TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 1, pp. 141-167, March 2001

# MARKOV PROCESSES AND DIFFUSION EQUATIONS ON UNBOUNDED INTERVALS

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**Abstract.** This paper deals with some Feller semigroups acting on a particular weighted function space on $[0, +\infty[$ whose generators are degenerate elliptic second order differential operators. We show that these semigroups are the transition semigroups associated with suitable Markov processes on $[0, +\infty]$. Furthermore, by means of a sequence of discrete-type positive operators we introduced in a previous paper, we evaluate the expected value and the variance of the random variables describing the position of the processes and we give an approximation formula (in the weak topology) of the distribution of the position of the processes at every time, provided the distribution of the initial position is given and possesses finite moment of order two.

## 0. Introduction

In a previous paper [3], we studied the differential operator $A : D(A) \to W_2^0$ defined by

$$Au(x) := \begin{cases} \alpha(x)\ u''(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $\alpha$ is a continuous function on $[0, +\infty[$, differentiable at $0$ and such that $0 < \alpha_0 \leq \alpha(x)/x \leq \alpha_1$ $(x \geq 0)$ for some $\alpha_0, \alpha_1 \in \mathbb{R}$, $D(A)$ denotes the subspace of all functions $u \in W_2^0 \cap C^2(]0, +\infty[)$ such that

$$\lim_{x \to 0^+} \alpha(x)\, u''(x) = \lim_{x \to +\infty} \frac{\alpha(x)\, u''(x)}{1 + x^2} = 0,$$

Received May 18, 1999; revised December 22, 1999.
Communicated by S.-Y. Shaw.
2000 *Mathematics Subject Classification*: 47D06, 60J35, 41A36.
*Key words and phrases*: Diffusion equation, Feller semigroup, Markov process, positive linear operator, weighted function space.

and $W_2^0$ is the Banach space of all continuous functions $f$ on $[0,+\infty[$ such that $\lim_{x\to+\infty} f(x)/(1+x^2)=0$, endowed with the norm

$$\|f\| = \sup_{x\geq 0} \frac{|f(x)|}{1+x^2} \qquad (f\in W_2^0).$$

Among other things, we showed that $(A, D(A))$ generates a positive $C_0$-semigroup on $W_2^0$.

In this paper, we show that this semigroup is, indeed, the transition semigroup associated with a suitable right-continuous normal Markov process with state space $[0,+\infty]$ whose paths have left-hand limit a.s.

In fact, this result is a consequence of a general study we carry out in a preliminary section where we investigate the interplay between Feller semigroups on weighted function spaces on locally compact spaces having a countable base, Markov transition functions and Markov processes.

The particular form of the differential operator $A$, the boundary conditions included in its domain $D(A)$ and the space $W_2^0$ imply, respectively, that the probability that the process reaches 0 after a finite laps of time is strictly positive, the process, when reaches 0 for the first time, sticks there forever and, finally, the random variables describing the position of the process at every time have finite variance.

In the final part of the paper, by using a particular sequence of discrete-type positive operators we introduced and studied in a previous paper [2], we evaluate the expected value and the variance of the random variables associated with the process and we give an approximation formula (in the weak topology) of the distributions of the position of the process at every time provided the distribution of the initial position is given and possesses finite moment of order two.

## 1. Notation and Preliminaries

In this section, we shall introduce the function spaces we shall deal with and we shall recall some definitions and properties concerning Markov processes and transition functions.

- *Weighted function spaces*

Throughout the paper we shall denote by $X$ a locally compact Hausdorff space which has a countable base. These hypotheses guarantee in particular that $X$ is metrizable and countable at infinity, and that its one-point-compactification $X_\infty := X\cup\{\infty\}$, where $\infty$ denotes the point at infinity of $X$, is metrizable too.

We shall denote by $\mathcal{B}(X)$ the $\sigma$-algebra of all Borel sets of $X$ and by $M^+(X)$ (resp., $M_b^+(X)$) the cone of all positive (resp., positive and bounded) Borel measures on $X$.

We shall also denote by $C(X)$ the vector space of all real-valued continuous functions on $X$, and by $K(X)$ the subspace of $C(X)$ of all functions whose support is compact.

Moreover, we shall consider the following vector subspaces of $C(X)$: the space $C_b(X)$ of all real-valued bounded continuous functions on $X$, the space $C_*(X)$ of all functions $f \in C(X)$ such that $\lim\limits_{x\to\infty} f(x)$ exists in $\mathbb{R}$, and the space $C_0(X)$ of all functions $f \in C(X)$ such that $\lim\limits_{x\to\infty} f(x) = 0$. The spaces $C_b(X)$, $C_*(X)$ and $C_0(X)$, endowed with the natural order and the norm $\|\cdot\|_\infty$ defined, on each of them, by

$$\|f\|_\infty := \sup_{x\in X} |f(x)|, \tag{1.1}$$

become Banach lattices.

Throughout the paper we shall fix a function $w$ such that

$$w \in C_0(X) \quad \text{and} \quad w(x) > 0 \quad \text{for every } x \in X, \tag{1.2}$$

and we shall consider the space $C_b^w(X)$ of all functions $f \in C(X)$ such that $wf \in C_b(X)$. This space becomes a Banach lattice if endowed with the natural order and the norm $\|\cdot\|_w$ defined by

$$\|f\|_w := \|wf\|_\infty \qquad (f \in C_b^w(X)). \tag{1.3}$$

Our interest will be mainly devoted to the space $C_0^w(X)$, which is the Banach sublattice of $C_b^w(X)$ of all functions $f \in C(X)$ such that $wf \in C_0(X)$.

We note that, by virtue of hypothesis (1.2), one has

$$\|f\|_w \le \|w\|_\infty \|f\|_\infty \qquad \text{for every } f \in C_b(X), \tag{1.4}$$

and the following inclusions hold true:

$$K(X) \subset C_0(X) \subset C_*(X) \subset C_b(X) \subset C_0^w(X) \subset C_b^w(X) \subset C(X). \tag{1.5}$$

We also recall that for every positive linear form $T : C_0^w(X) \to \mathbb{R}$, there exists a unique regular Borel measure $\mu \in M_b^+(X)$ such that $wf \in L^1(X, \mathcal{B}(X), \mu)$ and

$$T(f) = \int_X fw\,d\mu \qquad \text{for every } f \in C_0^w(X) \tag{1.6}$$

(see [11, Theorem 5.42]).

Furthermore, the spaces $C_*(X)$ and $C(X_\infty)$ can be naturally identified. More precisely, if for every $f \in C_*(X)$ we denote by $\tilde{f}$ its continuous extension on $X_\infty$ defined by

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in X, \\ \lim\limits_{x\to\infty} f(x) & \text{if } x = \infty, \end{cases} \tag{1.7}$$

then the application from $C_*(X)$ into $C(X_\infty)$ which maps $f$ into its extension $\tilde{f}$ is an isometric isomorphism of Banach lattices.

More generally, every function $f : X \to \mathbb{R}$ can be extended on $X_\infty$ by setting

$$f^*(x) := \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x = \infty. \end{cases} \tag{1.8}$$

Obviously, $\tilde{f} = f^*$ for every $f \in C_0(X)$ and, moreover, $f^*$ is $\mathcal{B}(X_\infty)$-measurable if $f$ is $\mathcal{B}(X)$-measurable.

If $\mu$ denotes a Borel measure on $\mathcal{B}(X_\infty)$ such that $\mu(\{\infty\}) = 0$, we shall say that a $\mathcal{B}(X)$-measurable function $f : X \to \mathbb{R}$ is $\mu$-integrable if $f^*$ is $\mu$-integrable. In this case, we shall set

$$\int_X f d\mu := \int_{X_\infty} f^* d\mu. \tag{1.9}$$

In this case, if we denote by $\text{rest}_{\mathcal{B}(X)}\, \mu$ the restriction of $\mu$ on $\mathcal{B}(X)$, we also have that $f$ is $\text{rest}_{\mathcal{B}(X)}\, \mu$- integrable and

$$\int_X f d\left(\text{rest}_{\mathcal{B}(X)}\mu\right) = \int_X f d\mu. \tag{1.10}$$

Now we present a result concerning the convergence of integrals with respect to finite Borel measures, which will be useful in Section 3.

We shall denote by $\mathbf{1}$ the constant function of value 1.

**Proposition 1.1.** *Let $w$ be a function satisfying hypothesis* (1.2) *such that $w(x) \le 1$ for every $x \in X$, and let $(\mu_n)_{n\ge1}$ and $\mu$ be in $M_b^+(X)$. Let us suppose that*

( i ) $\lim\limits_{n\to\infty} \int_X \mathbf{1}\, d\mu_n = \int_X w\, d\mu$

and

( ii ) $\lim\limits_{n\to\infty} \int_X g\, d\mu_n = \int_X gw\, d\mu \;\; for\ every\ g \in K(X).$

*If $f \in C_b^w(X)$ and if*

(iii) $K := \sup_{n\ge1} \int_X |f|^p d\mu_n < +\infty$ *for some* $p \in \mathbb{R}, p > 1,$

*then*

$$\lim_{n\to\infty} \int_X f\ d\mu_n = \int_X fw\ d\mu.$$

*Proof.* We preliminarily note that each $f \in C_b^w(X)$ satisfying (iii) belongs to $L^1(X, \mathcal{B}(X), \mu_n)$ for every $n \ge 1$ since $\mu_n$ is finite.

