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(p,q)-PROPERTIES OF A GENERALIZED RIESZ POTENTIALS GENERATED BY THE GENERALIZED SHIFT OPERATORS

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Abstract. In this paper, the inequality of Hardy-Littlewood-Sobolev type is established for the generalized Riesz potentials generated by the generalized shift operator with the functions in Sobolev spaces $W_{p,v}^m(\mathbf{R}_n^+)$.

1. INTRODUCTION

It is well known that the fractional integrals $I_+^\alpha f = C_{n,\alpha} f * |x|^{\alpha-n}$ are bounded operators from $L_p(\mathbf{R}_n)$ to $L_q(\mathbf{R}_n)$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ (see [10]). $(p,q)-$properties of the classical Riesz potentials were reported in [2,3,6]. Moreover, the generalized Riesz potentials, generated by the generalized shift operator, are bounded operators from $L_{p,\nu}(\mathbf{R}_n^+)$ to $L_{q,\nu}(\mathbf{R}_n^+)$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n+2|\nu|}$ [11].

Sobolev functions play a significant role in many fields of analysis. In recent years, the Lebesgue spaces L_p and the corresponding Sobolev spaces W_p^m have attraction more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations. One of the most important results for Sobolev functions is so-called Sobolev's embedding theorem[1,3-5,8,9]. $(p,q,l)-$properties of the Riesz potentials on the Sobolev spaces $W_p^m(\mathbf{R}_n^+)$ were studied in [4]. The aim of this work is to define the generalized Riesz potentials generated by the generalized shift operator which acts on functions in the Sobolev spaces $W_{p,v}^m(\mathbf{R}_n^+)$ and to study its (p,q)-properties for these potentials. These properties can be described as a theorem of the Hardy-Littlewood-Sobolev type [10].

Now, we give some notations and definitions. $L_{p,v} = L_{p,v}(\mathbf{R}_n^+)$ and $W_{p,v}^m(\mathbf{R}_n^+)$ are defined with respect to the Lebesgue measure $\left(\prod_{i=1}^n x_i^{2v_i}\right) dx$ as below:

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$$L_{p,v}=L_{p,v}(\mathbf{R}_n^+)=\left\{f:\,\|f\|_{p,v}=\left(\int\limits_{\mathbf{R}_n^+}|f(x)|^p\left(\prod_{i=1}^n x_i^{2v_i}\right)dx\right)^{\frac{1}{p}}<\infty\right\},$$

$$W_{p,v}^m(\mathbf{R}_n^+)=\left\{f\in L_{loc}^1(\mathbf{R}_n^+):\,\|f\|_{W_{p,v}^m}\right.$$

$$\left.=\left(\sum_{0\le|k|\le m}\int\limits_{\mathbf{R}_n^+}\left|D^kf(x)\right|^p\left(\prod_{i=1}^n x_i^{2v_i}\right)dx\right)^{\frac{1}{p}}<\infty\right\},$$

where $1\le p<\infty,\ v=(v_1,...,v_n),\ v_1>0,...,v_n>0, |v|=v_1+...+v_n$, $D^k=\frac{\partial^{|k|}}{\partial x_1^{k_1}\partial x_2^{k_2}...\partial x_n^{k_n}},\ |k|=k_1+...+k_n$ and $\mathbf{R}_n^+=\{x:\ x=(x_1,...,x_n),\ x_1>0,...,x_n>0\}$.

We will say that the linear operator G, acting on Sobolev space $W_{p,v}^m(\mathbf{R}_n^+)$, has a weak type (p,q,m) if for any positive μ the inequality

$$mes\left\{x\in\mathbf{R}_n^+:|Gf(x)|>\mu\right\}\le C_{p,q,m,v}\left(\frac{\|f\|_{W_{p,v}^m}}{\mu}\right)^q$$

holds, where $mesE$ denote the Lebesgue measure of a set E and $C_{p,q,m,v}$ is a constant independent from f.

