TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 1, pp. 1-17, March 2001

SIMILARITY PROBLEMS AND LENGTH

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Abstract. This is a survey of the author's recent results on the Kadison and Halmos similarity problems and the closely connected notion of "length" of an operator algebra.

1. Introduction

We start by recalling a well-known conjecture formulated by Kadison [19] in 1955.

Kadison's similarity problem. *Let A be a unital C^*-algebra and let $u : A \to B(H)$ (H Hilbert) be a unital homomorphism ($i.e.$, we have $u(1) = 1$ and $u(ab) = u(a)u(b) \forall a, b \in A$). Show that if u is bounded, then u is similar to a $*$-homomorphism, i.e., $\exists\ \xi : H \to H$ invertible such that $u_\xi : a \mapsto \xi^{-1}u(a)\xi$ is a $*$-homomorphism ($=$ C^*-representation).*

Explicitly, the conclusion means that

$$(\forall\ a \in A) \qquad \xi^{-1}u(a^*)\xi = (\xi^{-1}u(a)\xi)^*;$$

when this holds, Kadison calls u "orthogonalizable". Many partial results are known, mainly due to Erik Christensen ([4-7]) and Uffe Haagerup ([15]). In particular, they established (see [6] and [15]) this conjecture for *cyclic* homomorphisms, *i.e.,* when u admits a cyclic vector h in H (= a vector h such that $\overline{u(A)h} = H$) or, more generally, when u admits a finite cyclic set $h_1, \ldots, h_n$ (so that we have $\overline{u(A)h_1 + \cdots + u(A)h_n} = H$).

In addition, the Kadison conjecture is known in the following cases:

Received April 26, 2000.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: 46L07, 47L55, 47D99.
Key words and phrases: Similarity problem, completely bounded map, length.
*Supported in part by the NSF and the Texas Advanced Research Program 010366-163. This is an expanded version of a lecture given as keynote speaker at the January 2000 ICMAA in Taiwan.

(i) A is commutative.

(ii) A is the unitization, denoted by $\widetilde{\mathcal{K}}$, of the C^*-algebra, denoted by $\mathcal{K}$, of all compact operators on ℓ_2, or more generally when A is nuclear (see [5]).

(iii) $A = B(\mathcal{H})$ or, more generally, when A has no tracial states (see [15]).

(iv) $A = \widetilde{\mathcal{K}} \otimes B$ with B an arbitrary unital C^*-algebra.

(v) A is a II_1-factor with Murray and von Neumann's property Γ (see [7]); for instance, when A is the so-called hyperfinite II_1-factor (= infinite tensor product of 2×2 matrices with normalized trace).

In sharp contrast, the conjecture is still open when A is the reduced C^*-algebra of the free group with at least 2 generators, or even when

$$A = \left(\oplus \sum_{n \geq 1} M_n \right)_\infty = \{x = (x_n) \mid x_n \in M_n, \quad \sup \|x_n\| < \infty\}.$$

Kadison formulated his conjecture as the C^*-algebraic version of a well-known problem (at the time of his writing): Are all uniformly bounded group representations similar to unitary representations (= unitarizable)? While a counterexample to that question was soon found ([14]; see also [37] for more recent results on this theme), Kadison's conjecture remained open. Recently, it became entirely clear that his conjecture is equivalent to another important open question, the derivation problem, itself a crucial problem in the cohomology theory of operator algebras (cf. [39]).

Derivation problem. *Let $\pi : A \to B(H)$ be a $*$-homomorphism (= representation) on a C^*-algebra A. Let $\delta : A \to B(H)$ be a π-derivation (i.e., $\delta(ab) = \pi(a)\delta(b) + \delta(a)\pi(b)$). Show that the boundedness of δ (which is actually automatic here) implies that δ is inner, which means: $\exists\ T \in B(H)$ such that $\delta(a) = \pi(a)T - T\pi(a) \forall\ a \in A$. We set*

$$\delta_T(a) = \pi(a)T - T\pi(a).$$

The connection between the two problems is simple. Intuitively derivations appear as "infinitesimal generators" for homomorphisms. More elementarily, if δ is as above then

$$u(a) = \begin{pmatrix} \pi(a) & \delta(a) \\ 0 & \pi(a) \end{pmatrix}$$

is a homomorphism into $B(H \oplus H) = M_2(B(H))$. Kirchberg [21] recently proved that the C^*-algebras which satisfy the derivation problem are exactly the same as those which satisfy Kadison's conjecture, but it is still open whether this class is that of all C^*-algebras!

We now turn to a key notion to study these problems: "complete boundedness" (see [26]).

