

**ON AN EXTENSION OF THE KATO-VOIGT PERTURBATION
THEOREM FOR SUBSTOCHASTIC SEMIGROUPS
AND ITS APPLICATION**

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Abstract. The aim of this paper is to discuss some recent developments of the perturbation method introduced first by Kato for the Kolmogoroff equation and later extended by Voigt and Arlotti to deal with a range of problems related to the solvability of the so-called Master Equation. The paper consists of two parts. In the first we recall and unify some abstract results on generation of substochastic semigroups. In the second we perform a detailed analysis of an equation from the polymer degradation theory to demonstrate a number of possible generation cases.

1. INTRODUCTION

In many branches of the applied sciences we are interested in the evolution of the density function $(t, \xi) \mapsto u(t, \xi)$, where t is the time and $\xi \in \Omega$ is an element of some state space – it could be the velocity or energy in the kinetic theory, the number of bacteria in a sample in the population theory, mass in the coagulation-fragmentation theory. The function u is then interpreted as the probability (density) of finding an individual which at the time t has the property ξ . If we assume that the probability at time t is completely determined by the probability at an earlier time t_0 (that is, we are dealing with a Markov process), then the time evolution of u is described by the master equation (e.g., [11])

$$(1.1) \quad \begin{aligned} \partial_t u(t, \xi) &= \int_{\Omega} [k(\xi', \xi)u(t, \xi') - k(\xi, \xi')u(t, \xi)] d\mu_{\xi'} \\ &= \int_{\Omega} k(\xi', \xi)u(t, \xi') d\mu_{\xi'} - u(t, \xi) \int_{\Omega} k(\xi, \xi') d\mu_{\xi'}, \end{aligned}$$

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where $d\mu_\xi$ is an appropriate measure in the state space, and $k(\xi', \xi)$ is the transition rate (probability density of transitions per unit time) of the change between the states ξ' and ξ (that is, the probability of the change from ξ' to ξ to occur in the time interval dt is approximately $k(\xi', \xi)d\xi dt$).

An intrinsic property of the above process is that all the particles must be accounted for or, in other words, the total number of particles doesn't change:

$$(1.2) \quad \int_{\Omega} u(t, \xi) d\mu_\xi = \int_{\Omega} u(0, \xi) d\mu_\xi,$$

for any time t . Therefore from the physical point of view, the natural spaces for studying such problems are L_1 spaces (or l_1 if ξ takes only discrete values). Equation (1.2) can be checked formally by integrating (1.1) and using the symmetry of the transition rate.

The process discussed above (or a combination of such processes) may take place simultaneously with an evolution in the physical space so that in general we will be concerned with evolution equations of the form

$$(1.3) \quad \partial_t u = A_0 u + A_1 u + B u,$$

where A_0 typically is the free streaming operator: $A_0 u = -\mathbf{v} \cdot \nabla_x u$, or the diffusion operator, whereas $A_1 u = -m u$ with a nonnegative, measurable, and almost everywhere finite function m , and B is an integral operator.

The first equation of this type, namely the Kolmogoroff equation, was first studied in the semigroup theory framework by Kato in his seminal paper [6]. His results were extended to a more abstract framework and extensively applied to a range of problems arising mainly in the kinetic theory by Voigt, Arlotti, Desch, Mokhtar-Karroubi, and recently by the author (see, e.g., [13, 1, 2, 9, 3, 4]).

An interesting feature of these problems is that the semigroup $(G(t))_{t \geq 0}$ solving the Cauchy problem related to (1.3) is constructed by a limiting procedure which doesn't allow to control the domain of the generator; in the general case one can only prove that the generator T of $(G(t))_{t \geq 0}$ is an extension of $A_0 + A_1 + B$. In most applications it is not enough, as typically we are interested in the cases when the solution semigroup $(G(t))_{t \geq 0}$ is a transition (or stochastic) semigroup, that is, when $\|G(t)x\| = \|x\|$ for all $t \geq 0$ and $x \geq 0$ (see also Remark 2.1). In a number of applications (see [6]) this could happen if and only if $T = \overline{A_0 + A_1 + B}$; generally the last equality is a sufficient condition for $(G(t))_{t \geq 0}$ to be stochastic. Thus it is important to be able to determine when $T = \overline{A_0 + A_1 + B}$. We shall present some sufficient conditions, which are modifications of those introduced in [2], and apply them to the fragmentation model [14, 8] which will be introduced in Section 4. We shall see that for some values of the parameter the generator T of the semigroup is the closure of the operator appearing on the right-hand side of the equation (and

that this result is sharp, that is, $T \neq A_0 + A_1 + B$). On the other hand, in the last section of the paper we shall demonstrate that for a range of parameter values the generator T is a proper extension of $\overline{A_0 + A_1 + B}$ and that the semigroup generated by T is not stochastic.

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2. ABSTRACT GENERATION RESULTS

In this section, we shall formulate and provide proofs of several results which are scattered in the literature [13, 1, 2], and which are very often considered only in a specific context.

Let (Ω, μ) be a measure space. By X we denote the Banach space $L_1(\Omega, \mu)$ endowed with the standard norm $\|\cdot\|$. For any subspace $Z \subset X$, by Z_+ we denote the cone of nonnegative (a.e.) elements of Z . Let $(G(t))_{t \geq 0}$ be a strongly continuous semigroup on X . We say that $(G(t))_{t \geq 0}$ is a *substochastic semigroup* if for each $t \geq 0$, $G(t) \geq 0$ and $\|G(t)\| \leq 1$. It is called a *stochastic semigroup* if additionally $\|G(t)f\| = \|f\|$ for X_+ .

In accordance with the discussion in the introduction, we shall consider two linear operators in X : $A (= A_0 + A_1)$ and B , which have the following properties:

1. $(A, D(A))$ generates a substochastic semigroup denoted by $(G_A(t))_{t \geq 0}$,
2. $D(B) \supset D(A)$ and $Bf \geq 0$ for any $f \in D(B)_+$,
3. for any $f \in D(A)_+$,

$$(2.1) \quad \int_{\Omega} (Af + Bf) d\mu \leq 0.$$

Let us observe that the above list yields the following properties of the operators involved.

Lemma 2.1. (i) *The operator $B(\lambda I - A)^{-1}$ is a bounded positive operator in X .*

(ii) *If $Af = -mf$ for some a.e. positive measurable m , then for any $f \in D(A)$,*

$$(2.2) \quad \|Bf\| \leq \|Af\|.$$

Proof. (i) Since A generates a contraction semigroup, the resolvent is well-defined for $\lambda > 0$. From the properties of B we see that $B(\lambda I - A)^{-1}$ is defined on X and positive, and hence is bounded [12, Theorem II. 5.3].

(ii) From (2.1) we obtain for $f \in D(A)_+$,

$$\|Bf\| = \int_{\Omega} Bf d\mu = \int_{\Omega} mf d\mu = \|Af\|.$$

Let us take arbitrary $f \in D(A)$, then also $|f| \in D(A)$, $|Bf| \leq B|f|$ and

$$\|Bf\| \leq \|B|f|\| = \int_{\Omega} B|f| d\mu = \int_{\Omega} m|f| d\mu = \int_{\Omega} |mf| d\mu = \|Af\|. \quad \blacksquare$$

The next lemma is a minor generalization of Lemma 1.2 of [13], and is also more general than Lemma 1 of [1].