Moreover, we note that, since

$$\int_X \mathbf{1}\, d\mu = \mu(X) = \sup_{g\in K(X), 0\le g\le \mathbf{1}} \int_X g\ d\mu,$$

for every fixed $\varepsilon > 0$ there exists $g \in K(X)$, $0 \le g \le \mathbf{1}$, such that

$$\int_X (\mathbf{1} - g) d\mu \le \min\left\{ \frac{\varepsilon}{3(\|f\|_w + 1)}, \left(\frac{\varepsilon}{3K^{\frac{1}{p}}}\right)^q \right\}, \tag{1}$$

where $q$ denotes the conjugate exponent of $p$, i.e., $q = p/(p-1)$.

Furthermore, for every $n \ge 1$, from (iii) we obtain that $|f|^p \in L^1(X, \mathcal{B}(X), \mu_n)$ and, since $0 \le (\mathbf{1} - g)^q \le \mathbf{1} - g \in L^1(X, \mathcal{B}(X), \mu_n)$, we have that $(\mathbf{1} - g)^q \in L^1(X, \mathcal{B}(X), \mu_n)$. Taking the above remarks into account, from (iii) and Hölder's inequality we get

$$\begin{aligned}
\left|\int_X f\ d\mu_n - \int_X fw\, d\mu\right| &\le \left|\int_X f d\mu_n - \int_X fg d\mu_n\right| \\
&+ \left|\int_X fg\, d\mu_n - \int_X fgw\, d\mu\right| + \left|\int_X fgw\, d\mu_n - \int_X fw\, d\mu\right| \\
&\le \int_X |f|(\mathbf{1} - g) d\mu_n + \left|\int_X fg\, d\mu_n - \int_X fgw\, d\mu\right| \\
&+ \int_X |fw|(\mathbf{1} - g) d\mu \le \left(\int_X |f|^p d\mu_n\right)^{\frac{1}{p}} \left(\int_X (\mathbf{1} - g)^q d\mu_n\right)^{\frac{1}{q}} \\
&+ \left|\int_X fg\, d\mu_n - \int_X fgw\, d\mu\right| + \|f\|_w \int_X (\mathbf{1} - g) d\mu \\
&\le K^{\frac{1}{p}} \left(\int_X (\mathbf{1} - g) d\mu_n\right)^{\frac{1}{q}} + \left|\int_X fg\, d\mu_n - \int_X fgw\, d\mu\right| + \|f\|_w \int_X (\mathbf{1} - g) d\mu.
\end{aligned} \tag{2}$$

On the other hand, from (i), (ii), (1) and the hypothesis on $w$ we infer that

On the other hand, from (i), (ii), (1) and the hypothesis on $w$ we infer that

$$\lim_{n\to\infty} \int_X (\mathbf{1} - g) d\mu_n = \int_X (\mathbf{1} - g) w d\mu \le \int_X (\mathbf{1} - g) d\mu \le \left(\frac{\varepsilon}{3K^{\frac{1}{p}}}\right)^q \tag{3}$$

and

$$\lim_{n\to\infty} \int_X fg d\mu_n = \int_X fgw d\mu.$$

Hence, there exists $\nu \in \mathbb{N}$ such that, for every $n \ge \nu$, we have

$$\int_X (\mathbf{1} - g) d\mu_n \le \left(\frac{\varepsilon}{3K^{\frac{1}{p}}}\right)^q \tag{4}$$

and

$$(5) \qquad \left|\int_X fg\,d\mu_n - \int_X fgw\,d\mu\right| \le \frac{\varepsilon}{3}.$$

Consequently, by inserting (3), (4) and (5) in (2), we obtain

$$\left|\int_X f\,d\mu_n - \int_X fw\,d\mu\right| \le \epsilon \qquad \text{for every } n \ge \nu,$$

and hence the result follows. ■

**Remark 1.2.** Note that, as the above proof shows, Proposition 1.1 holds true simply by assuming that $w \in C_b(X)$ and $0 \le w \le \mathbf{1}$.

- *Transition functions and Markov processes*

Now we recall some definitions concerning Markov processes.

A family of functions $P_t : X \times \mathcal{B}(X) \to \mathbb{R}\,(t \ge 0)$ is called a Markov transition function on $X$, and we shall briefly denote it by $(P_t)_{t\ge 0}$, if it satisfies the following conditions:

1) $P_t(x,\cdot)$ is a positive Borel measure on $\mathcal{B}(X)$ and $P_t(x,X) \le 1$ for every $t \ge 0$ and $x \in X$;

2) $P_t(\cdot,B)$ is a Borel measurable function for every $t \ge 0$ and $B \in \mathcal{B}(X)$;

3) $P_0(x,\{x\}) = 1$ for every $x \in X$;

4) $P_{s+t}(x,B) = \displaystyle\int_X P_s(\cdot,B)\,dP_t(x,\cdot)$ for every $s,t \ge 0$, $x \in X$ and $B \in \mathcal{B}(X)$ (the Chapman- Kolmogorov equation).

Moreover, $(P_t)_{t\ge 0}$ is called normal if

$$(1.11) \qquad \lim_{t\to 0^+} P_t(x,X) = 1 \qquad \text{for every } x \in X;$$

the family is also said to be uniformly stochastically continuous on $X$ if for every $\varepsilon > 0$ and for every compact $K \subset X$ we have

$$(1.12) \qquad \lim_{t\to 0^+} \sup_{x\in K} [1 - P_t(x, U_\varepsilon(x))] = 0,$$

where $U_\varepsilon(x) := \{y \in X | d(x,y) < \varepsilon\}$ (here $d$ denotes the distance on $X$).

We also recall that a Markov process with state space $X$ is a quadruple $(\Omega, U, (P^x)_{x\in X}, (Z_t)_{t\ge 0})$, where $(\Omega, U)$ denotes a measurable space, $(P^x)_{x\in X}$ a family of probability measures on $U$, and $(Z_t)_{t\ge 0}$ a family of $(U, \mathcal{B}(X))$- random variables from $\Omega$ into $X$ such that the function $x \longmapsto P^x(A)$ is Borel measurable for every

$A \in U$ and $P^x\{Z_{s+t} \in B|U_s\} = P^{Z_s}\{Z_t \in B\}$ $P^x$- a.s. for every $s, t \geq 0$, $x \in X$ and $B \in \mathcal{B}(X)$.

Here $U_s$ denotes the $\sigma$-algebra generated by $(Z_u)_{0\leq u\leq s}$, $P^{Z_s}\{Z_t \in B\}$ the random variable $\omega \longmapsto P^{Z_s(\omega)}\{Z_t \in B\}$ and $P^x\{Z_{s+t} \in B|U_s\}$ is the conditional probability of $\{Z_{s+t} \in B\}$ given $U_s$.

The Markov process $(\Omega, U, (P^x)_{x\in X}, (Z_t)_{t\geq 0})$ is called normal if

$$P^x\{Z_0 = x\} = 1 \qquad \text{for every } x \in X. \tag{1.13}$$

Intuitively, we may think of a particle which moves in $X$ randomly after an experiment $\omega \in \Omega$. Then $Z_t(\omega)$ represents the position of the particle at time $t \geq 0$. If $B$ is a Borel subset of $X$, $P^x\{Z_t \in B\}$ is the probability that the particle starting at position $x$ will be found in the set $B$ at time $t$.

Moreover, for every $\omega \in \Omega$ the mapping $t \longmapsto Z_t(\omega)$ is called a path of the process and the Markov process $(\Omega, U, (P^x)_{x\in X}, (Z_t)_{t\geq 0})$ is called right-continuous if, for every $x \in X$, one has

$$P^x\{\omega \in \Omega | t \longmapsto Z_t(\omega) \text{ is right-continuous on } [0, +\infty[\} = 1. \tag{1.14}$$

We shall also need to consider the random variable $\zeta : \Omega \to \tilde{\mathbb{R}}$ defined by

$$\zeta(\omega) := \inf\{t \in [0, +\infty[ | Z_t(\omega) = \infty\} \qquad (\omega \in \Omega) \tag{1.15}$$

(with the convention $\inf \emptyset = +\infty$), called the life-time of the process.

At this point we recall how it is possible to associate to each Markov process a Markov transition function, and to the last one a semigroup of operators satisfying suitable properties (for more details, see [1] and [13]).

If $(\Omega, U, (P^x)_{x\in X}, (Z_t)_{t\geq 0})$ is a Markov process with state space $X$, by putting

$$\begin{aligned} P_t(x, B) := P^x(Z_t \in B) = P^x_{Z_t}(B) \text{ for every } t \geq 0, x \in X \\ \text{and } B \in \mathcal{B}(X), \end{aligned} \tag{1.16}$$

where $P^x_{Z_t}$ denotes the distribution of the random variable $Z_t$, we obtain a Markov transition function $(P_t)_{t\geq 0}$ on $X$ (and, conversely, each normal Markov transition function on $X$ corresponds to some Markov process with state space $X$ [13, Theorem 9.1.6]).

