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THE POTENTIAL AND CONSISTENCY PROPERTY FOR MULTI-CHOICE SHAPLEY VALUE

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Abstract. In this article, we complete the proof that the extended Shapley value has w-consistent property proposed by Hsiao, Yeh and Mo [4]. Then we suggest an axiomatization which is the parallel of Hart and Mas-Colell's [1] axiomatization of the Shapley value by applying the w-consistency property.

1. Introduction

Motivated by calculating the power indices of players in different levels of joint military actions, in [2] and [3], Hsiao and Raghavan extended the traditional cooperative game to a multi-choice cooperative game and extended the traditional Shapley value to a multi-choice Shapley value. We call the multi-choice Shapley value the **H&R Shapley value**.

In [3], Hsiao and Raghavan give weights(discriminations) to actions instead of players. The H&R Shapley value is symmetric among players and asymmetric among actions, therefore, the H&R Shapley value is an extention of both the symmetric and the asymmetric Shapley values.

In [1], Hart and Mas-Colell were the first to introduce the potential approach to TU games. In consequence, they proved that the Shapley value [9] can result as the vector of marginal contributions of a potential. The potential approach is also shown to yield a characterization for the Shapley value, particularly in terms of an internal consistency property.

The H&R Shapley value is monotone, transferable utility invariant, dummy free and independent of non-essential players, please see [2] and [5] for details. In 1991, when Hsiao and Raghavan presented [3] in the 2rd International Conference on

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Game Theory at Stony-Brook, Shapley suggested that we should study the consistent property of the H&R Shapley value.

The property of consistency is essentially equivalent to the existence of a potential function. Following Shapley's advice, in [4], Hsiao, Yeh and Mo defined the potential function for multi-choice TU games and found an explicit formula of the potential function. Moreover, they defined the w-reduced games with respect to an action vector and a solution of multi-choice TU games. Also, they showed that the H&R Shapley value is w-consistent and showed the coincidence of the H&R Shapley value and the vector of marginal contributions of a potential. However, the authors did not characterize the $H\&R$ Shapley value.

In [7], Liao tried to provide an axiomatization which is the parallel of Hart and Mas-Colell axiomatization of the Shapley value by applying the w-consistency property. However, Liao did not finish the job. In this article, we suggest an axiomatization which is the parallel of Hart and Mas-Colell's [1] axiomatization of the Shapley value by applying the w-consistency property.

The main contribution of this article is to complete the proof that the extended Shapley value has w-consistent property proposed by Hsiao, Yeh and Mo [4]. Moreover, we show that some of the results in [8] are special cases of the results in [4].

2. Definitions and Notations

Slightly extending [3], we have the following definitions and notations.

Let U be the universe of players. Let $N \subseteq U$ be a set of players and let $m = (m_i)_{i \in N}$ be the vector that describes the number of activity levels for each player, at which he can actively participate. For $i \in U$, we set $M_i = \{0, 1, \cdots, m_i\}$ as the action space of player i, where the action 0 means not participating, and $M_i^+ = M_i \setminus \{0\}$.

Note that we set $M_i = \{0, 1, \cdots, m_i\}$ as the action space of player i is just for convenience, any totally ordered set $\{\sigma_0, \sigma_1, \cdots, \sigma_{m_i}\}$ can do the same job.

For $N \subseteq U$, $N \neq \emptyset$, let $M^N = \prod_{i \in N} M_i$ be the product set of the action spaces for players N. Denote 0_N the zero vector in $\mathbb{R}^N$.

A **multi-choice TU game** is a triple (N, m, v), where N is a non-empty and finite set of players, m is the vector that describes the number of activity levels for each player, and $v : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $x = (x_i)_{i \in N} \in M^N$ the worth that the players can obtain when each player i plays at activity level $x_i \in M_i$ with $v(0_N) = 0$. If no confusion can arise a game (N, m, v) will sometimes be denoted by its characteristic function v. Denote the class of all multi-choice TU games by MC. Given $(N, m, v) \in MC$ and $x \in M^N$, we write (N, x, v) for the multi-choice TU subgame obtained by restricting v to $\{y \in M^N \mid y_i \leq x_i \ \forall i \in N\}$ only.

Given $(N, m, v) \in MC$, let $L^{N,m} = \{(i, j) \mid i \in N, j \in M_i^+\}$. Let $w : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$ be a non-negative function such that $w(0) = 0$ and for all $j \leq l$, $w(0) < w(j) \leq w(l)$, then w is called a weight function.

Remark 1. For the traditional asymmetric Shapley value, Shapley gives weights (discriminations) to the players. For our `H&R Shapley value`, we do not give weights(discriminations) to the players. However, as we allow players to have more than two choices, we should expect some differences due to actions. We use a weight function w to modify the differences due to actions.

It is well-known that the traditional Shapley value has applications in many fields such as economics, political sciences, accounting, and even military sciences. Of course, our extended Shapley value also has the same applications as the traditional Shapley value does.

