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SOME CHARACTERIZATIONS OF NULL, PSEUDO NULL AND PARTIALLY NULL RECTIFYING CURVES IN MINKOWSKI SPACE-TIME

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Abstract. In this paper, we define rectifying curves in Minkowski space-time and characterize null, pseudo null and partially null rectifying curves in terms of their curvatures. Also, we give some explicit equations of null, pseudo null and partially null rectifying curves in E^4_1.

1. Introduction

Rectifying curves are introduced by B. Y. Chen in [4] as space curves whose position vector always lies in its rectifying plane. The rectifying plane of an arbitrary curve $\alpha(s)$ in the Euclidean 3-space, is orthogonal to the principal normal vector $N(s)$ and therefore represents the ortogonal complement of $N(s)$. This implies that the rectifying plane is spanned by the tangent vector $T(s)$ and the binormal vector $B(s)$. Consequently, the position vector α of a rectifying curve in E^3, satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s),$$

for some differentiable functions $\lambda(s)$ and $\mu(s)$. The Euclidean rectifying curves have many interesting geometric properties. For example, if α is a rectifying curve in E^3, then the ratio of its torsion and its curvature is a non-constant linear function in arclength s ([4]). On the other hand, there is a simple relationship between the rectifying curves and the centrodes (i.e. the curves given by the Darboux vector). Also, the rectifying curves can be studied as the extremal curves (see [5]).

In Minkowski 3-space, the rectifying curves have similar geometric properties as in the Euclidean 3-space. Some characterizations of spacelike, timelike and null rectifying curves, lying fully in Minkowski 3-space, are given in [6].

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In this paper, we characterize null, pseudo null and partially null rectifying curves in Minkowski space-time. Firstly, we define the rectifying space of an arbitrary curve in E_1^4, and then we define a rectifying curve in E_1^4 as a curve whose position vector always lies in its rectifying space. In particular, we give a necessary and sufficient conditions for the null curves to be rectifying and obtain some explicit equations of null, pseudo null and partially null rectifying curves in Minkowski space-time.

2. Preliminaries

The Minkowski space-time E_1^4 is the Euclidean 4-space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_1^4. Since g is indefinite metric, recall that a vector $v \in E_1^4$ can be *spacelike* if $g(v,v) > 0$ or $v = 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$. In particular, the norm (length) of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. Next, recall that an arbitrary curve $\alpha(s)$ in E_1^4, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([8]). Recall that a spacelike curve in E_1^4 is called *pseudo null curve* or *partially null curve*, if respectively its principal normal vector is null or its first binormal vector is null ([2]). A null curve α is parameterized by *arclength function* s, if $g(\alpha''(s), \alpha''(s)) = 1$ ([1]). In particular, a pseudo null or a partially null curve α has *unit speed*, if $g(\alpha'(s), \alpha'(s)) = 1$.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along a curve α in E_1^4, consisting of the tangent, the principal normal, the first binormal and the second binormal vector fields. Depending on the causal character of α, the Frenet equations have the following forms.

Case (a). If α is null curve, the Frenet equations are given by ([1, 9]):

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & -\kappa_1 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ -\kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{1}$$

where the first curvature $\kappa_1(s) = 0$, if $\alpha(s)$ is straight line, or $\kappa_1(s) = 1$ in all other cases. Therefore, such curve has two curvatures $\kappa_2(s)$ and $\kappa_3(s)$ and the following equations hold:

$$g(T,T) = g(B_1, B_1) = 0, \quad g(N,N) = g(B_2, B_2) = 1,$$

$$g(T,N)=g(T,B_2)=g(N,B_1)=g(N,B_2)=g(B_1,B_2)=0,\quad g(T,B_1)=1.$$

Case (b). If α is pseudo null curve, the Frenet formulas are ([2, 9]):