Furthermore, if $(P_t)_{t\geq 0}$ is a Markov transition function on $X$, by putting, for every $t \geq 0$,

$$T(t)f(x) := \int_X f dP_t(x, \cdot) \tag{1.17}$$

for every $x \in X$ and for every Borel measurable and bounded function $f : X \to \mathbb{R}$, we obtain a positive semigroup $(T(t))_{t\geq 0}$ of contractions (with respect to the sup-norm) on the space of real-valued Borel measurable and bounded functions on

$X$. The semigroup $(T(t))_{t\geq 0}$ is called the transition semigroup associated with the Markov transition function $(P_t)_{t\geq 0}$ or, equivalently, with the Markov process $(\Omega, U, (P^x)_{x\in X}, (Z_t)_{t\geq 0})$, if $(P_t)_{t\geq 0}$ is given by (1.16). In the last case, we can also write

$$T(t)f(x) = \int_X f\, dP^x_{Z_t} = \int_\Omega f \circ Z_t \, dP^x = E_x(f(Z_t)), \tag{1.18}$$

where $E_x(f(Z_t))$ denotes the expected value (with respect to $P^x$) of the random variable $f(Z_t)$.

Finally, we recall that a semigroup $(T(t))_{t\geq 0}$ on $C_b(X)$ is called a Feller semigroup if it is strongly continuous on $C_b(X)$, $T(t)$ is positive and $\|T(t)\| \leq 1$ for every $t \geq 0$, while a Markov transition function $(P_t)_{t\geq 0}$ on $X$ is called a Feller function (resp., a $C_0$-function) if the semigroup $(T(t))_{t\geq 0}$ given by (1.17) leaves invariant the space $C_b(X)$ (resp., the space $C_0(X)$).

For more details on the deep relations between Markov processes, Markov transition functions and Feller semigroups, see, e.g., [1, 7, 13].

## 2. Positive Semigroups on Weighted Function Spaces, Feller Functions and Markov Processes

In this section, we shall study the relationship between semigroups on the space $C_0^w(X)$ and Markov processes with state space $X_\infty$.

Let $(T(t))_{t\geq 0}$ be a positive semigroup on $C_0^w(X)$ and, for every $t \geq 0$ and $x \in X$, let us consider the positive linear form $T_{x,t}$ on $C_0^w(X)$ defined by

$$T_{x,t}(f) := T(t)f(x) \qquad (f \in C_0^w(X)). \tag{2.1}$$

By virtue of (1.6), there exists a unique regular Borel measure $\mu_{x,t} \in M_b^+(X)$ such that $wf \in L^1(X, \mathcal{B}(X), \mu_{x,t})$ and

$$T(t)f(x) = T_{x,t}(f) = \int_X fw d\mu_{x,t} \qquad \text{for every } f \in C_0^w(X). \tag{2.2}$$

As regards the measure $w\mu_{x,t}$, we note that it is also finite, since $\mathbf{1} \in C_0^w(X)$ and, from (2.2), it follows that

$$(w\mu_{x,t})(X) = \int_X w d\mu_{x,t} = T(t)\mathbf{1}(x) \in \mathbb{R}. \tag{2.3}$$

At this point we prove the following result.

**Theorem 2.1.** *Let $(T(t))_{t\geq 0}$ be a positive semigroup on $C_0^w(X)$. Then the following statements are equivalent*:

a) (i) $T(t)(C_*(X)) \subset C_*(X)$ *and* $T(t)\mathbf{1}=\mathbf{1}$ *for every* $t \geq 0$;

(ii) $\lim\limits_{t\to 0^+} \|T(t)f - f\|_\infty = 0$ *for every* $f \in C_*(X)$.

b) *There exists a uniformly stochastically continuous normal Feller function* $(P_t)_{t\geq 0}$ *on* $X_\infty$ *such that*

(i) $P_t(x, \{\infty\}) = 0$ *for every* $t \geq 0$ *and* $x \in X$;

(ii) *if* $t \geq 0$ *and* $x \in X$, *then each* $f \in C_*(X)$ *is* $P_t(x,\cdot)$ - *integrable and*

$$T(t)f(x) = \int_X f dP_t(x,\cdot) = \int_{X_\infty} \tilde{f} dP_t(x,\cdot). \tag{2.4}$$

*Furthermore, if* a) *or* b) *holds*, *then*

1) *for every* $t \geq 0$ *and* $x \in X$, *one has that* $\mathrm{rest}_{\mathcal{B}(X)} P_t(x,\cdot) = w\mu_{x,t}$, *where* $\mu_{x,t}$ *is the finite regular Borel measure associated with* $(T(t))_{t\geq 0}$ *by* (2.2); *in particular*, $1/w$ *is* $P_t(x,\cdot)$ - *integrable*;

2) *if* $t \geq 0$ *and* $x \in X$, *then each* $f \in C_0^w(X)$ *is* $P_t(x,\cdot)$ - *integrable and*

$$T(t)f(x) = \int_X f w d\mu_{x,t} = \int_X f dP_t(x,\cdot) = \int_{X_\infty} f^* dP_t(x,\cdot); \tag{2.5}$$

3) *there exists a right-continuous normal Markov process* $(\Omega,\, U,\, (P^x)_{x\in X_\infty},\, (Z_t)_{t\geq 0})$ *with state space* $X_\infty$, *whose paths have left-hand limits on* $[0,\zeta[$ *a.s.*, *such that*

$$P^x_{Z_t}(B) = P_t(x,B) \qquad \text{for every } t \geq 0, x \in X \text{ and } B \in \mathcal{B}(X_\infty), \tag{2.6}$$

*i.e.*, $P^x_{Z_t} = P_t(x,\cdot)$. *Thus* $P^x\{Z_t = \infty\} = 0$ *and* $1/w$ *as well as all* $f \in C_0^w(X)$ *are* $P^x_{Z_t}$ - *integrable and*

$$T(t)f(x) = \int_X f dP^x_{Z_t} = E_x(f^*(Z_t)). \tag{2.7}$$

*Proof.* a)$\Longrightarrow$b): By virtue of the identification between $C_*(X)$ and $C(X_\infty)$ shown in the previous section and the hypotheses (i) and (ii), $(T(t))_{t\geq 0}$ can be regarded as a Feller semigroup on $C(X_\infty)$. Consequently (see, e.g., [1, Theorem 1.6.14]), $(T(t))_{t\geq 0}$ becomes the transition semigroup of a uniformly stochastically continuous normal Feller function $(P_t)_{t\geq 0}$ on $X_\infty$ (see (1.11) and (1.12)) such that

$$\widetilde{T(t)}f(x) = \int_{X_\infty} \tilde{f} dP_t(x,\cdot) \qquad \text{for every } t \geq 0, x \in X_\infty \text{ and } f \in C_*(X).$$

Here $\tilde{f}$ and $\widetilde{T(t)}f$ denote, respectively, the continuous extension on $X_\infty$ of $f$ and $T(t)f$ defined by (1.7). In particular, if $x \in X$ one gets

$$T(t)f(x) = \int_{X_\infty} \tilde{f} dP_t(x,\cdot). \tag{1}$$

In order to prove (i), it will be sufficient to show that $P_t(x, X) = 1$ for every $t \geq 0$ and $x \in X$.

We preliminarily note that, since $X$ is also countable at infinity, there exists an increasing sequence $(u_n)_{n\geq 1}$ in $K(X)$ such that each $u_n$ is positive and $\sup_{n\geq 1} u_n = \mathbf{1}$. Moreover, denoting by $\mathbf{1}_X$ the characteristic function of $X$ in $X_\infty$, one also has that $\sup_{n\geq 1} \tilde{u}_n = \mathbf{1}_X$. For given $t \geq 0$ and $x \in X$, taking (1) and formula (2.2) into account, from the Beppo Levi Theorem it follows that

$$
\begin{aligned}
P_t(x, X) &= \int_{X_\infty} \mathbf{1}_X dP_t(x, \cdot) = \sup_{n\geq 1} \int_{X_\infty} \tilde{u}_n dP_t(x, \cdot) \\
&= \sup_{n\geq 1} T(t)u_n(x) = \sup_{n\geq 1} \int_X u_n w d\mu_{x,t} \\
&= \int_X \left(\sup_{n\geq 1} u_n\right) w d\mu_{x,t} = \int_X w \, d\mu_{x,t} = T(t)\mathbf{1}(x) = 1.
\end{aligned}
$$

Hence (i) is proved.

In order to show (ii), we observe that if $f \in C_*(X)$, then $\tilde{f} = f^* = f$ on $X$, i.e., $P_t(x, \cdot)$ - a.e. and, consequently, $f^*$ is $P_t(x, \cdot)$ - integrable and, for every $t \geq 0$ and $x \in X$,

$$
\int_{X_\infty} f^* dP_t(x, \cdot) = \int_{X_\infty} \tilde{f} dP_t(x, \cdot). \tag{2}
$$

Hence $f$ is $P_t(x, \cdot)$ - integrable and, by virtue of (1) and (2), for every $t \geq 0$ and $x \in X$ we have

$$
T(t)f(x) = \int_{X_\infty} \tilde{f} dP_t(x, \cdot) = \int_{X_\infty} f^* dP_t(x, \cdot) = \int_X f dP_t(x, \cdot).
$$

And this completes the proof of part b).

b)$\Longrightarrow$a): From the hypothesis b) it follows that, if we consider for every $t \geq 0$ the operator $S(t) : C(X_\infty) \to C(X_\infty)$ defined by

$$
S(t)f(x) := \int_{X_\infty} f dP_t(x, \cdot) \qquad (f \in C(X_\infty), x \in X_\infty), \tag{3}
$$

then $(S(t))_{t\geq 0}$ is a Feller semigroup on $C(X_\infty)$ (see, e.g., [1, Theorem 1.6.14] and [13, Remark 9.1.10 and Theorem 9.2.6]).