Lemma 2.2. *For any $f \in D(A)$, the function $t \mapsto BG_A(t)f$ is continuous and*

$$(2.3) \quad \int_0^t \|BG_A(s)f\| ds \leq \|f\| - \|G_A(t)f\|.$$

Proof. For $f \in D(A)$ it follows as in the proof of Lemma 1.2 (a) in [13] that

$$\int_0^t \|BG_A(s)f\| ds \leq \| |f| \| - \|G_A(t)|f|\|.$$

However, $\| |f| \| = \|f\|$ and, since $G_A(t)|f| \geq |G_A(t)f|$, we obtain $-\|G_A(t)|f|\| \leq -\|G_A(t)f\|$, so that the lemma is proved. \blacksquare

The following gives an abstract version of Theorem 4 of [1].

Theorem 2.1. *Under the above assumptions, there exists a smallest substochastic semigroup $(G(t))_{t \geq 0}$ generated by an extension T of $A + B$. This semigroup satisfies the integral equation*

$$(2.4) \quad G(t)f = G_A(t)f + \int_0^t G(t-s)BG_A(s)f ds$$

for any $f \in D(A)$ and $t \geq 0$, and can be also obtained by the Phillips-Dyson expansion

$$(2.5) \quad G(t)f = \sum_{n=0}^{\infty} S_n(t)f, \quad f \in X,$$

where the iterates $S_n(t)$ are defined through (2.7).

Proof. In [13] the author proved, generalizing the idea of Kato [6], that the semigroup $(G(t))_{t \geq 0}$ can be obtained as a strong limit of the semigroups $(G_r(t))_{t \geq 0}$ generated by $A+rB$, when $0 < r \rightarrow 1^-$. Moreover, if $r \leq r'$, then $G_r(t) \leq G_{r'}(t)$ for each t . For $r < 1$, the operator rB is a Miyadera perturbation of A with bound r , and therefore

$$(2.6) \quad G_r(t)f = G_A(t)f + r \int_0^t G_r(t-s)BG_A(s)f ds.$$

Since also $rG_r(t) \leq r'G_{r'}(t)$ if $r \leq r'$, using the Lebesgue monotone convergence theorem (in $L_1(\Omega \times [0, t])$) we ascertain that the right-hand side of (2.6) converges to $G_A(t)f + \int_0^t G(t-s)BG_A(s)f ds$ as $r \rightarrow 1^-$ for all $f \in X_+$, thus, by linearity, for all $f \in X$. This proves (2.6).

To obtain the Phillips-Dyson expansion of the semigroup $(G(t))_{t \geq 0}$, we define recursively the following operators:

$$(2.7) \quad \begin{aligned} S_0(t)f &= G_A(t)f, \\ S_n(t)f &= \int_0^t S_{n-1}(t-s)BG_A(s)f ds, \quad n > 0, \end{aligned}$$

for $f \in D(A)$ and $t \geq 0$. It follows that the Dyson-Phillips iterates for the perturbation rB are given by $rS_n(t)$, thus, as in the Miyadera-Voigt theory, the families $\{S_n(t)\}_{t \geq 0}$ can be extended to strongly continuous families of bounded (contractive) positive operators in X . Moreover, for any $0 < r < 1$ the semigroup $(G_r(t))_{t \geq 0}$ is given by

$$G_r(t)f = \sum_{n=0}^{\infty} r^n S_n(t)f,$$

where the series is uniformly convergent on bounded t -intervals. Since $S_n(t)$ are positive, for $f \in X_+$ and $0 < r' \leq r < 1$ we have $r'^n S_n(t)f \leq r^n S_n(t)f$, thus

by the monotone convergence theorem (in $l^1(X)$) the series above is convergent to $\sum_{n=0}^{\infty} S_n(t)f$ in X . Thus for all $f \in X_+$ we have

$$G(t)f = \sum_{n=0}^{\infty} S_n(t)f.$$

The extension by linearity gives (2.5). ■

Remark 2.1. The drawback of Theorem 2.1 is that it doesn't provide any characterization of the domain of the generator T . The ideal situation would be, of course, if $D(T) = D(A) = D(A+B)$. However, as we mentioned in the introduction, the case when $T = \overline{A+B}$ is also physically acceptable since in such a case the semigroup is still stochastic provided $A+B$ is formally conservative, that is, the assumption (2.1) is replaced by

$$\int_{\Omega} (A+B)u d\mu = 0$$

for $u \in D(A+B) = D(A)$. If $T = \overline{A+B}$, then for $u \in D(T)$ there exists a sequence $(u_n)_{n \in \mathbb{N}}$ of elements of $D(A)$ such that $u_n \rightarrow u$ and $(A+B)u_n \rightarrow Tu$ in X as $n \rightarrow \infty$. Thus

$$\int_{\Omega} T u d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} (A+B)u_n d\mu = 0.$$

This in turn shows that if $0 \leq \overset{\circ}{u} \in D(T)$, then $0 \leq u(t) = G(t)\overset{\circ}{u} \in D(T)$ for any $t \geq 0$ and

$$\frac{d}{dt} \|u(t)\| = \int_{\Omega} \frac{du(t)}{dt} d\mu = \int_{\Omega} T G(t) \overset{\circ}{u} u d\mu = 0.$$

On the other hand, if T is a bigger extension of $A+B$, then the above property may not hold and there may be a loss of particles in the evolution. We shall see that such a situation is possible in Proposition 5.2. ■

The next proposition is related to the above discussion. We note that for a particular case of Kolmogoroff's equation similar results, though through a different method, have been proved in [6].

Proposition 2.1. *Under the assumption of this section we have:*

(i) the generator T is characterized by

$$(2.8) \quad (I - T)^{-1}f = \sum_{n=0}^{\infty} (I - A)^{-1}[B(I - A)^{-1}]^n f$$

for every $f \in X$.

(ii) If $\sum_{n=0}^{\infty} [B(I - A)^{-1}]^n f$ converges for any $f \in X$, then $T = A + B$.

(iii) If $\lim_{n \rightarrow \infty} \|[B(I - A)^{-1}]^n f\| = 0$ for any $f \in X$, then $T = \overline{A + B}$.

Proof. (i) Taking the Laplace transform of (2.5) with $\lambda = 1$, we obtain

$$(2.9) \quad (I - T)^{-1}f = \sum_{n=0}^{\infty} \mathcal{L}(S_n(t)f)(1).$$

Now, for $f \in D(A)$,

$$\mathcal{L}(S_n(t)f)(1) = \mathcal{L}(S_{n-1}(t)f)(1)\mathcal{L}(BG_A(t)f)(1),$$

where we used Corollary C.17 of [5] (applicable by Lemma 2.2). The next step requires some care as we don't know whether B is a closed operator. However, using the boundedness of $B(I - A)^{-1}$ we have

$$\mathcal{L}(BG_A(t)f)(1) = \mathcal{L}(B(I - A)^{-1}G_A(t)(I - A)f)(1) = B(I - A)^{-1}f.$$

Therefore, for any $f \in D(A)$,

$$\mathcal{L}(S_0(t)f)(1) = (I - A)^{-1}f,$$

$$\mathcal{L}(S_n(t)f)(1) = \mathcal{L}(S_{n-1}(t))B(I - A)^{-1}f = (I - A)^{-1}[B(I - A)^{-1}]^n f,$$

and since $\mathcal{L}(S_n(t)f)(1)$ is a bounded operator, we can extend the above to the whole of X . Combining this with (2.9), we obtain (2.8).