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 \\ 0 & \kappa_3 & 0 & -\kappa_2 \\ -\kappa_1 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2}$$

where the first curvature $\kappa_1(s)=0$, if α is straight line, or $\kappa_1(s)=1$ in all other cases. Such curve has two curvatures $\kappa_2(s)$ and $\kappa_3(s)$ and the following conditions are satisfied:

$$g(T,T)=g(B_1,B_1)=1,\quad g(N,N)=g(B_2,B_2)=0,$$

$$g(T,N)=g(T,B_1)=g(T,B_2)=g(N,B_1)=g(B_1,B_2)=0,\quad g(N,B_2)=1.$$

Case (c). If α is partially null curve, the Frenet formulas read ([2, 9]):

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 & 0 \\ 0 & -\kappa_2 & 0 & -\kappa_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{3}$$

where the third curvature $\kappa_3(s)=0$ for each s. Such curve has two curvatures $\kappa_1(s)$ and $\kappa_2(s)$ and lies fully in a lightlike hyperplane of E_1^4. In particular, the following equations hold

$$g(T,T)=g(N,N)=1,\quad g(B_1,B_1)=g(B_2,B_2)=0,$$

$$g(T,N)=g(T,B_1)=g(T,B_2)=g(N,B_1)=g(N,B_2)=0,\quad g(B_1,B_2)=1.$$

Let α be arbitrary curve in E_1^4. We define the rectifying space of α as the orthogonal complement $N^{\perp}$ of its principal normal vector field N. Therefore, the rectifying space $N^{\perp}$ is given by

$$N^{\perp}=\{w\in E_1^4\,|\,g(w,N)=0\}.$$

Next we define a rectifying curve in E_1^4 as a curve whose position vector always lies in its rectifying space. In particular, if α is a null curve, then N is the space-like vector field and hence the rectifying space $N^{\perp}$ is the 3-dimensional timelike subspace of E_1^4, spanned by $\{T,B_1,B_2\}$. If α is a pseudo null curve, then N is the null vector field and consequently the rectifying space $N^{\perp}$ is the 3-dimensional lightlike subspace of E_1^4, spanned by $\{T, N, B_1\}$. In particular, if α is a partially null curve, then it lies fully in a lightlike hyperplane of E_1^4, spanned by $\{T, N, B_1\}$. Since N is the spacelike vector field, the rectifying space $N^{\perp}$ is the 2-dimensional lightlike subspace of E_1^4, spanned by $\{T, B_1\}$.

Consequently, the position vector α of the null, pseudo null and partially null rectifying curves, satisfies respectively the equations

$$\alpha(s) = a(s)T(s) + b(s)B_1(s) + c(s)B_2(s),$$

$$\alpha(s) = a(s)T(s) + b(s)N(s) + c(s)B_1(s),$$

$$\alpha(s) = a(s)T(s) + b(s)B_1(s),$$

where $a(s)$, $b(s)$ and $c(s)$ are arbitrary differentiable functions.

3. Null Rectifying Curves

The null curves in E_1^4 with the third curvature $\kappa_3(s) = 0$ for each s, lie fully in the timelike hyperplane of E_1^4, i.e. in Minkowski 3-space ([1, 3]). Accordingly, the characterization of null rectifying curves in E_1^4 with $\kappa_3(s) = 0$ is equivalent to characterization of null rectifying curves lying fully in E_1^3 (see [6]). In this section we characterize null rectifying curves lying fully in E_1^4, with curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$ and $\kappa_3(s) \neq 0$. Note that the second curvature $\kappa_2(s)$ can be equal to zero or different from zero. For example, the null curve with constant curvatures $\kappa_3(s) \neq 0$ and $\kappa_2(s)$, is a null helix, lying in pseudosphere in E_1^4 ([3, 9]).