If $f \in C_*(X)$, for every $t \geq 0$ and $x \in X$, by virtue of the hypotheses and (3) we can write

$$
T(t)f(x) = \int_{X_\infty} \tilde{f} dP_t(x, \cdot) = S(t)\tilde{f}(x), \tag{4}
$$

and, hence, $T(t)f = S(t)\tilde{f}_{|X}$ and, consequently, $T(t)f \in C_*(X)$ and $\widetilde{T(t)f} = S(t)\tilde{f}$.

Moreover, from (4) it follows that

$$T(t)\mathbf{1}(x) = S(t)\tilde{\mathbf{1}}(x) = \int_{X_\infty} \tilde{\mathbf{1}} dP_t(x, \cdot) = P_t(x, X_\infty) = 1,$$

which proves (i).

As regards (ii), for every $f \in C_*(X)$ and $t \geq 0$, denoting by $\|\cdot\|$ the sup-norm on $C(X_\infty)$, one has

$$\|T(t)f - f\|_\infty = \|\widetilde{T(t)f} - \tilde{f}\| = \|S(t)\tilde{f} - \tilde{f}\| \to 0 \qquad (t \to 0^+).$$

And hence part a) is completely proved.

Let us suppose, now, that one, and hence both properties a) or b) are satisfied. We remark that, for every fixed $t \geq 0$ and $x \in X$, $w\mu_{x,t}$ and $\mathrm{rest}_{\mathcal{B}(X)} P_t(x, \cdot)$ are both regular Borel measures in $M_b^+(X)$, and that, by virtue of the representation formula (2.2), hypothesis b)(ii) and identity (2), one has

$$\int_X fw d\mu_{x,t} = T(t)f(x) = \int_{X_\infty} \tilde{f} dP_t(x, \cdot) = \int_X fd\left(\mathrm{rest}_{\mathcal{B}(X)} P_t(x, \cdot)\right)$$

for every $f \in K(X)$.

By the uniqueness in the Riesz representation theorem (see, e.g., [1, Theorem 1.2.4]), the above two measures coincide, and then 1) is proved.

As regards 2), we note that every function $f \in C_0^w(X)$ is Borel-measurable and, from (2.2) and part 1) just proved, it follows that

$$\begin{aligned}\int_{X_\infty} |f^*| dP_t(x, \cdot) &= \int_X |f^*| dP_t(x, \cdot) \\ &= \int_X |f| d\left(\mathrm{rest}_{\mathcal{B}(X)} P_t(x, \cdot)\right) \\ &= \int_X |f| w d\mu_{x,t} < +\infty,\end{aligned}$$

i.e., $f^*$ is $P_t(x, \cdot)$ - integrable. Hence, $f$ is $P_t(x, \cdot)$- integrable and, from (2.2) and 1) one has

$$\begin{aligned}\int_X f dP_t(x, \cdot) &= \int_{X_\infty} f^* dP_t(x, \cdot) = \int_X f^* dP_t(x, \cdot) \\ &= \int_X fd\left(\mathrm{rest}_{\mathcal{B}(X)} P_t(x, \cdot)\right) = \int_X fw d\mu_{x,t} = T(t)f(x),\end{aligned}$$

which proves 2).

Finally, part 3) is a direct consequence of a result of [7] (see, also, [1, Theorem 1.6.14]). ■

**Remark 2.2.** As the final part of the proof shows, the Markov process described in part 3) of the above theorem depends on the restriction of the semigroup $(T(t))_{t\geq 0}$ on $C_*(X)$ only.

Therefore, if $\rho \in C_0(X)$ is another weight satisfying (1.2) and if $(S(t))_{t\geq 0}$ is another positive semigroup on $C_0^{\rho}(X)$ satisfying part a) of Theorem 2.1 and such that

$$S(t) = T(t) \quad \text{on } C_*(X) \text{ for every } t \geq 0,$$

then their corresponding Markov processes are the same.

If in part a) of the above theorem one replaces $C_*(X)$ with $C_0(X)$, then one gets a Feller $C_0$- function on $X$.

**Theorem 2.3.** *Let $(T(t))_{t\geq 0}$ be a positive semigroup on $C_o^w(X)$. Then the following statements are equivalent*:

a) (i) $T(t)(C_0(X)) \subset C_0(X)$ *and* $T(t)$**1=1** *for every* $t \geq 0$;

(ii) $\lim\limits_{t\to 0^+} \|T(t)f - f\|_\infty = 0$ *for every* $f \in C_0(X)$.

b) *There exists a uniformly stochastically continuous normal Feller $C_0$-function $(P_t)_{t\geq 0}$ on $X$ such that*

(i) *for every $s > 0$ and for every compact $K \subset X$ one has*

$$\lim_{x\to\infty} \sup_{0\leq t\leq s} P_t(x, K) = 0;$$

(ii) *for every $t \geq 0$, $x \in X$ and $f \in C_0(X)$ one has*

$$T(t)f(x) = \int_X f dP_t(x, \cdot).$$

*Moreover, if* a) *or* b) *holds*, *then*

1) *for every $t \geq 0$ and $x \in X$ one has that $P_t(x,\cdot) = w\mu_{x,t}$, where $\mu_{x,t}$ is the finite regular Borel measure associated with $(T(t))_{t\geq 0}$ by* (2.2); *in particular*, $1/w$ *is $P_t(x,\cdot)$ - integrable*;

2) *if $t \geq 0$ and $x \in X$, then each $f \in C_0^w(X)$ is $P_t(x,\cdot)$ - integrable and*

$$T(t)f(x) = \int_X f dP_t(x, \cdot) = \int_X f w d\mu_{x,t};$$

3) *there exists a right-continuous normal Markov process* $(\Omega,\ U,\ (P^x)_{x\in X},\ (Z_t)_{t\geq 0})$ *with state space $X$, whose paths have left-hand limits on $[0, \zeta[$ a.s., such that*

$$P^x_{Z_t}(B) = P_t(x, B) \qquad \textit{for every } t \geq 0, x \in X \textit{ and } B \in \mathcal{B}(X).$$

*Therefore*, $1/w$ *and each* $f \in C_0^w(X)$ *are* $P_{Z_t}^x$ *- integrable and*

$$T(t)f(x) = \int_X f dP_{Z_t}^x = E_x(f(Z_t)).$$

*Proof.* As regards the equivalence between a) and b), apply Theorem 9.2.6 of [13]. As regards the remaining statements, take Theorem 9.1.9 of [13] into account, and adapt the proof of the corresponding part of Theorem 2.1. ■

We conclude this section by presenting a simple characterization of properties (i) and (ii) of part a) of Theorems 2.1 and 2.3, respectively, in terms of the infinitesimal generator of the semigroup.

**Proposition 2.4.** *Let* $(T(t))_{t\geq 0}$ *be a positive semigroup on* $C_0^w(X)$ *with infinitesimal generator* $(A, D(A))$. *Then properties* (i) *and* (ii) *of part* a) *of Theorem* 2.1 (*Theorem* 2.3, *respectively*) *are satisfied if and only if the following conditions hold true*:

1) $\mathbf{1} \in D(A)$ *and* $A(\mathbf{1}) = 0$;

2) *There exists a subspace* $D_0$ *of* $D(A) \cap C_*(X)$ (*of* $D(A) \cap C_0(X)$, *resp.*) *which is dense in* $C_*(X)$ (*in* $C_0(X)$, *resp.*) *for the uniform norm such that*

I) $A(D_0) \subset C_*(X)$ $(A(D_0) \subset C_0(X),\ resp.)$;

II) $A_{|D_0}$ *is closable in* $C_*(X)$ (*in* $C_0(X)$, *resp.*) *and its closure is the infinitesimal generator of a Feller semigroup on* $C_*(X)$ (*on* $C_0(X)$, *resp.*).

*Proof.* If $T(t)\mathbf{1}$=$\mathbf{1}$ for every $t \geq 0$, then $\lim\limits_{t\to 0^+}(T(t)\mathbf{1}$-$\mathbf{1})/t = 0$ and hence $\mathbf{1} \in D(A)$ and $A(\mathbf{1}) = 0$. Conversely, if $\mathbf{1} \in D(A)$ and $A(\mathbf{1}) = 0$, then the two functions $u(t) := \mathbf{1}$ and $T(t)\mathbf{1}(t \geq 0)$ are both solutions of the Cauchy problem $\dot{u}(t) = Au(t)(t \geq 0)$, $u(0) = \mathbf{1}$. Therefore $T(t)\mathbf{1} = u(t) = \mathbf{1}$ for each $t \geq 0$.

Assume now that properties (i) and (ii) of part a) of Theorem 2.1 are satisfied and set $\tilde{T}(t) := T(t)_{|C_*(X)}(t \geq 0)$. Then $(\tilde{T}(t))_{t\geq 0}$ is a Feller semigroup on $C_*(X)$ and let $(\tilde{A}, D(\tilde{A}))$ be its infinitesimal generator.

If $u \in D(\tilde{A})$, then $\lim\limits_{t\to 0^+}(\tilde{T}(t)u - u)/t$ exists with respect to $\|.\|_\infty$ and hence with respect to $\|.\|_w$. So, $u \in D(A)$ and $Au = \tilde{A}u$.

Thus conditions I) and II) follow by setting $D_0 := D(\tilde{A})$.

Conversely, assume that conditions 1) and 2) hold true. Set $A^* := A_{|D_0} : D_0 \to C_*(X)$ and denote by $(\tilde{A}, D(\tilde{A}))$ its closure in $C_*(X)$. Finally, denote by $(\tilde{T}(t))_{t\geq 0}$ the Feller semigroup on $C_*(X)$ generated by $\tilde{A}$.