Definition. Let $E \subset B(H)$ and $F \subset B(K)$ be operator spaces, and consider a map

$$\begin{array}{ccc} B(H) & & B(K) \\ \cup & & \cup \\ E & \xrightarrow{u} & F \end{array}$$

For any $n \geq 1$, let $M_n(E) = \{(x_{ij})_{i,j \leq n} \mid x_{ij} \in E\}$ be the space of $n \times n$ matrices with entries in E. In particular, we have a natural identification $M_n(B(H)) \simeq B(\ell_2^n(H))$, where $\ell_2^n(H)$ means $\underbrace{H \oplus H \oplus \cdots \oplus H}_{n \text{ times}}$. Thus, we may equip $M_n(B(H))$ and a fortiori its subspace $M_n(E) \subset M_n(B(H))$ with the norm induced by $B(\ell_2^n(H))$. Then, for any $n \geq 1$, the linear map $u : E \to F$ allows to define a linear map $u_n : M_n(E) \longrightarrow M_n(F)$ by setting

$$u_n \begin{pmatrix} & \vdots & \\ \dots & x_{ij} & \cdots \\ & \vdots & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \dots & u(x_{ij}) & \dots \\ & \vdots & \end{pmatrix}.$$

A map $u : E \to F$ is called *completely bounded* (in short c.b.) if

$$\sup_{n \geq 1} \|u_n\|_{M_n(E) \to M_n(F)} < \infty.$$

We define

$$\|u\|_{cb} = \sup_{n \geq 1} \|u_n\|_{M_n(E) \to M_n(F)}$$

and we denote by $cb(E, F)$ the Banach space of all c.b. maps from E into F equipped with the c.b. norm.

This concept is fundamental in the currently very actively developed theory of operator spaces; see [38].

Theorem 1 (Haagerup 1983, [15]). *In the situation of Kadison's similarity problem, u is similar to a $*$-homomorphism if and only if u is c.b. Moreover, we have*

$$\|u\|_{cb} = \inf\{\|\xi^{-1}\| \, \|\xi\| \mid u_\xi \ * -\text{homomorphism}\}.$$

For derivations, the analogous result is the following.

Theorem 2 (Christensen 1977, [8]). *In the derivation problem, δ is inner if and only if δ is c.b. Moreover, we have*

$$\|\delta\|_{cb} = \inf\{2\|T\| \mid \delta = \delta_T\}.$$

Vern Paulsen generalized Haagerup's result to the non-self-adjoint case:

Theorem 3 (Paulsen 1984, [27]). *Let A be a unital operator algebra ($i.e.$, we assume only that A is a closed subalgebra of $B(\mathcal{H})$ with $I \in A$). Consider again a homomorphism $u : A \to B(H)$. Then $\|u\|_{cb} < \infty$ if and only if u is similar to a completely contractive homomorphism, i.e., $\exists\ \xi : H \to H$ invertible such that $u_\xi : a \mapsto \xi^{-1}u(a)\xi$ satisfies $\|u_\xi\|_{cb} = 1$. Moreover, we have*

$$\|u\|_{cb} = \inf\{\|\xi\|\ \|\xi^{-1}\| \mid \|u_\xi\|_{cb} = 1\},$$

and this infimum is attained.

It is easy to see that if A is a C^*-algebra, then

$$\|u\|_{cb} = 1 \Leftrightarrow \|u\| = 1 \Leftrightarrow u \text{ is a } *\text{-homomorphism.}$$

This explains why Theorem 3 contains Theorem 1 as a special case. The preceding result leads us naturally to enlarge our investigation to the non-self-adjoint case as follows.

Generalized Similarity Problem. *Which unital operator algebras A have the following property denoted by* (SP) ?

(SP) Any bounded homomorphism $u : A \to B(H)$ (H an arbitrary Hilbert space) is c.b.

Loosely speaking, this property (SP) could be described as "automatic complete boundedness" in analogy with the field of automatic continuity for homomorphisms between Banach algebras (see [13]).

Example. The most natural example of a non-self-adjoint algebra is the disc algebra $A = A(\mathbb{D})$ which can be described as the completion of the set of all polynomials P for the norm

$$\|P\|_\infty = \sup\{|P(z)| \mid z \in \partial\mathbb{D}\}.$$

We consider $A(\mathbb{D})$ as an operator subalgebra of the commutative C^*-algebra $C(\partial\mathbb{D})$.

Consider a *fixed* operator $x \in B(H)$. Let

$$u^x : P \mapsto P(x) \in B(H)$$

be the homomorphism of evaluation at this fixed x. Then u^x is bounded if and only if x is polynomially bounded, *i.e.*, $\exists C$ such that

$$\forall P \qquad \|P(x)\| \le C\|P\|_\infty. \tag{1}$$

On the other hand, it follows from Paulsen's similarity criterion (Theorem 3 above) that u^x is c.b. if and only if x is similar to a contraction (which means $\exists \xi : H \to H$ invertible such that $\|\xi^{-1}x\xi\| \le 1$). Indeed, when $\|x\| \le 1$, von Neumann's classical inequality shows that (1) holds with $C = 1$ and actually also (Sz.-Nagy's dilation) that $\|u^x\|_{cb} = 1$. Thus it is the same to ask whether $A(\mathbb{D})$ satisfies (SP) or to ask whether any polynomially bounded operator x is similar to a contraction. This was a well-known problem originally formulated by Halmos in a landmark 1970 paper [18]. We have recently given a counterexample as follows.