However, the weight function w has different meanings in different fields. In military sciences, we may treat $w(j)$`s'` as parameters to modify the differences due to different levels of military actions.

Given $(N, m, v) \in MC$ and a weight function w for the actions, a solution on MC is a map ψ^w assigning to each $(N, m, v) \in MC$ an element

$$\psi^w(N, m, v) = \left(\psi^w_{i,j}(N, m, v)\right)_{(i,j)\in L^{N,m}} \in \mathbb{R}^{L^{N,m}}.$$

Here $\psi^w_{i,j}(N, m, v)$ is the power index or the value of the player i when he takes action j to play game v.

For convenience, given a $(N, m, v) \in MC$ and a solution ψ on MC, we define $\psi_{i,0}(N, m, v) = 0$ for all $i \in N$.

To state the H&R Shapley value, some more notations will be needed. Given $S \subseteq N$, let $|S|$ be the number of elements in S, $S^c = N \setminus S$ and let $e^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e_i^S(N)$ satisfies

$$e_i^S(N) = \begin{cases} 1 & \text{if } i \in S\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Note that if no confusion can arise $e_i^S(N)$ will be denoted by e_i^S.

Given $(N, m, v) \in MC$ and a weight function w, for any $x \in M^N$ and $i \in N$, we define $\|x\|_w = \sum_{i\in N} w(x_i)$, $\|x\| = \sum_{i\in N} x_i$ and $M_i(x; m) = \{i \mid x_i \neq m_i, i \neq j\}$.

From Hsiao and Raghavan [2], the **H&R Shapley value** γ^w is obtained by

$$\gamma^w_{i,j}(N, m, v) = \sum_{k=1}^{j} \sum_{\substack{x_i=k, x\neq 0_N \\ x\in M^N}}$$

$$\Big[\sum_{T\subseteq M_i(x;m)}(-1)^{|T|}\frac{w(x_i)}{\|x\|_w+\sum\limits_{r\in T}[w(x_r+1)-w(x_r)]}\Big]$$
$$\cdot\Big[v(x)-v(x-e^{\{i\}})\Big].$$

Given $(N, m, v) \in MC$ and a solution ψ on MC, if there exist $x, y \in M^N$, such that $x \neq y$, $v(x) \neq v(y)$,

$$\sum_{i\in N}\psi_{i,x_i}(N,m,v)=v(x)$$

and

$$\sum_{i\in N}\psi_{i,y_i}(N,m,v)=v(y),$$

then we say that (N, m, v) is multiple-efficient with respect to ψ.

The game (N, m, v) is called a non-essential multi-choice game, if $v(x) = \sum\limits_{i\in N} v(x_i e^{\{i\}})$ for all $x \in M^N$

Obviously, a non-essential multi-choice game is multiple-efficient with respect to the H&R Shapley value. Moreover, it is easy to find an essential multi-choice game which is multiple-efficient with respect to the H&R Shapley value. A characterization of games which are multiple-efficient with respect to the H&R Shapley value will be written in a separate paper.

Remark 2. The `H&R Shapley value` not only can be regarded as a value, but also can be regarded as a power index. When we regard the `H&R Shapley value` $\gamma^w_{i,j}(N, m, v)$ as a power index, it makes sense for player i , for every j. Please see [6] for example.

When we regard the `H&R Shapley value` $\gamma^w_{i,j}(N, m, v)$ as a value, player i may be interested in only one action level j^*, then $\gamma^w_{i,j}(N, m, v)$ makes no sense for player i whenever $j \neq j^*$.

However, we leave the definition of the `H&R Shapley value` as it is, not only because it can be a power index, but also because we want our `H&R Shapley value` applied to all kinds of multi-choice TU games such as multiple-efficient games with respect to our `H&R Shapley value`.

3. Potential

Following [4], we rewrite the definition of the potential function for multi-choice TU games. Furthermore, we complete the proof of the coincidence of the

H&R Shapley value and the vector of marginal contributions of a potential on the multi-choice TU games.

For $x \in \mathbb{R}^N$, we write x_S to be the restriction of x at S for each $S \subseteq N$. Given a $(N, m, v) \in MC$ and $x \in M^N$, let $i \in N$ and $j \in M_i$, for convenience we introduce the substitution notations x_{-i} to stand for $x_{N\setminus\{i\}}$. Moreover, $(x_{-i}, j) = y \in \mathbb{R}^N$ be defined by $y_{-i} = x_{-i}$ and $y_i = j$. Let $x, y \in \mathbb{R}^N$, we say $y \leq x$ if $y_i \leq x_i$ for all $i \in N$.

Given $x \in M^N$, we denote $S(x) = \{i | x_i \neq 0\}$ to be the set of players that take actions with levels higher than zero.