Let $\alpha(s)$ be a null rectifying curve in E_1^4, parameterized by arclength s. Then its position vector satisfies the equation

$$\alpha(s) = a(s)T(s) + b(s)B_1(s) + c(s)B_2(s), \tag{4}$$

for some differentiable functions $a(s)$, $b(s)$ and $c(s)$. Differentiating (4) with respect to s and by applying (1), we obtain system of equations

(4)

$$a'(s) - c(s)\kappa_3(s) = 1,\ a(s) - b(s)\kappa_2(s) = 0,\ b'(s) = 0,\ b(s)\kappa_3(s) + c'(s) = 0.$$

By using (4), we find that the tangential component of the position vector α is given by $g(\alpha, T) = b(s)$. From the third equation of (5) we get $b(s) = b_0$, $b_0 \in R$. Consequently, we may distinguish two cases: (I) $g(\alpha, T) = 0$ and (II) $g(\alpha, T) \neq 0$. With respect to this two possibilities, we obtain the following theorems.

Theorem 1. *Let $\alpha(s)$ be a null curve in E_1^4, parameterized by arclength s and with curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$ and $\kappa_3(s) \neq 0$. If α is a rectifying curve with the tangential component $g(\alpha, T) = 0$, then the following statements hold:*

(i) *α lies in pseudosphere $S^3_1(r)$, $r \in R^+_0$;*

(ii) *the third curvature $\kappa_3(s)$ is non-zero constant;*

(iii) *the first binormal and the second binormal component of the position vector α are respectively given by $g(\alpha, B_1) = 0$ and $g(\alpha, B_2) = -1/\kappa_3(s)$;*

Conversely, if $\alpha(s)$ is a null curve in E^4_1, parameterized by arclength s, with curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$, $\kappa_3(s) \neq 0$ and one of statements (i), (ii) or (iii) holds, then α is a rectifying curve.

Proof. First assume that $\alpha(s)$ is a null rectifying curve in E^4_1, parameterized by arclength s, with curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$, $\kappa_3(s) \neq 0$ and with $g(\alpha, T) = 0$. Then its position vector is given by (4). Moreover, from (4) we easily obtain $g(\alpha, T) = b(s) = 0$, so the system of equations (5) reduces to

$$-c(s)\kappa_3(s) = 1, \quad a(s) = 0, \quad b(s) = 0, \quad c'(s) = 0. \tag{6}$$

Since $c(s)\kappa_3(s) \neq 0$, we get $c(s) \neq 0$. Moreover, relation (6) implies

$$c(s) = c_0, \quad c_0 \in R_0. \tag{7}$$

Hence by relations (4), (6) and (7) we find that α has equation

$$\alpha(s) = c_0 B_2(s), \quad c_0 \in R_0. \tag{8}$$

Then (8) yields $g(\alpha, \alpha) = c_0^2$, which means that α lies in pseudosphere $S^3_1(r)$ with center at the origin and of radius $r = c_0 \in R^+_0$. This proves statement (i). Next, relations (6), (7) and (8) imply $\kappa_3(s) = -1/c_0$ =constant, $g(\alpha, B_1) = 0$ and $g(\alpha, B_2) = -1/\kappa_3(s)$, which proves (ii) and (iii).

Conversely, assume that $\alpha(s)$ is a null curve in E^4_1, parameterized by arclength s and with curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$ and $\kappa_3(s) \neq 0$. If (i) holds, then $g(\alpha, \alpha) = r^2$, $r \in R^+_0$. Differentiating the previous equation with respect to s and using (1), we find $g(\alpha, N) = 0$, which means that α is a rectifying curve. Next, if (ii) holds, putting $\kappa_3(s) = -1/c$, $c \in R_0$ and by applying (1), we easily obtain $\frac{d}{ds}(\alpha(s) - cB_2(s)) = 0$. Thus α is congruent to a rectifying curve. Finally, if (iii) holds, by taking the derivative of the equation $g(\alpha, B_1) = 0$ with respect to s and using (1), we easily get $g(\alpha, N) = 0$. Consequently, α is a rectifying curve. ∎

Theorem 2. *Let $\alpha(s)$ be a null curve in E^4_1, parameterized by arclength s and with curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$ and $\kappa_3(s) \neq 0$. If α is a rectifying curve with the tangential component $g(\alpha, T) \neq 0$, then the following statements hold:*

(i) *The distance function $\rho = ||\alpha||$ satisfies $\rho^2 = |c_1 s + c_2|$, $c_1 \in R_0$, $c_2 \in R$;*

(ii) *The first binormal and the second binormal components of the position vector* α *are respectively given by* $g(\alpha, B_1) = b_0\kappa_2(s)$, $g(\alpha, B_2) = (b_0\kappa_2'(s) - 1)/\kappa_3(s)$, *where* $b_0 \in R_0$.