Note that $D(\tilde{A}) \subset D(A)$ and $A_{|D(\tilde{A})} = \tilde{A}$, because $A_{|D(A)\cap C_*(X)}$ is closed with respect to the uniform norm as well.

Fix now $u_0 \in D(\tilde{A})$ and consider the following Cauchy problems:

(1) $$\dot{u}(t) = \tilde{A}u(t) = A\,u(t) \text{ in } C_*(X), t \geq 0, u(0) = u_0,$$

(2) $$\dot{v}(t) = A\,v(t) \text{ in } C_0^w(X), t \geq 0, v(0) = u_0.$$

The function $u(t) := \tilde{T}(t)u_0(t \geq 0)$ is a solution of (1) and hence of (2) and so $\tilde{T}(t)u_0 = T(t)u_0$ for every $t \geq 0$.

Since $D(\tilde{A})$ is dense in $C_*(X)$ and the operators $\tilde{T}(t)$ and $T(t)$ are continuous from $(C_*(X),\ \|.\|_\infty)$ into $(C_*(X),\ \|.\|_w)$, we infer that $\tilde{T}(t) = T(t)$ on $C_*(X)$ for each $t \geq 0$ and so part a) of Theorem 2.1 follows.

The respective part of the statement follows by replacing $C_*(X)$ with $C_0(X)$ in the above proof. ■

## 3. Markov Processes Associated with Diffusion Equations on $[0, +\infty[$

This is the main section of the paper. We shall construct a Markov process associated with a class of degenerate diffusion equations on $[0, +\infty[$ and we shall study some properties of it.

Consider the locally compact space $X = [0, +\infty[$ and the weight $w_2$ defined by

(3.1) $$w_2(x) := \frac{1}{1+x^2} \qquad (x \geq 0).$$

The corresponding space $C_0^w(X)$ will be denoted by $W_2^0$, and we shall consider it endowed with the norm $\|\cdot\|_2$ defined by

(3.2) $$\|f\|_2 := \sup_{x \geq 0} \frac{|f(x)|}{1+x^2} \qquad (f \in W_2^0)$$

(see (1.2) and (1.3)).

For a given function $\alpha \in C([0, +\infty[)$, let us consider the differential operator $A : D(A) \to W_2^0$ defined by

(3.3) $$Af(x) := \begin{cases} \alpha(x)f''(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

for every $f \in D(A)$, where the domain $D(A)$ of $A$ is the subspace of all functions $f \in W_2^0 \cap C^2(]0, +\infty[)$ satisfying the conditions

(3.4) $$\lim_{x \to 0^+} \alpha(x)f''(x) = 0 \text{ and } \lim_{x \to +\infty} \frac{\alpha(x)}{1+x^2} f''(x) = 0.$$

In [3, Corollary 4.4] we proved that, under the following additional assumptions:

(3.5) $\alpha$ is differentiable at $0$,

(3.6) there exist $\alpha_0, \alpha_1 \in \mathbb{R}$ such that $0 < \alpha_0 \leq \dfrac{\alpha(x)}{x} \leq \alpha_1$ for every $x \geq 0$,

the differential operator $(A, D(A))$ generates a positive $C_0$-semigroup $(T(t))_{t\geq 0}$ on $W_2^0$ such that $T(t)\mathbf{1}=\mathbf{1}$ for every $t \geq 0$.

Moreover,

(3.7) $T(t)(C_*([0,+\infty[)) \subset C_*([0,+\infty[)$ for every $t \geq 0$

and

(3.8) $(T(t))_{t\geq 0}$ is strongly continuous on $C_*([0,+\infty[)$

(see [3, Remark 4.5]).

Hence, from Theorem 2.1 we immediately obtain the following result.

**Theorem 3.1.** *Let $(T(t))_{t\geq 0}$ be the semigroup generated on the space $W_2^0$ by the differential operator $(A, D(A))$ defined by* (3.3) *and* (3.4) *under the conditions* (3.5) *and* (3.6).

*Then there exists a right-continuous normal Markov process $(\Omega, U, (P^x)_{0\leq x\leq +\infty}, (Z_t)_{t\geq 0})$ with state space $[0,+\infty]$, whose paths have left-hand limits on $[0,\zeta[$ a.s., such that*

1) $P^x\{Z_t = +\infty\} = 0$ *for every $t \geq 0$ and $x \geq 0$;*

2) *for every $t \geq 0$ and $x \geq 0$, there exists a unique regular Borel measure $\mu_{x,t} \in M_b^+([0,+\infty[)$ such that*

$$\operatorname{rest}_{\mathcal{B}([0,+\infty[)} P^x_{Z_t} = w_2 \mu_{x,t}.$$

*Thus, $1/w_2$ is $P^x_{Z_t}$- integrable, i.e., $\int_0^{+\infty} u^2 dP^x_{Z_t}(u) < +\infty$. Moreover, each $f \in W_2^0$ is $P^x_{Z_t}$- integrable and*

(3.9) $$T(t)f(x) = \int_0^{+\infty} f w_2 \, d\mu_{x,t} = \int_0^{+\infty} f \, dP^x_{Z_t} = E_x(f^*(Z_t))$$

*for every $t \geq 0$ and $x \geq 0$.*

The parabolic equation associated with the differential operator (3.3), i.e., the differential equation

(3.10) $$\frac{\partial u}{\partial t}(x,t) = \alpha(x) \frac{\partial^2 u}{\partial t^2}(x,t) \qquad (x \geq 0, t \geq 0),$$

is a degenerate diffusion equation and represents the Kolmogorov's backward equation of the normal Markov process described in the above theorem; the variance instantaneous velocity and the mean instantaneous velocity of the process at a position $x \in [0, +\infty]$ are $\alpha(x)$ and zero, respectively.

According to the terminology introduced by [8], 0 is an exit boundary point for the process. From a probabilistic point of view, this means that the probability that a particle (the process) located in $]0, +\infty[$ will reach 0 after a finite lap of time is strictly positive.

The first of the two conditions of (3.4) implies that when the particle reachs 0 for the first time, it sticks there forever [13, §§ 1.7, 1.8].

The second condition implies that $Z_t$ is $P^x$- square integrable, i.e., its variance is finite.

Our next aim is to determine the expected value $E_x(Z_t)$ and the variance $\mathrm{Var}_x(Z_t)$ of the random variables $Z_t$ with respect to $P^x$, and to give an approximation formula of the distribution of the position at every time. In particular, we shall prove the following results.

**Theorem 3.2.** *Under the same hypotheses of Theorem* 3.1, *for every* $t \geq 0$ *and* $x \geq 0$ *one gets*

( i ) $E_x(Z_t) = x$;

( ii ) $\mathrm{Var}_x(Z_t) = \alpha_0 tx$ *if there exists* $\alpha_0 \in \mathbb{R}$ *such that* $\alpha(x) = (\alpha_0/2)x\ (x \geq 0)$;

(iii) $c\lambda_0 tx \leq \mathrm{Var}_x(Z_t) \leq ctx$, *where* $\lambda_0 := \inf_{x>0}(2/c)(\alpha(x)/x)$ *and* $c := \sup\limits_{x>0} 2(\alpha(x)/x)$.

As regards the approximation of the distribution of the position we shall show that the Markov process considered in this section can be interpreted as a limit of random walks, provided $\alpha(x) = (\alpha_0/2)x\ (x \geq 0)$.

For every $n \geq 1$, set

$$\Delta_n := \left\{ \frac{h}{n} | \quad h \in \mathbb{N} \right\} \tag{3.11}$$

and for every $x \in \Delta_n$, consider the Borel measure $\rho_{n,x}$ on $\Delta_n$ defined by

$$\rho_{n,x} := \sum_{h=0}^{\infty} e^{-nx} \frac{(nx)^h}{h!} \varepsilon_{\frac{h}{n}}, \tag{3.12}$$

where $\varepsilon_{h/n}$ denotes the unit mass at $h/n$.

Consider the random walk

$$\left( \Delta_n, \frac{1}{n}, (\rho_{n,x})_{x \in \Delta_n} \right) \tag{3.13}$$

having $\Delta_n$ as support, $1/n$ as basic time-interval and $(\rho_{n,x})_{x\in\Delta_n}$ as one-step transition probability distributions. This means that, if a particle is at a position $x \in \Delta_n$ at time $h/n (h \in \mathbb{N})$, then it remains at $x$ during the interval $[h/n, (h+1)/n[$, and at the time $(h+1)/n$ it jumps so that the probability that it goes to any Borel subset $B$ of $\Delta_n$ is $\rho_{n,x}(B)$.

The family $(\rho_{n,x})_{x\in\Delta_n}$ determines a linear operator $T_n$ from the space $\mathcal{B}_2^*(\Delta_n)$ of all real-valued Borel measurable functions $f$ on $\Delta_n$ such that $\sup\limits_{h\geq 0}(w_2 f)(h/n) < +\infty$ into itself, which is defined as

$$T_n(f)(x) := \int_{\Delta_n} f\,d\rho_{n,x} \qquad (f \in \mathcal{B}_2^*(\Delta_n),\, x \in \Delta_n). \tag{3.14}$$

The operator $T_n$ is called the one-step transition operator of the random walk.