Theorem 4 (1997, [31]). *For any $c > 1$, there is a unital homomorphism $u : A(D) \to B(\ell_2)$ (necessarily of the form $P \mapsto P(x)$ for some x in $B(\ell_2)$) such that $\|u\| \le c$ but $\|u\|_{cb} = \infty$.*

The proof of the polynomial boundedness was simplified in [22] and [11].

Although this solves the somewhat prototypical case of $A(\mathbb{D})$, it leaves open the following question: Is it true that any uniform algebra (*i.e.*, a unital subalgebra of $C(K)$ for some compact set K) which is proper (*i.e.*, A separates the points of K and $A \ne C(K)$) fails (SP)?

See [23] for a partial result on this. Actually, when K is a domain in $\mathbb{C}$ with at least 2 holes, it is already unknown in general whether $\|u\| = 1$ implies $\|u\|_{cb} = 1$! The case of a single hole is covered by [1]. See also [12] and [29] for more on this theme.

Remarks.

(i) See [24, 25] for some recent progress on conditions for an operator to be similar to a normal operator.

(ii) The recent paper [KLM] contains the following striking example: For any $c > 1$ there is a power bounded operator on ℓ_2 which is not similar to any operator with powers bounded by c. The corresponding statement for polynomial boundedness seems open: Given $c > 1$, is there a polynomially bounded operator which is not similar to any operator polynomially bounded by c ?

We now turn to the notion of length which seems closely connected to the generalized similarity problem. The "length" that we have in mind is analogous to the following situation: Consider a unital semigroup S and a unital generating subset

$B \subset S$. It is natural to say that B generates S with length $\leq d$ if any x in S can be written as a product $x = b_1 b_2 \dots b_d$ with each b_i in B. We will use a somewhat "dual" viewpoint on the "length" based on homomorphisms. Our main idea can be illustrated in a rather transparent way on the above simple model of semigroups as follows. Assume that B generates S with length $\leq d$. Then any homomorphism $\pi : S \to B(H)$ (*i.e.*, $\pi(st) = \pi(s)\pi(t)$ and $\pi(1) = 1$) which is bounded on B with $\sup_{b \in B} \|\pi(b)\| \leq c$ must be bounded on the whole of S with $\sup_{s \in S} \|\pi(s)\| \leq c^d$.

Conversely, assume that we know that for some $\alpha \geq 0$ and $\kappa \geq 0$, all homomorphisms $\pi : S \to B(H)$ satisfy, for some $c > 1$, the following implication:

$$\sup_{b \in B} \|\pi(b)\| \leq c \Rightarrow \sup_{s \in S} \|\pi(s)\| \leq \kappa c^{\alpha}.$$

Then it is rather easy to see that B necessarily generates S with length $\leq [\alpha]$ (integral part of α), so that we can replace α by $[\alpha]$ and κ by 1.

We call this a "dual" viewpoint because it is reminiscent of the fact that the closed convex hull C of a subset $B \subset E$ of a Banach space is characterized by the implication

$$\sup_{b \in B} f(b) \leq 1 \Rightarrow \sup_{s \in C} f(s) \leq 1$$

for all continuous real linear forms f.

Although this is a wild analogy, we feel that our results on the length are a kind of "nonlinear" analog of this very classical duality principle for convex hulls.

In [32], we study various analogs of this concept of length for operator algebras or even for general Banach algebras. Surprisingly little seems to have been known up to now. We will now review the main results of our papers.

Definition. An operator algebra $A \subset B(\mathcal{H})$ is said to be of length $\leq d$ if there is a constant K such that, for any n and any x in $M_n(A)$, there is an integer $N = N(n, x)$ and scalar matrices $\alpha_0 \in M_{n,N}(\mathbb{C})$, $\alpha_1 \in M_N(\mathbb{C}), \dots, \alpha_{d-1} \in M_N(\mathbb{C})$, $\alpha_d \in M_{N,n}(\mathbb{C})$ together with diagonal matrices $D_1, \dots, D_d$ in $M_N(A)$ satisfying

$$\begin{cases} x = \alpha_0 D_1 \alpha_1 D_2 \dots D_d \alpha_d, \\ \displaystyle\prod_0^d \|\alpha_i\| \prod_1^d \|D_i\| \leq K\|x\|. \end{cases}$$

We denote by $\ell(A)$ the smallest d for which this holds and we call it the "length" of A (so that A has length $\leq d$ is indeed the same as $\ell(A) \leq d$).