(ii) If the series converges in X , then we clearly have

$$(I - T)^{-1}f = \sum_{n=0}^{\infty} (I - A)^{-1}[B(I - A)^{-1}]^n f = (I - A)^{-1} \sum_{n=0}^{\infty} [B(I - A)^{-1}]^n f,$$

which shows that $D(A) \supset D(T)$. Since $D(A) \subset D(T)$, we have the equality.

(iii) Assume that for any $f \in X$, $\|(B(I - A)^{-1})^n f\| \rightarrow 0$ as $n \rightarrow \infty$. Denote

$$z_N = (I - A)^{-1} \sum_{n=0}^N (B(I - A)^{-1})^n f \in D(A).$$

Clearly, $z_N \rightarrow z = (I - T)^{-1}f \in D(T)$ in X . Now,

$$\begin{aligned} (I - (A + B))z_N &= \sum_{n=0}^N (B(I - A)^{-1})^n f - B(I - A)^{-1} \sum_{n=0}^N (B(I - A)^{-1})^n f \\ &= \sum_{n=0}^N (B(I - A)^{-1})^n f - \sum_{n=1}^{N+1} (B(I - A)^{-1})^n f \\ &= f - (B(I - A)^{-1})^N f. \end{aligned}$$

Thus, if the assumption is satisfied, then $(I - (A + B))z_N$ converges in X and consequently $z \in D(\overline{A + B})$. Since z is an arbitrary member of $D(T)$, we obtain $D(T) \subset D(\overline{A + B})$. On the other hand, T , as a generator, is a closed extension of $A + B$, and thus $\overline{A + B} \subset T$. ■

From this theorem we see that the extension T of $A + B$ generates a substochastic semigroup. In particular, this is a semigroup of contractions and, consequently, T is a dissipative operator. However, any restriction of a dissipative operator is also dissipative, and thus $A + B$ is dissipative. This allows a simple corollary.

Corollary 2.1. *Under the assumptions of Theorem 2.1, we have:*

1. $T = \overline{A + B}$ if and only if the range of $\lambda - (A + B)$, $Rg(\lambda - (A + B))$, is dense in X for some (all) $\lambda > 0$.
2. If additionally $A + B$ is a closed operator, then $T = A + B$.

Proof. 1) Let us assume that $Rg(\lambda - (A + B))$ is dense in X . By [5, Theorem 3.15], (Lumer–Phillips Theorem), this statement is equivalent to saying that $\overline{A + B}$ is a generator (that $A + B$ is densely defined follows from the fact that A itself is a generator). Since T a closed extension of $A + B$ (as a generator), we must have $\overline{A + B} \subset T$. Since both are generators, they must coincide. The converse is obvious.

Item 2 follows immediately. ■

3. SUFFICIENT CONDITION FOR $T = \overline{A + B}$

In this section we shall sketch an approach of [2] (with some modification due to the author, first given for a specific situation in [4]) which enables a better characterization of the generator T .

As the first step we extend all the discussed operators in the following way.

Let E denote the set of all the extended real-valued measurable functions defined on Ω ; clearly $X \subset E$. We define the subset $F \subset E$ by the following condition: $f \in F$ if and only if for every nonnegative and nondecreasing sequence of functions $(f_n)_{n \in \mathbb{N}}$ satisfying $\sup_n f_n = |f|$ we have $\sup_n (I - A)^{-1} f_n \in X$.

Before proceeding any further we adopt the assumption that $f \in D(B)$ if and only if $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ both belong to $D(B)$. Through B we construct another subset of E , say G , defined as the set of all functions $f \in X$ such that for any nonnegative, nondecreasing sequence $(f_n)_{n \in \mathbb{N}}$ of elements of $D(B)$ such that $\sup_n f_n = |f|$, we have $\sup_n Bf_n < +\infty$ almost everywhere. It is easy to check that $D(A) \subseteq G \subseteq X \subseteq F \subseteq E$.

It can be proved that we can correctly define the mapping $L : F_+ \rightarrow X_+$ by

$$Lf = \sup_n (I - A)^{-1} f_n,$$

where $0 \leq f_n \leq f_{n+1}$ for any n , and $\sup_n f_n = f$.

To proceed, in [2] the assumptions (1)–(3) of Section 2 were supplemented by two more:

(4) The mapping L is injective.

(5) If $f \in G_+$ and $(f'_n)_{n \in \mathbb{N}}$, $(f''_n)_{n \in \mathbb{N}}$ are two nondecreasing sequences of elements of $D(A)_+$ satisfying $f = \sup_n f'_n = \sup_n f''_n$, then $\sup_n Bf'_n = \sup_n Bf''_n$. The common value will be denoted by Bf .

We extend the mappings L and B onto F and G , respectively, by linearity.

The above construction of L is not always easy to apply. We shall interpret it from the point of view of Sobolev towers (see, e.g., [5, Section II.5]). This interpretation will also show that the assumption (4) above is superfluous.

Thus, according to [5], we define the space X^{-1} as the completion of X with respect to the norm

$$\|f\|_{-1} = \|(I - A)^{-1} f\|_X.$$

Then the semigroup $(G_A(t))_{t \geq 0}$ extends continuously to a semigroup $(G_{A,-1}(t))_{t \geq 0}$ in X^{-1} , which is generated by the closure of A in X^{-1} . This closure, denoted by A_{-1} , is defined on the domain $D(A_{-1}) = X \subset X_{-1}$. The resolvent extends then by density to a bounded one-to-one operator $R(\lambda, A_{-1}) : X^{-1} \rightarrow X^{-1}$ with the range exactly equal to X . We have the following lemma.

Lemma 3.1. *The operator L is a restriction of $R(1, A_{-1})$. As a consequence, the assumption (4) is satisfied.*

Proof. Let $g \in X_+$ satisfy $g = Lf$. This means that $g = \sup_n (I - A)^{-1} f_n$ for a nondecreasing sequence of nonnegative functions $f_n \in X_+$ such that $\sup_n f_n = f$.

Since $(I - A)^{-1}$ is a positive operator, the sequence $((I - A)^{-1}f_n)_{n=0}^{\infty}$ is also nondecreasing and $g \geq (I - A)^{-1}f_n$ for any $n \in \mathbb{N}$. Because g is integrable, we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} (I - A)^{-1}f_n d\mu = \int_{\Omega} g d\mu$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} |g - (I - A)^{-1}f_n| = \int_{\Omega} g d\mu - \lim_{n \rightarrow \infty} \int_{\Omega} (I - A)^{-1}f_n d\mu = 0.$$

This shows that $((I - A)^{-1}f_n)_{n \in \mathbb{N}}$ converges in X and therefore $g = \mathbb{L}f = R(1, A_{-1})f$. The extension for arbitrary f is done by linearity. ■

Corollary 3.1. *If $Af = -mf$, where m is a nonnegative, measurable, and almost everywhere finite function, then*

$$(3.1) \quad \mathbb{F} = X^{-1} = \{f \in \mathbb{E}; (1 + m)^{-1}f \in X\},$$

and $\mathbb{L}f = (1 + m)^{-1}f$.