If a Borel probability measure $\mu$ on $\Delta_n$ gives the distribution of the initial position of the particle, then the Borel probability measure $T_n(\mu)$ on $\Delta_n$ defined by

$$T_n(\mu)(B) := \int_{\Delta_n} \rho_{n,x}(B)\,d\mu(x) = \int_{\Delta_n} T_n(\mathbf{1}_B)\,d\mu \quad (B \in \mathcal{B}(\Delta_n)) \tag{3.15}$$

(here $\mathbf{1}_B$ denotes the characteristic function of $B$), gives the distribution for its position after the first jump and, for $p \geq 2$, $T_n^p(\mu)$ is the distribution for its position after the $p$th jump, where

$$T_n^p(\mu) := T_n\left(T_n^{p-1}(\mu)\right). \tag{3.16}$$

(For more details on random walks, see, e.g., [6, pp. 190-191] and [14, Sect. 6].)

Finally, if $\nu \in M_b^+([0,+\infty[)$, for every $t \geq 0$ we shall denote by $T(t)(\nu)$ the Borel measure on $[0,+\infty[$ such that

$$\int_0^{+\infty} f d(T(t)(\nu)) = \int_0^{+\infty} T(t)(f) d\nu \tag{3.17}$$

for every $f \in C_b([0,+\infty[)$.

If $\nu$ is a probability measure which gives the distribution of the initial position of the particle whose motion is described by the Markov process described in Theorem 3.1, then $T(t)(\nu)$ gives the distribution of the position at time $t$.

**Theorem 3.3.** *Assume that* $\alpha(x) = (\alpha_0/2)x$ $(x \geq 0)$. *Then the sequence of random walks* (3.13) $W_2^0$*-converges to the Markov process described in Theorem* 3.1 *in the following sense*:

(i) *For every Borel probability measure* $\mu \in M^+([0,+\infty[)$ *such that* $1/w_2$ *is* $\mu$*-integrable* (*i.e.*, *the second moment of* $\mu$ *is finite*), *there exists a sequence* $(\mu_n)_{n\geq 1}$

*of Borel probability measures in* $M^+([0,+\infty[)$ *such that* $1/w_2$ *is* $\mu_n$*- integrable and* Supp $\mu_n \subset \Delta_n$ *for each* $n \geq 1$, *and*

(3.18) $$\mu = \lim_{n\to\infty} \mu_n \qquad weakly.$$

(ii) *For every* $\mu \in M_b^+([0,+\infty[)$ *and for every sequence* $(\mu_n)_{n\geq 1}$ *in* $M_b^+([0,+\infty[)$ *as above*, *for every* $t \geq 0$ *and for every sequence* $(k(n))_{n\geq 1}$ *of positive integers such that* $\lim\limits_{n\to\infty} k/(n)/n = ct$ (*for instance*, $k(n) = [cnt]$, $n \geq 1$, *where* $[\cdot]$ *denotes the integer part*), *we have*

(3.19) $$T(t)(\mu) = \lim_{n\to\infty} T_n^{k(n)}(\mu_n) \qquad weakly.$$

In order to present the proofs of Theorems 3.2 and 3.3, it is necessary to recall some results concerning a sequence of positive linear operators of discrete type introduced and studied in [2, 3, 5], to which we refer for further details.

In [3] we give, in particular, a representation of the semigroup $(T(t))_{t\geq 0}$ in terms of the iterates of the operators mentioned above, and such a representation will be essential in order to obtain the results contained in Theorem 3.2.

- *The positive approximation process*

We first point out that, if the function $\alpha$ satisfies the assumptions (3.5) and (3.6), then we can write

(3.20) $$\alpha(x) = c\frac{x\lambda(x)}{2} \qquad (x \geq 0),$$

where

(3.21) $$c := \sup_{x>0} 2\frac{\alpha(x)}{x}$$

and

(3.22) $$\lambda(x) := \begin{cases} \frac{2}{c}\frac{\alpha(x)}{x} & \text{if } x > 0, \\ \frac{2}{c}\,\alpha'(0 & \text{if } x = 0. \end{cases}$$

Hence $\lambda \in C([0,+\infty[)$ and $0 < (2/c)\alpha_0 \leq \lambda(x) \leq 1$ for every $x \geq 0$.

For every $\beta > 0$, we denote by $E_\beta$ the subspace of $C([0,+\infty[)$ of all functions $f \in C([0,+\infty[)$ such that $\sup_{x\geq 0} e^{-\beta x}|f(x)| < +\infty$. The space $E_\beta$, endowed with the natural order and the norm $\|\cdot\|_\beta$ defined by

(3.23) $$\|f\|_\beta := \sup_{x\geq 0}\, e^{-\beta x}|f(x)| \quad (f \in E_\beta),$$

is a Banach lattice.

We also set $E_\infty := \cup_{\beta>0} E_\beta$.

Given a function $\lambda \in C_b([0,+\infty[)$ such that $0 \le \lambda(x) \le 1 (x \ge 0)$, we set

$$M_{n,\lambda}(f)(x) := \sum_{p=0}^{n} \binom{n}{p} \lambda(x)^p (1-\lambda(x))^{n-p} e^{-px} \sum_{h=0}^{\infty} \frac{(px)^h}{h!} f\left(\frac{h}{n} + \left(1-\frac{p}{n}\right)x\right) \tag{3.24}$$

for every $f \in E_\infty$, $x \ge 0$ and $n \ge 1$.

The operators $M_{n,\lambda}$ are positive, linear and continuous on several subspaces of $E_\infty$. In particular, they map $W_2^0$ into itself, are continuous, and, for every $f \in W_2^0$,

$$\lim_{n\to\infty} M_{n,\lambda}(f) = f \quad \text{in } W_2^0 \tag{3.25}$$

(see [3, Corollary 2.5]).

In [3, Corollary 4.4]), we also showed that, if $\alpha \in C([0,+\infty[)$ satisfies conditions (3.5) and (3.6), and if $(T(t))_{t\ge 0}$ is the semigroup generated on the space $W_2^0$ by the differential operator $(A, D(A))$ defined by (3.3) and (3.4), then for every sequence $(k(n))_{n\ge 1}$ of positive integers such that $\lim_{n\to\infty} k(n)/n = ct$, one has

$$T(t)f = \lim_{n\to\infty} M_{n,\lambda}^{k(n)} f \quad \text{in } W_2^0, \tag{3.26}$$

and hence uniformly on compact subsets of $[0,+\infty[$, where $M_{n,\lambda}^{k(n)}$ denotes the iterate of order $k(n)$ of the operator $M_{n,\lambda}$.

We remark that, when $\lambda = \mathbf{1}$, the operators $M_{n,\lambda}$ become the Favard- Szász-Mirakjan operators $M_n$ defined by

$$M_n(f)(x) := \sum_{h=0}^{\infty} e^{-nx} \frac{(nx)^h}{h!} f\left(\frac{h}{n}\right) \quad (f \in E_\infty, x \ge 0, n \ge 1). \tag{3.27}$$

On the other hand, the operators $M_{n,\lambda}$ can be expressed in terms of the operators $M_n$ via the formula

$$M_{n,\lambda}(f)(x) = \sum_{p=0}^{n} \binom{n}{p} \lambda(x)^p (1-\lambda(x))^{n-p} M_p(f_{n,p,x})(x) \tag{3.28}$$

for every $f \in E_\infty$, $x \ge 0$ and $n \ge 1$. Here $M_0$ denotes the identity operator and $f_{n,p,x}$ is the function defined by

$$f_{n,p,x}(t) := f\left(\frac{p}{n}t + \left(1-\frac{p}{n}\right)x\right) \quad (t \ge 0). \tag{3.29}$$

For each $p \in \mathbb{N}$, $p \geq 1$, we set $e_p(x) := x^p \,(x \geq 0)$. One can prove (see [4, Lemma 3] and [3, p. 324]) that, for every $n \geq 1$,

$$M_n(e_1) = e_1, \qquad M_{n,\lambda}(e_1) = e_1, \tag{3.30}$$

$$M_n(e_2) = e_2 + \frac{e_1}{n}, \qquad M_{n,\lambda}(e_2) = e_2 + \frac{\lambda}{n} e_1, \tag{3.31}$$

$$M_n(e_3) = e_3 + \frac{3}{n}\, e_2 + \frac{e_1}{n^2}, \tag{3.32}$$

$$M_n(e_4) = e_4 + \frac{6}{n}\, e_3 + \frac{a}{n^2}\, e_2 + \frac{e_1}{n^3} \quad \text{(with } a > 0\text{)}. \tag{3.33}$$

First of all, we estimate, from above and from below, the iterates of order $p$ of the operators $M_n$ evaluated on the functions $e_2$, $e_3$ and $e_4$.

**Lemma 3.4.** *For every $n, p \geq 1$ one gets*
( i ) $M_n^p(e_2) = e_2 + \frac{p}{n}\, e_1$;
( ii ) $0 \leq M_n^p(e_3) \leq e_3 + 3\frac{p}{n}\, e_2 + \frac{3}{2}\frac{p^2}{n^2}\, e_1$;
(iii) $0 \leq M_n^p(e_4) \leq e_4 + 6\,\frac{p}{n}\, e_3 + b\,\frac{p(p+1)}{n^2}\, e_2 + b\,\frac{p(p+1)(2p+1)}{3n^3} e_1$,
*where $b$ is a constant independent of $n$ and $p$.*

*Proof.* Assertion (i) follows from formulas (3.30) and (3.31) and from the linearity of each $M_n$.