(iii) *the third curvature is given by*

$$\kappa_3(s) = \frac{b_0\kappa_2'(s) - 1}{\sqrt{2(b_0 s - b_0^2\kappa_2(s) + c_0)}},$$

where $b_0 \in R_0$, $c_0 \in R$. *Conversely, if* $\alpha(s)$ *is a null curve in* E_1^4, *parameterized by arclength* s, *with curvatures* $\kappa_1(s) = 1$, $\kappa_2(s)$, $\kappa_3(s) \neq 0$ *and one of statements* (i), (ii), *or* (iii) *holds, then* α *is a rectifying curve.*

Proof. First assume that $\alpha(s)$ is a null rectifying curve in E_1^4, parameterized by arclength s, with curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$, $\kappa_3(s) \neq 0$ and with $g(\alpha, T) \neq 0$. Then its position vector is given by (4). Relation (4) implies $g(\alpha, T) = b(s) \neq 0$. Moreover, from the third and the second equation of (5) we get

$$b(s) = b_0, \quad a(s) = b_0\kappa_2(s), \tag{9}$$

where $b_0 \in R_0$. The last equation of (5) implies $c'(s) = -b_0\kappa_3(s) \neq 0$ and hence $c(s) \neq 0$. Multiplying the first equation of (5) with b_0 and the last equation of (5) with $c(s)$ and adding, we obtain differential equation

$$c(s)c'(s) + b_0^2\kappa_2'(s) - b_0 = 0.$$

Integration of the previous equation gives

$$c(s) = \pm\sqrt{2(b_0 s - b_0^2\kappa_2(s) + c_0)}, \tag{10}$$

whereby $c_0 \in R_0$ and $b_0 s - b_0^2\kappa_2(s) + c_0 > 0$. Accordingly, relations (4), (9) and (10) imply that the curve α has the equation

$$\alpha(s) = b_0\kappa_2(s)T(s) + b_0 B_1(s) + \sqrt{2(b_0 s - b_0^2\kappa_2(s) + c_0)}B_2(s). \tag{11}$$

From (11) we easily find $\rho^2 = |g(\alpha,\alpha)| = |c_1 s + c_2|$, $c_1 \in R_0$, $c_2 \in R$, which proves statement (i). Next, (4) and (9) imply $g(\alpha, B_1) = a(s) = b_0\kappa_2(s)$, while (4), (9) and the first equation of (5) yields $g(\alpha, B_2) = c(s) = (b_0\kappa_2'(s) - 1)/\kappa_3(s)$. This proves statement (ii). Since the first equation of (5) implies $\kappa_3(s) = (a'(s) - 1)/c(s)$, by using (9) and (10) we find

$$\kappa_3(s) = \frac{b_0\kappa_2'(s) - 1}{\sqrt{2(b_0 s - b_0^2\kappa_2(s) + c_0)}},$$

which proves statement (iii).