Given $(N, m, v) \in MC$ and a weight function w, we define a function P_w : $MC \longrightarrow I\!R$ which associates a real number $P_w(N, m, v)$. Subsequently, we define the following operators :

$$D^{i,j} P_w(N, m, v) = w(j) \cdot \left[P_w\big(N, (m_{-i}, j), v\big) - P_w\big(N, (m_{-i}, j-1), v\big)\right]$$

and

$$H_{i,x_i} = \sum_{l=1}^{x_i} D^{i,l}.$$

Definition 1. A function $P_w : MC \longrightarrow I\!R$ with $P_w(N, 0_N, v) = 0$ is called w**-potential function** if it satisfies the following condition :
Given $(N, m, v) \in MC$ and a weight function w,

$$\sum_{i \in S(m)} H_{i,m_i} P_w(N, m, v) = v(m).$$

Remark 3. Please note that one can not get an explicit formula of the above w-`potential function` easily by just observing the explicit formula of the potential given in [1]. In [4] Hsiao, Yeh and Mo got an explicit formula of the above w-`potential function` by using an extended Möbius inversion formula.

Let $x, y \in \mathbb{R}^N$, we say $y \leq x$ if $y_i \leq x_i$ for all $i \in N$. The analogue of unanimity games for multi-choice games are **minimal effort games** (N, m, u_N^x), where $x \in M^N$, $x \neq 0_N$, defined by

$$u_N^x(y) = \begin{cases} 1 & \text{if } y \geq x \text{ ;} \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in M^N$. In [2] it is known that for all $(N, m, v) \in MC$, it holds that $v = \sum_{\substack{x \in M^N \\ x \neq 0_N}} a^x(v)\, u_N^x$, where $a^x(v) = \sum_{S \subseteq S(x)} (-1)^{|S|}\, v(x - e^S)$.

The following is a theorem in [4] which is an extension of Theorem A in [1]. Here we complete the proof.

Theorem 1. *The potential of a multi-choice cooperative game is unique. Furthermore, given a weight function w and $(N, m, v) \in MC$, the H&R Shapley value γ^w and the w-potential P_w have the following relationship. For all $(i, j) \in L^{N,m}$,*

$$\gamma^w_{i,j}(N, m, v) = H_{i,j} P_w(N, m, v).$$

Proof. Given $(N, m, v) \in MC$ and a weight function w, Hsiao, Yeh and Mo [4] proved that the w-potential of a multi-choice cooperative game is unique, and

$$P_w(N, m, v) = \sum_{\substack{y \leq m, \\ y \neq 0_N}} \frac{1}{\| y \|_w} a^y(v). \cdots (1.1)$$

For all $(i, j) \in L^{N,m}$,

$$\begin{aligned}
& H_{i,j} P_w(N, m, v) \\
&= \sum_{k=1}^{j} D_{i,k} P_w(N, m, v) \\
&= \sum_{k=1}^{j} w(k) \cdot \Big[P_w(N, (m_{-i}, k), v) - P_w(m_{-i}, k-1), v) \Big] \\
(1) \quad &= \sum_{k=1}^{j} w(k) \cdot \Big[\sum_{\substack{y \leq (m_{-i}, k) \\ y \neq 0_N}} \frac{1}{\| y \|_w} a^y(v) - \sum_{\substack{y \leq (m_{-i}, k-1) \\ y \neq 0_N}} \frac{1}{\| y \|_w} a^y(v) \Big] \\
&= \sum_{k=1}^{j} w(k) \cdot \Big[\sum_{y \in M^N, y_i = k} \frac{1}{\| y \|_w} a^y(v) \Big] \\
&= \sum_{k=1}^{j} w(k) \cdot \Big[\sum_{y \in M^N, y_i = k} \frac{1}{\| y \|_w} \cdot \big[\sum_{T \subseteq S(y)} (-1)^{|T|} \cdot (v(y - \sum_{r \in T} e^{\{r\}})) \big] \Big].
\end{aligned}$$

Consider $\sum_{T \subseteq S(y)} (-1)^{|T|} \cdot (v(y - e^T))$.

$$\begin{aligned}
& \sum_{T \subseteq S(y)} (-1)^{|T|} \cdot (v(y - e^T)) \\
&= \sum_{T \subseteq S(y), i \in T} (-1)^{|T|} \cdot (v(y - e^T)) + \sum_{T \subseteq S(y), i \notin T} (-1)^{|T|} \cdot (v(y - e^T)) \\
(2) \quad &= \sum_{T \subseteq S(y), i \notin T} \Big[(-1)^{|T|+1} \cdot (v(y - e^{T \cup \{i\}})) + (-1)^{|T|} \cdot (v(y - e^T)) \Big] \\
&= \sum_{T \subseteq S(y), i \notin T} (-1)^{|T|+1} \cdot \Big[(v(y - e^{T \cup \{i\}}) - v(y - e^T)) \Big].
\end{aligned}$$

Let $z = y - e^T$. By (2), we have that

$$(3) \qquad \sum_{T \subseteq S(y)} (-1)^{|T|} \cdot (v(y - e^T)) = \sum_{T \subseteq S(y), i \notin T} (-1)^{|T|} \cdot \Big[v(z) - v(z - e^{\{i\}}) \Big].$$