Equivalently, we may reformulate this using infinite matrices: If we view as usual $M_n(A) \subset M_{n+1}(A)$ via the mapping $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, and if we let $\mathcal{K}(A) =$

$\overline{\cup M_n(A)}$ be the completion of the union with the natural extension of the norm, then it is easy to check that $\ell(A) \le d$ if and only if any x in $\mathcal{K}(A)$ can be written as

$$x = \alpha_0 D_1 \alpha_1 \dots D_d \alpha_d$$

with α_i in $\mathcal{K}(\mathbb{C})$ and D_i diagonal in $\mathcal{K}(A)$. (The constant K automatically exists by the open mapping theorem.)

Our central result is as follows.

Theorem 5 (1999, [32]). *A unital operator algebra A satisfies* (SP) *if and only if $\ell(A) < \infty$. Moreover, if*

$$d(A) = \inf\{\alpha \ge 0 \mid \exists\, K\ \forall\, u\ \|u\|_{cb} \le K\|u\|^{\alpha}\}$$

(*here, of course, u denotes an arbitrary unital homomorphism from A to $B(H)$*), *then*

$$d(A) = \ell(A)$$

and the infimum defining $d(A)$ is attained.

Proof of $d(A) \le \ell(A)$. This is the easy direction. The converse is much more involved. Assume $\ell(A) \le d$. Consider x in $M_n(A)$. Recall that $\|u\|_{cb} = \sup\{\|u_n(x)\|_{M_n(B(H))} \mid n \ge 1,\ \|x\|_{M_n(A)} \le 1\}$. Consider a factorization of the above form:

$$x = \alpha_0 D_1 \dots D_d \alpha_d$$

with α_i "scalar" and D_i "diagonal". We have then

$$u_n(x) = \alpha_0 u_N(D_1)\alpha_1 \dots u_N(D_d)\alpha_d,$$

and hence

$$\|u_n(x)\| \le \prod \|\alpha_i\| \prod \|u_N(D_i)\|.$$

But clearly since the D_i's are diagonal, $\|u_N(D_i)\| \le \|u\|\ \|D_i\|$. Hence

$$\|u_n(x)\| \le \|u\|^d \prod \|\alpha_i\| \prod \|D_i\|,$$

which yields (recalling the meaning of $\ell(A) \le d$)

$$\|u\|_{cb} \le K\|u\|^d.$$ ■

Remark 6. Let us briefly return to the derivation problem. If A is a C^*-algebra, Kirchberg's argument in [21], as slightly improved in [32], shows that if we have

$$\|\delta\|_{cb} \le \alpha\|\delta\| \qquad (2)$$

for all π and all π-derivations $\delta : A \to B(H)$ then we have $\|u\|_{cb} \le \|u\|^\alpha$ for all u as in Theorem 5. Therefore $\ell(A)$ is less or equal to the integral part of α. This leads us to conjecture that, in the C^*-case, the best possible α in (2) is always an integer. Also when A is an infinite-dimensional C^*-algebra, we have no example of A for which the best K such that: $\forall u \ \|u\|_{cb} \le K\|u\|^{d(A)}$ is > 1, but we believe such examples exist (we suspect $A = B(H) \oplus \ell_\infty$ might be such an example).

It is easy to see that if $I \subset A$ is a closed two-sided ideal then $\ell(A/I) \le \ell(A)$ and also that $\ell(A) \le \max\{\ell(I), \ell(A/I)\}$. If A is a C^*-algebra, we have

$$\ell(A) = \max\{\ell(I), \ell(A/I)\}.$$

To show $\ell(I) \le \ell(A)$ we merely use the fact (due to Arveson; see, *e.g.*, [40]) that there is a "quasi-central approximate unit" in I, *i.e.*, a net (a_i) in the unit ball of I such that for any x in I we have $xa_i \to x$ and $a_ix \to x$ and moreover (quasi-centrality) $a_ia - aa_i \to 0$ for any a in A.

In particular, for all finite sets $A_1, \dots, A_n$ of operator algebras we have

$$\ell(A_1 \oplus ... \oplus A_n) = \max\{\ell(A_i) \mid 1 \le i \le n\}.$$

The case of infinite direct sums is discussed in [36].