Conversely, suppose that $\alpha(s)$ is a null curve in E_1^4, parameterized by arclength s and with curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$ and $\kappa_3(s) \neq 0$. If (i) holds, differentiating the equation $g(\alpha, \alpha) = \pm(c_1 s + c_2)$ with respect to s and by using (1), we get $g(\alpha, N) = 0$, which means that α is a rectifying curve. If (ii) holds, in a similar way we conclude that α is a rectifying curve. If (iii) holds, by applying (1) we easily find that

$$\frac{d}{ds}\Big(\alpha(s) - b_0\kappa_2(s)T(s) - b_0 B_1(s) - \sqrt{2(b_0 s - b_0^2\kappa_2(s) + c_0)}B_2(s)\Big) = 0.$$

Up to isometries of E_1^4, it follows that α is a rectifying curve. ■

Theorem 3. *Let $\alpha(s)$ be a null curve in E_1^4, parameterized by arclength s, with curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$, $\kappa_3(s) \neq 0$ and with spacelike position vector. Then α is a rectifying curve with the tangential component $g(\alpha, T) \neq 0$ if and only if, up to parametrization, it is given by*

$$\alpha(t) = ce^t y(t), \quad c \in R_0^+, \tag{12}$$

where $y(t)$ is a unit speed timelike curve lying in $S_1^3(1)$.

Proof. Let us first assume that $\alpha(s)$ is a null rectifying curve in E_1^4, parameterized by arclength s, with curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$, $\kappa_3(s) \neq 0$, with spacelike position vector and with $g(\alpha, T) \neq 0$. By theorem 2, the distance function $\rho = ||\alpha||$ satisfies $\rho^2 = g(\alpha, \alpha) = c_1 s + c_2$, $c_1 \in R_0$, $c_2 \in R$. We may take $c_1 \in R_0^+$. Let us define a curve $y(s)$ by $y(s) = \alpha(s)/\rho(s)$. Since $g(y, y) = 1$, the curve y lies in pseudosphere $S_1^3(1)$ with center at the origin and of radius 1. Moreover, we have

$$\alpha(s) = y(s)\sqrt{c_1 s + c_2}. \tag{13}$$

By taking the derivative of (13) with respect to s, we find

$$T(s) = \frac{c_1}{2\sqrt{c_1 s + c_2}} y(s) + y'(s)\sqrt{c_1 s + c_2}. \tag{14}$$

¿From (14) we get

$$0 = g(T(s), T(s)) = \frac{c_1^2}{4(c_1 s + c_2)} + g(y'(s), y'(s))(c_1 s + c_2),$$

and hence

$$g(y'(s), y'(s)) = -\frac{c_1^2}{4(c_1 s + c_2)^2}. \tag{15}$$

Consequently, y is a timelike curve. By using (15), we obtain that $||y'(s)|| = c_1/2(c_1 s + c_2)$. Denote by $t = \int_0^s ||y'(u)||\, du$ the arclength parameter of the curve y. It follows that $t = (1/2)\ln(c_0(c_1 s + c_2))$, $c_0 \in R_0^+$ and thus $c_1 s + c_2 = e^{2t}/c_0$. Substituting this in (13), yields (12).

Conversely, suppose that α is a null curve in E_1^4 with the curvatures $\kappa_1(s) = 1$, $\kappa_2(s)$, $\kappa_3(s) \neq 0$, with spacelike position vector and given by (12), where $y(t)$ is a unit speed timelike curve lying in pseudosphere $S_1^3(1)$. We may reparametrize the curve α by $t = (1/2)\ln|c_1 s + c_2|$, where s is arclength function of α, $c_1 \in R_0$ and $c_2 \in R$. Substituting parameter t in (12), we obtain $\alpha(s) = cy(s)\sqrt{|c_1 s + c_2|}$ and therefore $\rho^2 = ||\alpha||^2 = c^2|c_1 s + c_2|$. According to theorem 2, we conclude that α is a rectifying curve. ∎

4. Pseudo Null and Partially Null Rectifying Curves

In this section, we find parameter equations of pseudo null and partially null rectifying curves in E_1^4.

