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ON PERTURBATIONS AND EXTENSIONS OF ISOMETRIC OPERATORS*

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Abstract. In this paper, some recent advances and open problems on perturbations and extensions of the isometric operators are presented. It is shown that for the spaces $B(L^\beta(\mu) \to L^\beta(\nu))$ $(0 < \beta < 1)$, $B(L^\infty(\mu) \to L^\infty(\nu))$ (with some special measure spaces), $B(E_{(2)} \to L^1(\mu))$, $B(E_{(n)} \to C(\Omega))$ and $B(X \to L^\infty(\mu))$, where X is a uniformly smooth Banach space with dim$X$=1 or ∞ and (Ω, μ) is a purely atomic or purely non-atomic finite measure space and so on, the answer to the problem of the isometric approximations is positive. However, for the spaces $B(l^1 \times l^\infty)$, $B(L^1(\mu) \to L^\infty(\nu))$ and $B(L^1(\mu) \to C_b(\Delta))$, the answer is negative.

In practice, the strictly (theorized) "equidistant operator" does not come into existence. For example, we say "the rotation of a rigid body" (which is a linear isometric operator), but it practically does not have the "point" around which the rigid body rotates; and there is no strict "translation" (which is a non-linear isometric operator) by the famous principle of uncertainty in the classical physics. So the so-called "equidistant" is always to have its errors, that is, what can be seen in practice is the "almost equidistant" or "almost isometric" operator.

1. The Non-linear (w)-perturbations of Surjective Isometries

How to describe the "almost isometric" operator? First, let us introduce the non-linear (w)-ϵ-isometric operator which was defined in 1945 [12].

Definition. Let E and E_1 be two normed spaces. Then, an operator T of E to E_1 is called (w)-ϵ-isometric if it satisfies

$$| \, \|Tx - Ty\| - \|x - y\| \, | \le \epsilon. \quad \forall x, y \in E.$$

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In particular, if T satisfies $\|Tx - Ty\| = \|x - y\|$ ($\forall x, y \in E$), then T is called isometric.

Hyers-Ulam problem (*The* (w)*-stability problem for the "surjective" isometries*). Is there a constant $\alpha(E, E_1)$ such that for each surjective (w)-ϵ-isometric T of E onto E_1 there exists a surjective isometric V with $\|Tx - Vx\| \le \alpha(E, E_1)\epsilon$, $\forall x \in E$? Furthermore, if the answer is affirmative, how is the best possible value of $\alpha(E, E_1)$) found?

The above problem was studied by many famous mathematicians (e.g., see [5]) and was finally solved by Gevirtz on Banach spaces [10]. In addition, Omladič and Šemrl obtain a sharper result that for each surjective (w)-ϵ-isometric T with $T(0) = 0$, there exists a unique linear surjective isometric V such that $\|Tx - Vx\| \le 2\epsilon$, $\forall x \in E$.

Question. How about the F-spaces for the above problem?

2. The Linear (s)-perturbations of the Isometries

The (s)-ϵ-operator and (s)-stability problems for the linear isometries were investigated by the famous mathematicians Pelczynski and Michael [15] in 1966. In their paper, an operator $T \in B(l^\infty_{(m)} \to l^\infty_{(n)})$ is called ϵ-isometric if $\|x\| \le \|Tx\| \le (1 + \epsilon)\|x\|$, $\forall x \in l^\infty_{(m)}$, and it was obtained that for each ϵ-isometric T, there exists an isometric V such that $\|T - V\| \le \epsilon$.