Remark 7. Let $H = \ell_2$. One useful way to apply Theorem 5 is as follows: Given a *d-linear* map $w : A^d \to B(H)$, we may consider all the possible ways to "factorize" w so that there exist linear bounded maps $v_i : A \to B(H)$ such that

$$(\forall (a_1, \dots, a_d) \in A^d) \qquad w(a_1, a_2, \dots, a_d) = v_1(a_1)v_2(a_2)\dots v_d(a_d).$$

Then we set

$$|||w|||_d = \inf\left\{\prod_{i=1}^d \|v_i\|\right\},$$

where the product runs over all possible ways to "factorize" w as above. Then let $v : A \to B(H)$ be a *linear* map. Assume that we have a finite set of d-linear maps w_p as before such that

$$(\forall\ a_i \in A \quad (1 \le i \le d)) \qquad v(a_1a_2\dots a_d) = \sum_p w_p(a_1, \dots, a_d).$$

Then we set

$$\|v\|_{[d]} = \inf\left\{\sum_p |||w_p|||_d\right\},$$

where the infimum runs over all possible ways to write as $v = \sum_p w_p$. Then if $\ell(A) \le d$, it is a simple exercise to show that for any *linear* $v : A \to B(H)$ we have

$$\|v\|_{cb} \le K\|v\|_{[d]}.$$

Thus Theorem 5 allows to strengthen the property (SP): Not only homomorphisms are c.b. but also all linear maps v for which we have $\|v\|_{[d]} < \infty$. Actually, it is possible to show that $w \mapsto |||w|||_d$ is subadditive but we will not really need this. This norm $\| \ \|_{[d]}$ is closely connected with the notion of "multilinear c.b. map" introduced by E. Christensen and A. Sinclair (see [9, 10]).

Examples. If $1 < \dim A < \infty$, then $d(A) = 1$, so from now on we assume $\dim A = \infty$. We can now review the examples of C^*-algebras listed previously:

(i) If A is commutative, then $d(A) = 2$.

(ii) If $A = \widetilde{\mathcal{K}}$ or if A is nuclear, then also $d(A) = 2$.

(iii) If $A = B(H)$, then $d(A) = 3$.

(iv) If $A = \widetilde{\mathcal{K}} \otimes B$ with B an arbitrary unital C^*-algebra, then $2 \le d(A) \le 3$.

(v) If A is a II_1-factor with property Γ, then $3 \le d(A) \le 5$.

Notes: (i) and (ii) are due to J. Bunce and E. Christensen (see [5]). In (iii), ≤ 3 is proved in [15] while ≥ 3 is proved in [32] (see below). (iv) is essentially in [15]. Finally, concerning (v), Christensen proved in [7] that $d(A) \le 44$, but the estimate was reduced in [36]. It was also observed in [36] that (as pointed out by N. Ozawa) Anderson's construction in [2] remains valid on any II_1 factor, thus yielding $d(A) \ge 3$ for any II_1 factor A by the same argument as in [32].

The class of algebras with $d(A)(= \ell(A))$ equal to 2 is closely related to that of "amenable Banach algebras" (see, *e.g.*, [30]). A von Neumann algebra $M \subset B(\mathcal{H})$ is called amenable (= injective) if there is a projection $P : B(\mathcal{H}) \to M$ with $\|P\| = 1$. It is known that a C^*-algebra A is nuclear ($\Leftrightarrow$ amenable by [16]) if and only if for every representation (= $*$-homomorphism) $\pi : A \to B(H)$, the von Neumann algebra $M_\pi = \pi(A)''$ generated by π is amenable (= injective). This motivates the following

Definition. A C^*-algebra is called *seminuclear* if for any representation $\pi : A \to B(H)$ generating a semifinite von Neumann algebra $\pi(A)''$, the generated algebra $\pi(A)''$ is injective.

Theorem 8 [32]. *For a C^*-algebra A, $d(A) \le 2$ implies that A is seminuclear.*

It is an open problem whether in general seminuclear implies nuclear. However, if A is either the reduced or the full C^*-algebra of a discrete group G, then

$$A \text{ nuclear } \Leftrightarrow A \text{ semi-nuclear } \Leftrightarrow G \text{ amenable.}$$

The preceding result shows that $d(B(H)) > 2$, since otherwise $B(H)$ would be semi-nuclear, which contradicts [2]. Hence, we have $d(B(H)) \geq 3$. Actually, using the length $\ell(B(H))$ instead, we can obtain a very simple proof that $d(B(H)) = 3$, as follows.

Proof of $\ell(B(H)) \leq 3$. This very direct proof comes from [36]. Fix $n \geq 1$. Let W_1 and W_2 be any two $n \times n$ unitary matrices such that

$$\forall\ i,j \qquad |W_1(i,j)| = |W_2(i,j)| = n^{-1/2}.$$

Then, for any x in the unit ball of $M_n(B(H))$ (with $\dim H = \infty$) there are diagonal matrices D_1, D_2, D_3 also in the unit ball of $M_n(B(H))$ such that

$$x = D_1 W_1 D_2 W_2 D_3.$$

The proof of this is very simple. Let S_i, $i = 1, \dots, n$, be isometries on H with orthogonal ranges so that

$$\forall\ i,j \qquad S_i^* S_j = \delta_{ij} I.$$

Then let

$$D_1(i,i) = S_i^* \quad \text{and} \quad D_3(j,j) = S_j$$

and moreover

$$D_2(k,k) = n \sum_{i,j} \overline{W_1(i,k)} S_i x_{ij} S_j^* \overline{W_2(k,j)}.$$

It is an easy exercise (left to the reader) to check the announced properties. ∎

By Theorems 1 and 5, we have:

Proposition 9. *The Kadison similarity problem has a positive answer for all unital C^*-algebras A if and only if there is an integer d_0 such that $\ell(A) \leq d_0$ for any C^*-algebra A.*

Unfortunately, up to now, the highest known value of $\ell(A)$ for a C^*-algebra is 3, but we conjecture that there are examples of arbitrarily large length. However, in the *non-self-adjoint* case, we have recently been able to prove the following.