Denote by T^y the generalized shift operator acting according to the law

$$T^yf(x)=C_v\int\limits_0^\pi...\int\limits_0^\pi f\left(\sqrt{x_1^2+y_1^2-2x_1y_1\cos\alpha_1},...,\sqrt{x_n^2+y_n^2-2x_ny_n\cos\alpha_n}\right)$$
$$\times\left(\prod_{i=1}^n\sin^{2v_i-1}\alpha_id\alpha_i\right),$$

where $x,y\in\mathbf{R}_n^+$, $C_v=\prod\limits_{i=1}^n\frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$ [5].

The convolution operator determined by the T^y is as follows,

$$(f*\varphi)(y)=c_v\int\limits_{\mathbf{R}_n^+}f(y)T_x^y\varphi(x)(\prod_{i=1}^n y_i^{2v_i})dy. \tag{1}$$

The convolution (1) known as a B-convolution. We note the following properties of the B-convolution and operator T^y as described in [7, 11].

(i) $f*\varphi=\varphi*f$

(ii) $\|f*\varphi\|_{p,\nu} \leq \|f\|_{p,\nu} \cdot \|\varphi\|_{p,\nu}$, $1 \leq p\,, r \leq \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$

(iii) $T^y.1 = 1$

(iv) If $f(x), g(x) \in C(\mathbf{R}_n^+)$, $g(x)$ is a bounded function all $x \in \mathbf{R}_n^+$ and

$$\int\limits_{\mathbf{R}_n^+} |f(x)| \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx < \infty,$$

then

$$\int\limits_{\mathbf{R}_n^+} T_x^y f(x) g(y) (\prod_{i=1}^{n} y_i^{2v_i}) dy = \int\limits_{\mathbf{R}_n^+} f(y) T_x^y g(x) (\prod_{i=1}^{n} y_i^{2v_i}) dy.$$

(v) $|T_x^y f(x)| \leq \sup\limits_{x \geq 0} |f(x)|$.

Let m be a natural number, α_k are real numbers and $0 < \alpha_k < n$ for multi index k such that $0 < |k| < m$, where $|k| = k_1 + k_2 + ... + k_n$. Let also accept $f \in W_{p,v}^m(\mathbf{R}_n^+)$,$1 \leq p < \infty$ and consider the integral,

$$(2) \quad (R_{\alpha,m,v} f)(x) = \sum_{0 \leq |k| \leq m} C_k \int\limits_{\mathbf{R}_n^+} \left[D^k f(y)\right] T^y |x|^{\alpha_k - n - 2|v|} (\prod_{i=1}^{n} y_i^{2v_i}) dy$$

where $x \in \mathbf{R}_n^+$ and C_k are real constants.

It is obvious that the $R_{\alpha,m,v} f$ is the generalized Riesz potential generated by the generalized shift operator for $m = 0$. In addition, the integral $R_{\alpha,m,v} f$ is the classical Riesz potential when $m = 0$, $|\nu| = 0$ and $\cos\alpha_i = 1$, $i = 1, 2, ..., n$.

Lemma 1. *Let $f \in L_{p,v}(\mathbf{R}_n^+)$, $1 \leq p < \infty$. Then, we have the following inequality*

$$|T^y f(x)|^p \leq T^y |f(x)|^p \ , \quad \text{for} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Proof. Let

$$F(\alpha, x, y) = f\left(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\alpha_1}, ..., \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\alpha_n}\right).$$

From Hölder's inequality, we obtain

$$\begin{aligned} |T^y f(x)|^p &= \left| C_v \int_0^\pi ... \int_0^\pi F(\alpha, x, y) \prod_{i=1}^{n} (\sin^{2v_i - 1} \alpha_i d\alpha_i) \right|^p \\ &\leq \left[\left(C_v \int_0^\pi ... \int_0^\pi |F(\alpha, x, y)|^p \prod_{i=1}^{n} (\sin^{2v_i - 1} \alpha_i d\alpha_i) \right) \right] \end{aligned}$$

$$\times\left[\left(C_v\int\limits_0^\pi ...\int\limits_0^\pi \prod_{i=1}^n(\sin^{2v_i-1}\alpha_i d\alpha_i)\right)^{\frac{1}{p'}}\right]^p$$
$$\leq T^y\left(|f(x)|^p\right).$$