Since

$$\begin{cases} y = z + e^T \\ y \neq 0_N \\ i \notin T \end{cases}, \text{ we have that } \{T \subseteq S(y), i \notin T\} = \{T \subseteq M_i(z; m)\}.$$

Hence (3) can be written as

$$(4) \qquad \sum_{T \subseteq M_i(z;m)} (-1)^{|T|} \cdot \big[v(z) - v(z - e^{\{i\}}) \big].$$

By (4), (1) can be written as

$$\begin{aligned}
&\sum_{k=1}^{j} \sum_{\substack{z_i = k, z \neq 0_N \\ z \in M^N}} \Big[\sum_{T \subseteq M_i(z;m)} (-1)^{|T|} \frac{w(z_i)}{\|z + e^T\|_w} \Big] \cdot \Big[v(z) - v(z - e^{\{i\}}) \Big] \\
&= \sum_{k=1}^{j} \sum_{\substack{z_i = k, z \neq 0_N \\ z \in M^N}} \Big[\sum_{T \subseteq M_i(z;m)} (-1)^{|T|} \frac{w(z_i)}{\|z\|_w + \sum_{r \in T} [w(z_r + 1) - w(z_r)]} \Big] \\
&\quad \cdot \Big[v(x) - v(x - e^{\{i\}}) \Big] \\
&= \gamma^w_{i,j}(N, m, v).
\end{aligned}$$

Hence, for all $(i, j) \in L^{N,m}$, $H_{i,j} P_w(N, m, v) \ = \ \gamma^w_{i,j}(N, m, v)$. ∎

By Theorem 1 and (1.1), the H&R Shapley value could be provided as an alternative form, i.e.,

$$\gamma^w_{i,j}(N, m, v) = \sum_{k=1}^{j} w(k) \cdot \Big[\sum_{\substack{y_i = k, \\ y \in M^N}} \frac{1}{\| y \|_w} \cdot a^y(v) \Big].$$

4. w-Consistency Property

Following [4], we rewrite the definitions of the w-reduced game and of the w-consistency property of solutions for the multi-choice games. We will complete the proof that the H&R Shapley value satisfies w-consistency property.

Given $(N, m, v) \in MC$, a weight function w and its solution,

$$\psi^w(N, m, v) = (\psi^w_{i,j}(N, m, v))_{(i,j)\in L^{N,m}}.$$

For each $z \in M^N$, we define an action vector $z^* = (z^*_i)_{i\in N}$ where

$$\begin{cases} z^*_i = m_i & \text{if } \; z_i < m_i \\ z^*_i = 0 & \text{if } \; z_i = m_i. \end{cases}$$

Furthermore, we define a new game $v^{\psi^w}_z$ such that

$$v^{\psi^w}_z(y) = v(y \vee z^*) - \sum_{k\in S(z^*)} \psi^w_{k,m_k}(N, (y \vee z^*), v) \;\; \text{for all } y \leq z.$$

We call $(N, z, v^{\psi^w}_z)$ a w-**reduced game** of v with respect to z and the solution ψ^w, where $(y \vee z^*)_i = \max\{y_i, z^*_i\}$ for all $i \in N$.

Remark 4. Every subset S of players $S \subseteq N$ can be represented by an action vector $e^S(N)$, therefore reducing the number of players is a special case of reducing the number of action levels. In short, if $z_i = 0$ in the action vector z then reducing the number of action levels is reducing the number of players. Therefore, we prefer reducing the number of action levels rather than reducing the number of players.

Definition 2. Given a weight function w, a solution ψ^w on MC is w-**consistent** **(**w**-CON)** if for all $(N, m, v) \in MC$,

$$\psi^w_{i,j}(N, m, v) = \psi^w_{i,j}(N, z, v^{\psi^w}_z) \;\; \text{for all } i \in N\backslash S(z^*) \text{ and for all } j \leq z_i.$$

For each $z \in M^N$, we define a set $H(z) = \{i|z_i = m_i\}$, then $H(z) \cap S(z^*) =$ and $H(z) \cup S(z^*) = N$. Hence $H(z) = N \setminus S(z^*)$. Therefore, a solution ψ^w on MC is w-**consistent (**w**-CON)** if for all $(N, m, v) \in MC$,

$$\psi^w_{i,j}(N, m, v) = \psi^w_{i,j}(N, z, v^{\psi^w}_z) \;\; \text{for all } i \in H(z) \text{ and for all } j \leq z_i.$$

However, for computational convenience, we leave Definition 2 alone.

Remark 5. Comparing our definition of w-consistency for multi-choice solutions with the definition of consistency for traditional solutions in [1], readers may easily see that our definition of w-consistency is a nature extension of the consistency defined in [1] by Hart and Mas-Colell.