Proof. Since $(I - A)^{-1}f = (1 + m)^{-1}f$, by the definition of \mathbb{F} , $f \in \mathbb{F}$ provided $\sup_n (1 + m)^{-1}f_n \in X$ for any nondecreasing sequence of nonnegative functions f_n such that $\sup_n f_n = |f|$. ■

For $h \in D(T)$, let us denote $g = (I - T)h \in X$. Using (2.8) and replacing the operators involved there by their extensions, we see that h can be expressed by

$$(3.2) \quad h = \sum_{k=0}^{\infty} \mathbb{L}(\mathbb{B}\mathbb{L})^k g.$$

Let us denote, for any $g \in X$,

$$f_n = \sum_{k=0}^n (\mathbb{B}\mathbb{L})^k g$$

and

$$h_n = \mathbb{L}f_n.$$

We note that for a nonnegative g we can consider limits of both sequences in the sense of monotonic convergence almost everywhere, as \mathbb{L} and \mathbb{B} are positive operators. We denote by f and h the respective limits provided they exist. In this case we obtain $\mathbb{L}f = h$. The result of [2] which we shall apply here reads as follows.

Theorem 3.1. *Let all the above assumptions be satisfied. If for any $g \in X_+$ the limit of the following real convergent sequence satisfies*

$$(3.3) \quad \lim_{n \rightarrow \infty} \int_{\Omega} A_0 h_n d\mu \leq \int_{\Omega} (h - f + Bh) d\mu,$$

then $T = \overline{A + B}$. ■

This theorem is not immediately useful as the function h is not explicitly known. However, the following corollary is useful in many instances.

Corollary 3.2. *If $Af = A_1f = -mf$ for some measurable, nonnegative, and almost everywhere finite function m , and for any $H \in X_+$ such that $-mH + BH \in X$ we have*

$$(3.4) \quad \int_{\Omega} (-mH + BH) d\mu \geq 0,$$

then $T = \overline{A + B}$.

Proof. Inequality (3.4) coincides with (3.3) re-written in the context of this corollary. Hence if we prove then that any h defined by (3.2) satisfies the assumptions imposed on H , then (3.4) will yield (3.3). Clearly, $h \in X_+$. Next, since $f_n \in X$, we have $h_n \in D(A) \subset D(B)$ and $Bh_n = f_{n+1} - g$. Thus $\sup_n Bh_n = f - g$. On the other hand, we have $f = \sup_n f_n$ and $h = \sup_n h_n = \sup_n (I - A)^{-1} f_n = Lf \in X$. Thus $f \in F = X^{-1}$ and from the assumption on m it follows that f is bounded almost everywhere. This implies that $h \in G$ and $Bh = f - g$. Therefore, $-mh + Bh = h - f + f - g = h - g \in X$. Hence, if (3.4) is satisfied for any H stipulated in the corollary, then (3.3) is satisfied for any h defined by (3.2) and the proof is complete. ■

Remark 3.1. The theory sketched in the previous two sections has been successfully applied to a number of problems ranging from the linear Boltzmann equation with external field in the context of both standard [1, 2] and extended [3] kinetic theory, Kolmogoroff's and cell growth equations [2] and kinetic-diffusion equations arising in the theory of hydrodynamic limits in the extended kinetic theory [4]. In the next section we shall apply it to the fragmentation equation.

4. FRAGMENTATION MODEL

An equation of a simple fragmentation model (polymer degradation) [14] can be written as:

$$(4.1) \quad \partial_t u = K_\alpha u = A_\alpha u + B_\alpha u,$$

where, for $x > 0$,

$$A_\alpha u = -x^\alpha u(x)$$

and

$$(B_\alpha u)(x) = 2 \int_x^\infty y^{\alpha-1} u(y) dy.$$

Here u is the number density of particles with mass x . The parameter α is an arbitrary real number. The problem splits into three distinct cases depending on whether $\alpha = 0$, $\alpha < 0$, $\alpha > 0$. When $\alpha = 0$, the operators K_α and B_α are bounded and the problem is easy; one can obtain even the closed form solution to (4.1) (cf. [7]). Therefore in the analysis below we shall always assume that $\alpha \neq 0$.

The basic space in our considerations will be the space

$$X_1 = L_1([0, \infty[, x dx)$$

with the natural norm

$$\|f\| = \int_0^\infty x |f(x)| dx.$$

The natural domain for K_α is then

$$D(K_\alpha) = D(A_\alpha) = D_\alpha = \{f \in X_1; x^\alpha f \in X_1\}.$$

The reason for the introduction of the space X_1 is that for $0 \leq u \in D_\alpha$,

$$\begin{aligned} \int_0^\infty (B_\alpha u)(x) x dx &= 2 \int_0^\infty x \left(\int_x^\infty u(y) y^{\alpha-1} dy \right) dx \\ &= 2 \int_0^\infty y^{\alpha-1} u(y) \left(\int_0^y x dx \right) dy = \int_0^\infty (-A_\alpha u)(x) x dx. \end{aligned}$$

This equation in the present context expresses the mass conservation law.

We see immediately that we can apply Theorem 2.1 to get the following result.

Theorem 4.1. *Let $\alpha \in \mathbb{R}$ be arbitrary. There exists a smallest substochastic semigroup $(G_\alpha(t))_{t \geq 0}$ in X_1 generated by an extension T_α of K_α . ■*

Below we shall prove that the generator of $(G_\alpha(t))_{t \geq 0}$ coincides with the closure of K_α if $\alpha > 0$ and identify the generator (which turns out to be a proper extension of $\overline{K_\alpha}$) for $\alpha < 0$.

Since the operator A_α is the multiplication by $-x^\alpha$, we can apply Corollary 3.2. Thus the space F coincides with the first associated space of the Sobolev tower, that is, explicitly

$$F = X^{-1} = L_1(\mathbb{R}_+, x(1+x^\alpha)^{-1}dx).$$

The set G is the set of functions $f \in X_1$ for which the function

$$F(x) = (B_\alpha f)(x) = 2 \int_x^\infty y^{\alpha-1} f(y) dy$$

is finite almost everywhere. Clearly, the only point where $F(x)$ could be infinite is $x = 0$. Now Corollary 3.2 reads that if for any $0 \leq H \in X_1$ such that $-x^\alpha H + B_\alpha H \in X_1$ we have

$$\int_0^\infty (-x^\alpha H + B_\alpha H) x dx \geq 0,$$

then $T_\alpha = \overline{K_\alpha}$. Technical details relevant to the problem at hand are given below.