Lemma 2. *Let $f\in L_{p,v}(\mathbf{R}_n^+)$ and $1\leq p<\infty$. Then, we have*

$$\|T^y f\|_{p,v}\leq\|f\|_{p,v}\,.$$

Proof. From Lemma 1, we have

$$\begin{aligned}\|T^y f\|_{p,v}^p &= \int\limits_{\mathbf{R}_n^+}|T^y f(x)|^p\,(\prod_{i=1}^n y_i^{2v_i})dy\\ &\leq\int\limits_{\mathbf{R}_n^+}T^y\,|f(x)|^p\,(\prod_{i=1}^n y_i^{2v_i})dy.\end{aligned}$$

If we consider the properties (**iii**) and (**iv**) of the operator T^y, then we have the following inequality

$$\|T^y f\|_{p,\nu}\leq\left(\int\limits_{\mathbf{R}_n^+}|f(y)|^p\,(\prod_{i=1}^n y_i^{2v_i})dy\right)^{\frac{1}{p}}=\|f\|_{p,\nu}\,.$$

We prove the following Hardy-Littlewood Sobolev type theorem for (2).

Theorem 1. *Let $1\leq p<q<\infty,\alpha_{\max}=\max\limits_{0\leq|k|\leq m}\alpha_k$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha_{\max}}{n+2|v|}$. Then,*

(a) *The integral (2) absolutely convergence for almost every x.*

(b) *If $p>1$, then*

$$\|R_{\alpha,m,v}f\|_{q,v}\leq C_{p,q,m,v}\,\|f\|_{W_{p,v}^m}$$

where $C_{p,q,m,v}$ is the constant independent from f.

(c) *If $f\in W_{p,v}^m(\mathbf{R}_n^+)$, then for any $\mu>0$*

$$mes\left\{x:|R_{\alpha,m,v}f|>\mu\right\}\leq C_{p,q,m,v}\left(\frac{\|f\|_{W_{p,v}^m}}{\mu}\right)^q$$

that is, the mapping $f\to R_{\alpha,m,v}f$ is of weak type $(1,q,m)$.

Proof. Let us define for any positive μ, $C(\mu) = \max\limits_{0\leq|k|\leq m} (\mu^{\alpha_k-\alpha_{\max}}, \mu^{\alpha_k-\alpha_{\min}})$ where $\alpha_{\max} = \max\limits_{0\leq|k|\leq m} \alpha_k$ and $\alpha_{\min} = \min\limits_{0\leq|k|\leq m} \alpha_k$. Then it can be easy seen that,

$$|x|^{\alpha_k-n-2|v|} = \begin{cases} C(\mu)\,|x|^{\alpha_{\min}-n-2|v|} & \text{if } |x| \leq \mu \\ C(\mu)\,|x|^{\alpha_{\max}-n-2|v|} & \text{if } |x| > \mu. \end{cases} \tag{3}$$

Now we can be define the following two kernels with $A_1 = \max\limits_{0\leq|k|\leq m} \alpha_k\,(C_k)\,C(\mu)$

$$K_1(y) = \begin{cases} A_1\,|x|^{\alpha_{\min}-n-2|v|} & \text{if } |x| \leq \mu \\ 0 & \text{if } |x| > \mu \end{cases} \tag{4}$$

$$K_\infty(y) = \begin{cases} 0 & \text{if } |x| \leq \mu \\ A_1\,|x|^{\alpha_{\max}-n-2|v|} & \text{if } |x| > \mu. \end{cases} \tag{5}$$