Remark 6. Hwang and Liao [8] defined a reduced game only for the $H\&R$ `Shapley value` with symmetric form as following : For $S \subseteq N$, we denote $S^c =$

$N \setminus S$ and 0_S the zero vector in $\mathbb{R}^S$. Given a solution ψ, a game $(N, m, v) \in MC$ and $S \subseteq N$, the **reduced game** $\Big(N, (m_S, 0_{S^c}), v^{\psi}_{S,m}\Big)$ with respect to ψ, S and m is defined by

$$v^{\psi}_{S,m}(x, 0_{S^c}) = v(x, m_{S^c}) - \sum_{i \in S^c} \psi_{i,m_i}\Big(N, (x, m_{S^c}), v\Big) \text{ for all } x \in M^S.$$

Furthermore, they defined the consistency property only for the `H&R Shapley value` with symmetric form as following : A solution ψ on MC satisfies **consistency(CON)** if for all $(N, m, v) \in MC$ and all $S \subseteq N$,

$$\psi_{i,j}\Big(N, (m_S, 0_{S^c}), v^{\psi}_{S,m}\Big) = \psi_{i,j}(N, m, v) \text{ for all } i \in S \text{ and } j \in M_i^+.$$

Clearly, the reduced game defined by Hwang and Liao [8] is a special case of w-reduced game defined by Hsiao, Yeh and Mo [4]. Formally, given $(N, m, v) \in MC$, a solution ψ on MC and $S \subseteq N$. Let $z = (m_S, 0_{S^c})$, by definitions of v^{ψ}_z and $v^{\psi}_{S,m}$, we have that $v^{\psi}_z(y) = v^{\psi}_{S,m}(y)$ for all $y \leq z = (m_S, 0_{S^c})$. Hence, if a solution satisfies w-CON, then it satisfies CON.

Lemma 1. *Given $(N, m, v) \in MC$ and a weight function w, let $(N, z, v^{\gamma^w}_z)$ be the w-reduced game of (N, m, v) with respect to γ^w and $z \in M^N$. Obviously, z can be written by $z = (m_S, z_{S^c})$ for some $S \subseteq N$. If $v = \sum_{y \in M^N, y \neq 0_N} a^y(v) \cdot u^y_N$, then $v^{\gamma^w}_z$ can be expressed to be*

$$v^{\gamma^w}_z = \sum_{y \leq z, y \neq 0_N} a^y(v^{\gamma^w}_z) \cdot u^y_N,$$

where for all $y \leq z$,

$$a^y(v^{\gamma^w}_z) = \begin{cases} \displaystyle\sum_{t \leq m_{S^c}} \frac{\|(y_S, 0_{S^c})\|_w}{\|(y_S, 0_{S^c})\|_w + \|(t, 0_S)\|_w} \cdot a^{(y_S, t)}(v) & \text{if } y = (y_S, 0_{S^c}) \ , \\ 0 & \text{if } y \leq z \text{ with } |S(y_{S^c}, 0_S)| \neq 0. \end{cases}$$

Proof. Let $(N, m, v) \in MC$, a weight function w and $z \in M^N$. Obviously, z can be written by $z = (m_S, z_{S^c})$ for some $S \subseteq N$, where $z_i \neq m_i$ for all $i \in S^c$. Clearly, $z^* = (0_S, m_{S^c})$. For any $y \leq z$,

$$\begin{aligned}
v_z^{\gamma^w}(y) &= v(y \vee z^*) - \sum_{k \in S(z^*)} \gamma^w_{k,m_k}\Big(N, (y \vee z^*), v\Big) \\
&= v(y_S, m_{S^c}) - \sum_{k \in S^c} \gamma^w_{k,m_k}\Big(N, (y_S, m_{S^c}), v\Big) \\
&= \sum_{k \in S(y_S, 0_{S^c})} \gamma^w_{k,y_k}\Big(N, (y_S, m_{S^c}), v\Big) \\
&= \sum_{k \in S(y_S, 0_{S^c})} \sum_{l=1}^{y_k} \sum_{\substack{z \le (y_S, m_{S^c}) \\ z_k = l}} \frac{w(l)}{\|z\|_w} \cdot a^z(v) \\
&= \sum_{k \in S(y_S, 0_{S^c})} \sum_{\substack{z \le (y_S, m_{S^c}) \\ z_k = 1}} \frac{w(1)}{\|z\|_w} \cdot a^z(v) + \sum_{k \in S(y_S, 0_{S^c})} \sum_{\substack{z \le (y_S, m_{S^c}) \\ z_k = 2}} \frac{w(2)}{\|z\|_w} \cdot a^z(v) \\
&\quad + \cdots + \sum_{k \in S(y_S, 0_{S^c})} \sum_{\substack{z \le (y_S, m_{S^c}) \\ z_k = y_k}} \frac{w(y_k)}{\|z\|_w} \cdot a^z(v) \\
&= \sum_{k \in S(y_S, 0_{S^c})} \sum_{\substack{p \le y_S \\ p_k = 1}} \sum_{t \le m_S} \frac{w(1)}{\|(p, 0_{S^c})\|_w + \|(t, 0_S)\|_w} \cdot a^{(p,t)}(v) + \cdots \\
&\quad + \cdots + \sum_{k \in S(y_S, 0_{S^c})} \sum_{\substack{p \le y_S \\ p_k = y_k}} \sum_{t \le m_S} \frac{w(y_k)}{\|(p, 0_{S^c})\|_w + \|(t, 0_S)\|_w} \cdot a^{(p,t)}(v) \\
&= \sum_{k \in S(y_S, 0_{S^c})} \sum_{\substack{p \le y_S \\ p_k = 1}} \sum_{t \le m_S} \frac{w(1)}{\|(p, 0_{S^c})\|_w + \|(t, 0_S)\|_w} \cdot a^{(p,t)}(v) + \cdots \\
&\quad + \cdots + \sum_{k \in S(y_S, 0_{S^c})} \sum_{\substack{p < y_S \\ p_k = y_k}} \sum_{t \le m_S} \frac{w(y_k)}{\|(p, 0_{S^c})\|_w + \|(t, 0_S)\|_w} \cdot a^{(p,t)}(v) \\
&\quad + \sum_{t \le m_S} \frac{\|(y_S, 0_{S^c})\|_w}{\|(y_S, 0_{S^c})\|_w + \|(t, 0_S)\|_w} \cdot a^{(y_S,t)}(v).
\end{aligned}$$

