{2}{k}.$$

Thus $d(T) \geq 2/k$ and we would have

$$k\psi_{5,X}(k^2) \geq 2$$

similarly as in Example III.

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