From (3), (4) and (5), we have the following inequality

$$\begin{aligned} |(R_{\alpha,m,v}f)(x)| \leq\ & \sum_{0\leq|k|\leq m} \int\limits_{\mathbf{R}_n^+} \left[D^k f(y)\right] T^y K_1(x) (\prod_{i=1}^{n} y_i^{2v_i}) dy \\ & + \sum_{0\leq|k|\leq m} \int\limits_{\mathbf{R}_n^+} \left[D^k f(y)\right] T^y K_\infty(x) (\prod_{i=1}^{n} y_i^{2v_i}) dy \\ =\ & I_1(x) + I_2(x). \end{aligned}$$

Applying the generalized Minkowsky inequality for integrals and using the properties **(iii)** and **(iv)** of the operator T^y, we obtain that

$$\|I_1\|_{L_{p,v}} \leq \|K_1\|_{L_{1,v}} \|f\|_{W_{p,v}^m} .$$

From the definition of K_1, we have

$$\|K_1\|_{L_{1,v}} = A_1 \int\limits_{|x|<\mu} |x|^{\alpha_{\min}-n-2|v|} (\prod_{i=1}^{n} x_i^{2v_i}) dx = A_2 \mu^{\alpha_{\min}} < \infty. \tag{6}$$

Here, the constant A_2 consists the value of integral coordinates angles. Therefore, $I_1 \in L_{p,v}$ and is finite almost everywhere.

For the integral I_2 we have the following inequality by the Hölder inequality

$$I_2(x) \leq \|K_\infty\|_{p',v} \|f\|_{W_{p,v}^m} .$$

From the definition of K_∞, we have

$$\|K_\infty\|_{p'} = A_1 \left(\int\limits_{|y|_\lambda \geq \mu} |x|^{(\alpha_{\max}-n-2|v|)p'} (\prod_{i=1}^{n} x_i^{2v_i}) dx \right)^{1/p'} \tag{7}$$

$$= A_3 \mu^{(\alpha_{\max}-n-2|v|)p'+n+2|v|}.$$

Here, since $(\alpha_{\max} - n - 2|v|)\, p' + n + 2|v| < 0$ (which is equal to $q < \infty$), $\|K_\infty\|_{p',v}$ is finite. This means that the integral I_2 is also finite and the part (a) of the theorem is proved.

c. Assume without loss of generality that $\|f\|_{W^m_{p,v}} = 1$ and rewrite the potential $R_{\alpha,m,v}f$ in the following form

$$R_{\alpha,m,v}f(x) = R^1_{\alpha,m,v}f(x) + R^2_{\alpha,m,v}f(x) \tag{8}$$

where $R^1_{\alpha,m,v}f(x)$ and $R^2_{\alpha,m,v}f(x)$ are the potentials generated by the kernels $r_1(x)$ and $r_2(x)$ respectively

$$r_1(x) = \begin{cases} |x|^{\alpha_k-n-2|v|} & , |x| \leq \mu \\ 0 & , |x| > \mu \end{cases} \tag{9}$$

$$r_2(x) = \begin{cases} 0 & , |x| \leq \mu \\ |x|^{\alpha_k-n-2|v|} & , |x| > \mu. \end{cases}$$

Then for any positive λ, we have the following inequality

$$mes\{x : |R_{\alpha,m,v}f(x)| > \lambda\} \leq mes\left\{x : \left|R^1_{\alpha,m,v}f(x)\right| > \frac{\lambda}{2}\right\} + mes\left\{x : \left|R^2_{\alpha,m,v}f(x)\right| > \tfrac{\lambda}{2}\right\}. \tag{10}$$

Denoting $E_1 = \left\{x : \left|R^1_{\alpha,m,v}f(x)\right| > \frac{\lambda}{2}\right\}$, we see that

$$mesE_1 \leq \frac{2^p}{\lambda^p} \int\limits_{E_1} \left|R^1_{\alpha,m,v}f(x)\right|^p (\prod_{i=1}^{n} x_i^{2v_i}) dx. \tag{11}$$