Continuing in this way, we have that

$$(1^o) \qquad v_z^{\gamma^w}(y) = \sum_{x \le y} \sum_{t \le m_S} \frac{\|(x_S, 0_{S^c})\|_w}{\|(x_S, 0_{S^c})\|_w + \|(t, 0_S)\|_w} \cdot a^{(x_S,t)}(v).$$

By definition of $v_z^{\gamma^w}$, for any

$$(2^o) \qquad y \le z, v_z^{\gamma^w}(y) = v_z^{\gamma^w}(y_S, 0_{S^c}).$$

Set

$$\bar{a^y}(v) = \begin{cases} \displaystyle\sum_{t \le m_{S^c}} \frac{\|(y_S, 0_{S^c})\|_w}{\|(y_S, 0_{S^c})\|_w + \|(t, 0_S)\|_w} a^{(y_S,t)}(v) & \text{if } y = (y_S, 0_{S^c}) \ , \\ 0 & \text{if } y = (y_S, y_{S^c}) \text{ with } |S(y_{S^c})| \neq 0. \end{cases}$$

By (1^o) and (2^o), $v_z^{\gamma^w} = \sum\limits_{\substack{y \neq 0_N \\ y \leq z}} \bar{a^y}(v) \cdot u_N^y = \sum\limits_{\substack{y \leq z, y \neq 0_N \\ |S(y_s, 0_{s^c})| = 0}} \bar{a^y}(v) \cdot u_N^y$.

Let $a^y(v_z^{\gamma^w}) = \bar{a^y}(v)$. We have that $v_z^{\gamma^w}$ can be expressed to be $v_z^{\gamma^w} = \sum\limits_{\substack{y \leq z \\ y \neq 0_N}} a^y(v_z^{\gamma^w}) \cdot u_N^y$ where for all $y \leq z$,

$$a^y(v_z^{\gamma^w}) = \begin{cases} \sum\limits_{t \leq m_{S^c}} \dfrac{\|(y_S, 0_{S^c})\|_w}{\|(y_S, 0_{S^c})\|_w + \|(t, 0_S)\|_w} a^{(y_S, t)}(v) & \text{if } y = (y_S, 0_{S^c}) \ , \\ 0 & \text{if } y \leq z \text{ with } |S(y_{S^c}, 0_S)| \neq 0. \end{cases}$$

Here, we complete the following theorem stated in [4]. ■

Theorem 2. *The solution γ^w is w-consistent.*

Proof. Let $(N, m, v) \in MC$, a weight function w. Obviously, z can be written by $z = (m_S, z_{S^c})$ where $z_i \neq m_i$ for all $i \in S^c$. By Lemma 1, for all $i \in N \backslash S(z^*) = S$ and for all $j \leq z_i$,

$$\begin{aligned} \gamma_{i,j}^w(N, z, v_z^{\gamma^w}) &= \sum_{k=1}^{j} \sum_{\substack{x \leq z \\ x_i = k}} \frac{w(k)}{\|x\|_w} \cdot a^x(v_z^{\gamma^w}) \\ &= \sum_{k=1}^{j} \sum_{\substack{x \leq z, x_i = k \\ |S(x_{S^c}, 0_S)| = 0}} \frac{w(k)}{\|(x_S, 0_{S^c})\|_w} \cdot a^{(x_S, 0_{S^c})}(v_z^{\gamma^w}) \\ &= \sum_{k=1}^{j} \sum_{\substack{x \leq z, x_i = k \\ |S(x_{S^c}, 0_S)| = 0}} \frac{w(k)}{\|(x_S, 0_{S^c})\|_w} \\ &\qquad \sum_{t \leq m_{S^c}} \frac{\|(x_S, 0_{S^c})\|_w}{\|(x_S, 0_{S^c})\|_w + \|(0_S, t)\|_w} \cdot a^{(x_S, t)}(v) \\ &= \sum_{k=1}^{j} \sum_{\substack{y \leq m \\ y_i = k}} \frac{w(k)}{\|y\|_w} \cdot a^y(v) \\ &= \gamma_{i,j}^w(N, m, v). \end{aligned}$$