Proposition 4.1. *If $\alpha > 0$, then $T_\alpha = \overline{K_\alpha}$.*

Proof. Since $H \in X_1$, $-x^\alpha H \in L_1([0, n], x dx)$ for any n and, since by assumption $-x^\alpha H + B_\alpha H \in X_1$, we have also $B_\alpha H \in L_1([0, n], x dx)$. Therefore

$$\begin{aligned} & \int_0^\infty (-x^\alpha H(x) + (B_\alpha H)(x)) x dx \\ &= \lim_{n \rightarrow \infty} \int_0^n (-x^\alpha H(x) + (B_\alpha H)(x)) x dx \\ &= \lim_{n \rightarrow \infty} \left(\int_0^n (-x^\alpha H(x)) x dx + 2 \int_n^\infty \left(\int_0^n x dx \right) y^{\alpha-1} H(y) dy \right. \\ & \quad \left. + 2 \int_0^n \left(\int_0^y x dx \right) y^{\alpha-1} H(y) dy \right) \\ &= \lim_{n \rightarrow \infty} n^2 \int_n^\infty y^{\alpha-1} H(y) dy. \end{aligned}$$

Since the limit is finite by the definition of H , it must be nonnegative and the statement is proved. ■

If we look now at the case $\alpha < 0$, then a similar argument shows that

$$\begin{aligned} & \int_0^{\infty} (-x^\alpha H(x) + (B_\alpha H)(x)) x dx \\ &= \lim_{\epsilon \rightarrow 0} \int_\epsilon^{\infty} (-x^\alpha H(x) + (B_\alpha H)(x)) x dx \\ &= \lim_{\epsilon \rightarrow \infty} \left(\int_\epsilon^{\infty} (-x^\alpha H(x)) x dx + 2 \int_\epsilon^{\infty} \left(\int_\epsilon^y x dx \right) y^{\alpha-1} H(y) dy \right) \\ &= - \lim_{\epsilon \rightarrow \infty} \epsilon^2 \int_\epsilon^{\infty} y^{\alpha-1} H(y) dy. \end{aligned}$$

Here, unfortunately, the limit can be either zero or negative, which is inconclusive. Although it is possible to find a function H satisfying the assumptions of Corollary 3.2, and for which the above limit is strictly negative (H should behave as 1 when x is close to zero), we don't know whether such a function belongs to $D(T)$.

Remark 4.1. This short section shows some advantages as well as limitations of the method presented earlier in this paper. Apart from the last inconclusive result, we don't know whether Proposition 4.1 gives the best answer – in principle it is still possible that $T_\alpha = K_\alpha$. Fortunately, for the model discussed in this section it is possible to find explicitly the resolvent of the generator and study it directly to find answers to all the relevant questions. As we shall see in the next section, the results obtained here are the best possible: for $\alpha > 0$ indeed $T_\alpha = \overline{K_\alpha} \neq K_\alpha$, and for $\alpha < 0$ the generator T_α is a proper extension of $\overline{K_\alpha}$.

5. DIRECT STUDY OF THE FRAGMENTATION MMODEL

For reasons which will become clear later (see also the proof of Theorem 2.1), we shall consider a more general operator

$$K_{\alpha,r} = A_\alpha + rB_\alpha$$

for $0 < r \leq 1$. From the preceding theory, for any α and $0 < r \leq 1$ there is an extension $T_{\alpha,r}$ of $K_{\alpha,r}$ which is the generator of a semigroup, say, $(G_{\alpha,r}(t))_{t \geq 0}$.

We shall adopt the convention that for any operator $C_{\alpha,r}$ depending on r , $C_{\alpha,1} = C_\alpha$.

From Section 2 it follows that for $r < 1$, rB_α is a Miyadera perturbation of A_α and thus the Miyadera-Voigt theory implies that $T_{\alpha,r} = K_{\alpha,r}$, with the domain D_α , generating $(G_{\alpha,r}(t))_{t \geq 0}$.

Our next aim is to identify the resolvent of $T_{\alpha,r}$. We start by solving the equation

$$\lambda u + x^\alpha u - 2r \int_x^\infty y^{\alpha-1} u(y) dy = f.$$

Proceeding at this moment purely formally, we differentiate the above equation with respect to x to get

$$(\lambda + x^\alpha)u' + (\alpha + 2r)x^{\alpha-1}u = f',$$

and, after standard manipulations, we obtain

$$(5.1) \quad u = \frac{f}{\lambda + x^\alpha} + \frac{1}{(\lambda + x^\alpha)^{1+2r/\alpha}} \left(C - 2r \int_0^x f(s)(\lambda + s^\alpha)^{-1+2r/\alpha} s^{\alpha-1} ds \right).$$

The multiplying function $(\lambda + x^\alpha)^{-1}$ is bounded for any α . Precisely, we have for $\alpha > 0$,

$$(5.2) \quad (\lambda + x^\alpha)^{-1} = \begin{cases} O(x^{-\alpha}) & \text{as } x \rightarrow +\infty, \\ O(1) & \text{as } x \rightarrow 0^+, \end{cases}$$

and for $\alpha < 0$

$$(5.3) \quad (\lambda + x^\alpha)^{-1} = \begin{cases} O(x^{-\alpha}) & \text{as } x \rightarrow 0^+, \\ O(1) & \text{as } x \rightarrow +\infty. \end{cases}$$

Let us also write down the asymptotics of the kernel of the integral operator. Since we know that xf is integrable, we shall rather investigate $(\lambda + x^\alpha)^{-1+2r/\alpha} x^{\alpha-2}$. We have for $\alpha > 0$,

$$(5.4) \quad (\lambda + x^\alpha)^{-1+2r/\alpha} x^{\alpha-2} = \begin{cases} O(x^{2(r-1)}) & \text{as } x \rightarrow +\infty, \\ O(x^{\alpha-2}) & \text{as } x \rightarrow 0^+, \end{cases}$$

and for $\alpha < 0$

$$(5.5) \quad (\lambda + x^\alpha)^{-1+2r/\alpha} x^{\alpha-2} = \begin{cases} O(x^{\alpha-2}) & \text{as } x \rightarrow +\infty, \\ O(x^{2(r-1)}) & \text{as } x \rightarrow 0^+. \end{cases}$$

Hence, the first term in (5.1) doesn't create any problems. Let us consider first the case with $\alpha > 0$. By (5.2), the multiplying function $(\lambda + x^\alpha)^{-1-2r/\alpha}$ is $O(x^{-\alpha-2r})$ as $x \rightarrow \infty$, and bounded as $x \rightarrow 0^+$. Since u should be integrable with the weight

$x^{1+\alpha}$, we see that it is necessary to require that $C = 2 \int_0^\infty f(s)(\lambda + s^\alpha)^{-1+2r/\alpha} s^{\alpha-1} ds$ and the solution takes the form

$$(5.6) \quad u = \frac{f}{\lambda + x^\alpha} + \frac{2r}{(\lambda + x^\alpha)^{1+2r/\alpha}} \int_x^\infty f(s)(\lambda + s^\alpha)^{-1+2r/\alpha} s^{\alpha-1} ds.$$

Let us suppose now that $\alpha < 0$. The multiplying function $(\lambda + x^\alpha)^{-1-2r/\alpha}$ is $O(x^{-\alpha-2r})$ as $x \rightarrow 0^+$, and bounded away from zero as $x \rightarrow +\infty$. We have to have $u \in X_1$, and thus as above the only way to ensure this is to make the bracket in (5.1) vanish at infinity, which produces the same solution given by (5.6).