Theorem 1. *Let $\alpha(s)$ be unit speed pseudo null rectifying curve in E_1^4 with the curvature $\kappa_1(s) = 1$. Then α is a planar curve.*

Proof. Let us suppose that $\alpha(s)$ is unit speed pseudo null rectifying curve in E_1^4, with the curvature $\kappa_1(s) = 1$. Then its position vector satisfies the equation

$$\alpha(s) = a(s)T(s) + b(s)N(s) + c(s)B_1(s),$$

for some differentiable functions $a(s)$, $b(s)$ and $c(s)$. Differentiating the previous equation with respect to s and by applying (2), we obtain system of equations

$$a'(s) = 1,\ a(s) + b'(s) + c(s)\kappa_3(s) = 0,\ b(s)\kappa_2(s) + c'(s) = 0,\ c(s)\kappa_2(s) = 0. \tag{16}$$

The last equation of (16) implies $\kappa_2(s) = 0$ or $c(s) = 0$. If $\kappa_2(s) = 0$ for each s, α is a planar curve. If $c(s) = 0$, the system of equations (16) reduces to

$$a'(s) = 1 \quad a(s) + b'(s) = 0 \quad b(s)\kappa_2(s) = 0.$$

Since $b'(s) = -a(s) \neq 0$, it follows that $b(s) \neq 0$ and consequently $\kappa_2(s) = 0$, which completes the proof of the theorem. ∎

Theorem 2. *Let $\alpha(s)$ be unit speed pseudo null curve in E_1^4, with curvature $\kappa_1(s) = 1$. Then α is a rectifying curve if and only if it is given by*

$$\alpha(s) = s^2 Q + sP, \tag{17}$$

where $P, Q \in E_1^4$ are constant vectors satisfying equations $g(P,P) = 1$, $g(Q,Q) = 0$ and $g(P,Q) = 0$.

Proof. First suppose that $\alpha(s)$ is a unit speed pseudo null rectifying curve in E_1^4, with curvature $\kappa_1(s) = 1$. According to theorem 1, α is a planar curve and hence $\kappa_2(s) = 0$ for each s. Then Frenet equations (2) imply $N'(s) = 0$, so $N(s) = \alpha''(s)$ =constant. Integration of the previous equation gives $\alpha(s) = \frac{s^2}{2}Q_0 + sP_0 + R_0$, where $P_0, Q_0, R_0 \in E_1^4$ are constant vectors. Moreover, from the condition $g(T,T) = 1$, we get $g(P_0,P_0) = 1$, $g(Q_0,Q_0) = 0$, $g(P_0,Q_0) = 0$. Up to a translation of E_1^4, we may take $R_0 = (0,0,0,0)$. Putting $P = P_0$ and $Q = \frac{Q_0}{2}$, we obtain that α is given by (17).

Conversely, assume that a unit speed pseudo null curve α in E_1^4 with curvature $\kappa_1(s) = 1$, is given by (17). Since $T(s) = \alpha'(s) = 2sQ + P$ and $N(s) = T'(s) = 2Q$, we easily find that (17) can be written in the form $\alpha(s) = sT(s) - \frac{s^2}{2}N(s)$, which means that α is a rectifying curve. ∎

Theorem 3. *Let $\alpha(s)$ be unit speed partially null curve in E_1^4, with curvature $\kappa_3(s) = 0$. Then α is a rectifying curve if and only if it is a straight line.*

Proof. Let $\alpha(s)$ be unit speed partially null rectifying curve in E_1^4, with curvature $\kappa_3(s) = 0$. Then the position vector of the curve satisfies the equation

$$\alpha(s) = a(s)T(s) + b(s)B_1(s), \tag{18}$$

for some differentiable functions $a(s)$ and $b(s)$. Differentiating (18) with respect to s and by using (3), we obtain system of equations

$$a'(s) = 1, \quad a(s)\kappa_1(s) = 0, \quad b'(s) = 0. \tag{19}$$

It follows that $a(s) \neq 0$, so the second equation of (19) implies $\kappa_1(s) = 0$. Hence α is a straight line.

Conversely, assume that $\alpha(s)$ is a unit speed partially null straight line. Since the position vector α of the curve is collinear with tangent vector $T(s)$, it follows that α lies in the plane spanned by $\{T, B_1\}$. Therefore, α is a rectifying curve. ∎

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