■

5. Characterization

In this section, we will prove the main result of this article, say, Theorem 3. We provide an axiomatization which is the parallel of Hart and Mas-Colell axiomatization of the Shapley value by applying consistency. To state the axiomatization, some more definitions will be needed. Let ψ be a solution on MC and w be a weight function.

• **Efficiency (EFF)**: For all $(N, m, v) \in MC$, $\sum_{i \in S(m)} \psi^w_{i,m_i}(N, m, v) = v(m)$.

• **Weak efficiency (WEFF)**: For all $(N, m, v) \in MC$ with $S(m) = \{i\}$ for some $i \in N$, ψ^w satisfies EFF.

• **Standard for two-person game (ST)**: For all $(N, m, v) \in MC$ with $|S(m)| \leq 2$, $\psi^w = \gamma^w$.

• **Independence of individual expansions (IIE)**[1]: For all $(N, m, v) \in MC$ and all $(i, j) \in L^{N,m}$,

$$\psi^w_{i,j}\Big(N, (m_{-i}, j), v\Big) = \psi^w_{i,j}\Big(N, (m_{-i}, j+1), v\Big) = \cdots = \psi^w_{i,j}(N, m, v).$$

• **Weak independence of individual expansions (WIIE)**: For all $(N, m, v) \in MC$ with $S(m) = \{i\}$ for some $i \in N$ and all $(i, j) \in L^{N,m}$, ψ^w satisfies IIE.

Remark 7. Given a weight function w, if a solution ψ^w satisfies w-CON and ST, then $\psi^w = \gamma^w$ for all $(N, m, v) \in MC$ with $|S(m)| = 1$. The proof is similar to the TU-case by adding a "dummy" player to one-person problem, this is left to the readers. Hence, if ψ^w satisfies w-CON and ST, it satisfies WEFF and WIIE.

Lemma 2. *Given a weight function w, if a solution ψ^w satisfies ST and w-CON, then it satisfies EFF and IIE.*

Proof. Suppose that ψ^w satisfies ST and w-CON, by Remark 7, ψ^w satisfies WEFF and WIIE. Hwang and Liao [8] proved that if a solution satisfies WEFF, WIIE and CON, then it satisfies EFF and IIE. By Remark 6, the proof is completed. ■

Here, Liao prove the following theorem stated in [7].

Theorem 3. *Given a weight function w, a solution ψ^w satisfies ST and w-CON if and only if $\psi^w = \gamma^w$.*

Proof. Given a weight function w, by definition of ST, γ^w satisfies ST. By Theorem 2, γ^w satisfies w-CON.

To prove uniqueness, suppose that the solution ψ^w on MC satisfies ST and w-CON. By Lemma 2, ψ^w satisfies EFF and IIE. Given $(N, m, v) \in MC$. The proof proceeds by induction on the number $\|m\|$. Assume that $\|m\| = 1$ and $S(m) = \{i\}$. By EFF of ψ^w and γ^w, $\psi^w_{i,1}(N, m, v) = v(m) = \gamma^w_{i,1}(N, m, v)$. Assume that $\psi^w(N, m, v) = \gamma^w(N, m, v)$ if $\|m\| \leq l - 1$, where $l \geq 2$.

---

[1] This axiom was proposed by Hwang and Liao [8]. They characterized the $H\&R$ Shapley value with symmetric form by apply efficiency, balanced contributions and IIE.

The case $\|m\| = l$: First we show that $\psi^w_{i,m_i}(N,m,v) = \gamma^w_{i,m_i}(N,m,v)$ for all $i \in S(m)$. Two cases may be distinguish :

Case 1. Assume that $|S(m)| \leq 2$. By ST of ψ^w, $\psi^w_{i,m_i}(N,m,v) = \gamma^w_{i,m_i}(N,m,v)$ for all $i \in S(m)$.