Thus we introduce the operators

$$(5.7) \quad R_{\alpha,r}(\lambda)f = R_1(\lambda)f + R_2(\lambda)f = \frac{f}{\lambda + x^\alpha} + \frac{2r}{(\lambda + x^\alpha)^{1+2r/\alpha}} \int_x^\infty f(s)(\lambda + s^\alpha)^{-1+2r/\alpha} s^{\alpha-1} ds.$$

In all the considerations below we assume that $\lambda > 0$.

Lemma 5.1. *The operators $R_{\alpha,r}(\lambda)$ extend to bounded operators on X_1 .*

Proof. We need to focus only on R_2 . Changing the order of integration, we obtain

$$\|R_2(\lambda)f\| \leq \int_0^\infty \left(\int_0^s \frac{x}{(\lambda + x^\alpha)^{1+2r/\alpha}} dx \right) |sf(s)| (\lambda + s^\alpha)^{-1+2r/\alpha} s^{\alpha-2} ds.$$

We shall analyse the behaviour of the inner integral for cases $\alpha > 0$ and $\alpha < 0$ separately. Let first $\alpha > 0$. Then the inner integrand behaves as x close to zero and as $x^{-\alpha-2r+1}$ as $x \rightarrow \infty$. Thus for small s the inner integral is of order of s^2 , and since by (5.4) the kernel $(\lambda + s^\alpha)^{-1+2r/\alpha} s^{\alpha-2}$ behaves there as $s^{\alpha-2}$, we obtain the integrability at zero. For large s , the inner integral is of order of $s^{-\alpha-2r+2}$ and since the kernel behaves as s^{2r-2} , we obtain that the integral is finite.

Let us look now at $\alpha < 0$. The inner integrand is of order of $x^{-\alpha-2r+1}$ as x is close to zero, and thus the integral is of order of $s^{-\alpha-2r+2}$. Since by (5.5) the kernel is of order of s^{2r-2} at zero, the product is of order of $s^{-\alpha}$ and bounded, as $\alpha < 0$; thus we obtain the integrability at zero. Next, for large x the inner integrand behaves as x ; thus the integral increases as s^2 , but the kernel is of order of $s^{\alpha-2}$. Hence again the product is bounded and the integral is finite. ■

Lemma 5.2. *For each $0 < r \leq 1$ and all α , the operator $R_{\alpha,r}(\lambda)$ is a left inverse of $\lambda - K_{\alpha,r}$.*

Proof. We have to prove that for any $u \in D_\alpha$ we have $R_{\alpha,r}(\lambda)(\lambda + A_\alpha - rB_\alpha)u = u$. Clearly, $R_1(\lambda)(\lambda + A_\alpha)u = u$. After some calculations, we obtain

$$\begin{aligned} -rR_2(\lambda)B_\alpha u &= \frac{-4r^2}{(\lambda + x^\alpha)^{1+2r/\alpha}} \int_x^\infty s^{\alpha-1}(\lambda + s^\alpha)^{-1+2r/\alpha} \left(\int_s^\infty y^{\alpha-1}u(y)dy \right) ds \\ &= \frac{-2r}{(\lambda + x^\alpha)^{1+2r/\alpha}} \int_x^\infty y^{\alpha-1}u(y) \left((\lambda + y^\alpha)^{2r/\alpha} - (\lambda + x^\alpha)^{2r/\alpha} \right) dy \\ &= -R_2(\lambda)(\lambda + A_\alpha)u + R_1(\lambda)(rB_\alpha)u. \end{aligned}$$

Thus

$$\begin{aligned} &R_{\alpha,r}(\lambda)(\lambda + A_\alpha - rB_\alpha)u \\ &= R_1(\lambda)(\lambda + A_\alpha)u - R_1(\lambda)(rB_\alpha)u + R_2(\lambda)(\lambda + A_\alpha)u - R_2(\lambda)(rB_\alpha)u \\ &= u - rR_1(\lambda)B_\alpha u + R_2(\lambda)(\lambda + A_\alpha)u - (-rR_1(\lambda)B_\alpha u + R_2(\lambda)(\lambda + A_\alpha)u) \\ &= u \end{aligned}$$

and the lemma is proved. ■

Proposition 5.1. *For any $\alpha \in \mathbb{R}$ and $0 < r \leq 1$, the operator $R_{\alpha,r}(\lambda)$ is the resolvent of the generator of the semigroup $(G_{\alpha,r}(t))_{t \geq 0}$. Thus we have*

$$(5.8) \quad T_{\alpha,r} = \lambda - (R_{\alpha,r}(\lambda))^{-1}.$$

Proof. For $r < 1$ the statement is clear as rB_α is the Miyadera perturbation of A_α . In particular,

$$K_{\alpha,r} = T_{\alpha,r} = \lambda - (R_{\alpha,r}(\lambda))^{-1}.$$

Let now $r = 1$. As in the proof of Theorem 2.1 (or [13]), the semigroup $(G_\alpha(t))_{t \geq 0}$ can be obtained as

$$G_\alpha(t)f = \lim_{r \rightarrow 1^-} G_{\alpha,r}(t)f, \quad f \in X_1.$$

Moreover, for $r \leq r'$ we have $G_{\alpha,r}(t) \leq G_{\alpha,r'}(t)$ for any $t \geq 0$. This implies immediately that for $f \in X_{1,+}$,

$$R_{\alpha,r'}(\lambda)f = \int_0^\infty e^{-\lambda t}G_{\alpha,r'}(t)f dt \geq \int_0^\infty e^{-\lambda t}G_{\alpha,r}(t)f dt = R_{\alpha,r}(\lambda)f,$$

so that by (5.7) the resolvents $R_{\alpha,r}(\lambda)f$ monotonically converge for all f to $R_\alpha(\lambda)f$. Using again the monotonicity we obtain

$$\|R_{\alpha,r'}(\lambda)f - R_\alpha(\lambda)f\|_{X_1} = \int_0^\infty ((R_\alpha(\lambda)f)(x) - (R_{\alpha,r'}(\lambda)f)(x))x dx,$$

and the right-hand side tends to zero by the Lebesgue monotone convergence theorem. Thus, for any $f \in X_{1,+}$, $R_{\alpha,r}(\lambda)f \rightarrow R_\alpha(\lambda)f$ as $r \rightarrow 1^-$ in X_1 norm. On the other hand, it is clear that the resolvents $R_{\alpha,r}(\lambda)f$ converge to the resolvent $R(\lambda)$ of $(R_\alpha(t))_{t \geq 0}$ in X_1 . Thus, $R(\lambda)f = R_\alpha(\lambda)f$ for all positive f and by linearity this extends to the whole of X_1 . ■

Lemma 5.3. *For $0 < r < 1$ and all α we have*

$$(5.9) \quad R_{\alpha,r}(\lambda)X_1 = D_\alpha.$$

If $r = 1$ and $\alpha \neq 0$, then

$$(5.10) \quad R_\alpha(\lambda)X_1 \neq D_\alpha.$$

Moreover, there exists a dense subspace $\hat{X}_1 \subset X_1$ such that $R_\alpha(\lambda)\hat{X}_1 \subset D_\alpha$.