Theorem 10 ([34]). *For any integer $d \geq 1$, there is a (non-self-adjoint) operator algebra A_d such that $\ell(A_d) = d$.*

Problem. *Are there uniform algebras with arbitrarily large finite length*?

For uniform algebras, no example with $2 < \ell(A) < \infty$ is known. However, it is proved in [32] that any proper uniform algebra A must satisfy $\ell(A) > 2$. It is also unknown whether there are Q-algebras (= quotients of uniform algebras) A with $2 < \ell(A) < \infty$.

Sketch of proof of Theorem 10. *The algebras A_d are not at all "pathological"; they are the "obvious" ones: the maximal operator algebras generated by a sequence of contractions (x_n) to which we impose the relations*

$$(\mathcal{R}_d) \qquad x_{n_1}x_{n_2}\dots x_{n_{d+1}} = 0$$

for any $(d+1)$-tuple of integers $(n_1, \dots, n_{d+1})$. However, while the proof that $d(A_d) \leq d$ is then quite easy, the fact that $\ell(A_d) > d-1$ has turned out to be much harder to prove. The proof given in [34] uses crucially Gaussian random matrices and specifically a recent difficult estimate due to Haagerup and Thorbjørnsen [17]. We will only give a brief description of the argument from [34]. Let $P = P(X_1, X_2, \dots)$ be a polynomial of degree $\leq d$ in *noncommuting* (formal) variables $X_1, X_2, \dots$. We introduce the norm

$$\|P\|_{A_d} = \sup\{\|P(x_1, x_2, \dots)\|\}, \tag{3}$$

where the supremum runs over all sequences of contractions in $B(\ell_2)$ satisfying $(\mathcal{R}_d)$. It is easy to check that this is a norm of the set of polynomials P with degree $\leq d$. We denote by A_d the completion of the set of P's equipped with this norm. Clearly, this defines an operator algebra naturally embedded into $\bigoplus_x B(H_x)$, where $H_x = \ell_2$ and where $x = (x_n)_{n\geq 1}$ runs over the set of all possible sequences of contractions satisfying $(\mathcal{R}_d)$. In order to show that $\ell(A_d) > d-1$, the next lemma is crucial. To state it we first need a specific notation.

Notation. Let $H = \ell_2$. Let $m \geq 1$ and $d \geq 1$ be fixed integers. We will denote by $C(m,d)$ the smallest constant C for which the following holds: If $\{x_i \mid i \in [m]^d\}$ in $B(H)$ satisfies

$$\forall \lambda_i \in \mathbb{C} \qquad \left\|\sum \lambda_i x_i\right\| \leq \sup_{\substack{X_i\in B(H)\\ \|X_i\|\leq 1}} \left\{\left\|\sum \lambda_i X_{i_1}X_{i_2}\dots X_{i_d}\right\|\right\}, \tag{4}$$

then $\exists\ \hat{x}_k \in B(H)$ $(1 \leq k \leq m)$ with $\|\hat{x}_k\| \leq 1$ such that

$$\forall\ i \in [m]^d \qquad x_i = C\hat{x}_{i_1}\hat{x}_{i_2}\dots\hat{x}_{i_d}. \tag{4)$'$}$$

Lemma 11. *For any $m \geq 1$ and $d \geq 1$, we have*

$$\delta_d m^{\frac{d-1}{2}} \leq C(m,d) \leq m^{\frac{d-1}{2}},$$

where $\delta_d > 0$ is a constant independent of m.

Example. In the case $d = 2$, this means the following: If $x_{ij} \in B(H)$ $(i,j = 1,...,m)$ satisfy

$$(\forall\ \lambda_{ij} \in \mathbb{C}) \qquad \left\| \sum_{ij \leq m} \lambda_{ij} x_{ij} \right\| \leq \sup_{\|X_i\| \leq 1} \left\| \sum \lambda_{ij} X_i X_j \right\|,$$

then x_{ij} can be factorized as

$$x_{ij} = C \hat{x}_i \hat{x}_j \quad \text{with} \quad \|\hat{x}_i\| \leq 1, \tag{5}$$

but in general the best possible C will be $\simeq \sqrt{m}$. This case is rather easy to prove given the state of the art. However, already the case $d = 3$ is more delicate, and, as we already mentioned, the case of an arbitrary d requires the upper estimates given in [17] which are highly nontrivial. An easier proof of the lower bound (which is the difficult part) in Lemma 11 would be most welcome.