Case 2. Assume that $|S(m)| \geq 3$. Let $i,k \in S(m)$ and $z = (m_i, m_k, 0_{-ik})$. By induction hypotheses and w-CON of ψ^w and γ^w,

$$\begin{aligned} & \psi^w_{i,m_i}(N,m,v) - \gamma^w_{i,m_i}(N,m,v) \\ \text{(a)} \qquad = \ & \psi^w_{i,m_i}(N,z,v_z^{\psi^w}) - \gamma^w_{i,m_i}(N,z,v_z^{\gamma^w}) \\ = \ & \gamma^w_{i,m_i}(N,z,v_z^{\psi^w}) - \gamma^w_{i,m_i}(N,z,v_z^{\gamma^w}) \end{aligned}$$

By ST of γ^w,

$$\text{(b)} \qquad \begin{aligned} (a) = & \sum_{t=1}^{m_i} \Big[\sum_{l \in M_k} \Big(\big(\frac{w(t)}{w(t)+w(l)}\big) \cdot a^{(t,l,0_{-ik})}(v_z^{\psi^w}) \Big) \Big] \\ & - \sum_{t=1}^{m_i} \Big[\sum_{l \in M_k} \Big(\big(\frac{w(t)}{w(t)+w(l)}\big) \cdot a^{(t,l,0_{-ik})}(v_z^{\gamma^w}) \Big) \Big]. \end{aligned}$$

By induction hypotheses and definitions of w-reduced game, we obtain that $v_z^{\psi^w}$ and $v_z^{\gamma^w}$ may differ only by z. So we have that

$$\begin{aligned} (b) = & \big(\frac{w(m_i)}{w(m_i)+w(m_k)}\big) \cdot \Big[v_z^{\psi^w}(z) - v_z^{\gamma^w}(z) \Big] \\ = & \big(\frac{w(m_i)}{w(m_i)+w(m_k)}\big) \cdot \Big[\big(\psi^w_{i,m_i}(N,m,v) - \gamma^w_{i,m_i}(N,m,v)\big) \\ & + \big(\psi^w_{k,m_k}(N,m,v) - \gamma^w_{k,m_k}(N,m,v)\big) \Big]. \end{aligned}$$

If $\frac{w(m_i)}{w(m_i)+w(m_k)} = 1$, then $\big(\psi^w_{k,m_k}(N,m,v) - \gamma^w_{k,m_k}(N,m,v)\big) = 0$ for all $k \neq i$. That is implying $\psi^w_{k,m_k}(N,m,v) = \gamma^w_{k,m_k}(N,m,v)$ for all $k \neq i$. By EFF of ψ^w and γ^w, $\psi^w_{i,m_i}(N,m,v) = \gamma^w_{i,m_i}(N,m,v)$. If $\frac{w(m_i)}{w(m_i)+w(m_k)} \neq 1$, then

$$\begin{aligned} & w(m_k) \cdot \big(\psi^w_{i,m_i}(N,m,v) - \gamma^w_{i,m_i}(N,m,v)\big) \\ & = w(m_i) \cdot \big(\psi^w_{k,m_k}(N,m,v) - \gamma^w_{k,m_k}(N,m,v)\big). \end{aligned}$$

By EFF of ψ^w and γ^w,

$$\|m\|_w \cdot \Big[\psi^w_{i,m_i}(N,m,v) - \gamma^w_{i,m_i}(N,m,v) \Big] = w(m_i) \cdot \Big[v(m) - v(m) \Big] = 0.$$

That is $\psi^w_{i,m_i}(N,m,v)=\gamma^w_{i,m_i}(N,m,v)$.

By Cases 1 and 2, $\psi^w_{i,m_i}(N,m,v)=\gamma^w_{i,m_i}(N,m,v)$ for all $i\in S(m)$.
It remains to show that $\psi^w_{i,j}(N,m,v)=\gamma^w_{i,j}(N,m,v)$ for all $(i,j)\in L^{N,m}$ with $j\neq m_i$. Since ψ^w satisfies IIE, for all $(i,j)\in L^{N,m}$ with $j\neq m_i$, by induction hypotheses and IIE of ψ^w and γ^w,

$$\psi^w_{i,j}(N,m,v)=\psi^w_{i,j}\Big(N,(m_{-i},j),v\Big)=\gamma^w_{i,j}\Big(N,(m_{-i},j),v\Big)=\gamma^w_{i,j}(N,m,v).$$

Hence $\psi^w_{i,j}(N,m,v)=\gamma^w_{i,j}(N,m,v)$ for all $(i,j)\in L^{N,m}$ with $j\neq m_i$. ∎

The following examples show that each of the axioms used in Theorem 2 is logically independent of the remaining axioms.

Example 1. Given a weight function w, define a solution ψ^w on MC by for all $(N,m,v)\in MC$ and for all $(i,j)\in L^{N,m}$,

$$\psi^w_{i,j}(N,m,v)=0.$$

It's easy to verify that ψ^w satisfies w-CON, but it violates ST.

Example 2. Given a weight function w, define a solution ψ^w on MC by for all $(N,m,v)\in MC$ and for all $(i,j)\in L^{N,m}$,

$$\psi^w_{i,j}(N,m,v)=\begin{cases}\gamma^w_{i,j}(N,m,v) & \text{if } |S(m)|\leq 2\\ \gamma^w_{i,j}(N,m,v)-\varepsilon & \text{otherwise .}\end{cases}$$

where $\varepsilon\in\mathbb{R}\setminus\{0\}$.
It's easy to verify that ψ^w satisfies ST, but it violates w-CON.

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