Proof. Let first $r < 1$. Then from Proposition 5.1 we see that $K_{\alpha,r}$ defined on D_α is surjective. Since $R_{\alpha,r}(\lambda)$ is a left inverse of $\lambda - K_{\alpha,r}$, it must be the resolvent and (5.9) is proved.

Let us consider now the case $r = 1$ and $\alpha > 0$. Suppose $0 \leq f \in X_1$; then we obtain as above

$$(5.11) \quad \|x^\alpha R_2(\lambda)f\|_{X_1} = \int_0^\infty \left(\int_0^s \frac{x^{1+\alpha}}{(\lambda + x^\alpha)^{1+2/\alpha}} dx \right) (sf(s))(\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-2} ds.$$

It is easy to see that for large s the inner integral behaves as $\ln s$. As in (5.4) we see that

$$(\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-2} = (\lambda s^{-\alpha} + 1)^{-1+2/\alpha} \rightarrow 1, \text{ as } s \rightarrow \infty.$$

Thus it is bounded away from zero for large s . Therefore, there exist functions in X_1 for which the above integral is divergent (for example, one can take $sf(s) = 1/(s \ln^2 s)$). Since all the functions are nonnegative, the original integral is also divergent by Tonelli's theorem. It is also clear that if f has the support in $[0, n]$,

$n < \infty$, then the above integral is convergent. Therefore there exists a dense subspace \hat{X}_1 such that $R_\alpha(\lambda)\hat{X}_1 \subset D_\alpha$.

Let us consider now the case with $r = 1$ and $\alpha < 0$. Then

$$(5.12) \quad \|x^\alpha R_2(\lambda)f\|_{X_1} = \int_0^\infty \frac{x^{\alpha+1}}{(\lambda+x^\alpha)^{1+2/\alpha}} \int_x^\infty (sf(s))(\lambda+s^\alpha)^{-1+2/\alpha} s^{\alpha-2} ds dx,$$

and we see that the term $x^{\alpha+1}/(\lambda+x^\alpha)^{1+2/\alpha}$ behaves as x^{-1} at zero (and as $x^{\alpha+1}$ at infinity) and therefore the above integral is divergent unless at least $\int_0^\infty (sf(s))(\lambda+s^\alpha)^{-1+2/\alpha} s^{\alpha-2} ds = 0$. ■

Lemma 5.4. *For arbitrary $\alpha \in \mathbb{R}$, if for some $f \in X_1$ we have $R_\alpha(\lambda)f \in D_\alpha$, then $(\lambda - K_\alpha)R_\alpha(\lambda)f = f$. Therefore,*

- (a) *if $\alpha > 0$, then the range of $(\lambda - K_\alpha, D_\alpha)$ is dense in X_1 , and*
- (b) *if $\alpha < 0$, then the range of $(\lambda - K_\alpha, D_\alpha)$ is not closed; its closure is a subspace of X_1 of codimension 1.*

Proof. From Lemma 5.3 we know that have $R_\alpha(\lambda)$ is a left inverse to $\lambda - K_\alpha$. Thus, if $R_\alpha(\lambda)f \in D_\alpha$, then

$$R_\alpha(\lambda)(\lambda - K_\alpha)R_\alpha(\lambda)f = R_\alpha(\lambda)f.$$

From Proposition 5.1, we know that $R_\alpha(\lambda)$ is a resolvent, and hence it is an injective operator, which yields $(\lambda - K_\alpha)R_\alpha(\lambda)f = f$.

Let $r = 1$ and $\alpha > 0$. In Lemma 5.3, we showed that there exists a dense subspace \hat{X}_1 for which we have $R_\alpha(\lambda)\hat{X}_1 \subset D(K_\alpha)$. The first part of this lemma gives $(\lambda - K_\alpha)R_\alpha(\lambda)\hat{X}_1 = \hat{X}_1 \subset Rg(\lambda - K_\alpha)$ (the range of $(\lambda - K_\alpha, D_\alpha)$), and consequently the range of $\lambda - K_\alpha$ is dense in X_1 (and not equal to X_1).

Finally, let us consider the case with $\alpha < 0$ and $r = 1$. Suppose $f \in Rg(\lambda - K_\alpha)$, then $f = (\lambda - K_\alpha)g$ for some $g \in D_\alpha$. Lemma 5.2 implies then that $R_\alpha(\lambda)f \in D_\alpha$, but by Lemma 5.3 we must have $\int_0^\infty (sf(s))(\lambda+s^\alpha)^{-1+2/\alpha} s^{\alpha-2} ds = 0$. In fact, since the function $(sf(s))(\lambda+s^\alpha)^{-1+2/\alpha} s^{\alpha-2}$ is integrable on $[0, \infty[$, the function $F(x) = \int_x^\infty (sf(s))(\lambda+s^\alpha)^{-1+2/\alpha} s^{\alpha-2} ds$ is continuous and therefore it has a limit at zero, $F(0)$. If this limit is nonzero, then by continuity $F(x)$ is of constant sign

at some interval $[0, \delta]$, and since $x^{\alpha+1}(\lambda + x^\alpha)^{-1-2/\alpha}$ is positive and of order of x^{-1} as x is close to zero, we obtain that the integral

$$(5.13) \quad \int_0^\infty \frac{x^{\alpha+1}}{(\lambda + x^\alpha)^{1+2/\alpha}} F(x) dx$$

diverges. Thus $F(0) = 0$ or, in other words, for f to belong to $Rg(\lambda - K_\alpha)$, it is necessary that $\int_0^\infty (sf(s))(\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-2} ds = 0$. Hence, f must belong to the subspace of codimension 1 which is annihilated by the functional generated by the function $(\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-1}$; this subspace will be denoted by \tilde{X}_1 . (Note that for $\alpha < 0$ this function belongs to $X_1^* = \{f; f \text{ measurable, } x^{-1}f \in L_\infty([0, \infty])\}$). However, $\tilde{X}_1 \neq Rg(\lambda - K_\alpha)$. This can be established by noting that if $F(x)$ behaves as $1/\ln x$ at zero, then the integral in (5.13) diverges as $\int_0^{0.5} (1/x \ln x) dx = -\infty$.