Remark 12. Given $\{x_i \mid i \in [m]^d\}$ in $B(H)$ satisfying (4), we can define a *linear* map

$$v : A_d \to B(H)$$

by setting $v(X_{i_1} X_{i_2} \ldots X_{i_d}) = x_{i_1 i_2 \ldots i_d}$ with $1 \leq i_1, i_2, \ldots, i_d \leq m$ and $v(X_{i_1} X_{i_2} \ldots X_{i_k}) = 0$ in all other cases. Then it can be shown, using the factorization of multilinear cb maps of Christensen-Sinclair and Paulsen-Smith (see [34]) that $\|v\|_{cb}$ is equal to the smallest constant C such that $(4)'$ holds.

We now wish to sketch how Lemma 11 is used to prove that $\ell(A_d) > d - 1$. To lighten the exposition, we will restrict to the simplest case: $d = 3$. So we will show that Lemma 11 implies $\ell(A_3) > 2$. We will show that if $\ell(A_3) \leq 2$ then $C(m,d) \leq K\sqrt{m}$ for some K, but this will contradict Lemma 11 for $d = 3$ since $(d-1)/2 = 1 > 1/2$, whence the conclusion that $\ell(A_3) > 2$.

Now assume $\ell(A_3) \leq 2$. Let $\{x_{i_1 i_2 i_3} \mid i \in [m]^3\}$ be as in the definition of $C(m,d)$ for $d = 3$. For convenience, we extend the function $(i_1, i_2, i_3) \mapsto x_{i_1 i_2 i_3}$ to the whole of $\mathbb{N}^3$ by setting it equal to zero outside $[1, \ldots, m]^3$.

We will use Remark 7.

Let $v : A_3 \to B(H)$ be the linear map defined by $v(1) = 0$, $v(X_i) = 0$, $v(X_{i_1} X_{i_2}) = 0$ and finally:

$$v(X_{i_1} X_{i_2} X_{i_3}) = x_{i_1 i_2 i_3}.$$

It is easy to see using (3) and (4) that $\|v\| \le 1$.

We claim that (4) implies (with the notation of Remark 7) $\|v\|_{[2]} \le 2 + 2\sqrt{m}$.

We will use the following notation: We consider the disjoint union

$$\Omega = \phi \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3,$$

and we set

$$\begin{array}{rcl} X^{\phi} & = & 1, \\ X^{i} & = & X_i \quad \text{if} \quad i \in \mathbb{N} \\ X^{ij} & = & X_i X_j \quad \text{if} \quad (i,j) \in \mathbb{N}^2 \\ X^{ijk} & = & X_i X_j X_k \quad \text{if} \quad (i,j,k) \in \mathbb{N}^3. \end{array}$$

For $i \in \Omega$ we set $|i| = 0$ if $i = \phi$, and $|i| = k$ if $i \in \mathbb{N}^k$.

With this notation, any polynomial P in A_3 can be written as a finite sum

$$P = \sum_{i \in \Omega} \lambda_i(P) X^i$$

with $\lambda_i(P) \in \mathbb{C}$. We have then $v(X^i) = x_i$ for all i in Ω, and hence $\forall\, P_1, P_2 \in A_3$,

$$v(P_1 P_2) = \sum_{i,j \in \Omega} \lambda_i(P_1) \lambda_j(P_2) x_{ij},$$

where ij denotes now the multi-index of length ≤ 6 obtained by putting j after i. We set $|ij| = |i| + |j|$. But since $x_{ij} = 0$ unless $|i| + |j| = 3$ we find a decomposition of v as follows:

(6) $$v(P_1 P_2) = \sum_{(\alpha\beta) \in J} w_{\alpha\beta}(P_1, P_2),$$

where the sum runs over the set J of all pairs (α, β) in [0, 1, 2, 3] such that $\alpha + \beta = 3$, and where $w_{\alpha\beta}$ are bilinear forms on $A_3 \times A_3$ defined by setting

$$w_{\alpha\beta}(P_1, P_2) = \sum_{\substack{|i| = \alpha \\ |j| = \beta}} \lambda_i(P_1) \lambda_j(P_2) x_{ij}.$$

Using (4), it is easy to see that if (α, β) is either (3, 0) or (0, 3), then, with the notation of Remark 7, $|||w_{\alpha\beta}|||_2 \le 1$.