In general, we can define the (s)-ϵ-isometric operator as follows:

Definition. Let E and E_1 be normed spaces. Then an operator $T \in B(E \to E_1)$ is called (s)-ϵ-isometric ($\epsilon \ge 0$) if

$$(1 - \epsilon)\|x\| \le \|Tx\| \le (1 + \epsilon)\|x\|, \quad \forall x \in E.$$

Thus we can pose the following stability problem:

The (s)*-stability problem for the linear isometries.* Let E and E_1 be normed (or some F-normed) spaces. Then, is there a funtion $\delta(\epsilon)$ defined on (0,1) such that $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$ and for each (s)-ϵ-isometric $T \in B(E \to E_1)$ there exists an isometric $V \in B(E \to E_1)$ with $\|T - V\| \le \delta(\epsilon)$?

So, it is evident that the "(w)-almost equidistant" is essentially a form of "absolute error" and the "(s)-almost equidistant" is described by the form of "relative error".

In 1980, Ding resolved the (s)-stability problem for the linear isometries in the spaces $B(E_{(m)} \to E_{(n)})$, where $E_{(m)}$ and $E_{(n)}$ are two finite-dimensional normed spaces, and $B(L^1_+)$ (i.e., we consider the above (s)-stability problem on the "positive cone" of $L^1[0,1]$ [6]).

Naturally, it is very interesting and quite difficult for us to consider the same problem in the infinite-dimensional spaces because Huang has obtained the following results [11]:

(1) For any Banach space E with dim $E \geq 2$, there exists a Banach space E_1 such that for every $0 < \epsilon < 1$ there is an ϵ-isometric $T \in B(E \to E_1)$ and for the E_1 the answer to the above (s)-stability problem is negative in $B(E \to E_1)$.

(2) There exists a separable Banach space E_1 such that for every finite-dimensional $E_{(n)}$ with $n \geq 2$ there is an ϵ-isometric $T \in B(E_{(n)} \to E_1)$ and for the $E_{(n)}$ the answer to the above (s)-stability problem is negative in $B(E_{(n)} \to E_1)$.

Ding and his graduate students have done much work on the (s)-stability problem of linear isometries, which is called the "isometric approximation problem" (or IAP for short). For example, the affirmative answer to the above problem has been obtained in the spaces $B(L^p_+(\mu) \to L^p_+(\nu))$ $(1 < p < \infty)$, $B(L^\beta(\mu) \to L^\beta(\nu))$ $(0 < \beta < 1)$ and $B(L^\infty(\mu) \to L^\infty(\nu))$ with some special measure spaces as well as $B(X \to l^\infty(\Gamma))$ and $B(X \to L^\infty(\mu))$, where X is a uniformly smooth Banach space with dim X=1 or ∞ and (Ω, μ) is a purely non-atomic finite measure space. Moreover, the negative answers to the above problem have been gained in the spaces $B(l^1 \times l^\infty)$, $B(L^1(\mu) \to L^\infty(\nu))$ and $B(L^1(\mu) \to C_b(\Delta))$ when $L^1(\mu)$ is infinite-dimensional and can be isometrically embedded into $L^\infty(\nu)$ (Δ is an arbitrary interval in $\mathbb{R}$) and so on (e.g., see [7]).

Also, the excellent work on this problem has been done for the spaces $B(C(\Omega_1) \to C(\Omega_2))$ and $B(L^p(\mu) \to L^p(\nu))$ $(1 \leq p < \infty)$ by Benyamini [3, 4] and Alspach [2].

The newest work concerning the operators from finite-dimensional to infinite-dimensional spaces on this problem has been done by Wang [21] and Zhan [27]. They have got the affirmative answers for the spaces $B(E_{(n)} \to C(\Omega))$ when $B(E^*_{(n)})$ is polyhedral or Ω has only finitely many isolated points, and $B(E_{(2)} \to L^1(\mu))$ when (Ω, μ) is a non-atomic measure space, respectively.

Question. What about the spaces $B(L^p \to C)$ and $B(L^\infty(\mu) \to L^\infty(\nu))$ when (Ω, μ) is an arbitrary measure space?