As regards (ii), we first note that, from (i) and formula (3.32) it follows that

$$M_n^{p+1}(e_3) = M_n^p(M_n(e_3)) = M_n^p(e_3) + \frac{3}{n}\left(e_2 + \frac{p}{n} e_1\right) + \frac{e_1}{n^2}. \tag{1}$$

Now we prove (ii) by induction on $p$.

If $p = 1$, (ii) follows directly from (3.32). If we suppose that (ii) is true for $p$, then, taking (1) into account, we get

$$\begin{aligned} M_n^{p+1}(e_3) &\leq e_3 + 3\frac{p}{n} e_2 + \frac{3}{2}\frac{p^2}{n^2} e_1 + \frac{3}{n}\left(e_2 + \frac{p}{n} e_1\right) + \frac{e_1}{n^2} \\ &= e_3 + 3\frac{p+1}{n} e_2 + \frac{1}{n^2}\left(\frac{3}{2}p^2 + 3p + 1\right) e_1 \\ &\leq e_3 + 3\frac{p+1}{n} e_2 + \frac{3}{2}\frac{(p+1)^2}{n^2} e_1. \end{aligned}$$

As regards (iii), from (ii) and formulas (3.31) and (3.33) it follows that

$$\begin{aligned} M_n^{p+1}(e_4) =& M_n^p(M_n(e_4)) = M_n^p(e_4) + \frac{6}{n}M_n^p(e_3) + \frac{a}{n^2}\left(e_2 + \frac{p}{n}e_1\right) \\ (2) \qquad & + \frac{e_1}{n^3} \leq M_n^p(e_4) + \frac{6}{n}\left(e_3 + 3\frac{p}{n}e_2 + \frac{3}{2}\frac{p^2}{n^2}e_1\right) \\ & + \frac{a}{n^2}\left(e_2 + \frac{p}{n}e_1\right) + \frac{e_1}{n^3}. \end{aligned}$$

We set $b := \max\{a, 18\}$ and we prove (iii) by induction on $p$.

If $p = 1$, (iii) follows from (3.32). If we suppose that (iii) is true for some $p$, then, taking (2) into account, we have

$$\begin{aligned} M_n^{p+1}(e_4) \leq & \, e_4 + 6\frac{p}{n}e_3 + b\frac{p(p+1)}{n^2}e_2 + b\frac{p(p+1)(2p+1)}{3n^3}e_1 \\ & + \frac{6}{n}\left(e_3 + 3\frac{p}{n}e_2 + \frac{3}{2}\frac{p^2}{n^2}e_1\right) + \frac{a}{n^2}\left(e_2 + \frac{p}{n}e_1\right) + \frac{e_1}{n^3} \leq e_4 \\ & + 6\frac{p+1}{n}e_3 + \frac{b}{n^2}(p(p+1) + p + 1)e_2 \\ & + \frac{b}{3n^3}\left(p(p+1)(2p+1) + 3p^2 + 3p + 3\right)e_1 = e_4 + 6\frac{p+1}{n}e_3 \\ & + \frac{b}{n^2}(p+1)^2 e_2 + \frac{b}{3n^3}\left(p(p+1)(2p+1) + 3p^2 + 3p + 3\right)e_1 \\ & \leq e_4 + 6\frac{p+1}{n}e_3 + b\frac{(p+1)(p+2)}{n^2}e_2 \\ & + b\frac{(p+1)(p+2)(2(p+1)+1)}{3n^3}e_1, \end{aligned}$$

since $p(p+1)(2p+1) + 3p^2 + 3p + 3 \leq (p+1)(p+2)(2(p+1)+1)$ for every $p \geq 1$. ■

In order to quickly prove the main result of this section, we need to estimate the iterates of the operators $M_{n,\lambda}$ evaluated on the functions $e_2$ and $e_4$. To do it, we shall need the following result, which is of independent interest.

In the sequel, $\lambda$ will denote a fixed continuous function on $[0, +\infty[$ such that $0 \leq \lambda(x) \leq 1$ for every $x \geq 0$.

**Proposition 3.5.** *Let $f \in E_\infty$ be a convex function. Then, for every $n \geq 1$,*

(i) $f \leq M_{n+1,\lambda}(f) \leq M_{n,\lambda}(f)$;

(ii) *if* $\alpha \in C_b([0, +\infty[)$, $0 \leq \alpha \leq \lambda$, *one gets* $M_{n,\alpha}(f) \leq M_{n,\lambda}(f)$.

*Proof.* (i) By virtue of formulas (1.18) and (1.19) of [2] and of Theorem 2 of [12], it is easy to show that $M_{n+1,\lambda}(f) \le M_{n,\lambda}(f)$ for every $n \ge 1$ (see, also, [9, Theorem 3]). As regards the first inequality, it follows from the second one and from part 1) of Theorem 2.3 of [2].

(ii) In [2, formula (1.8)], we give the following representation of the operators $M_{n,\lambda}$:

$$(1)\quad M_{n,\lambda}(f)(x) = \int_0^{+\infty} \cdots \int_0^{+\infty} f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\rho_{x,\lambda}(x_1) \cdots d\rho_{x,\lambda}(x_n),$$

which holds true for every $f \in E_\infty$, $x \ge 0$ and $n \ge 1$. Here $\rho_{x,\lambda}$ denotes the distribution on $\mathbb{R}$ defined by $\rho_{x,\lambda} := \lambda(x)\pi_x + (1 - \lambda(x))\varepsilon_x$, where $\pi_x$ denotes the Poisson distribution on $\mathbb{R}$ with parameter $x$ (with the convention $\pi_0 = \varepsilon_0$).

If we take $n = 1$ and $\lambda = \mathbf{1}$, then by using (i) and (1), for every $x \ge 0$ and for every convex function $f \in E_\infty$, we obtain

$$(2)\qquad f(x) \le M_{1,\mathbf{1}}(f)(x) = M_1(f)(x) = \int_0^{+\infty} f d\pi_x.$$

Moreover, the function $f_{z,a} : \mathbb{R} \to \mathbb{R}$ defined, for every fixed $z \ge 0$ and $a \in [0,1]$ by $f_{z,a}(t) := f(at + (1-a)z)$ $(t \ge 0)$, is convex too. Hence, again, for every $x \ge 0$ we obtain

$$(3)\qquad f_{z,a}(x) \le M_1(f_{z,a})(x) = \int_0^{+\infty} f_{z,a} d\pi_x.$$

At this point it is sufficient to use (3) and to follow the proof of Theorem 6.1.13 of [1], replacing $\mu_x^T$ with $\pi_x$ and $\nu_{x,\alpha}^T$ with $\rho_{x,\alpha}$. ∎

Taking the above proposition into account, we are able to prove the next result, whose corollary will be essential in the proof of Theorem 3.2.

**Proposition 3.6.** *Let $f \in E_\infty$ be a convex function. Then, for every $\alpha,\ \beta \in \mathbb{R}$ such that $0 < \alpha \le \lambda(x) \le \beta \le 1$ $(x \ge 0)$, one has*

$$M_{n,\alpha}^p(f) \le M_{n,\lambda}^p(f) \le M_{n,\beta}^p(f)$$

*for every $n, p \ge 1$.*

*Proof.* We fix a convex function $f \in E_\infty$ and we preliminarily note that $M_{n,\lambda}(f)$ is a convex function for every $n \ge 1$ if and only if $\lambda$ is constant (see [5, Theorem 2.3]). We prove the result by induction on $p$. If $p = 1$, the assertion is part (ii) of

Proposition 3.5. If the assertion is true for some $p$, taking part (ii) of Proposition 3.5 into account, we obtain, for every $n \geq 1$,

$$M_{n,\alpha}^{p+1}(f) = M_{n,\alpha}\left(M_{n,\alpha}^{p}(f)\right) \leq M_{n,\lambda}\left(M_{n,\alpha}^{p}(f)\right) \leq M_{n,\lambda}\left(M_{n,\lambda}^{p}(f)\right) = M_{n,\lambda}^{p+1}(f)$$

and

$$M_{n,\lambda}^{p+1}(f) = M_{n,\lambda}\left(M_{n,\lambda}^{p}(f)\right) \leq M_{n,\lambda}\left(M_{n,\beta}^{p}(f)\right) \leq M_{n,\beta}\left(M_{n,\beta}^{p}(f)\right) = M_{n,\beta}^{p+1}(f).$$

■

**Corollary 3.7.** *If* $\lambda_0 := \inf_{x\geq 0}\lambda(x) > 0$, *then for every* $n,\, p \geq 1$ *we have*
(i) $M_{n,\lambda}^{p}(e_4) \leq M_n^p(e_4)$;
(ii) $M_{n,\lambda_0}^{p}(e_2) \leq M_{n,\lambda}^{p}(e_2) \leq M_n^p(e_2)$, *i.e.*,

$$e_2 + \frac{p}{n}\lambda_0 e_1 \leq M_{n,\lambda}^{p}(e_2) \leq e_2 + \frac{p}{n}e_1.$$

*Proof.* (i) The result follows directly from Proposition 3.6, the convexity of $e_4$ and the equality $M_{n,\mathbf{1}} = M_n$ $(n \geq 1)$.

(ii) The first part again follows from Proposition 3.6, the convexity of $e_2$ and the inequalities $0 < \lambda_0 \leq \lambda(x) \leq 1$ $(x \geq 0)$.

As regards the second part, it is a consequence of the first one and of the fact that $M_{n,\alpha}^{p}(e_2) = e_2 + p\frac{\alpha}{n}e_1$ for every $n,\, p \geq 1$ and $\alpha \in ]0,1]$ because of (3.31). ■

- *Proofs of the main results*

We are now ready to present the proof of Theorems 3.2 and 3.3.