Here, applying the generalized Minkowsky inequality for integrals and using the definition of kernel $r_1(x)$ we obtain

$$\int\limits_{E_1} \left|R^1_{\alpha,m,v}f(x)\right|^p (\prod_{i=1}^{n} x_i^{2v_i}) dx < C^* \mu^{p\alpha_{\max}}$$

where C^* is a constant depend on C_k and p. Using this inequality in (11) we have

$$mesE_1 \leq \frac{2^p}{\lambda^p} C^* \mu^{p\alpha_{\max}}. \tag{12}$$

Let $E_2 = \left\{ x : \left| R^2_{\alpha,m,v} f(x) \right| > \frac{\lambda}{2} \right\}$ for the second term of (8). Then applying the Hölder inequality, we have

$$\left| R^2_{\alpha,m,v} f(x) \right| \leq M \mu^{\frac{n+2|v|}{q}}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha_k}{n+2\,|v|}$$

where M is a constant depending on α_k and p'. Therefore choosing

$$2M\mu^{\frac{n+2|v|}{q}} = \lambda. \tag{13}$$

Thus, $mes(E_2) = 0$ is obtained.

Choosing $\mu > \max\left(1, \left(\frac{2M}{\lambda}\right)^{\frac{q}{n+2|n|}}\right)$ and using (12) and (13) in (10) we obtain

$$mes\left\{ x : |(R_{\alpha,m,v} f)(x)| > \lambda \right\} \leq C_{p,q,m,v} \left(\frac{\|f\|_{w^m_{p,v}}}{\lambda} \right).$$

Consequently, if $1 \leq p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha_k}{n+2|v|}$, then $(R_{\alpha,m,v} f)$ has a weak type $(W^m_{p,v}, L_{q,v})$ in the sense of our definition.

b. To prove this part of theorem we use the Marcinkiewicz interpolation theorem in [10]. We consider the following potential

$$(Z_{\alpha,m,v} g)(x) = \int\limits_{\mathbf{R}^+_n} g(y) T^y \,|x|^{\alpha_k - n - 2|v|} \,(\prod_{i=1}^{n} y_i^{2v_i}) dy$$

where $g(y) = \sum\limits_{0 \leq |k| \leq m} C_k D^k f(y) \in L_{p,v}(\mathbf{R}^+_n)$. In the same way as in **(a)** and **(c)** we can obtain the following inequality

$$mes\left\{ x : |Z_{\alpha,m,v} g| > \mu \right\} \leq C_{p,q,m,v} \left(\frac{\|g\|_{p,\nu}}{\mu} \right)^q$$

which holds for any $\mu > 0$ and $1 \leq p < q < \infty$. Using the Marcinkiewicz interpolation theorem, we have the following inequality for this potential

$$\|Z_{\alpha,m,v} g\|_{p,v} \leq C_{p,q,m,v} \, \|g\|_{q,\nu}.$$

From $g(y) = \sum\limits_{0 \le |k| \le m} C_k D^k f(y)$, we have the following inequality

$$\|Z_{\alpha,m,v} g\|_{p,v} \le C_{p,q,m,v} \left\| \sum_{0 \le |k| \le m} C_k D^k f(y) \right\|_{q,\nu} \le C_{p,q,m,v} \|f\|_{W^m_{p,v}} .$$

Now it is obvious that for $f \in W^m_{p,v}(\mathbf{R}^+_n)$

$$|R_{\alpha,m,v} f(x)| \le Z_{\alpha,m,v} g(x),$$

where $g(y) = \sum\limits_{0 \le |k| \le m} C_k \left| D^k f(y) \right|$. Therefore we obtain the inequality

$$\|R_{\alpha,m,v} f\|_{q,v} \le C_{p,q,m,v} \|f\|_{w^m_{p,v}} \quad \text{for } \frac{1}{q} = \frac{1}{p} - \frac{\alpha_{\max}}{n + 2|v|}.$$

The proof of part **(b)** is completed.

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