3. The Other Kind of Non-linear (w)-perturbations of Isometries

In 1967, Figiel obtained the following result [9]: Let E and E_1 be two Banach spaces, V an isometric operator of E into E_1 with $V(0) = 0$. Then there exists a surjective $S \in B(\overline{\text{span}\, V(E)} \to E)$ with $\|S\| = 1$ such that $S \circ V = I_d$ (identity operator).

In 1995, Qian proposed another (w)-perturbation of isometries connecting the above Figiel theorem with the Hyers-Ulam problem as follows:

Another (w)-stability problem for the non-linear isometries. Let E and E_1 be two Banach spaces. Then, is there a constant $\beta(E, E_1)$ with the property that for each (w)-ϵ-isometric T of E into E_1 with $T(0) = 0$ there exists a surjective $S \in B(\overline{\text{span}\,T(E)} \to E)$ with $\|S\| = 1$ such that

$$\|(S \circ T)(x) - x\| \leq \beta(E, E_1)\epsilon, \quad \forall x \in E?$$

He obtained the affirmative answers for the spaces $B(L^p(\mu) \to L^p(\nu))$ $(1 < p < \infty)$ and $B(E \to H)$, where H is a Hilbert space [18].

I personally think that the new stability problem for isometries is certain to attract some mathematicians to work on it in the future.

4. The Affine Isometric Extension

The classical Mazur-Ulam theorem tells us that a surjective isometry V between two real normed spaces is affine (i.e., $V(\lambda x + (1-\lambda)y) = \lambda V(x) + (1-\lambda)V(y)$, $\forall x, y \in E$, $\lambda \in \mathbb{R}$). So, it is important to determine the isometry of an operator on the whole space by its isometry on some region of the space or its preserving some distance properties.

In 1987, Tingley [20] posed the problem of isometric extension from the unit sphere as follows:

Isometric extension problem from the unit sphere. Let E and E_1 be two Banach spaces. If V_0 is a surjective isometry between the two unit spheres $S(E)$ and $S(E_1)$, does V_0 have an isometric affine extension?

In Tingley's paper [20], he has got the affirmative answer to the assertion $V_0(-x) = -V_0(x)$ $(\forall x \in S(E))$ when the spaces are finite-dimensional. In 1992, Ma, one of my graduate students, first considered this problem in infinite-dimensional spaces. She got the same affirmative answer to the above assertion when the spaces are strictly convex Banach spaces [13]. Then, my other students also got the affirmative answers to this problem in the $C_0(\Omega)$ type spaces (Ω is a locally compact Hausdorff space which has at least two points and for each $x \in C_0(\Omega)$ the set $\{t|\ |x(t)| \geq \alpha,\ t \in \Omega\}$ is compact as $\alpha > 0$ [22]), the abstract L^p spaces $(1 < p < \infty,\ p \neq 2)$ [26] and the l^p-sum of $C_0(\Omega, E)$ type [23] and so on. The newest result has been obtained in some subspaces of $L^1(\Omega, X)$, where (Ω, μ) is a σ-finite measure space and is not atomic while X is strictly convex [28].

Since 1991, I have been considering the similar problem in the F-normed spaces. In 1996, we got some results presented in the following theorem [8]:

Theorem. *Let E, E_1 be F-normed spaces, G an "open connected" subset of E and T an isometry from G onto an "open" set $T(G) \subset E_1$. Assume that one of the following conditions is satisfied*:

(1) *E is locally bounded and $\|2^{-n}(Tx - Ty)\| = \|2^{-n}(x - y)\|$ $(\forall x, y \in G,\ n \in N)$.*

(2) *E is locally bounded, $\phi(t) = \|tx\|$ is increasing in $\mathbb{R}^+$ for every $x \in E$ and there exist $\{\delta_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{R}^+$ with $\inf_{\lambda\in\Lambda} \delta_\lambda = 0$ such that $\|\delta_\lambda(Tx - Ty)\| = \|\delta_\lambda(x - y)\|$ $(\forall x, y \in G,\ \lambda \in \Lambda)$.*