The remaining possibilities in J are only (2, 1) and (1, 2). But if $(\alpha, \beta) = (1, 2)$ for instance, we can write

$$w_{\alpha\beta}(P_1, P_2) = \left(\sum_{|i|=1} \lambda_i(P_1) e_{1i} \otimes I \right) \left(\sum_{k=1}^{m} e_{k1} \otimes \sum_{|j|=2} \lambda_j(P_2) x_{kj} \right)$$

(here we identify $B(H)$ with $B(H)\overline{\otimes}B(H)$ and denote by (e_{ki}) the standard matrix units in $B(H)$). Using this, one can check rather easily that if $(\alpha, \beta) = (2,1)$ or $(1,2)$ then $|||w_{\alpha\beta}|||_2 \le \sqrt{m}$.

Thus using (6) we obtain our claim that $\|v\|_{[2]} \le 2 + 2\sqrt{m}$.

Then if we assume $\ell(A_3) \le 2$, Remark 7 ensures that $\|v\|_{cb} \le K(2 + 2\sqrt{m})$.

Now by Remark 12 this implies that $\{x_i \mid i \in [m]^3\}$ satisfies (4)$'$ with $C \le K(2 + \sqrt{m})$. Thus we conclude that $C(m, 3) \le K(2 + \sqrt{m})$ but this obviously contradicts Lemma 11 with $d = 3$. Thus we have shown, by this contradiction, that $\ell(A_3) > 2$. ∎

The notion of length is quite natural in the more general context of a Banach algebra B generated by a family of subalgebras $B_i \subset B$ $(i \in I)$. For simplicity, we will restrict ourselves to the case of a pair of subalgebras $B_1 \subset B$, $B_2 \subset B$. In this case, we say that B_1, B_2 generate B with length $\le d$ if there is a bounded subset of the union $C \subset B_1 \cup B_2$ such that every x in the unit ball of B belongs to the closed convex hull of the union $\cup_{j=1}^{d} C^j$, where

$$C^j = \{x_1 x_2 \ldots x_j \mid x_k \in C \quad \forall k = 1, ..., j\}.$$

Assuming that this holds, let $u : B \mapsto \beta$ be a continuous homomorphism into another Banach algebra β. It is then easy to check that

$$\|u\| \le K \sum_{j=1}^{d} \max\{\|u_{|B_1}\|, \|u_{|B_2}\|\}^j,$$

where K is a constant (depending only on d and the size of the subset C).

Thus if B, β and u are all unital, we obtain (since all the norms are now ≥ 1)

$$\|u\| \le dK \max\{\|u_{|B_1}\|, \|u_{|B_2}\|\}^d.$$

In the converse direction, assuming again B_1, B_2 and B all unital, let $\mathrm{alg}(B_1, B_2)$ denote the algebra generated by B_1 and B_2, which we assume is dense in B. Assume that every unital homomorphism $u : \mathrm{alg}(B_1, B_2) \to \beta$ into an arbitrary unital Banach algebra β such that $\|u_{|B_1}\| < \infty$ and $\|u_{|B_2}\| < \infty$ is actually bounded and satisfies

$$\|u\| \le K(\max\{\|u_{|B_1}\|, \|u_{|B_2}\|\})^\alpha,$$

where K and $\alpha \ge 0$ are independent of u and β.

Then it follows (see [32, §8]) that B_1, B_2 generate B with length at most equal to the integral part of α. For example, let A be a unital operator algebra, and let $B = \mathcal{K}(A)$. We may consider the subalgebra $B_1 \subset B$ formed of all the diagonal matrices (viewing the elements of $\mathcal{K}(A)$ as bi-infinite matrices with coefficients in A) and we let $B_2 = \mathcal{K}(\mathbb{C})$.

It is then easy to check that $\ell(A) \leq d$ implies that B_1, B_2 generate B with length $\leq 2d+1$. Conversely, if B_1, B_2 generate B with length $\leq m$, then $\ell(A) \leq \left[\frac{m+1}{2}\right]$.

Remark. The slight discrepancy appearing here comes from the fact that in the products appearing in the subset C^d we do not specify that the first term of the product must lie in B_2 or B_1 while in the corresponding definition of $\ell(A)$ the analogous term must be in B_2. This difficulty can be circumvented: One should then consider homomorphisms u : alg$(B_1, B_2) \to \beta$ such that $\|u_{|B_2}\| = 1$ and study the inequality $\|u\| \leq K\|u_{|B_1}\|^\alpha$. See [33] for more variations on this theme.

The case study of $\ell(A)$ suggests to examine many other examples of the same kind, for instance, the pair $B_1 = \mathcal{K}(A_1)$, $B_2 = \mathcal{K}(A_2)$, where $A_1 \subset A$, $A_2 \subset A$ are two closed subalgebras. In particular, we may consider the case where A is the maximal tensor product of two unital C^*-algebras C_1, C_2, namely, we take $A = C_1 \otimes_{\max} C_2$ with $A_1 = C_1 \otimes 1$ and $A_2 = 1 \otimes C_2$. All these cases are studied in [33], to which we refer the reader for several illustrating examples and more information.

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