On the other hand, we can prove that $\tilde{X}_1 = \overline{Rg(\lambda - K_\alpha)}$. Firstly, we observe that if $f \in \tilde{X}_1$ is continuous at $x = 0$, then $f \in Rg(\lambda - K_\alpha)$. In fact, in this case F' is continuous at zero and the claim follows from

$$\int_0^\delta \frac{1}{x} F(x) dx = \int_0^\delta \frac{F(x) - F(0)}{x} dx < +\infty$$

for sufficiently small $\delta > 0$. Having this in mind, for a given $f \in \tilde{X}_1$ we construct the sequence

$$f_n(x) = \begin{cases} f(x) & \text{for } x > n^{-1}, \\ -2n(\lambda + x^\alpha)^{1-2/\alpha} x^{-\alpha+1} \int_{n^{-1}}^\infty f(s)(\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-1} ds & \text{for } (2n)^{-1} \leq x \leq n^{-1}, \\ 0 & \text{for } 0 \leq x < (2n)^{-1}. \end{cases}$$

We see that

$$\int_0^\infty f_n(x)(\lambda + x^\alpha)^{-1+2/\alpha} x^{\alpha-1} dx = 0.$$

Moreover, since $f_n(x) = 0$ for $0 \leq x \leq (2n)^{-1}$, $f_n \in Rg(\lambda - K_\alpha)$. We shall prove that $f_n \rightarrow f$ in X_1 as $n \rightarrow \infty$. We have

$$\begin{aligned} \|f - f_n\|_{X_1} &\leq \int_0^{n^{-1}} x |f(x)| dx \\ &+ 2n \left(\int_{(2n)^{-1}}^{n^{-1}} n^{-1} x (\lambda + x^\alpha)^{1-2/\alpha} x^{-\alpha+1} dx \right) \left(\int_{n^{-1}}^\infty |f(s)| (\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-1} ds \right). \end{aligned}$$

The first term tends to zero since $f \in X_1$. In the second term we see that the integrand $x(\lambda+x^\alpha)^{1-2/\alpha}x^{-\alpha+1}$ is bounded for x close to zero, and thus the product tends to zero due to the behaviour of $\int_{n^{-1}}^\infty |f(s)|(\lambda+s^\alpha)^{-1+2/\alpha}s^{\alpha-1}ds$. Hence, $\overline{Rg(\lambda - K_\alpha)} = \tilde{X}_1$. This ends the proof of the lemma. ■

Corollary 5.1. *Let $\alpha \neq 0$. The operator K_α is closable but not closed. If $\alpha > 0$, then the extension T_α of K_α which generates the semigroup of $(G_\alpha(t))_{t \geq 0}$ is given by*

$$T_\alpha = \overline{K_\alpha}.$$

If $\alpha < 0$, then T_α is a proper extension of $\overline{K_\alpha}$.

Proof. It is clear that K_α is densely defined. By the comment before Corollary 2.1, we know that K_α is dissipative for any α . Thus from [5, Proposition II.3.14 (iii-iv)] and the lemma above, we infer that K_α is closable but not closed. Hence, for neither $\alpha \neq 0$, K_α is the generator of a semigroup.

Assume now that $\alpha > 0$. Since we have proved that $Rg(\lambda - K_\alpha)$ is a proper dense subspace of X_1 , using Corollary 2.1 we obtain that $T_\alpha = \overline{A_\alpha + B_\alpha}$ is the generator.

Consider next the case $\alpha < 0$. Using again Proposition II.3.14 (iv) of *op. cit.*, we see that $\overline{Rg(\lambda - K_\alpha)} = Rg(\lambda - \overline{K_\alpha})$. By the lemma above, $\overline{Rg(\lambda - K_\alpha)} \neq X_1$. Hence in this case $\overline{K_\alpha}$ cannot be m -dissipative and therefore the generator T_α must be a proper extension of $\overline{K_\alpha}$. ■

In the last step we shall prove that the semigroup $(G_\alpha(t))_{t \geq 0}$, $\alpha < 0$, is not a stochastic semigroup. This result could also be proved by modifying an argument of [6]. but in our opinion the direct proof offered here has some instructive values.

Proposition 5.2. *Let $\alpha < 0$. For any $f \in X_{1,+}$, $f \neq 0$, there exists $t > 0$ such that*

$$(5.14) \quad \|G_\alpha(t)f\| \neq \|f\|.$$

Therefore, $(G_\alpha(t))_{t \geq 0}$ is not a stochastic semigroup.

Proof. Let $f \in X_{1,+}$, $f \neq 0$. If in (5.14) there was an equality for all $t \geq 0$, then using the additivity of the L_1 norm we would have

$$(5.15) \quad \|R_\alpha(1)f\| = \int_0^\infty e^{-t} \|G_\alpha(t)f\| dt = \|f\|.$$

On the other hand,

(5.16)

$$\|R_\alpha(1)f\| = \int_0^\infty \frac{xf(x)dx}{1+x^\alpha} + 2 \int_0^\infty \left(\frac{x}{(1+x^\alpha)^{1+2/\alpha}} \int_x^\infty f(s)(1+s^\alpha)^{-1+2/\alpha} s^{\alpha-1} ds \right) dx.$$

Evaluating the second integral, we obtain

$$\begin{aligned} & 2 \int_0^\infty \left(f(s)(1+s^\alpha)^{-1+2/\alpha} s^{\alpha-1} \int_0^s \frac{x}{(1+x^\alpha)^{1+2/\alpha}} dx \right) ds \\ &= \int_0^\infty f(s)(1+s^\alpha)^{-1+2/\alpha} s^{\alpha-1} \left(\frac{s^2}{(1+s^\alpha)^{2/\alpha}} - 1 \right) ds \\ &= \int_0^\infty f(s)(1+s^\alpha)^{-1} s^{\alpha+1} ds - \int_0^\infty f(s)(1+s^\alpha)^{-1+2/\alpha} s^{\alpha-1} ds. \end{aligned}$$

Inserting the above into (5.16) and simplifying, we obtain

$$\|R_\alpha(1)f\| = \int_0^\infty xf(x)dx - \int_0^\infty f(x)(1+x^\alpha)^{-1+2/\alpha} x^{\alpha-1} dx < \|f\|,$$

which contradicts (5.15). ■

REFERENCES

1. L. Arlotti, The Cauchy problem for the linear Maxwell-Boltzmann equation, *J. Differential Equations* **69** (1987), 166-184.
2. L. Arlotti, A perturbation theorem for positive contraction semigroups on L^1 -spaces with applications to transport equations and Kolmogorov's differential equations, *Acta Appl. Math.* **23** (1991), 129-144.
3. J. Banasiak, Mathematical properties of inelastic scattering models in kinetic theory, *Math. Modells Methods Appl. Sci.* **10** (2000), 163-186.
4. J. Banasiak, On a diffusion-kinetic equation arising in extended kinetic theory, *Math. Methods Appl. Sci.* **23**(14) (2000), 1237-1256.
5. K.-J. Engel, R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, Springer Verlag, New York, 1999.
6. T. Kato, On the semi-groups generated by Kolmogoroff's differential equations, *J. Math. Soc. Japan* **6** (1954), 1-15.

7. W. Lamb, private communication.
8. D. J. McLaughlin, W. Lamb and A. C. McBride, A semigroup approach to fragmentation models, *SIAM J. Math. Anal.* **28** (1997), 1158-1172.
9. M. Mokhtar-Kharroubi, *Mathematical Topics in Neutron Theory: New Aspects*, World Scientific, Singapore, 1997.
10. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag, New York, 1983.
11. H. Risken, *The Fokker-Planck Equation*, 2nd ed., Springer Verlag, Berlin, 1989.
12. H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer Verlag, New York, 1974.
13. J. Voigt, On substochastic C_0 -semigroups and their generators, *Transport Theory Statist. Phys.* **16** (1987), 453-466.
14. R. M. Ziff and E. D. McGrady, The kinetics of cluster fragmentation equation and polymerization, *J. Phys. A* **18** (1985), 3027-3037.

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