(3) *E is "unbounded" (i.e., $\lim\limits_{t\to\infty} \|tx\| = \infty, \forall x \neq 0$) and $\|2^n(Tx - Ty)\| = \|2^n(x - y)\|$ $(\forall x, y \in G,\ n \in \mathbb{N})$.*

(4) *E is "unbounded", $\phi(t) = \|tx\|$ is increasing in $\mathbb{R}^+$ for every $x \in E$ and there exist $\{\gamma_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{R}^+$ with $\sup_{\lambda\in\Lambda} \gamma_\lambda = \infty$ such that $\|\gamma_\lambda(Tx - Ty)\| = \|\gamma_\lambda\|$ $(\forall x, y \in G,\ \lambda \in \Lambda)$.*

Then T can be extended (uniquely) to an affine homeomorphism from E onto F. Moreover, the affine extension is also an isometry if, in addition, (like (4)) there exist $\{t_n\} \subset \mathbb{R}^+$ with $t_n \to \infty$ such that $\|t_n(Tx - Ty)\| = \|t_n(x - y)\|$ $(\forall x, y \in G,\ n \in \mathbb{N})$.

The isometric extension problem from some distance preserving property was proposed by Aleksandrov [1] as follows:

Aleksandrov problem. Under what conditions is a mapping of a metric space into itself an isometry preserving the unit distance?

Assume that E and E_1 are metric spaces. Then a mapping $f : E \to E_1$ is called an isometry if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in E$. Moreover, $f(x)$ is said to have the distance one preserving property (or DOPP for short) if $d(x, y) = 1$ implies $d(f(x), f(y)) = 1$, and the strong distance one preserving property (or SDOPP for short) if $d(x, y) = 1$ if and only if $d(f(x), f(y)) = 1,\ \forall x, y \in E$.

Even if E and E_1 are normed spaces, the above problem is also not easy to answer. For example, the following question has not been solved yet.

Question. Is there a mapping f from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that f satisfies DOPP but is not an isometry?

Mielnik and Rassias [16] gane an example showing that there exist two normed spaces E and E_1 as well as a mapping $f : E \to E_1$ such that f satisfies DPP but is not an isometry [16]. In general, a mapping f is not an isometry even if it satisfies SDOPP and is surjective as well as continuous. Wang [24] gave a simple and beautiful counterexample in the normed space $(E \oplus \mathbb{R})_\infty$, where E is an arbitrary normed space.

Rassias and Semrl [19] proved the following result: Let E and E_1 be two real normed spaces with $\max(\dim E, \dim E_1) > 1$. If f is a surjective and non-expansive mapping (i.e., $\|fx - fy\| \leq \|x - y\|$, $\forall x, y \in E$) satisfying SDOPP, then f is an isometry (so it is also affine by Mazur-Ulam theorem).

Let E and E_1 be two real normed spaces and E_1 strictly convex. In 1997, Ma [14] obtained that f is an isometry (so it is also affine by Baker's theory) if f is a non-expansive mapping (that is, only the condition $\|x - y\| \leq 1$ is assumed here) satisfying DOPP. She also got some similar results in p-normed spaces $(0 < p < 1)$.

During the last few months, Xiang Shuhuang has done much work on the above problem. For example, when E and E_1 are two real Hilbert spaces, he obtained that $f : E \to E_1$ is an affine isometry if $\dim E \geq 2$ and $f : E \to E_1$ preserves the two distances 1 and $\sqrt{3}$, or $\dim E \geq 3$ and f preserves the two distances 1 and $\sqrt{2}$, or $\dim X \geq 2$ and f preserves the three distances 1, a and $n\sqrt{4 - a^2}$ $(0 \leq a \leq 2,\ n \in \mathbb{N})$.

Thus, it is also a difficult problem to find the conditions for the concrete classical Banach spaces or Frechet spaces, under which a mapping satisfying DOPP or SDOPP is also an isometry.

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