*Proof of Theorem* 3.2. Let us fix $t \geq 0$, $x \geq 0$ and a sequence $(k(n))_{n\geq 1}$ of positive integers such that $\lim\limits_{n\to\infty} k(n)/n = ct$, where $c$ is the constant defined by (3.21).

Since $e_1 \in W_2^0$ and $M_{n,\lambda}(e_1) = e_1$ for every $n \geq 1$, by virtue of (3.9) and formula (3.26) applied to the function $e_1$, one gets

$$E_x(Z_t) = \int_0^{+\infty} e_1 dP_{Z_t}^x = T(t)e_1(x) = \lim_{n\to\infty} M_{n,\lambda}^{k(n)}(e_1)(x) = e_1(x) = x,$$

which proves (i).

As regards the determination of $\mathrm{Var}_x(Z_t)$, we first remark that

$$\mathrm{Var}_x(Z_t) = \int_0^{+\infty} e_2\, dP_{Z_t}^x - (E_x(Z_t))^2 < +\infty$$

since $\int_0^{+\infty} e_2 \; dP^x_{Z_t} < +\infty$.

We begin to prove that $\int_0^{+\infty} e_2 w_2 \; d\mu_{x,t} = \lim_{n\to\infty} M^{k(n)}_{n,\lambda}(e_2)(x)$.

For every $n \geq 1$, let us consider the discrete Borel measure $\mu_{n,x} \in M^+_b([0,+\infty[)$ such that

$$\int_0^{+\infty} g \, d\mu_{n,x} = M^{k(n)}_{n,\lambda}(g)(x) \quad \text{for every } g \in E_\infty,$$

and let us note that, by virtue of the representation formula (3.26) and of Theorem 3.1 we have that

$$\int_0^{+\infty} \mathbf{1} \, d\mu_{n,x} = 1 = T(t)\mathbf{1}(x) = \int_0^{+\infty} w_2 \, d\mu_{x,t} \tag{1}$$

since $M_{n,\lambda}(\mathbf{1}) = \mathbf{1}$, and

$$\lim_{n\to\infty} \int_0^{+\infty} g \, d\mu_{n,x} = \lim_{n\to\infty} M^{k(n)}_{n,\lambda}(g)(x) = T(t)g(x) = \int_0^{+\infty} g w_2 \, d\mu_{x,t} \tag{2}$$

for every $g \in K([0,+\infty[)$, since $K([0,+\infty[) \subset W^0_2$.

On the other hand, from part (iii) of Lemma 3.4 and part (i) of Corollary 3.7, it follows that

$$\begin{aligned}\int_0^{+\infty} e_2^2 \, d\mu_{n,x} = \int_0^{+\infty} e_4 \, d\mu_{n,x} &= M^{k(n)}_{n,\lambda}(e_4)(x) \leq M^{k(n)}_{n}(e_4)(x) \leq e_4(x) \\ &+6\frac{k(n)}{n}e_3(x) + b\frac{k(n)(k(n)+1)}{n^2}e_2(x) \\ &+b\frac{k(n)(k(n)+1)(2k(n)+1)}{3n^3}e_1(x) \\ &\leq e_4(x) + \alpha e_3(x) + \beta e_2(x) + \gamma e_1(x),\end{aligned} \tag{3}$$

where $\alpha$, $\beta$ and $\gamma$ are positive constants independent of $n$, whose existence is guaranteed by the fact that $\sup_{n\geq 1} k(n)/n < +\infty$.

Hence, by virtue of (1), (2) and (3), we can apply Proposition 1.1 to the case $w = w_2$, $f = e_2$, $\mu = \mu_{x,t}$, $\mu_n = \mu_{n,x}$ and $p = 2$, and we obtain

$$\int_0^{+\infty} e_2 w_2 \, d\mu_{x,t} = \lim_{n\to\infty} \int_0^{+\infty} e_2 \, d\mu_{n,x} = \lim_{n\to\infty} M^{k(n)}_{n,\lambda}(e_2)(x). \tag{4}$$

Now we proceed to prove statements (ii) and (iii).

(ii) From the hypothesis on the function $\alpha$, i.e., $\lambda = \mathbf{1}$ and $c = \alpha_0$ (cf. (3.20), (3.21) and (3.22)) and from part (i) of Lemma 3.4, it follows that

$$M^{k(n)}_{n,\lambda}(e_2) = M^{k(n)}_{n}(e_2) = e_2 + \frac{k(n)}{n}e_1,$$

and hence, by formula (4),

$$(5) \qquad \int_0^{+\infty} e_2 w_2 \, d\mu_{x,t} = \lim_{n\to\infty} \left( e_2(x) + \frac{k(n)}{n} e_1(x) \right) = e_2(x) + \alpha_0 t e_1(x).$$

Consequently, taking (i) and (5) into account, we get

$$\mathrm{Var}_x(Z_t) = \int_0^{+\infty} e_2 w_2 \, d\mu_{x,t} - e_2(x) = \alpha_0 t e_1(x) = \alpha_0 t x.$$

(iii) We note that $\lambda_0 = \inf_{x\geq 0} \lambda(x) > 0$ because of (3.22); furthermore, by virtue of part (ii) of Corollary 3.7, we can write

$$e_2 + \frac{k(n)}{n}\lambda_0 e_1 \leq M_{n,\lambda}^{k(n)}(e_2) \leq e_2 + \frac{k(n)}{n} e_1 \quad \text{for every } n \geq 1,$$

from which, taking (4) into account, we get

$$(6) \qquad c\lambda_0 t x \leq \int_0^{+\infty} e_2 w_2 \, d\mu_{x,t} - x^2 \leq ctx,$$

and so the result follows. ■

*Proof of Theorem* 3.3. For every $n \geq 1$, let $X_n$ be the Banach space of all real-valued continuous functions $f : \Delta_n \to \mathbb{R}$ such that $\lim_{h\to\infty} (w_2 f)(h/n) = 0$, endowed with the norm

$$\varphi_n(f) := \sup_{x\in\Delta_n} w_2(x)|f(x)| \quad (f \in X_n).$$

Moreover, consider the linear operator $P_n : W_2^0 \to X_n$ defined as

$$P_n(f) := f_{|\Delta_n} \quad \left(f \in W_2^0\right).$$

Since every open subset of $[0, +\infty[$ intersects $\Delta_n$ for sufficiently large $n$, $(X_n)_{n\geq 1}$ forms a sequence of Banach spaces approximating $W_2^0$ (see [14, Sect. 2] and, also, [10, Sect. 3.6]).

So part (i) follows from [14, Lemma 3.1 and p. 895] because, by virtue of Theorem 5.42 of [11], every Borel measure $\mu \in M_b^+([0, +\infty[)$ (having support contained in some $\Delta_n$, resp.) corresponds to a positive linear functional on $W_2^0$ (on $X_n$, resp.) if and only if $1/w_2$ is $\mu$- integrable.

Note that $T_n(X_n) \subset X_n$. Moreover, if $f \in W_2^0$, then for every $n \geq 1$,

$$T_n\left(f_{|\Delta_n}\right) = M_n(f)_{|\Delta_n}$$

(see (3.27)), and so we have

$$\begin{aligned}&\sup_{x\in\Delta_n} w_2(x)|T(t)f(x)-T_n^{k(n)}(f)(x)|\\&=\sup_{x\in\Delta_n} w_2(x)|T(t)f(x)-M_n^{k(n)}(f)(x)|\le\|T(t)f-M_n^{k(n)}(f)\|_2.\end{aligned}$$

We end the paper with the following result.We always refer to the semigroup and the Markov process described in Theorem 3.1.

**Proposition 3.8.** *The following statements hold true*:

(i) *a given function* $f\in W_2^0$ *is convex if and only if*

$$f(x)\le E_x(f^*(Z_t))\quad for\ every\ t\ge 0\ and\ x\ge 0,$$

*i.e., the* $P^x$*-expected value of* $f$ *on the* $(random)$ *position of the particle at every time* $t$, *provided it starts at position* $x$, *is greater that* $f(x)$;

(ii) *for every* $f\in W_2^0$, *we have*

$$\lim_{t\to+\infty}\frac{|T(t)f(x)|}{t}\le cx\|f\|_2\quad(x\ge 0),\tag{3.34}$$

*where* $c$ *is defined by* (3.21). *In particular*,

$$\|T(t)\|\le 1+\frac{c}{2}\,t.\tag{3.35}$$

*Proof.* Part (i) is a direct consequence of (3.9) and Theorem 2.1 of [5].

As regards part (ii), since $|f|\le\|f\|_2(1+e_2)$, from formula (6) of the proof of Theorem 3.2, for every $t\ge 0$ and $x\ge 0$, it follows that

$$\begin{aligned}|T(t)f(x)|&\le\int_0^{+\infty}|f|w_2d\mu_{x,t}\le\|f\|_2\int_0^{+\infty}(1+e_2)w_2d\mu_{x,t}\\&\le\left(1+x^2+ctx\right)\|f\|_2\end{aligned}$$

and (3.34) follows.

Finally, (3.35) follows from the above inequality because

$$\frac{|T(t)f(x)|}{1+x^2}\le\left(1+ct\frac{x}{1+x^2}\right)\|f\|_2\le\left(1+\frac{c}{2}t\right)\|f\|_2. \qquad\blacksquare$$

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